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# Regularity of minimal intrinsic graphs in 3-dimensional sub-Riemannian structures of step 2

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## Abstract

This work provides a characterization of the regularity of noncharacteristic intrinsic minimal graphs for a class of vector fields that includes non-nilpotent Lie algebras as the one given by Euclidean motions of the plane. The main result extends a previous one on the Heisenberg group, using similar techniques to deal with nonlinearities. This wider setting provides a better understanding of geometric constraints, together with an extension of the potentialities of specific tools as the lifting–freezing procedure and interpolation inequalities. As a consequence of the regularity, a foliation result for minimal graphs is obtained.

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## Résumé

Cet article donne une caractérisation de la régularité des graphes intrinsèques minimaux pour une classe de champs de vecteurs qui comprend les algèbres de Lie non nilpotentes comme celles obtenues dans le groupe des mouvements rigides du plan Euclidien. Le résultat essentiel étend un précédent résultat obtenu pour le groupe d'Heisenberg, en utilisant des techniques similaires pour traiter les non linéarités. Dans ce cadre plus général, on peut mieux comprendre les contraintes géométriques et les potentialités des outils spécifiques comme la procédure de « lifting–freezing » et les inégalités d'interpolation. A la fin un résultat de feuilletage pour les graphes minimaux est obtenu comme conséquence de la régularité.

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## 1. Introduction

We prove a regularity result for noncharacteristic intrinsic minimal graphs in a 3-dimensional sub-Riemannian contact structure. Such a sub-Riemannian structure is defined by the choice at every point of  $\mathbb{R}^3$  of a couple of vector fields  $X_1, X_3$ , which, together with their first order commutator

$$X_2 = [X_1, X_3] \tag{1.1}$$

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span the tangent space at any point. This condition is a special case of the well-known Hörmander type condition [29] and ensures that it is possible to endow the space with a sub-Riemannian metric (see [3]). However we will see that low-dimensional situation is particularly interesting for our problem, since minimal graphs in 3-dimensional Lie group satisfy a foliation result, which is not present in higher dimension.

As it is usual, we will call horizontal tangent bundle the bundle  $\mathbb{H}$  spanned at every point by  $X_1$  and  $X_3$ , and we will define on it a metric  $g_0$ , by requiring that  $\{X_1, X_3\}$  form a  $g_0$ -orthonormal basis of  $\mathbb{H}$ . For each  $\epsilon > 0$  we extend  $g_0$  to a Riemannian metric  $g_\epsilon$  by requiring that  $\{X_1, X_3, X_2\}$  form a  $g_\epsilon$ -orthonormal basis of  $\mathbb{H}$ .

The notion of regular surface in the sub-Riemannian setting has been introduced by Franchi, Serapioni and Serra Cassano, who (in [21,22]) extended to this context the celebrated blow up technique introduced by De Giorgi. They defined a regular surface  $M$  as the zero level set of a function  $f$ , whose intrinsic gradient  $(X_1 f, X_3 f)$  never vanishes. The horizontal tangent space to  $M$ , denoted by  $H\mathcal{M}$ , is the fiber bundle whose fibers  $H_p\mathcal{M}$  at any point  $p$  are given by the intersection between the Euclidean tangent space  $\mathcal{T}_p\mathcal{M}$  and the horizontal tangent space  $\mathbb{H}_p$ . The assumption of nonvanishing gradient on  $f$  ensures that  $H_p\mathcal{M}$  is spanned at every point by a single vector field, called horizontal tangent vector to  $M$  at the point  $p$ . Hence a surface of this type is not regular in the standard Euclidean sense and in general it will be a fractal (see [31]). In particular, the standard implicit function result does not hold in this case. However, up to a suitable change of coordinates, we can assume that the vector fields  $X_i$  are represented as

$$X_3 = \partial_{x_3}, \quad X_i = \sigma_i^j(x_1, x_2, x_3)\partial_{x_j}, \quad i = 1, 2,$$

and in these coordinates the surface  $M$  becomes an intrinsic graph of the form

$$\mathcal{M} = \{x_3 = u(x_1, x_2); (x_1, x_2) \in \Omega \subset \mathbb{R}^2\}. \tag{1.2}$$

It has been proved in [1] and [15] that the function  $u$  is differentiable along the vector field  $X_{1,u}$ , projection on  $\mathcal{T}_\Omega$  of the horizontal tangent vector to  $M$ :

$$X_{1,u} = X_1|_{\mathcal{M}} = \sigma_1^1(x_1, x_2, u(x_1, x_2))\partial_{x_1} + \sigma_1^2(x_1, x_2, u(x_1, x_2))\partial_{x_2}.$$

Besides, if the Lie derivative  $X_{1,u}u$  is continuous, the function  $u$  is called of class  $\mathcal{C}_u^1$ . In the particular case of the Heisenberg group a systematic study of properties of spaces  $\mathcal{C}_u^1$  has been carried out in [23,24,6,4].

### 1.1. Minimal surfaces

The first definition of minimal surfaces in this context is due to Garofalo and Nhieu [25], as first variation of the perimeter functional. Pauls in [35], studied the same notion as the elliptic regularization of the sub-Riemannian metric. A different, but equivalent notion was given by Cheng, Hwang, Malchiodi and Yang [11] in general CR manifolds. In particular the specific case of three-dimensional pseudo-hermitian manifolds has been outlined in [32]. Later, Danielli, Garofalo and Nhieu in [19] extended these concepts with a general notion of horizontal mean curvature in sub-Riemannian setting. Ritoré and Rosales, in [37] and [38], use a different approach in order to introduce a notion of constant mean curvature under a volume constraint. We also refer to [33,40] for different notions of first variation of the perimeter, and to [10] for a survey of the problem.

The condition of zero horizontal mean curvature can be expressed as follows in terms of the previously introduced vector field:

$$X_{1,u} \left( \frac{X_{1,u}u}{\sqrt{1 + |X_{1,u}u|^2}} \right) = 0 \tag{1.3}$$

(see [7,2,20]). Indeed Eq. (1.3) is the Euler–Lagrange equation for the sub-Riemannian area functional

$$\text{Lip}(\Omega) \ni u \mapsto P_{X_1, X_3}(E_u, \Omega \times \mathbb{R}) = \int_{\Omega} \sqrt{1 + |X_{1,u}u|^2} dx_1 dx_2,$$

where  $E_u \doteq \{(x_1, x_2, x_3) \in \Omega \times \mathbb{R} : x_3 < u(x_1, x_2); (x_1, x_2) \in \Omega\}$  and  $P_{X_1, X_3}(E_u, \Omega \times \mathbb{R})$  denotes the sub-Riemannian perimeter measure of  $E_u$  in  $\Omega \times \mathbb{R}$  associated to the system of vector fields  $X_1$  and  $X_3 \equiv \partial_{x_3}$ , according to the definition introduced in [25].

It is also well known (at least in the examples of the Heisenberg group and the roto-translation group, see [11,18, 27]) that for regular curves the mean curvature coincides with the curvature of the projection of the Legendrian leaves

on the horizontal plane. However the same result is known for Lipschitz continuous solution only in the Heisenberg case.

On the other side weaker definitions of minimal surfaces have been proposed (see [1,11–13,25,34,35]). Explicit examples of nonregular surfaces (in the Euclidean sense) have been provided by [34–36] and in [13]. Hence the problem of regularity is particularly interesting, and by now only a few results are known in the Heisenberg group [7,6].

Here we plan to study regularity of Lipschitz continuous solutions in general 3D structures. Of course we try to adapt to this situation the typical instruments of nonlinear PDEs. The celebrated works of Hörmander [29] and Rothschild and Stein [39] started the study of linear operators defined in terms of vector fields, satisfying the bracket generating condition. We refer to the monographs [5] and [41] for an exhaustive presentation of the theory, which however, cannot be applied here for two main reasons. The first one is that our vector fields are nonlinear, since they explicitly depend on the solution. To our knowledge the only known results regarding Hörmander type nonlinear vector fields have been obtained in [8,42], while studying curvature equations in a completely different context. However not even these more delicate results can be applied to our situation, since we are dealing with only one vector field in a two-dimensional space. Clearly one vector field cannot generate the whole tangent space at any point, hence the Hörmander condition is violated at every point and because of this situation we can obtain a foliation result.

In order to prove regularity, we will follow the approach introduced to handle the Levi curvature equation in [16] or the  $\mathbb{H}^1$  curvature equation in [7]. We introduce an elliptic regularization

$$L_{\epsilon,u}u = X_{i,u} \left( \frac{X_{i,u}u}{\sqrt{1 + |\nabla_{\epsilon}u|^2}} \right) = 0, \tag{1.4}$$

where summation over repeated indices is intended. The regularization is performed adding the transverse direction scaled by a small parameter, so that we denote

$$X_{2,u} = \epsilon [\sigma_2^1(x_1, x_2, u(x_1, x_2))\partial_{x_1} + \sigma_2^2(x_1, x_2, u(x_1, x_2))\partial_{x_2}] = \epsilon X_2|_{\mathcal{M}},$$

and  $\nabla_{\epsilon}$  stands for the gradient  $(X_{1,u}, X_{2,u})$ . The class of minimal graphs under study is then that of vanishing viscosity solutions:

**Definition 1.1.** A function  $u \in \text{Lip}(\Omega)$  is said to be a vanishing viscosity solution of (1.3) if there exists a sequence  $(\epsilon_n)$  in  $\mathbb{R}^+$  with  $\lim_{n \rightarrow \infty} \epsilon_n = 0$  and a sequence  $(u_n)$  in  $C^\infty$  such that

1.  $L_{\epsilon_n, u_n}u_n = 0$  in  $\Omega$  for all  $n$ ;
2. the sequence  $(u_n)$  is uniformly bounded in  $\text{Lip}(\Omega)$ , i.e. there exists an  $l > 0$  such that  $\|u_n\|_{\text{Lip}(\Omega)} \leq l$ , and converges to  $u$  uniformly on compact subsets of  $\Omega$ .

We note that this approach will allow us, for all positive values of  $\epsilon$ , to work with smooth solutions, due to the uniform ellipticity of (1.4). This point will be of key importance since it permits to skip passing through density of smooth functions, which is not guaranteed in this general setting.

### 1.2. Main results

Throughout all the paper we will assume the solution  $u$  to be Euclidean Lipschitz, i.e. we set

$$Y_u = X_2|_{\mathcal{M}} = \frac{1}{\epsilon} X_{2,u},$$

and assume that

$$\|X_{1,u}u\|_{L^\infty(\Omega)} + \|Y_uu\|_{L^\infty(\Omega)} \leq M < \infty. \tag{1.5}$$

In order to adopt the previously described approach, and hence work with smooth approximating functions, we will also require that the coefficients  $\sigma_i^j$  be smooth, more precisely we will intend them as  $C^\infty$  functions.

Under this assumption we will prove the following result.

**Theorem 1.2.** Let  $u$  be a vanishing viscosity solution of Eq. (1.3). Then for all  $k > 0$  and all  $\alpha < 1$  it holds

$$X_{1,u}^k u \in C_{\text{loc}}^\alpha(\Omega). \tag{1.6}$$

The main regularity result, expressed by Theorem 1.2, provides  $C^\infty$  regularity along the horizontal direction. This in particular implies that Eq. (1.3) can be represented simply as  $X_{1,u}^2 u = 0$  pointwise everywhere in  $\Omega$ . The geometric meaning is expressed by Corollary 1.3, which ensure that minimal graphs are foliated into horizontal curves, along which they are smooth.

**Corollary 1.3.** *Let  $x_3 = u(x_1, x_2)$ ,  $(x_1, x_2) \in \Omega$  be a vanishing viscosity minimal graph. Then the flow of the vector field  $X_{1,u}$  yields a foliation of the domain  $\Omega$  by horizontal curves  $\gamma$ . More precisely, let  $(x_1, x_2) \in \Omega$  and denote by  $\gamma$  the integral curve of  $X_{1,u}$  passing through it. Then  $u$  is differentiable at  $(x_1, x_2)$  along  $\gamma$  and Eq. (1.3) reduces to*

$$\frac{d^2}{dt^2} u(\gamma(t)) = 0.$$

We remark that no regularity can be achieved in the transverse direction, since this does not play a role in the intrinsic curvature equation.

This result then extends to a very general setting results only known in the special case of the Heisenberg group. We explicitly note that the largest majority of the results known for intrinsic graphs only apply to nilpotent or homogeneous groups, while here this condition is not required. In this wider setting only the definition of regular surface is known. Adapting the proof in this contest is not purely an extension, but also provides a better understanding of the role of geometric constraints in the present approach to surfaces regularity, which in the end turns out not to rely on any nilpotency assumption.

We explicitly note that results of this type are not expected in higher dimension. In the specific case of the Heisenberg group, it is indeed known that 2-dimensional minimal surfaces foliates in regular curves, and are not regular as surfaces. In higher dimension the surfaces are  $C^\infty$  in the Euclidean sense and there is no foliation result.

### 1.3. Sketch of the proof

We first introduce Sobolev spaces associated to the vector fields  $X_{i,u}$ : we say that a function  $z$  belongs to the Sobolev space  $W_\epsilon^{1,p}$  if

$$z, \nabla_\epsilon z \in L^p,$$

and in this case we set

$$\|z\|_{W_\epsilon^{1,p}} = \|z\|_{L^p} + \|\nabla_\epsilon z\|_{L^p}.$$

Higher order Sobolev spaces are defined in a similar way. We explicitly note that this definition contains a linearization, since we now apply the vector fields  $X_{i,u}$  to any function  $z$ , not necessarily to the function  $u$ .

Accordingly, in Section 3 we study the regularity properties of solutions for such a linearization of Eq. (1.4). This is done applying a Moser-like iteration procedure, and to this end we have worked with the linear operator (with nonregular coefficients) that defines the equation satisfied by the derivatives of the solution to the linearized equation, namely

$$M_{\epsilon,u} z = X_{i,u} \left( \frac{a_{ij}(\nabla_\epsilon u)}{\sqrt{1 + |\nabla_\epsilon u|^2}} X_{j,u} z \right) = f$$

where the coefficients  $a_{ij}$  are regular and uniformly elliptic (see also definition (3.2)).

The Moser iteration technique has been extended for equation expressed in terms of vector fields, which satisfy Sobolev embedding theorem. This theorem is in general true for Hörmander vector fields, with regular or nonregular coefficients (see e.g. [30,9,26] and references therein). However we cannot hope to prove a Sobolev inequality uniform in  $\epsilon$ , since the equation does not naturally involve derivatives in the transverse direction: indeed the intrinsic geometry of these minimal graphs will induce a Legendrian foliation, so one cannot expect to gain Sobolev regularity when passing from one fiber of the foliation to the other. Hence following [14] and [7] we replace these type of inequalities with an interpolation inequality, Proposition 3.6. Another crucial ingredient in the Moser method is a Caccioppoli type inequality. This is then obtained in Lemma 3.8. Combining these two instruments with a bootstrap argument, we obtain estimates for a solution of the linear equation in terms of the coefficients of the equation itself in Theorem 3.1.

Unfortunately the regularity for the coefficients required by the cited theorem is not available under assumption (1.5). For this reason we prove in Section 4 a rather delicate Sobolev type estimate to start the iteration. This estimate will not be optimal, since for the previously recalled reason we cannot hope to obtain an intrinsic Sobolev inequality with optimal exponent. Moreover, we will need assumptions both on the intrinsic derivatives  $X_{1,u}u$  and on the transverse derivatives in direction  $Y_u$ . The main idea is to approximate, through a freezing and a lifting procedure, the vector fields  $X_{i,u}$  with Hörmander type vector fields of step three. We will choose the approximating vector fields in such a way that they do not span the space at step 2, so that the second derivatives along the approximating operator provide a good approximation of the corresponding derivatives along the vectors  $X_{i,u}$ . On the other side, since the approximating vector fields satisfy the Hörmander condition, we will be able to establish the required Sobolev inequality. This will be done in Theorem 4.1. We explicitly note that this freezing and lifting method is only vaguely reminiscent of the celebrated method of Rothschild and Stein [39]. Indeed they approximate Hörmander operators with nilpotent operators with the same bracket generating property. Here on the contrary the novelty of the idea is to adapt this method to non-Hörmander vectors fields.

Finally in Section 5 we obtain the regularity in the Sobolev spaces for viscosity solutions of the nonlinear equation (1.3), concluding the proof of Theorem 1.2.

## 2. Notations and technical facts

### 2.1. Significant examples of such structures

This setting include all three-dimensional simply connected Lie group  $\mathcal{G}$  whose Lie algebra  $\mathfrak{g}$  can be generated by two vector fields. The simplest noncommutative Lie algebra with two generator is the three-dimensional Heisenberg group, whose left invariant vector fields over  $\mathbb{R}^3$  read

$$X_1 = \partial_{x_1} + x_3 \partial_{x_2}, \quad X_2 = \partial_{x_2}, \quad X_3 = \partial_{x_3}.$$

A relevant example of non-nilpotent Lie group is given by the group  $E(2)$  of translations and rotations of the Euclidean plane. In this case the vectors  $X_1$  and  $X_2$  express translations in two orthogonal directions of  $\mathbb{R}^2$  while  $X_3$  describes a rotation. Hence they can be expressed as vector fields over  $\mathbb{R}^2_{x_1,x_2} \times S^1_{x_3}$  with components

$$X_1 = \cos(x_3)\partial_{x_1} + \sin(x_3)\partial_{x_2}, \quad X_2 = -\sin(x_3)\partial_{x_1} + \cos(x_3)\partial_{x_2}, \quad X_3 = \partial_{x_3}.$$

And the bracket relations are the following:

$$[X_1, X_3] = -X_2, \quad [X_1, X_2] = 0, \quad [X_2, X_3] = X_1.$$

This case is particularly important since the related minimal surfaces provide a concrete model for the geometric completion operated by the primary visual cortex, as introduced in [18]. Indeed an interesting perspective on minimal surfaces in this setting was recently obtained in [28].

Other notable Lie group included in the present investigation are  $SO(3)$ , or  $SL(2)$ , but we remark that the group structure is not an essential point in defining the setting.

### 2.2. Linear vector fields in the sub-Riemannian space

Let us first recall some properties of the linear vector fields  $X_i$ , defined in (1.1) on  $R^3$ . In order to obtain the asserted representation of the vector field  $X_3$  as a partial derivative, we need to introduce polarized exponential coordinates.

Indeed to any pair of points  $(x, y, z), (\bar{x}, \bar{y}, \bar{z}) \in \mathbb{R}^3$  sufficiently close we can associate coordinates  $(x_1, x_2, x_3) \in \mathbb{R}^3$  such that

$$(x, y, z) = \exp(x_3 X_3) \exp(x_1 X_1 + x_2 X_2)(\bar{x}, \bar{y}, \bar{z}). \tag{2.1}$$

Relation (2.1) simply means that if we take two curves  $\gamma_1, \gamma_2$  in  $\mathbb{R}^3$  such that

$$\gamma_1(0) = (\bar{x}, \bar{y}, \bar{z}), \quad \dot{\gamma}_1(t) = x_1 X_1(\gamma_1(t)) + x_2 X_2(\gamma_1(t)),$$

and

$$\gamma_2(0) = \gamma_1(1), \quad \dot{\gamma}_2(t) = x_3 X_3(\gamma_2(t))$$

then, due to the span property of the vector fields, we can always choose  $(x_1, x_2, x_3)$  such that  $(x, y, z) = \gamma_2(1)$ .

In these coordinates, by a structure result given by [15, Lemma 3.1], we have

**Remark 2.1.** The vector field  $X_3$  is expressed in the new coordinates as

$$X_3 = \partial_{x_3}$$

while for  $X_1, X_2$  we can write

$$\begin{aligned} X_1 &= \sigma_1^1(x_1, x_2, x_3)\partial_{x_1} + \sigma_1^2(x_1, x_2, x_3)\partial_{x_2} + \sigma_1^3(x_1, x_2, x_3)\partial_{x_3}, \\ X_2 &= [X_3, X_1] = \sigma_2^1(x_1, x_2, x_3)\partial_{x_1} + \sigma_2^2(x_1, x_2, x_3)\partial_{x_2} + \sigma_2^3(x_1, x_2, x_3)\partial_{x_3} \end{aligned}$$

where condition (1.1) sets the  $\sigma_2^i$ 's to  $\sigma_2^i(x_1, x_2, x_3) = \partial_{x_3}\sigma_1^i(x_1, x_2, x_3)$ .

The span condition for the vector fields can be written as

$$\text{rank} \begin{pmatrix} \sigma_1^1 & \sigma_1^2 \\ \partial_{x_3}\sigma_1^1 & \partial_{x_3}\sigma_1^2 \end{pmatrix} = 2,$$

i.e. we assume that

$$|\sigma_1^1(x_1, x_2, x_3)\partial_{x_3}\sigma_1^2(x_1, x_2, x_3) - \sigma_1^2(x_1, x_2, x_3)\partial_{x_3}\sigma_1^1(x_1, x_2, x_3)| > 0 \tag{2.2}$$

for all  $(x_1, x_2, x_3) \in \mathcal{G}$ .

We remark that we do not assume nilpotency, so that

$$\begin{aligned} [X_2, X_3] &= c_{2,3}^1 X_1 + c_{2,3}^2 X_2 + c_{2,3}^3 X_3, \\ [X_1, X_2] &= c_{1,2}^1 X_1 + c_{1,2}^2 X_2 + c_{1,2}^3 X_3 \end{aligned}$$

where the structure coefficients  $c_{j,k}^i(x_1, x_2, x_3)$  can well be nonzero.

### 2.3. Projected vector fields in $\mathbb{R}^2$

For what concerns the nonlinear vector fields  $X_{1,u}, X_{2,u}$ , their formal adjoints are given by

$$X_{i,u}^\dagger = -X_{i,u} - m_i(x_1, x_2), \tag{2.3}$$

where

$$\begin{aligned} m_1(x_1, x_2) &= \partial_{x_1}\sigma_1^1(x_1, x_2, u(x_1, x_2)) + \partial_{x_2}\sigma_1^2(x_1, x_2, u(x_1, x_2)) \\ &= \partial_i\sigma_1^i(x_1, x_2, u(x_1, x_2)) + \partial_3\sigma_1^i(x_1, x_2, u(x_1, x_2))\partial_{x_i}u(x_1, x_2) \\ &= \partial_i\sigma_1^i + [X_3, X_1]|_{\mathcal{M}u} = \partial_i\sigma_1^i + \frac{1}{\epsilon}X_{2,u}u, \\ m_2(x_1, x_2) &= \epsilon(\partial_{x_1}\sigma_2^1(x_1, x_2, u(x_1, x_2)) + \partial_{x_2}\sigma_2^2(x_1, x_2, u(x_1, x_2))) \\ &= \epsilon(\partial_i\sigma_2^i(x_1, x_2, u(x_1, x_2)) + \partial_3\sigma_2^j(x_1, x_2, u(x_1, x_2))\partial_{x_i}u(x_1, x_2)) \\ &= \epsilon(\partial_i\sigma_2^i + [X_3, X_2]|_{\mathcal{M}u}) = \epsilon\partial_i\sigma_2^i - \epsilon c_{2,3}^1 X_{1,u}u - c_{2,3}^2 X_{2,u}u, \end{aligned}$$

where  $c_{jk}^i$  stands now and in what follows for  $c_{jk}^i(x_1, x_2, u(x_1, x_2))$  and we note that, in agreement with the choice of  $X_{2,u} = \epsilon X_2|_{\mathcal{M}}$ , only  $m_2$  is of global order  $\epsilon$ .

Moreover, their commutator can be written explicitly:

$$\begin{aligned} [X_{1,u}, X_{2,u}] &= \epsilon[\sigma_1^j\partial_{x_j}, \sigma_2^k\partial_{x_k}] = \epsilon(\sigma_1^j(\partial_{x_j}\sigma_2^k)\partial_{x_k} - \sigma_2^k(\partial_{x_k}\sigma_1^j)\partial_{x_j}) \\ &= \epsilon(\sigma_1^j(\partial_j\sigma_2^k)\partial_{x_k} - \sigma_2^k(\partial_k\sigma_1^j)\partial_{x_j}) + \epsilon\sigma_1^j(\partial_3\sigma_2^k\partial_{x_j}u)\partial_{x_k} - \epsilon\sigma_2^k(\partial_3\sigma_1^j\partial_{x_k}u)\partial_{x_j} \\ &= \epsilon[X_1, X_2]|_{\mathcal{M}} + \epsilon(X_{1,u}u)[X_3, X_2]|_{\mathcal{M}} - (X_{2,u}u)X_2|_{\mathcal{M}} \\ &= \omega^j(x_1, x_2)X_{j,u}, \end{aligned}$$

where

$$\begin{aligned} \omega^1(x_1, x_2) &= \epsilon(c_{1,2}^1 - (X_{1,u}u)c_{2,3}^1), \\ \omega^2(x_1, x_2) &= c_{1,2}^2 - (X_{1,u}u)c_{2,3}^2 - \frac{1}{\epsilon}(X_{2,u}u). \end{aligned}$$

By  $\omega_{j,k}^i$  we will indicate the corresponding commutator coefficients, antisymmetric with respect to lower indices, i.e.  $\omega_{j,k}^i = -\omega_{k,j}^i$  and

$$\omega_{j,k}^i(x_1, x_2) = \begin{cases} 0 & \text{if } j = k, \\ \omega^i(x_1, x_2) & \text{if } j = 1, k = 2 \end{cases} \tag{2.4}$$

so that in general we can write

$$[X_{j,u}, X_{k,u}] = \omega_{j,k}^i X_{i,u}$$

and we note that this commutator is of global order  $\epsilon$ , in agreement with the choice of scaling  $X_{2,u}$  with  $\epsilon$ .

This implies in particular that, under hypothesis (1.5), there exists a finite positive constant  $C_M$  depending only on  $M$  and  $\Omega$  such that

$$\|m_1\|_{L^\infty(\Omega)} + \frac{1}{\epsilon}\|m_2\|_{L^\infty(\Omega)} + \frac{1}{\epsilon}\|\omega^1\|_{L^\infty(\Omega)} + \|\omega^2\|_{L^\infty(\Omega)} \leq C_M < \infty. \tag{2.5}$$

### 3. A priori estimates I: Caccioppoli inequalities

In this section we provide a priori Sobolev estimates for solutions  $u$  to Eq. (1.4). The adopted procedure relies on the equations satisfied by the derivatives  $z_1 = X_{k,u}u$ ,  $z_2 = X_{l,u}X_{k,u}u$  and  $z_Y = Y_uu$  of such solutions: these equations will be treated as linear equations with respect to  $z$ , considering  $u$  as part of the coefficients of the differential operator that defines the equations. The first two lemmata of the section show that they take the following divergence form

$$M_{\epsilon,u}z = X_{i,u} \left( \frac{a_{ij}(\nabla_\epsilon u)}{\sqrt{1 + |\nabla_\epsilon u|^2}} X_{j,u}z \right) = f \tag{3.1}$$

where the precise formulations of the nonhomogeneous term will be given and the coefficients  $a_{ij}$  are

$$a_{ij}(v) = \delta_{ij} - \frac{v_i v_j}{1 + |v|^2}, \tag{3.2}$$

and we note at they are symmetric, uniformly bounded and positive definite, i.e.

$$\frac{1}{1 + |v|^2} |\xi|^2 \leq a_{ij}(v)\xi_i \xi_j \leq |\xi|^2 \quad \forall \xi \in \mathbb{R}^2, \forall v \in \mathbb{R}^2. \tag{3.3}$$

The final result will be the following.

**Theorem 3.1.** *Let  $u$  be a function satisfying (1.5). Let  $z$  be a smooth solution of Eq. (3.1) and  $f$  be a given  $C^\infty(\Omega)$  function, then for any  $p \geq 3$  there exists a positive constant  $C$  depending only on  $p$ ,  $\Omega$  and the constant  $M$  of (1.5) but independent of  $\epsilon$  such that for any  $m \geq 1$  it holds*

$$\begin{aligned} \text{(i)} \quad & \|z\|_{W_\epsilon^{m+1,p+1/2}(\Omega)}^{p+1/2} \leq C \left( 1 + \|z\|_{W_\epsilon^{m,4p+2}(\Omega)}^{4p+2} + \|u\|_{W_\epsilon^{m+1,(4p+2)/3}(\Omega)}^{(4p+2)/3} \right. \\ & \left. + \|u\|_{W_\epsilon^{m+2,(2p+1)/3}(\Omega)}^{(2p+1)/3} + \|Yu\|_{W_\epsilon^{m,(2p+1)/3}(\Omega)}^{(2p+1)/3} + \|f\|_{W_\epsilon^{m,(4p+2)/7}(\Omega)}^{(4p+2)/7} \right), \\ \text{(ii)} \quad & \| |\nabla_\epsilon^{m+1} z|^{(p-1)/2} \|_{W_\epsilon^{1,2}(\Omega)}^2 \leq C \left( 1 + \|z\|_{W_\epsilon^{m,4p+2}(\Omega)}^{4p+2} + \|u\|_{W_\epsilon^{m+1,(4p+2)/3}(\Omega)}^{(4p+2)/3} \right. \\ & \left. + \|u\|_{W_\epsilon^{m+2,(2p+1)/3}(\Omega)}^{(2p+1)/3} + \|Yu\|_{W_\epsilon^{m,(2p+1)/3}(\Omega)}^{(2p+1)/3} + \|f\|_{W_\epsilon^{m,(4p+2)/7}(\Omega)}^{(4p+2)/7} \right). \end{aligned}$$

For the sake of simplicity, from now on we will abandon the subscript  $u$  in the nonlinear vector fields, and we will make use of the following short-hand notation

$$\mathcal{A}_{ij}(v) = \frac{a_{ij}(v)}{\sqrt{1 + |v|^2}}$$

so that we can write the operator in Eq. (3.1) as  $M_{\epsilon,u} = X_i \mathcal{A}_{ij}(\nabla_\epsilon u) X_j$ , or

$$M_{\epsilon,u} = X_i \mathcal{A}_{ij} X_j.$$

By differentiating Eq. (1.4), we have the following lemmata.

**Lemma 3.2.** *If  $u$  is a smooth solution of  $L_{\epsilon,u}u = 0$ , then  $z = X_k u$  is a solution of equation*

$$M_{\epsilon,u}z = f \tag{3.4}$$

where the term at the right-hand side is given by

$$f = -X_i \mathcal{A}_{ij} \omega_{k,j}^l X_l u - \omega_{k,i}^l X_l \left( \frac{X_i u}{\sqrt{1 + |\nabla_\epsilon u|^2}} \right)$$

and the constants  $\omega_{j,k}^l$  are given by (2.4).

**Proof.**

$$\begin{aligned} 0 &= L_{\epsilon,u}u = X_k L_{\epsilon,u}u = X_k X_i \left( \frac{X_i u}{\sqrt{1 + |\nabla_\epsilon u|^2}} \right) \\ &= [X_k, X_i] \left( \frac{X_i u}{\sqrt{1 + |\nabla_\epsilon u|^2}} \right) + X_i X_k \left( \frac{X_i u}{\sqrt{1 + |\nabla_\epsilon u|^2}} \right) \\ &= [X_k, X_i] \left( \frac{X_i u}{\sqrt{1 + |\nabla_\epsilon u|^2}} \right) + X_i \left[ \frac{X_i X_k u}{\sqrt{1 + |\nabla_\epsilon u|^2}} + \frac{[X_k, X_i]u}{\sqrt{1 + |\nabla_\epsilon u|^2}} \right. \\ &\quad \left. - (X_i u) \left( \frac{(X_j u)(X_j X_k u + [X_k, X_j]u)}{\sqrt{1 + |\nabla_\epsilon u|^2}^3} \right) \right] \\ &= [X_k, X_i] \left( \frac{X_i u}{\sqrt{1 + |\nabla_\epsilon u|^2}} \right) + X_i \left[ \frac{X_i z}{\sqrt{1 + |\nabla_\epsilon u|^2}} - \frac{(X_i u)(X_j u)}{\sqrt{1 + |\nabla_\epsilon u|^2}^3} X_j z \right] \\ &\quad + X_i \left[ \frac{[X_k, X_i]u}{\sqrt{1 + |\nabla_\epsilon u|^2}} - \frac{(X_i u)(X_j u)}{\sqrt{1 + |\nabla_\epsilon u|^2}^3} [X_k, X_j]u \right]. \quad \square \end{aligned}$$

**Lemma 3.3.** *If  $z$  is a smooth solution of  $M_{\epsilon,u}z = f$ , then  $s = X_k z$  is again a solution of the same equation, but with a different right-hand side, i.e.*

$$M_{\epsilon,u}s = \rho, \tag{3.5}$$

where  $\rho$  is given by

$$\rho = X_k f - (X_i X_k \mathcal{A}_{ij})(X_j z) - X_i \mathcal{A}_{ij} \omega_{k,j}^l X_l z - \omega_{k,i}^l X_l \mathcal{A}_{ij} X_j z,$$

and the constants  $\omega_{j,k}^l$  are given by (2.4).

**Proof.**

$$\begin{aligned} X_k f &= X_k X_i \mathcal{A}_{ij} X_j z = X_i X_k \mathcal{A}_{ij} X_j z + [X_k, X_i] \mathcal{A}_{ij} X_j z \\ &= X_i [(X_k \mathcal{A}_{ij})(X_j z) + \mathcal{A}_{ij} X_j X_k z + A_{ij} [X_k, X_j] z] + [X_k, X_i] \mathcal{A}_{ij} X_j z. \quad \square \end{aligned}$$

By direct computation and noting that, under hypothesis (1.5), there exists a positive constant  $C$  depending only on  $M$  of (1.5) such that



$$|X_i \mathcal{A}_{ij}| \leq C |\nabla_\epsilon^2 u|, \quad j = 1, 2, \tag{3.6}$$

we obtain the following corollaries.

**Corollary 3.4.** *There exists a positive constant C depending only on M such that the function f of Lemma 3.2 satisfies*

$$|f| \leq C (|\nabla_\epsilon u| + |\nabla_\epsilon^2 u|).$$

**Corollary 3.5.** *If u is a smooth solution of  $L_{\epsilon,u}u = 0$ , then  $v = Yu$  is a solution of equation*

$$M_{\epsilon,u}v = f_v$$

where  $f_v$  is such that there exists a positive constant C depending only on M such that

$$|f_v| \leq C (1 + |\nabla_\epsilon^2 u| + |\nabla_\epsilon v|).$$

From the study of Eqs. (3.4) and (3.5) we can obtain estimates for the derivatives of the solution to Eq. (1.4).

**Proposition 3.6** (First interpolation inequality). *Let us assume (1.5). Then for every  $p \geq 3$  there exists a  $\delta^* > 0$  and a positive constant C depending only on  $p, \Omega$  and  $M$  of (1.5) such that for every  $0 < \delta < \delta^*$  it holds*

$$\int |X_i z|^{p+1/2} \phi^{2p} \leq \frac{C}{\delta} \left( \int |z|^{4p+2} \phi^{2p} + 1 \right) + \delta \int |\nabla_\epsilon (|X_i z|^{(p-1)/2})|^2 \phi^{2p}$$

for every function  $z \in C^\infty(\Omega)$  and every cutoff function  $\phi$ .

**Proof.** In this proof we will not make use of the convention of summation over repeated indices;

$$\begin{aligned} I &= \int |X_i z|^{p+1/2} \phi^{2p} = \int X_i z |X_i z|^{p-1/2} \text{sign}(X_i z) \phi^{2p} = \int z X_i^\dagger [ |X_i z|^{p-1/2} \text{sign}(X_i z) \phi^{2p} ] \\ &= - \int m_i z |X_i z|^{p-1/2} \text{sign}(X_i z) \phi^{2p} - (p-1/2) \int z X_i^2 z |X_i z|^{p-3/2} \phi^{2p} \\ &\quad - 2p \int z |X_i z|^{p-1/2} \text{sign}(X_i z) \phi^{2p-1} X_i \phi \\ &\leq C_M I_1 + (p-1/2) I_2 + 2p I_3 \end{aligned}$$

where, by Young inequality and noting that

$$\int |X_i (|X_i z|^{(p-1)/2})|^2 \phi^{2p} = \frac{(p-1)^2}{4} \int |X_i^2 z|^2 |X_i z|^{p-3} \phi^{2p} \tag{3.7}$$

we have

$$\begin{aligned} I_1 &= \int |z| |X_i z|^{p-1/2} \phi^{2p} \leq \delta I + \frac{1}{\delta} \int |z|^{4p+2} \phi^{2p} + \frac{1}{\delta} \int \phi^{2p}, \\ I_2 &= \int |z| |X_i^2 z| |X_i z|^{p-3/2} \phi^{2p} \\ &\leq \frac{\delta}{p-1/2} \int |X_i (|X_i z|^{(p-1)/2})|^2 \phi^{2p} + \delta \frac{(p-1)^2}{4(p-1/2)} I + \frac{1}{\delta} \frac{4(p-1/2)}{(p-1)^2} \int |z|^{4p+2} \phi^{2p}, \\ I_3 &= \int |z| |X_i z|^{p-1/2} \phi^{2p-1} |X_i \phi| \leq \delta I + \frac{1}{\delta} \left( \int |z|^{4p+2} \phi^{2p} + \int |X_i \phi|^{2p} + \int \phi^{2p} \right). \end{aligned}$$

Thus

$$I \leq \left( C_M + 2p + \frac{(p-1)^2}{4} \right) \delta I + \frac{1}{\delta} \left( C_M + 2p + \frac{4(p-1/2)^2}{(p-1)^2} \right) \int |z|^{4p+2} \phi^{2p} + \delta \int |\nabla_\epsilon (|X_i z|^{(p-1)/2})|^2 \phi^{2p} + \frac{C_M + 2p}{\delta} \left( \int |X_i \phi|^2 + \int \phi^{2p} \right)$$

hence setting  $\delta^* = (C_M + 2p + \frac{(p-1)^2}{4})^{-1}$  we obtain the statement.  $\square$

For solutions to Eqs. (3.4) and (3.5) it is possible to estimate the last term in Proposition 3.6.

**Lemma 3.7.** *Let  $z$  be a smooth solution of Eq. (3.4) and let  $f$  be locally summable in  $\Omega$ , then for every  $p \geq 3$  there exists a positive constant  $C_1$  depending only on  $p$  and the constant  $M$  in (1.5) such that for any cutoff function  $\phi$  it holds*

$$\int |\nabla_\epsilon (|z|^{(p-1)/2})|^2 \phi^{2p} \leq C_1 \left( \frac{1}{\delta} \int |z|^{p-1} (\phi^2 + |\nabla_\epsilon \phi|^2) \phi^{2p-2} - \int f |z|^{p-3} z \phi^{2p} \right).$$

We explicitly note that we cannot estimate the last term with its absolute value, but we prefer keeping the minus sign, since a very delicate estimate will be needed in the next lemma.

**Proof.** The proof relies essentially on uniform boundedness and ellipticity of the coefficients  $a_{ij}$  (3.2). We start by multiplying both members of (3.4) by  $|z|^{p-3} z \phi^{2p}$  and integrating over  $\Omega$ , then integrate by parts using (2.3). At that point we make use of (3.3) and Lipschitz condition (2.5):

$$\begin{aligned} \int f |z|^{p-3} z \phi^{2p} &= \int (X_i A_{ij} X_j z) |z|^{p-3} z \phi^{2p} \\ &= -(p-2) \int A_{ij} (X_j z) (X_i z) |z|^{p-3} \phi^{2p} \\ &\quad - 2p \int A_{ij} (X_j z) |z|^{p-3} z (X_i \phi) \phi^{2p-1} - \int A_{ij} (X_j z) m_i |z|^{p-3} z \phi^{2p} \\ &\leq -\frac{p-2}{(1+C_M^2)^{3/2}} \int |\nabla_\epsilon z|^2 |z|^{p-3} \phi^{2p} \\ &\quad + 2p \int |z|^{p-3} z \phi^{2p-1} |\nabla_\epsilon z| |\nabla_\epsilon \phi| + \int |z|^{p-3} z \phi^{2p} |\nabla_\epsilon z| \sqrt{m_1^2 + m_2^2} \\ &\leq \left( (2p + \sqrt{2} C_M) \delta - \frac{p-2}{(1+C_M^2)^{3/2}} \right) \int |\nabla_\epsilon z|^2 |z|^{p-3} \phi^{2p} \\ &\quad + \frac{2p + \sqrt{2} C_M}{\delta} \int |z|^{p-1} (\phi^2 + |\nabla_\epsilon \phi|^2) \phi^{2p-2}, \end{aligned}$$

where the last transition is Young inequality. By (3.7) and choosing  $\delta$  sufficiently small we then end up with the desired claim.  $\square$

**Lemma 3.8.** *Let  $z$  be a smooth solution of Eq. (3.4) and let  $f$  be locally summable in  $\Omega$ , then for every  $p \geq 3$  there exists a positive constant  $C_2$  depending only on  $p$  and the constant  $M$  in (1.5) such that for any cutoff function  $\phi$  it holds*

$$\int |\nabla_\epsilon (|\nabla_\epsilon z|^{(p-1)/2})|^2 \phi^{2p} \leq C_2 \left( \int |\nabla_\epsilon z|^{p+1/2} \phi^{2p} + \int |\nabla_\epsilon^2 u|^{(4p+2)/3} \phi^{2p} + \int |\nabla_\epsilon Y u|^{(2p+1)/3} \phi^{2p} + \int |f|^{(4p+2)/7} \phi^{2p} + 1 \right).$$

**Proof.** By Lemma 3.3,  $s = X_k z$  satisfies Eq. (3.5), so that we can apply Lemma 3.7 to the left-hand side:

$$\begin{aligned} I &= \int |\nabla_\epsilon (|s|^{(p-1)/2})|^2 \phi^{2p} \\ &\leq C_1 \left( \frac{1}{\delta} \int |s|^{p-1} \phi^{2p} + \frac{1}{\delta} \int |s|^{p-1} |\nabla_\epsilon \phi|^2 \phi^{2p-2} - \int \rho |s|^{p-3} s \phi^{2p} \right) \\ &= C_1 \left( \frac{J' + J''}{\delta} - J \right). \end{aligned}$$

Let us now consider the last term at the right-hand side explicitly in terms of  $f$

$$\begin{aligned} J &= \int \rho |s|^{p-3} s \phi^{2p} \\ &= \int (X_k f) |s|^{p-3} s \phi^{2p} - \int (X_i (X_k \mathcal{A}_{ij})(X_j z)) |s|^{p-3} s \phi^{2p} \\ &\quad - \int (X_i \mathcal{A}_{ij} \omega_{k,j}^l(X_l z)) |s|^{p-3} s \phi^{2p} - \int \omega_{k,i}^l (X_l \mathcal{A}_{ij} X_j z) |s|^{p-3} s \phi^{2p} \end{aligned}$$

and integrate by parts

$$\begin{aligned} J &= (p-2) \int (X_k \mathcal{A}_{ij})(X_j z)(X_i s) |s|^{p-3} \phi^{2p} \\ &\quad + 2p \int (X_k \mathcal{A}_{ij})(X_j z) |s|^{p-3} s (X_i \phi) \phi^{2p-1} + \int m_i (X_k \mathcal{A}_{ij})(X_j z) |s|^{p-3} s \phi^{2p} \\ &\quad + (p-2) \int \mathcal{A}_{ij} \omega_{k,j}^l (X_l z)(X_i s) |s|^{p-3} \phi^{2p} \\ &\quad + 2p \int \mathcal{A}_{ij} \omega_{k,j}^l (X_l z) |s|^{p-3} s (X_i \phi) \phi^{2p-1} + \int m_i \mathcal{A}_{ij} \omega_{k,j}^l (X_l z) |s|^{p-3} s \phi^{2p} \\ &\quad + (p-2) \int \mathcal{A}_{ij} (X_j z) \omega_{k,i}^l (X_l s) |s|^{p-3} \phi^{2p} \\ &\quad + 2p \int \mathcal{A}_{ij} (X_j z) \omega_{k,i}^l |s|^{p-3} s (X_l \phi) \phi^{2p-1} + \int m_l \omega_{k,i}^l \mathcal{A}_{ij} (X_j z) |s|^{p-3} s \phi^{2p} \\ &\quad + \int \mathcal{A}_{ij} (X_j z) (X_l \omega_{k,i}^l) |s|^{p-3} s \phi^{2p} - (p-2) \int f (X_k s) |s|^{p-3} \phi^{2p} \\ &\quad - 2p \int f |s|^{p-3} s (X_k \phi) \phi^{2p-1} - \int m_k f |s|^{p-3} s \phi^{2p}. \end{aligned}$$

We will now work to isolate terms of type  $\int |\nabla_\epsilon z|^{p+1/2} \phi^{2p}$  or terms that can be reabsorbed into  $I, J'$  or  $J''$ . Indeed  $p + 1/2$  is the power needed for the estimate as it results from Proposition 3.6, and

$$\begin{aligned} J' + J'' &= \int |\nabla_\epsilon z|^{p-1} (\phi^2 + |\nabla \phi|^2) \phi^{2p-2} \\ &\leq \int |\nabla_\epsilon z|^{p+1/2} \phi^{2p} + \int \phi^{2p} + 2p \int (\phi^2 + |\nabla \phi|^2)^p \end{aligned} \tag{3.8}$$

so the other terms that appear in the statement result by pairing through Young inequality.

First we observe that there exists a positive constant  $C_A$  depending only on  $M$  such that

$$|X_k \mathcal{A}_{ij}| \xi_i \eta_j \leq C_A |\nabla_\epsilon^2 u| |\xi| |\eta| \quad \forall \xi, \eta \in \mathbb{R}^2 \tag{3.9}$$

indeed

$$|X_k \mathcal{A}_{ij}| \leq |X_k a_{ij}| + a_{ij} |\nabla_\epsilon^2 u| \leq |X_k X_i u X_j u| + 3a_{ij} |\nabla_\epsilon^2 u|.$$

Using (3.9) and Young inequality we can estimate the first term by

$$\begin{aligned} \left| \int (X_k A_{ij})(X_j z)(X_i s) |s|^{p-3} \phi^{2p} \right| &\leq C_A \int |\nabla_\epsilon^2 u| |\nabla_\epsilon s| |\nabla_\epsilon z| |s|^{p-3} \phi^{2p} \\ &\leq C_A \left[ \delta \frac{4}{(p-1)^2} I + \frac{1}{\delta} \left( \int |\nabla_\epsilon z|^{p+1/2} \phi^{2p} + \int |\nabla_\epsilon^2 u|^{(4p+2)/3} \phi^{2p} \right) \right]. \end{aligned}$$

All the other terms can be treated analogously, making use of assumption (2.5). We only turn our attention to the term that needs to be estimated with  $Yu$ , since it involves derivatives of the commutator coefficients:

$$\begin{aligned} &\left| \int A_{ij}(X_j z)(X_l \omega_{k,i}^l) |s|^{p-3} s \phi^{2p} \right| \\ &\leq \int |X\omega| |\nabla_\epsilon z|^{p-1} \phi^{2p} \\ &\leq \int |\nabla_\epsilon z|^{p+1/2} \phi^{2p} + \int |X\omega|^{(2p+1)/3} \phi^{2p} \\ &\leq \int |\nabla_\epsilon z|^{p+1/2} \phi^{2p} + 2^{(2p-2)/3} \int |\nabla_\epsilon Yu|^{(2p+1)/3} \phi^{2p} \\ &\quad + 2^{(2p-2)/3} (|c_{2,3}^1| + |c_{2,3}^2|)^{(2p+1)/3} \left( \int |\nabla_\epsilon^2 u|^{(4p+2)/3} \phi^{2p} + \int \phi^{2p} \right), \end{aligned}$$

where we have indicated  $X\omega = X_1\omega^1 + X_2\omega^2$ , and the last transition holds since

$$\begin{aligned} |X\omega|^p &= \left| \epsilon c_{2,3}^1 X_1 X_1 u + c_{2,3}^2 X_2 X_1 u + \frac{1}{\epsilon} X_2 X_2 u \right|^p \\ &\leq 2^{p-1} [(|c_{2,3}^1| + |c_{2,3}^2|)^p |\nabla_\epsilon^2 u|^p + |X_2 Yu|^p]. \end{aligned}$$

Keeping only the higher power of  $f$  in the last terms, we then end up with

$$\begin{aligned} I &\leq C \left( \delta I + \frac{J' + J''}{\delta} + \int |\nabla_\epsilon z|^{p+1/2} \phi^{2p} + \int |\nabla_\epsilon^2 u|^{(4p+2)/3} \phi^{2p} \right. \\ &\quad \left. + \int |\nabla_\epsilon Yu|^{(2p+1)/3} \phi^{2p} + \int |f|^{(4p+2)/7} \phi^{2p} + \int \phi^{2p} + \int |\nabla_\epsilon \phi|^2 \right), \end{aligned}$$

where  $C$  is a positive constant depending only on  $p, M$ . By (3.8), for  $\delta$  sufficiently small we end up with the desired claim.  $\square$

**Theorem 3.9.** *Let  $u$  be a function satisfying (1.5). Let  $z$  be a smooth solution of Eq. (3.1) and  $f$  be a given  $C^\infty(\Omega)$  function, then for any  $p \geq 3$  there exists a positive constant  $C$  depending only on  $p, \Omega$  and the constant  $M$  of (1.5), hence independent of  $\epsilon$ , such that it holds*

$$\begin{aligned} &\|z\|_{W_\epsilon^{1,p+1/2}(\Omega)}^{p+1/2} + \|\nabla_\epsilon z\|_{W_\epsilon^{1,2}(\Omega)}^{(p-1)/2} \\ &\leq C_3 \left( \|z\|_{L^{4p+2}(\Omega)}^{4p+2} + \|u\|_{W_\epsilon^{2,(4p+2)/3}(\Omega)}^{(4p+2)/3} + \|Yu\|_{W_\epsilon^{1,(2p+1)/3}(\Omega)}^{(2p+1)/3} + \|f\|_{L^{(4p+2)/7}(\Omega)}^{(4p+2)/7} + 1 \right). \end{aligned}$$

**Proof.** Applying Lemma 3.8 to Proposition 3.6 we get

$$\begin{aligned} \int |\nabla_\epsilon z|^{p+1/2} \phi^{2p} &\leq \frac{C}{\delta} \left( \int |z|^{4p+2} \phi^{2p} + 1 \right) + \delta C_2 \int |\nabla_\epsilon z|^{p+1/2} \phi^{2p} \\ &\quad + \delta C_2 \left( \int |\nabla_\epsilon^2 u|^{(4p+2)/3} \phi^{2p} + \int |\nabla_\epsilon Yu|^{(2p+1)/3} \phi^{2p} + \int |f|^{(4p+2)/7} \phi^{2p} + 1 \right) \end{aligned}$$

so that for any  $\delta < \frac{1}{C_2}$  we obtain

$$\begin{aligned} \int |\nabla_\epsilon z|^{p+1/2} \phi^{2p} &\leq C_3 \left( \int |z|^{4p+2} \phi^{2p} + \int |\nabla_\epsilon^2 u|^{(4p+2)/3} \phi^{2p} \right. \\ &\quad \left. + \int |\nabla_\epsilon Yu|^{(2p+1)/3} \phi^{2p} + \int |f|^{(4p+2)/7} \phi^{2p} + 1 \right). \end{aligned}$$

By inserting this last expression into Lemma 3.8 we have the estimate

$$\begin{aligned} & \int |\nabla_\epsilon z|^{p+1/2} \phi^{2p} + \int |\nabla_\epsilon (|\nabla_\epsilon z|^{(p-1)/2})|^2 \phi^{2p} \\ & \leq C_3 \left( \int |z|^{4p+2} \phi^{2p} + \int |\nabla_\epsilon^2 u|^{(4p+2)/3} \phi^{2p} + \int |\nabla_\epsilon Y u|^{(2p+1)/3} \phi^{2p} + \int |f|^{(4p+2)/7} \phi^{2p} + 1 \right). \end{aligned}$$

The statement of the theorem now follows by choosing a sequence of cutoff functions that converges to the characteristic function of the set  $\Omega$ .  $\square$

Iterating the previous theorem once, we obtain the following estimates.

**Theorem 3.10.** *Let  $u$  be a function satisfying (1.5). Let  $z$  be a smooth solution of Eq. (3.1) and  $f$  be a given  $C^\infty(\Omega)$  function, then for any  $p \geq 3$  there exists a positive constant  $C$  depending only on  $p$ ,  $\Omega$  and the constant  $M$  of (1.5) but independent of  $\epsilon$  such that it holds*

$$\begin{aligned} & \|z\|_{W_\epsilon^{2,p+1/2}(\Omega)}^{p+1/2} + \| |\nabla_\epsilon^2 z|^{(p-1)/2} \|_{W_\epsilon^{1,2}(\Omega)}^2 \\ & \leq C_4 \left( \|f\|_{W_\epsilon^{1,4p+2}(\Omega)}^{4p+2} + \|u\|_{W_\epsilon^{2,(4p+2)/3}(\Omega)}^{(4p+2)/3} + \|Y u\|_{W_\epsilon^{1,(2p+1)/3}(\Omega)}^{(2p+1)/3} + \|f\|_{W_\epsilon^{1,(4p+2)/7}(\Omega)}^{(4p+2)/7} + \|u\|_{W_\epsilon^{3,(2p+1)/3}(\Omega)}^{(2p+1)/3} + 1 \right). \end{aligned}$$

**Proof.** If we call  $s = X_k z$  then by Lemma 3.3  $M_{\epsilon,u,s} = \rho$ , where  $\rho$  is given by

$$\rho = X_k f - X_i (X_k \mathcal{A}_{ij}) X_j z - X_i \mathcal{A}_{ij} \omega_{k,j}^l X_l z - \omega_{k,i}^l X_l \mathcal{A}_{ij} X_j z.$$

From Theorem 3.9 we have

$$\begin{aligned} & \|s\|_{W_\epsilon^{1,p+1/2}(\Omega)}^{p+1/2} + \| |\nabla_\epsilon s|^{(p-1)/2} \|_{W_\epsilon^{1,2}(\Omega)}^2 \\ & \leq C_3 \left( \|s\|_{L^{4p+2}(\Omega)}^{4p+2} + \|u\|_{W_\epsilon^{2,(4p+2)/3}(\Omega)}^{(4p+2)/3} + \|Y u\|_{W_\epsilon^{1,(2p+1)/3}(\Omega)}^{(2p+1)/3} + \|\rho\|_{L^{(4p+2)/7}(\Omega)}^{(4p+2)/7} + 1 \right), \end{aligned}$$

where, using  $\lambda = (4p + 2)/7$  and denoting with  $C$  a positive constant depending only on  $p$ ,  $M$  that may change from line to line

$$\begin{aligned} \|\rho\|_{L^\lambda(\Omega)}^\lambda & \leq C \left( \|X_k f\|_{L^\lambda(\Omega)}^\lambda + \|X_i (X_k \mathcal{A}_{ij}) X_j z\|_{L^\lambda(\Omega)}^\lambda + \|X_i \mathcal{A}_{ij} \omega_{k,j}^l X_l z\|_{L^\lambda(\Omega)}^\lambda + \|\omega_{k,i}^l X_l \mathcal{A}_{ij} X_j z\|_{L^\lambda(\Omega)}^\lambda \right) \\ & \leq C \left( \|f\|_{W_\epsilon^{1,\lambda}(\Omega)}^\lambda + \| |\nabla_\epsilon^3 u| |\nabla_\epsilon z| \|_{L^\lambda(\Omega)}^\lambda + \| |\nabla_\epsilon^2 u| |\nabla_\epsilon^2 z| \|_{L^\lambda(\Omega)}^\lambda \right. \\ & \quad \left. \times \| |\nabla_\epsilon^2 u| |\nabla_\epsilon z| \|_{L^\lambda(\Omega)}^\lambda + \| |\nabla_\epsilon Y u| |\nabla_\epsilon z| \|_{L^\lambda(\Omega)}^\lambda + \| |\nabla_\epsilon^2 z| \|_{L^\lambda(\Omega)}^\lambda \right). \end{aligned}$$

Again, by Young inequality we can estimate each of these terms by

$$\begin{aligned} \| |\nabla_\epsilon^3 u| |\nabla_\epsilon z| \|_{L^{(4p+2)/7}(\Omega)}^{(4p+2)/7} & \leq \| |\nabla_\epsilon^3 u| \|_{L^{(2p+1)/3}(\Omega)}^{(2p+1)/3} + \| |\nabla_\epsilon z| \|_{L^{4p+2}(\Omega)}^{4p+2}, \\ \| |\nabla_\epsilon^2 u| |\nabla_\epsilon^2 z| \|_{L^{(4p+2)/7}(\Omega)}^{(4p+2)/7} & \leq \frac{1}{\delta} \| |\nabla_\epsilon^2 u| \|_{L^{(4p+2)/3}(\Omega)}^{(4p+2)/3} + \delta \| |\nabla_\epsilon^2 z| \|_{L^{p+1/2}(\Omega)}^{p+1/2}, \\ \| |\nabla_\epsilon^2 u| |\nabla_\epsilon z| \|_{L^{(4p+2)/7}(\Omega)}^{(4p+2)/7} & \leq \| |\nabla_\epsilon^2 u| \|_{L^{(4p+2)/3}(\Omega)}^{(4p+2)/3} + \| |\nabla_\epsilon z| \|_{L^{4p+2}(\Omega)}^{4p+2} + c, \\ \| |\nabla_\epsilon Y u| |\nabla_\epsilon z| \|_{L^{(4p+2)/7}(\Omega)}^{(4p+2)/7} & \leq \| |\nabla_\epsilon Y u| \|_{L^{(2p+1)/3}(\Omega)}^{(2p+1)/3} + \| |\nabla_\epsilon z| \|_{L^{4p+2}(\Omega)}^{4p+2}, \\ \| |\nabla_\epsilon^2 z| \|_{L^{(4p+2)/7}(\Omega)}^{(4p+2)/7} & \leq \delta \| |\nabla_\epsilon^2 z| \|_{L^{p+1/2}(\Omega)}^{p+1/2} + \frac{c}{\delta} \end{aligned}$$

that proves the statement for  $\delta < \frac{1}{C_3}$ .  $\square$

Iterating this last result we can then obtain Theorem 3.1.

**4. A priori estimates II: approximation with a Hörmander-type operator**

In this section we provide a new estimate of Sobolev type for solutions of Eq. (1.4), using the fact that the equation can be represented in the nondivergence form

$$N_{\epsilon,u}u = a_{ij}(\nabla_{\epsilon}u)X_{i,u}X_{j,u}u = 0, \tag{4.1}$$

where the coefficients  $a_{ij}$  are given by (3.2).

We first linearize this equation, i.e. we apply the operator  $N_{\epsilon,u}$  to a sufficiently regular arbitrary function  $z$ . This leads to

$$N_{\epsilon,u}z = a_{ij}(\nabla_{\epsilon}u)X_{i,u}X_{j,u}z = 0. \tag{4.2}$$

The aim of this section is to obtain some a priori bounds of Sobolev norms for solutions  $z$  to the linearized equation (4.2). In particular we will need to estimate  $L^p$  norm of second order derivatives of the function  $z$  in the direction of the vector fields  $X_1, X_2$ , which do not satisfy a Hörmander type condition uniform in  $\epsilon$ . Hence we will approximate these vector fields with  $C^\infty$  vector fields which, together with their commutators of order 2, do not span the space. Since at the moment we are not interested to higher regularity of the solution, we do not require that the new vector fields approximate the given one up to the third derivative, and we can require, at the opposite, that they span the space at step 3. Hence they can satisfy a Sobolev type inequality, which will be used to estimate second order derivatives of the solution of the given operator.

**Theorem 4.1.** *Let  $z$  be a classical solution of equation  $N_{\epsilon,u}z = 0$ , then*

- (i) *if there exists a constant  $\alpha \in ]0, 1[$  and a constant  $p > 10/3$  such that for any compact  $K \Subset \Omega$  there exists a positive constant  $C$  such that*

$$\|u\|_{C_\epsilon^{1,\alpha}(K)} + \|Yz\|_{L^p(K)} + \|Yz\|_{W_\epsilon^{1,2}(K)} + \|z\|_{W_\epsilon^{2,2}(K)} \leq C$$

*then for any compact set  $K_1 \Subset K$  there exists a positive constant  $C_1$  depending on  $K, C$  and  $M$  such that*

$$z \in W_\epsilon^{2,10/3}(K_1);$$

- (ii) *if, in addition, there exists a positive constant  $C'$  such that*

$$\|YX_kz\|_{L^4(K)} \leq C',$$

*and  $\alpha \geq 1/4$ , then for every  $p > 1$*

$$z \in W_\epsilon^{2,p}(K_1).$$

We have denoted by  $C_\epsilon^{1,\alpha}$  the class of functions whose  $X_{1,u}$  and  $X_{2,u}$  derivatives are Hölder continuous of order  $\alpha$ . They are in particular Euclidean Hölder continuous of order  $(1 + \alpha)$ , but the Hölder constant is not independent on  $\epsilon$ .

**4.1. Lifting and freezing**

We introduce the new vector fields

$$\tilde{X}_1 = X_1 + s^2Y, \quad \tilde{X}_2 = X_2, \quad \tilde{X}_3 = \partial_s \tag{4.3}$$

such that

$$[\tilde{X}_3, \tilde{X}_1] = \frac{2s}{\epsilon} \tilde{X}_2, \quad [\tilde{X}_3, [\tilde{X}_3, \tilde{X}_1]] = \frac{2}{\epsilon} \tilde{X}_2.$$

The related exponential coordinates based at  $(x_0, 0)$ , which we will call  $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ , are defined as

$$(x_1, x_2, s) = \exp_{(x_0,0)}(\tilde{e}_1\tilde{X}_1 + \tilde{e}_2\tilde{X}_2 + \tilde{e}_3\tilde{X}_3) \tag{4.4}$$

that is to say, given a curve  $\gamma(t)$  such that  $\gamma(0) = (x_0, 0)$ , whose derivative is given by the above vector field, i.e.  $\dot{\gamma}(t) = \tilde{e}_1 \tilde{X}_1(\gamma(t)) + \tilde{e}_2 \tilde{X}_2(\gamma(t)) + \tilde{e}_3 \tilde{X}_3(\gamma(t))$ , then the exponential coordinates in (4.4) satisfy  $(x_1, x_2, s) = \gamma(1)$ . They can be obtained by the following system:

$$\begin{cases} \dot{x}_1 = \tilde{e}_1 \sigma_1^1 + (\tilde{e}_1 s^2 + \epsilon \tilde{e}_2) \sigma_2^1, \\ \dot{x}_2 = \tilde{e}_1 \sigma_1^2 + (\tilde{e}_1 s^2 + \epsilon \tilde{e}_2) \sigma_2^2, \\ \dot{s} = \tilde{e}_3. \end{cases}$$

Integrating we have

$$\begin{cases} s = \tilde{e}_3, \\ (x - x_0)_i = \tilde{e}_1 I_1^i + \tilde{e}_1 s^2 J^i + \epsilon \tilde{e}_2 I_2^i, \end{cases}$$

where

$$I_j^i = \int_0^1 dt \sigma_j^i(x(t), u(x(t))),$$

$$J^i = \int_0^1 dt t^2 \sigma_2^i(x(t), u(x(t)))$$

and the system

$$\begin{cases} \tilde{e}_1 I_1^1 + \tilde{e}_1 s^2 J^1 + \epsilon \tilde{e}_2 I_2^1 = (x - x_0)_1, \\ \tilde{e}_1 I_1^2 + \tilde{e}_1 s^2 J^2 + \epsilon \tilde{e}_2 I_2^2 = (x - x_0)_2 \end{cases}$$

is solved implicitly by

$$\begin{cases} \tilde{e}_1 = \frac{(x - x_0)_1 I_2^2 - (x - x_0)_2 I_2^1}{(I_1^1 I_2^2 - I_1^2 I_2^1) + s^2 (J^1 I_2^2 - J^2 I_2^1)}, \\ \epsilon \tilde{e}_2 = \frac{(x - x_0)_2 (I_1^1 + s^2 J^1) - (x - x_0)_1 (I_1^2 + \epsilon s^2 J^2)}{(I_1^1 I_2^2 - I_1^2 I_2^1) + s^2 (J^1 I_2^2 - J^2 I_2^1)}, \end{cases} \tag{4.5}$$

where rank condition (2.2) ensures the well posedness of this solution, since  $I_1^1 I_2^2 \neq I_1^2 I_2^1$ .

We now introduce a freezing for the vector fields  $\tilde{X}_j$ . With the short-hand notation

$$\Sigma_j^k = \frac{\sigma_j^k(x_0)}{\sigma_1^1(x_0) \sigma_2^2(x_0) - \sigma_1^2(x_0) \sigma_2^1(x_0)}$$

we start approximating the exponential coordinates (4.5) in the following way

$$\begin{cases} e_{01} = \Sigma_2^2(x - x_0)_1 - \Sigma_2^1(x - x_0)_2, \\ \epsilon e_{02} = \Sigma_1^1(x - x_0)_2 - \Sigma_1^2(x - x_0)_1 \end{cases}$$

and define an analogue of the first order Taylor polynomial of a function  $h : \Omega \rightarrow \mathbb{R}$

$$\begin{aligned} P_{x_0} h(x) &= h(x_0) + e_{01}(x) \tilde{X}_1 h(x_0, 0) + e_{02}(x) \tilde{X}_2 h(x_0, 0) \\ &= h(x_0) + e_{01}(x) X_1 h(x_0) + e_{02}(x) X_2 h(x_0). \end{aligned} \tag{4.6}$$

The freed vector fields then read

$$X_{01} = P_{x_0} \sigma_1^i(x) \partial_{x_i} + s^2 Y_0, \quad X_{02} = \epsilon P_{x_0} \sigma_2^i(x) \partial_{x_i}, \quad X_{03} = \partial_s, \tag{4.7}$$

where

$$Y_0 = P_{x_0} \sigma_2^i(x) \partial_{x_i}.$$

These are Hörmander vector fields defining a step three stratified Lie algebra, whose first layer is spanned by  $\{X_{01}, X_{03}\}$ . We will denote by  $d_0$  the corresponding Carnot–Carathéodory distance, by  $d_{0\epsilon}$  the full Riemannian distance and by  $B_{0\epsilon}$  the balls relative to  $d_{0\epsilon}$ . Notice also that the homogeneous dimension of the space  $(\mathbb{R}^3, d_0)$  is 5.

**Remark 4.2.** It is well known that sub-Riemannian structures can be obtained with a limiting procedure from Riemannian counterparts [3]. More precisely, in this case we have that  $(\mathbb{R}^3, d_{0\epsilon})$  converge as a metric space in the Hausdorff–Gromov sense to  $(\mathbb{R}^3, d_0)$  as  $\epsilon$  goes to zero. In particular for any  $\xi, \eta \in \Omega \times (-1, 1)$  we have that  $d_{0\epsilon}(\xi, \eta) \rightarrow d_0(\xi, \eta)$ , that implies that given a threshold  $\epsilon_0$  sufficiently small, there exists a positive constant  $C = C(\epsilon_0)$  depending only on it such that for all  $0 < \epsilon < \epsilon_0$  it holds

$$|B_{0\epsilon}((x_0, 0), R)| \geq CR^5.$$

This means that for small  $\epsilon$  the volume growth for the Riemannian metric measure space can be considered the same as for the sub-Riemannian one.

**Lemma 4.3.** *If  $h : \Omega \rightarrow \mathbb{R}$  is a  $C_E^{1,\alpha}$ , i.e.  $h$  belongs to the Euclidean Hölder class  $(1, \alpha)$ , then there exists an  $\epsilon_0 > 0$ , a neighborhood  $U$  of  $x_0$  and a positive constant  $C$  depending on  $\epsilon_0, U$  but not on  $\epsilon$  such that*

$$|h(x) - P_{x_0}h(x)| \leq Cd_{0\epsilon}^{1+\alpha}(x, x_0) \tag{4.8}$$

for all  $x$  in  $U$  and all  $\epsilon < \epsilon_0$ .

**Proof.** We will first prove that

$$|h(x) - P_{x_0}h(x)| \leq Cd_E^{1+\alpha}(x, x_0) \tag{4.9}$$

where  $d_E$  stands for the Euclidean distance. Indeed this is sufficient for having (4.8), since by the hypothesis of  $u$  being  $C_E^{1,\alpha}$ , in a sufficiently small neighborhood of  $x_0$  the Euclidean distance is controlled by the Riemannian one with constant depending on the coefficients that define the vector fields (4.7), i.e. the derivatives of  $u$ . Moreover, since  $X_{01}$  contains the unscaled direction  $Y_0$ , once  $\epsilon$  is fixed under a given threshold  $\epsilon_0$  it does not interfere anymore on the constant.

Let us first calculate exponential coordinates  $(e_1, e_2, e_3)$  with respect to the vector fields (4.7), in a neighborhood of  $(x_0, 0)$ . They are then given by

$$(x_1, x_2, s) = \exp_{(x_0,0)}(e_1 X_{01} + e_2 X_{02} + e_3 X_{03}) \tag{4.10}$$

that is

$$\begin{cases} e_1 = \frac{(x - x_0)_1 I_{02}^2 - (x - x_0)_2 I_{02}^1}{(I_{01}^1 I_{02}^2 - I_{01}^2 I_{02}^1) + s^2 (J_0^1 I_{02}^2 - J_0^2 I_{02}^1)}, \\ e_2 = \frac{(x - x_0)_2 (I_{01}^1 + s^2 J_0^1) - (x - x_0)_1 (I_{01}^2 + s^2 J_0^2)}{(I_{01}^1 I_{02}^2 - I_{01}^2 I_{02}^1) + s^2 (J_0^1 I_{02}^2 - J_0^2 I_{02}^1)}, \end{cases}$$

where

$$I_{0j}^k = \int_0^1 dt P_{x_0} \sigma_j^k(x(t), u(x(t))) = \sigma_j^k(x_0, u(x_0)) + X_i \sigma_j^k(x_0, u(x_0)) \int_0^1 dt e_{0i}(x(t)),$$

$$J_0^i = \int_0^1 dt t^2 P_{x_0} \sigma_2^i(x(t), u(x(t))).$$

We now define in  $\Omega$  an integral curve of (4.7), connecting  $x_0$  with  $x$

$$\gamma(t) = \exp_{x_0}(t(e_1(x)X_{01} + e_2(x)X_{02}))$$



so that, by Mean Value Theorem, for all  $t \in (0, 1)$  there exists a  $t^* \in (0, t)$  such that

$$h(\gamma(t)) - h(x_0) = e_1(x)X_{01}h(\gamma(t^*)) + e_2(x)X_{02}h(\gamma(t^*)).$$

Thus, for all  $t \in (0, 1)$  we have

$$\begin{aligned} h(\gamma(t)) - P_{x_0}h(\gamma(t)) &= h(\gamma(t)) - h(x_0) - e_{01}(\gamma(t))\tilde{X}_1h(x_0, 0) - e_{02}(\gamma(t))\tilde{X}_2h(x_0, 0) \\ &= e_i(x)X_{0i}h(\gamma(t^*)) - e_{0i}(\gamma(t))\tilde{X}_i h(x_0, 0) \\ &= e_i(x)[X_{0i}h(\gamma(t^*)) - X_{0i}h(x_0)] \\ &\quad + e_i(x)[X_{0i}h(x_0) - \tilde{X}_i h(x_0, 0)] + [e_i(x) - e_{0i}(\gamma(t))]\tilde{X}_i h(x_0, 0), \end{aligned}$$

where in particular the second term is zero since  $X_{0i}h(x_0) = \tilde{X}_i h(x_0, 0)$ . For the first term, then, we observe that

$$\begin{aligned} |X_{01}h(\gamma(t^*)) - X_{01}h(x_0)| &\leq Cd_E^\alpha(x, x_0), \\ |X_{02}h(\gamma(t^*)) - X_{02}h(x_0)| &\leq \epsilon Cd_E^\alpha(x, x_0) \end{aligned}$$

because of the hypothesis of  $h$  being  $C^{1,\alpha}$  and the smoothness of the coefficients, so that the constant  $C$  depends also on the neighborhood of  $x_0$  chosen. Then

$$|e_i(x)||X_{0i}h(\gamma(t^*)) - X_{0i}h(x_0)| \leq Cd_E^{1+\alpha}(x, x_0).$$

For the third term, after introducing the notation

$$\mathcal{I}_{0j}^k = \frac{I_{0j}^k}{I_{01}^1 I_{02}^2 - I_{01}^2 I_{02}^1}$$

we have

$$\begin{cases} e_1(x, 0) = \mathcal{I}_{02}^2(x - x_0)_1 - \mathcal{I}_{02}^1(x - x_0)_2, \\ \epsilon e_2(x, 0) = \mathcal{I}_{01}^1(x - x_0)_2 - \mathcal{I}_{01}^2(x - x_0)_1 \end{cases}$$

so that

$$\begin{aligned} |e_1(x, 0) - e_{01}(\gamma(t))| &= |\mathcal{I}_{02}^2(x - x_0)_1 - \mathcal{I}_{02}^1(x - x_0)_2 - \Sigma_2^2(\gamma(t) - x_0)_1 + \Sigma_2^1(\gamma(t) - x_0)_2| \\ &\leq |\mathcal{I}_{02}^2(x - x_0)_1 - \Sigma_2^2(\gamma(t) - x_0)_1| + |\mathcal{I}_{02}^1(x - x_0)_2 - \Sigma_2^1(\gamma(t) - x_0)_2|, \\ \epsilon |e_2(x, 0) - e_{02}(\gamma(t))| &= |\mathcal{I}_{01}^1(x - x_0)_2 - \mathcal{I}_{01}^2(x - x_0)_1 - \Sigma_1^1(\gamma(t) - x_0)_2 - \Sigma_1^2(\gamma(t) - x_0)_1| \\ &\leq |\mathcal{I}_{01}^1(x - x_0)_2 - \Sigma_1^1(\gamma(t) - x_0)_2| + |\mathcal{I}_{01}^2(x - x_0)_1 - \Sigma_1^2(\gamma(t) - x_0)_1|. \end{aligned}$$

The generic term to control is then

$$\Delta_{ij}^k = |(x - x_0)_i \mathcal{I}_{0j}^k - (\gamma(t) - x_0)_i \Sigma_j^k|.$$

Explicitly, setting  $D_1 = I_{01}^1 I_{02}^2 - I_{01}^2 I_{02}^1$  and  $D_2 = \sigma_1^1(x_0)\sigma_2^2(x_0) - \sigma_1^2(x_0)\sigma_2^1(x_0)$

$$\begin{aligned} \Delta_{ij}^k &= \left| (x - x_0)_i \frac{\int_0^1 d\tau P_{x_0} \sigma_j^k(\gamma(\tau))}{D_1} - (\gamma(t) - x_0)_i \frac{\sigma_j^k(x_0)}{D_2} \right| \\ &\leq |x - x_0)_i| \left| \frac{\int_0^1 d\tau (P_{x_0} \sigma_j^k(\gamma(\tau)) - \sigma_j^k(\gamma(\tau)))}{D_1} \right| \\ &\quad + \left| (x - x_0)_i \int_0^1 d\tau \sigma_j^k(\gamma(\tau)) \right| \left| \frac{D_1 - D_2}{D_1 D_2} \right| \\ &\quad + \left| \frac{(x - x_0)_i \int_0^1 d\tau \sigma_j^k(\gamma(\tau)) - (\gamma(t) - x_0)_i \sigma_j^k(x_0)}{D_2} \right| \\ &= \frac{N_1}{D_1} + \frac{N_{12}}{D_1 D_2} + \frac{N_2}{D_2} \leq C(N_1 + N_{12} + N_2), \end{aligned}$$

where the last transition holds since rank condition (2.2) ensures that the  $D_i$ 's do not vanish, so that they can be bounded from below uniformly on compacts, and  $C$  is a constant depending on the compact set. Moreover

$$N_{12} \leq C d_E^2(x, x_0)$$

indeed

$$\begin{aligned} N_{12} \leq & \left| (x - x_0)_i \int_0^1 d\tau \sigma_j^k(\gamma(\tau)) \right| \left| \left( \left| X_i(\sigma_1^1 \sigma_2^2 - \sigma_1^2 \sigma_2^1)(x_0) \int_0^1 d\tau e_{0i}(\gamma(\tau)) \right| \right. \right. \\ & \left. \left. + \left| (X_i \sigma_1^1)(X_j \sigma_2^2) \int_0^1 d\tau e_{0i} \int_0^1 d\tau e_{0j} - (X_i \sigma_1^2)(X_j \sigma_2^1) \int_0^1 d\tau e_{0i} \int_0^1 d\tau e_{0j} \right| \right) \right| \end{aligned}$$

where

$$\left| (x - x_0)_i \int_0^1 d\tau \sigma_j^k(\gamma(\tau)) \right| \leq C d_E(x, x_0)$$

since the functions  $\sigma_j^k$  are smooth, so they and their derivatives are locally bounded, and

$$\left| \int_0^1 d\tau e_{01}(\gamma(\tau)) \right| \leq \int_0^1 d\tau |\Sigma_2^2(\gamma(\tau) - x_0)_1 - \Sigma_2^1(\gamma(\tau) - x_0)_2| \leq C d_E(x, x_0),$$

the same holds for  $e_{02}$ , where the  $1/\epsilon$  in the integrand is compensated by the  $\epsilon$  in the vector field.

Writing for  $N_1$  and  $N_2$

$$\begin{aligned} N_1 & \leq d_E(x, x_0) \int_0^1 d\tau |\sigma_j^k(\gamma(\tau)) - P_{x_0} \sigma_j^k(\gamma(\tau))|, \\ N_2 & \leq \int_0^1 d\tau |(x - x_0)_i \sigma_j^k(\gamma(\tau)) - (\gamma(t) - x_0)_i \sigma_j^k(x_0)| \end{aligned}$$

we obtain

$$\begin{aligned} |h(\gamma(t)) - P_{x_0} h(\gamma(t))| & \leq C \left( d_E^{1+\alpha}(x, x_0) + \sum_{ijk} \int_0^1 d\tau |(x - x_0)_i \sigma_j^k(\gamma(\tau)) - (\gamma(t) - x_0)_i \sigma_j^k(x_0)| \right. \\ & \left. + d_E(x, x_0) \sum_{jk} \int_0^1 d\tau |\sigma_j^k(\gamma(\tau)) - P_{x_0} \sigma_j^k(\gamma(\tau))| \right). \end{aligned} \tag{4.11}$$

Now if we set  $t = 1$ , inequality (4.11) reads

$$\begin{aligned} |h(x) - P_{x_0} h(x)| & \leq C \left( d_E^{1+\alpha}(x, x_0) + d_E(x, x_0) \sum_{jk} \int_0^1 d\tau |\sigma_j^k(\gamma(\tau)) - \sigma_j^k(x_0)| \right. \\ & \left. + d_E(x, x_0) \sum_{jk} \int_0^1 d\tau |\sigma_j^k(\gamma(\tau)) - P_{x_0} \sigma_j^k(\gamma(\tau))| \right) \\ & \leq C \left( d_E^{1+\alpha}(x, x_0) + d_E(x, x_0) \sum_{jk} \int_0^1 d\tau |\sigma_j^k(\gamma(\tau)) - P_{x_0} \sigma_j^k(\gamma(\tau))| \right) \end{aligned}$$

so that in order to complete the proof we need only to prove the following claim

$$|\sigma_j^k(\gamma(\tau)) - P_{x_0}\sigma_j^k(\gamma(\tau))| \leq d_E(x, x_0).$$

To do so, we make use of inequality (4.11) in the case  $h(x) = \sigma_\alpha^\beta(x)$ , where for  $x$  sufficiently close to  $x_0$  the last term at the right-hand side can be reabsorbed, so the claim follows by noting that

$$\begin{aligned} & |(x - x_0)_i \sigma_j^k(\gamma(\tau)) - (\gamma(t) - x_0)_i \sigma_j^k(x_0)| \\ & \leq |(x - x_0)_i| |\sigma_j^k(\gamma(\tau)) - \sigma_j^k(x_0)| + |(x - \gamma(t))_i| |\sigma_j^k(x_0)| \\ & \leq C d_E(x, x_0). \end{aligned}$$

This proves (4.9) and hence the full statement.  $\square$

**Remark 4.4.** The difference between the frozen vector fields and the original ones are expressed by

$$\begin{aligned} X_1 - X_{01} &= (\sigma_1^i(x) - P_{x_0}\sigma_1^i(x) - s^2\sigma_2^i(x))\partial_{x_i}, \\ X_2 - X_{02} &= \epsilon(\sigma_2^i(x) - P_{x_0}\sigma_2^i(x))\partial_{x_i}. \end{aligned}$$

In view of Lemma 4.3, and because of the choice of lifting up to step three, we get the estimate

$$\begin{aligned} |X_i z - X_{0i} z| &\leq d_{0\epsilon}^{1+\alpha}((x, s), (x_0, 0)) |\nabla_E z| \\ &\leq d_{0\epsilon}^{1+\alpha}((x, s), (x_0, 0)) (|\nabla_\epsilon z| + |Yz|). \end{aligned} \tag{4.12}$$

Indeed if we had  $s$  instead of  $s^2$  that would not hold true. This will be a key step for concluding the proof of Theorem 4.1.

#### 4.2. The fundamental solution of the frozen operator

If we extend to three dimensions the  $2 \times 2$  matrix  $a_{ij}(\nabla_\epsilon u(x_0))$  to a  $3 \times 3$  matrix keeping the same form (3.2) but with respect to the lifted vector fields (4.3), i.e. we define

$$a_{0ij} = a_{0ij}(\tilde{\nabla}u(x_0, 0)) = \delta_{ij} - \frac{\tilde{X}_i u(x_0, 0)\tilde{X}_j u(x_0, 0)}{1 + |\tilde{\nabla}u(x_0, 0)|^2}$$

then with the new vector fields (4.7) we can define a uniformly subelliptic operator acting on a smooth function  $z$  generally defined on  $\Omega \times (-1, 1)$

$$N_0 z \doteq \sum_{i,j \in \{1,3\}} a_{0ij} X_{0i} X_{0j} z \tag{4.13}$$

that, due to the uniform ellipticity of the matrix  $a_0$ , behaves like a sublaplacian with respect to the corresponding stratified Lie algebra. Its elliptic regularization contains also the vector field  $X_{02}$  and reads

$$N_{0\epsilon} z \doteq \sum_{i,j=1}^3 a_{0ij} X_{0i} X_{0j} z. \tag{4.14}$$

Operator (4.14) will be used in this section to approximate the original operator in nondivergence form (4.1). The main advantage in this approximation consists of the fact that the limit operator for (4.14) is subelliptic, while the limit operator for (4.1) is not.

We will make use of the fundamental solution to (4.14), which will be denoted by  $\Gamma_{0\epsilon}(\xi|\eta)$ . Its variables  $\xi, \eta$  are then in  $\Omega \times (-1, 1)$ . For the sake of simplicity we will also make use of the following notation

$$(\Gamma_{0\epsilon} * f)(\xi) = \int \Gamma_{0\epsilon}(\xi|\eta) f(\eta) d\eta,$$

and when a derivation is applied to  $\Gamma_{0\epsilon}(\xi|\eta)$ , we will mean it is made with respect to its first variable.

The following estimates on the fundamental solution are proved in [14].

**Theorem 4.5.** Let  $\Gamma_{0\epsilon}(\xi|\eta)$  be the fundamental solution of (4.14) and  $x_0 \in \Omega$ . Then for all compact sets  $K \subseteq \Omega \times (-1, 1)$ , for all  $k \in \mathbb{N}$  and every multiindex  $I = (i_1, \dots, i_k)$  with  $i_j \in \{1, 2, 3\}$  there exists a positive constant  $C$  independent of  $\epsilon$  such that

$$|\nabla_{0\epsilon}^I \Gamma_{0\epsilon}(\xi|\eta)| \leq C \frac{d_{0\epsilon}^{2-k}(\xi, \eta)}{|B_{0\epsilon}(\xi, d_{0\epsilon}(\xi, \eta))|}.$$

**Proposition 4.6.** Let  $f$  be in  $L^q(\Omega)$  and extend it to a function on  $\Omega \times (-1, 1)$  by setting  $f(x, s) \doteq f(x)$ . Let  $\mathcal{K} : [\Omega \times (-1, 1)]^2 \rightarrow \mathbb{R}$  be a kernel satisfying

$$|\mathcal{K}(\xi, \eta)| \leq C \frac{d_{0\epsilon}^k(\xi, \eta)}{|B_{0\epsilon}(\xi, d_{0\epsilon}(\xi, \eta))|}. \tag{4.15}$$

Then a function  $u(\xi)$  defined as

$$u(\xi) = \int \mathcal{K}(\xi, \eta) f(\eta) d\eta$$

and such that  $u(x, s) = u(x)$  satisfies

$$\|u\|_{L^r(\Omega)} \leq C \|f\|_{L^q(\Omega)}$$

with  $5 - \kappa q > 0$  and  $r = \frac{5q}{5 - \kappa q}$ .

The idea is that, by the homogeneity of the measure on stratified groups, the requirement (4.15) implies that  $\mathcal{K}$  is in  $L^s$  with  $s < \frac{Q}{Q - \kappa}$  where  $Q$  is the homogeneous dimension of the group, in this case  $Q = 5$ . An argument like the standard Young inequality for convolutions implies then  $\|u\|_r \leq C \|\mathcal{K}\|_s \|f\|_q$  with  $1/s + 1/q = 1 + 1/r$ , so  $r < \frac{Qq}{Q - \kappa q}$ .

For the sake of clarity, we reproduce here the main arguments of the proof, addressing the reader to [17] and [7] for the full detailed computation of representation formulae.

**Proof of Theorem 4.1.** Let  $z$  be a classical solution to  $N_{\epsilon, \mu} z = 0$ , hence if considered as a function on  $\Omega \times (-1, 1)$  it will not have dependence on the  $s$  variable, i.e.  $z(\xi) = z(x_0, 0)$ . Using the convention of summation over repeated indices, by definition of fundamental solution we have

$$\begin{aligned} z\phi &= \Gamma_{0\epsilon} * N_{0\epsilon} z\phi \\ &= \Gamma_{0\epsilon} * (zN_{0\epsilon}\phi + 2a_{0ij}(X_{0i}z)(X_{0j}\phi)) + \Gamma_{0\epsilon} * (N_{0\epsilon}z - N_{\epsilon}z)\phi, \end{aligned}$$

where the term involving the difference between the freezed and the original operator, since  $z$  does not depend on  $s$ , is

$$\begin{aligned} N_{0\epsilon}z - N_{\epsilon}z &= a_{0ij}X_{0i}X_{0j}z - a_{ij}(\nabla_{\epsilon}u)X_iX_jz \\ &= (a_{0ij} - a_{ij}(\nabla_{\epsilon}u))X_iX_jz + a_{0ij}(X_{0i}X_{0j}z - X_iX_jz), \end{aligned}$$

and in particular the last term can be written as

$$X_{0i}X_{0j}z - X_iX_jz = (X_{0i} - X_i)X_jz + X_{0i}(X_{0j} - X_j)z.$$

We then obtain the representation formula

$$\begin{aligned} z\phi &= \Gamma_{0\epsilon} * (zN_{0\epsilon}\phi + 2a_{0ij}(X_{0i}z)(X_{0j}\phi)) + \Gamma_{0\epsilon} * (\phi(a_{0ij} - a_{ij}(\nabla_{\epsilon}u))X_iX_jz) \\ &\quad + \Gamma_{0\epsilon} * (\phi a_{0ij}(X_{0i} - X_i)X_jz) + \Gamma_{0\epsilon} * (a_{0ij}(X_{0i}\phi)(X_{0j} - X_j)z) \\ &\quad + X_{0i}\Gamma_{0\epsilon} * (a_{0ij}\phi(X_{0j} - X_j)z). \end{aligned} \tag{4.16}$$

To get a  $W^{2,p}$  estimate of  $z$  we perform second derivatives on the above representation formula. Arguing as in [7], we can represent frozen second derivatives at the freezing point  $x_0$ , i.e.  $X_{0k}X_{0l}z\phi(x_0, 0)$ , simply by putting the derivatives on  $\Gamma_{0\epsilon}$ , hence obtaining a representation formula similar to the above one but with  $X_{0k}X_{0l}\Gamma_{0\epsilon}((x_0, 0), \eta)$ . Proceeding in this direction, the worst term to estimate is then the one corresponding to the last term in (4.16), since it contains third derivatives of  $\Gamma_{0\epsilon}$ . Making use of Remark 4.4, it becomes

$$\begin{aligned} & \int |X_{0k} X_{0l} X_{0i} \Gamma_{0\epsilon}((x_0, 0)|\eta)(a_{0ij} \phi(X_{0j} - X_j)z(\eta))| d\eta \\ & \leq C \int \mathcal{K}((x_0, 0), \eta)(|\nabla_\epsilon z(\eta)| + |Yz(\eta)|)\phi d\eta, \end{aligned}$$

where by (4.12) and Theorem 4.5

$$\mathcal{K} = |X_{0k} X_{0l} X_{0i} \Gamma_{0\epsilon}((x_0, 0)|\eta)|d_{0\epsilon}^{1+\alpha}(x, x_0) \leq C \frac{d_{0\epsilon}^\alpha((x_0, 0), \eta)}{|B_{0\epsilon}((x_0, 0), d_{0\epsilon}((x_0, 0), \eta))|}.$$

Applying Proposition 4.6 to this term and handling the others in the same way, we get the boundedness of the function  $x_0 \rightarrow |X_{0k} X_{0l} z(x_0)|$  in the  $L^r$  norm, with  $r = \frac{5q}{5-\alpha q}$  where  $q$  is by hypothesis bigger than  $10/3$ , so also  $r > 10/3$ .  $\square$

### 5. Uniform Sobolev regularity for the nonlinear approximating PDE

In this section we unify the results obtained up to now, coming back to the nonlinear regularized equation (1.4). We will prove that

**Theorem 5.1.** *Let  $u$  be a smooth solution to  $L_{\epsilon,u}u = 0$  in  $\Omega \subset \mathbb{R}^2$ , then for every open set  $\Omega_1 \Subset \Omega$ , every  $p \geq 3$  and every  $m \geq 2$  there exists a positive constant  $C$  depending on  $p, m, \Omega_1$  and the constant  $M$  of (1.5) but is independent of  $\epsilon$  such that*

$$\|u\|_{W_\epsilon^{m,p}(\Omega_1)} + \|Yu\|_{W_\epsilon^{m,p}(\Omega_1)} \leq C.$$

#### 5.1. First iteration step

##### 5.1.1. First Euclidean Caccioppoli inequality: $C^{1,\alpha}$ regularity

We start providing a regularity sufficient to apply Theorem 4.1. This can be obtained by an Euclidean Caccioppoli type estimate for solutions to (1.4), uniform in  $\epsilon$ , which allows to perform a classical Moser iteration.

**Lemma 5.2.** *Let  $u$  be a smooth solution of  $L_{\epsilon,u}u = 0$ , then for every  $q \geq 2$  there exists a positive constant  $C$  independent of  $\epsilon$  such that for any cutoff function  $\phi$  it holds*

$$\int |\nabla_E(|\nabla_\epsilon u|^{q/2})|^2 \phi^{2p} \leq C \int |\nabla_\epsilon u|^q (\phi^2 + |\nabla_E \phi|^2) \phi^{2p-2}.$$

**Proof.** We first prove the following claim:

$$\int |\nabla_\epsilon(|\nabla_\epsilon u|^{q/2})|^2 \phi^{2p} \leq C \int |\nabla_\epsilon u|^q (\phi^2 + |\nabla_E \phi|^2) \phi^{2p-2}. \tag{5.1}$$

In order to do this, we note that setting  $z = X_k u$  we can rewrite the conclusion of Lemma 3.7 as

$$\int |\nabla_\epsilon(|z|^{q/2})|^2 \phi^{2p} \leq C \left( \int |z|^q (\phi^2 + |\nabla_\epsilon \phi|^2) \phi^{2p-2} + \int |f||z|^{q-1} \phi^{2p} \right)$$

by choosing the exponent  $p$  of the function  $z$  as equal to  $q - 1$  while leaving unvaried the exponent of the cutoff function (the proof of Lemma 3.7 shows that the two exponents need not to be the same). Let us now look at the last term. By Corollary 3.4 we have

$$|f| \leq C(|\nabla_\epsilon u| + |\nabla_\epsilon z|)$$

so that

$$\begin{aligned} \int |f||z|^{q-1} \phi^{2p} & \leq C \left( \int |\nabla_\epsilon u|^q \phi^{2p} + \int |\nabla_\epsilon z||z|^{q-1} \phi^{2p} \right) \\ & \leq C \left[ \left(1 + \frac{1}{\delta}\right) \int |\nabla_\epsilon u|^q \phi^{2p} + \delta \int |\nabla_\epsilon z|^2 |z|^{q-2} \phi^{2p} \right) \end{aligned}$$

and

$$|\nabla_\epsilon z|^2 |z|^{q-2} = \frac{4}{q^2} |\nabla_\epsilon (|z|^{q/2})|^2.$$

Noting then that  $|\nabla_\epsilon \phi| \leq |\nabla_E \phi|$ , claim (5.1) is proved. The same argument holds for  $|Y(|\nabla_\epsilon u|^{q/2})|^2$ , which provides the full statement since for any given function  $h$

$$|\nabla_E h| \leq C(|\nabla_\epsilon h| + |Yh|). \quad \square \tag{5.2}$$

The previous lemma allows us to obtain a statement analogous to that of [8, Proposition 3.7], i.e.

**Theorem 5.3.** *Let  $u$  be a solution of  $L_{\epsilon,u}u = 0$  in  $\Omega \subset \mathbb{R}^2$ , then for every compact set  $K \Subset \Omega$  there exists a positive constant  $C$  such that*

$$\|u\|_{C_\epsilon^{1,\alpha}(K)} + \|u\|_{W_\epsilon^{2,2}(K)} + \|Yu\|_{W_\epsilon^{1,2}(K)} \leq C$$

where the class  $C_\epsilon^{1,\alpha}$  is the same as was defined for Theorem 4.1.

Indeed the previous Caccioppoli estimates works precisely on the  $\nabla_\epsilon$  derivatives of  $u$ , so  $C_\epsilon^{1,\alpha}$  is the natural class that can result from a Moser iteration.

We note that this is sufficient to apply Theorem 4.1, since when dealing with the nonlinear equation there is not anymore difference between “solution”  $z$  and “coefficients”  $u$ , so in particular the  $L_{loc}^p$  requirement for  $Yz$  is implied by the  $C^{1,\alpha}$  boundedness of  $u$ .

5.1.2. *Second Euclidean Caccioppoli inequality: higher integrability*

Proceeding analogously to the proof of Proposition 3.6 and Lemma 3.8, we obtain the following interpolation inequality and a priori Caccioppoli estimate.

**Proposition 5.4** (Second interpolation inequality). *For every  $p \geq 3$  there exists a positive constant  $C$  dependent only on  $p$  and the constant  $M$  in (1.5) such that for every function  $z \in C^\infty(\Omega)$ , every cutoff function  $\phi$  and every  $\delta > 0$  it holds*

$$\int |X_i z|^{p+1} \phi^{2p} \leq C \left[ \int (z^{p+1} \phi^{2p} + z^2 |X_i z|^{p-1} \phi^{2(p-1)} |X_i \phi|^2) + \int |X_i^2 z|^2 |X_i z|^{p-3} z^2 \phi^{2p} \right].$$

**Corollary 5.5.** *Let  $z$  be a smooth solution of Eq. (3.4) and let  $f$  be locally summable in  $\Omega$ , then for all  $p \geq 3$  there exists a positive constant  $C'_2$  depending only on  $p$  and the constant  $M$  in (1.5) such that for any cutoff function  $\phi$  it holds*

$$\begin{aligned} & \int |\nabla_\epsilon (|\nabla_\epsilon z|^{(p-1)/2})|^2 \phi^{2p} \\ & \leq C'_2 \left( \delta \int |\nabla_\epsilon z|^{p+1} \phi^{2p} + \frac{1}{\delta} \int |\nabla_\epsilon^2 u|^{p+1} \phi^{2p} + \frac{1}{\delta} \int |\nabla_\epsilon Yu|^{(p+1)/2} \phi^{2p} + \frac{1}{\delta} \int |f|^{(p+1)/2} \phi^{2p} + 1 \right). \end{aligned}$$

These two results allow to obtain a second Euclidean Caccioppoli estimate for solutions to the nonlinear elliptic regularized equation, which combined with the Euclidean Sobolev inequality will provide the first step of the iteration.

**Lemma 5.6.** *Let  $u$  be a smooth solution of  $L_{\epsilon,u}u = 0$ . Then there exists a positive constant  $C$  independent of  $\epsilon$  such that its derivatives  $z = X_k u$  satisfy*

$$\int |\nabla_E \nabla_\epsilon z|^2 \phi^6 \leq C \left( \int |\nabla_\epsilon z|^4 \phi^6 + 1 \right).$$

**Proof.** Let us indicate  $v = Yu$  and  $z = X_k u$  and denote by  $C$  a positive constant depending only on  $p, M$  allowed to change from line to line.

By Corollary 5.5, choosing  $p = 3$  and using Corollaries 3.4 and 3.5, we get

$$\int |\nabla_\epsilon^2 z|^2 \phi^6 \leq C \left( \delta \int |\nabla_\epsilon z|^4 \phi^6 + \frac{1}{\delta} \left( \int |\nabla_\epsilon^2 u|^4 \phi^6 + \int |\nabla_\epsilon v|^2 \phi^6 + 1 \right) \right),$$

$$\int |\nabla_\epsilon^2 v|^2 \phi^6 \leq C \left( \delta \int |\nabla_\epsilon v|^4 \phi^6 + \frac{1}{\delta} \left( \int |\nabla_\epsilon^2 u|^4 \phi^6 + 1 \right) \right).$$

Moreover, by Proposition 5.4 with  $p = 3$  and using assumption (2.5) we obtain

$$\int |\nabla_\epsilon z|^4 \phi^6 \leq C \left( \int |\nabla_\epsilon z|^2 \phi^4 |\nabla_\epsilon \phi|^2 + \int |\nabla_\epsilon^2 z|^2 \phi^6 \right),$$

$$\int |\nabla_\epsilon v|^4 \phi^6 \leq C \left( \int |\nabla_\epsilon v|^2 \phi^4 |\nabla_\epsilon \phi|^2 + \int |\nabla_\epsilon^2 v|^2 \phi^6 \right),$$

where the first term is bounded by Theorem 5.3, so that

$$\int |\nabla_\epsilon z|^4 \phi^6 \leq C \left( \int |\nabla_\epsilon^2 z|^2 \phi^6 + 1 \right),$$

$$\int |\nabla_\epsilon v|^4 \phi^6 \leq C \left( \int |\nabla_\epsilon^2 v|^2 \phi^6 + 1 \right) \tag{5.3}$$

that provide the two intrinsic Caccioppoli inequalities

$$\int |\nabla_\epsilon^2 z|^2 \phi^6 \leq C \left( \int |\nabla_\epsilon^2 u|^4 \phi^6 + 1 \right),$$

$$\int |\nabla_\epsilon^2 v|^2 \phi^6 \leq C \left( \int |\nabla_\epsilon^2 u|^4 \phi^6 + 1 \right). \tag{5.4}$$

Noting that

$$\begin{aligned} |\nabla_E X_j z| &\leq |X_1 X_j z| + |Y X_j z| \leq |\nabla_\epsilon^2 z| + |X_j Y X_k u| + \frac{1}{\epsilon} |\omega^l X_l z| \\ &\leq |\nabla_\epsilon^2 z| + |X_j X_k Y u| + \frac{1}{\epsilon} |X_j \omega^l X_l u| + \frac{1}{\epsilon} |\omega^l X_l z| \\ &\leq |\nabla_\epsilon^2 z| + |\nabla_\epsilon^2 v| + \frac{1}{\epsilon} |(X_j \omega^l) X_l u| + \frac{1}{\epsilon} |\omega^l X_j X_l u| + \frac{1}{\epsilon} |\omega^l X_l z|, \end{aligned}$$

where

$$\begin{aligned} \frac{1}{\epsilon} |(X_j \omega^l) X_l u| &= \left| \left( X_j \frac{\omega^1}{\epsilon} \right) X_1 u + (X_j \omega^2) Y u \right| \leq C |\nabla_\epsilon^2 u|, \\ \frac{1}{\epsilon} |\omega^l X_j X_l u| &= \left| \frac{\omega^1}{\epsilon} X_j X_1 u + \omega^2 X_j Y u \right| \leq C (|\nabla_\epsilon^2 u| + |\nabla_\epsilon v|), \\ \frac{1}{\epsilon} |\omega^l X_l z| &= \left| \frac{\omega^1}{\epsilon} X_1 z + \omega^2 Y z \right| \leq C (|\nabla_\epsilon z| + |\nabla_\epsilon v| + 1) \end{aligned}$$

so that

$$|\nabla_E X_j z| \leq C (|\nabla_\epsilon^2 z| + |\nabla_\epsilon^2 v| + |\nabla_\epsilon z| + |\nabla_\epsilon v| + |\nabla_\epsilon^2 u| + 1) \tag{5.5}$$

we end up with the desired claim.  $\square$

**Proposition 5.7.** *Let  $u$  be a solution of  $L_{\epsilon,u} u = 0$  in  $\Omega \subset \mathbb{R}^2$ , then for every  $\Omega_1 \Subset \Omega$  and every  $p \geq 1$  there exists a positive constant  $C$  independent of  $\epsilon$  such that*

- (i)  $\|u\|_{W_\epsilon^{2,p}(\Omega_1)} \leq C,$
- (ii)  $\|Yu\|_{W_\epsilon^{1,p}(\Omega_1)} \leq C.$

**Proof.** To prove (i) we make use of Lemma 5.6 together with the Euclidean Sobolev inequality in  $\mathbb{R}^2$ . Let us indicate  $z = X_k u$  and denote with  $\phi$  a cutoff function. Then we get, for any  $2 \leq r < \infty$

$$\begin{aligned} \left( \int (|\nabla_\epsilon z| \phi^3)^r \right)^{2/r} &\leq \int |\nabla_E(\nabla_\epsilon z \phi^3)|^2 \leq C \left[ \int |\nabla_\epsilon z|^4 \phi^6 + 1 \right] \\ &\leq C \left[ \left( \int (|\nabla_\epsilon z|^\alpha \phi^{3\alpha})^q \right)^{\frac{1}{q}} \left( \int (|\nabla_\epsilon z|^{4-\alpha} \phi^{6-3\alpha})^{\frac{q}{q-1}} + 1 \right)^{\frac{q-1}{q}} \right], \end{aligned}$$

where the last transition is Hölder inequality. We now choose  $\alpha$  and  $q$  such that  $\alpha q = r$  and  $1/q = 2/r$ , i.e.  $\alpha = 2$  and  $q = r/2$  so that

$$\left( \int (|\nabla_\epsilon z| \phi^3)^r \right)^{2/r} \leq C \left[ \left( \int (|\nabla_\epsilon z| \phi^3)^r \right)^{\frac{2}{r}} \left( \int_{\text{supp}(\phi)} |\nabla_\epsilon z|^{\frac{2r}{r-2}} \right)^{\frac{r-2}{r}} + 1 \right]. \tag{5.6}$$

By Theorem 5.3 we can apply Theorem 4.1(i), so that if we set  $r = 6$  we have  $2r/(r - 2) = 3$ , and

$$\int_{\text{supp}(\phi)} |\nabla_\epsilon z|^3 \leq \left( \int_{\text{supp}(\phi)} |\nabla_\epsilon z|^{\frac{10}{3}} \right)^{9/10} |\text{supp}(\phi)|^{1/10}$$

can be chosen arbitrarily small with the support of  $\phi$ . We emphasize that for this point it was crucial to have an integrability up to a power higher than 3. This could be obtained through estimates which could be applied with the full strength of the Hörmander setting by making use of the freezing technique.

Since estimate (5.6) holds for  $k = 1, 2$  we now obtain

$$\int |\nabla_\epsilon^2 u|^6 \phi^{18} \leq C.$$

By inequalities (5.3) and (5.4) we then have

$$\begin{aligned} \int |\nabla_\epsilon v|^4 \phi^6 &\leq \int |\nabla_\epsilon^2 v|^2 \phi^6 \leq C \left( \int |\nabla_\epsilon^2 u|^4 \phi^6 + 1 \right) \\ &\leq C \left( \int |\nabla_\epsilon^2 u|^6 \phi^{18} + \int_{\text{supp}(\phi)} |\nabla_\epsilon^2 u|^3 + 1 \right) \leq C. \end{aligned} \tag{5.7}$$

Now since

$$|\nabla_E \nabla_\epsilon u| \leq C (|\nabla_\epsilon^2 u| + |\nabla_\epsilon v| + 1)$$

we have proved that  $|\nabla_E \nabla_\epsilon u| \in L^4_{\text{loc}}$ , and consequently by the Euclidean Sobolev–Morrey inequality in  $\mathbb{R}^2$  we have that  $\nabla_\epsilon u$  is Euclidean Hölder continuous of order  $1/2$ . This way we can apply the conclusion of Theorem 4.1(ii) and obtain point (i).

For point (ii) we note that we can apply Theorem 3.9 to  $v = Yu$ :

$$\|v\|_{W^{1,p+1/2}(\Omega)}^{p+1/2} \leq C_3 \left( \|v\|_{L^{4p+2}(\Omega)}^{4p+2} + \|u\|_{W^{2,(4p+2)/3}(\Omega)}^{(4p+2)/3} + \|v\|_{W^{1,(2p+1)/3}(\Omega)}^{(2p+1)/3} + \|fv\|_{L^{(4p+2)/7}(\Omega)}^{(4p+2)/7} + 1 \right),$$

where at the right-hand side the first term is bounded by assumption (2.5), the second term is point (i) we proved, and the third and fourth terms can be bounded using Corollary 3.5 by

$$\|\nabla_\epsilon v\|_{L^{(2p+1)/3}(\Omega)}^{(2p+1)/3} + \|fv\|_{L^{(4p+2)/7}(\Omega)}^{(4p+2)/7} \leq C \|\nabla_\epsilon v\|_{L^p(\Omega)}^p$$

that is bounded for  $p = 4$  by (5.7) and then for all  $p$ .  $\square$



5.2. Proof of the main iteration step

**Lemma 5.8.** *Let us assume (1.5). Then there exists a positive constant  $C$  depending only on  $\Omega$  and  $M$  of (1.5) such that for every  $\delta$  sufficiently small it holds*

$$\int |\nabla_\epsilon z|^{7/3} \phi^4 \leq C \left( \delta \int |\nabla_\epsilon^2 z|^2 \phi^4 + \delta \int |\nabla_\epsilon z|^2 \phi^4 + \frac{1}{\delta} \int |z|^3 (|\nabla_\epsilon \phi|^3 + \phi^3) \phi + 1 \right)$$

for every function  $z \in C^\infty(\Omega)$  and every cutoff function  $\phi$ .

**Proof.** The claim follows by choosing  $p = 2$  in the following interpolation inequality

$$\begin{aligned} \int |\nabla_\epsilon z|^{p+1/3} \phi^{2p} &\leq C \left( \delta \int |\nabla_\epsilon (|\nabla_\epsilon z|^{p/2})|^2 \phi^{2p} + \delta \int |\nabla_\epsilon z|^p \phi^{2p} \right. \\ &\quad \left. + \frac{1}{\delta} \int |z|^{3p/2} (|\nabla_\epsilon \phi|^{3p/2} + \phi^{3p/2}) \phi^{p/2} + 1 \right) \end{aligned}$$

which can be proved in the same way as Proposition 3.6.  $\square$

**Proof of Theorem 5.1.** Let us denote  $z = X_k u$  and  $v = Yu$ . Then by Theorem 3.1(i) we have

$$\begin{aligned} \|z\|_{W^{m,p+1/2}(\Omega)}^{p+1/2} + \|v\|_{W^{m,p+1/2}(\Omega)}^{p+1/2} &\leq C_4 \left( \|z\|_{W^{m-1,4p+2}(\Omega)}^{4p+2} + \|v\|_{W^{m-1,4p+2}(\Omega)}^{4p+2} + \|fz\|_{W^{m-1,(4p+2)/7}(\Omega)}^{(4p+2)/7} \right. \\ &\quad \left. + \|fv\|_{W^{m-1,(4p+2)/7}(\Omega)}^{(4p+2)/7} + \|u\|_{W^{m,(4p+2)/3}(\Omega)}^{(4p+2)/3} \|u\|_{W^{m+1,(2p+1)/3}(\Omega)}^{(2p+1)/3} + 1 \right) \\ &\leq C \left( \|z\|_{W^{m-1,4p+2}(\Omega)}^{4p+2} + \|v\|_{W^{m-1,4p+2}(\Omega)}^{4p+2} \right. \\ &\quad \left. + \|z\|_{W^{m,(2p+1)/3}(\Omega)}^{(2p+1)/3} + \|v\|_{W^{m,(2p+1)/3}(\Omega)}^{(2p+1)/3} + 1 \right), \end{aligned} \tag{5.8}$$

where the last transition holds due to Corollaries 3.4 and 3.5 and Hölder inequality. Let us now rewrite the obtained inequality setting  $q = p + 1/2$ :

$$\|z\|_{W^{m,q}(\Omega)}^q + \|v\|_{W^{m,q}(\Omega)}^q \leq C \left( \|z\|_{W^{m-1,4q}(\Omega)}^{4q} + \|v\|_{W^{m-1,4q}(\Omega)}^{4q} + \|z\|_{W^{m,\frac{2}{3}q}(\Omega)}^{\frac{2}{3}q} + \|v\|_{W^{m,\frac{2}{3}q}(\Omega)}^{\frac{2}{3}q} + 1 \right).$$

By Proposition 5.7(i) and (ii) the first two terms at the right-hand side are bounded for  $m = 2$ , provided a restriction to a subsets  $\Omega_1$  compactly contained in  $\Omega$ . We want to prove now, by induction, that for all  $m \geq 2$

$$\|z\|_{W^{m,q}(\Omega_1)}^q + \|v\|_{W^{m,q}(\Omega_1)}^q \leq C \left( \|z\|_{W^{m,\frac{2}{3}q}(\Omega_1)}^{\frac{2}{3}q} + \|v\|_{W^{m,\frac{2}{3}q}(\Omega_1)}^{\frac{2}{3}q} + 1 \right). \tag{5.9}$$

The induction scheme reads as follows: given that for any  $m$  it holds

$$\|h\|_{m,q}^q \leq C \left( \|h\|_{m-1,4q}^{4q} + \|h\|_{m,\frac{2}{3}q}^{\frac{2}{3}q} + 1 \right) \tag{5.10}$$

and given the starting point  $m = 2$

$$\|h\|_{2,q}^q \leq C \left( \|h\|_{2,\frac{2}{3}q}^{2/3q} + 1 \right)$$

we want to prove that if for a fixed  $\bar{m}$  we have

$$\|h\|_{\bar{m},q}^q \leq C \left( \|h\|_{\bar{m},\frac{2}{3}q}^{\frac{2}{3}q} + 1 \right) \tag{5.11}$$

then it holds

$$\|h\|_{\bar{m}+1,q}^q \leq C \left( \|h\|_{\bar{m}+1,\frac{2}{3}q}^{\frac{2}{3}q} + 1 \right).$$

This is done using (5.10)

$$\|h\|_{\bar{m}+1,q}^q \leq C \left( \|h\|_{\bar{m},4q}^{4q} + \|h\|_{\bar{m}+1,\frac{2}{3}q}^{\frac{2}{3}q} + 1 \right)$$

and noting that (5.11) implies that  $\|h\|_{\frac{q}{m},q}^q$  is bounded for all  $q$  by bootstrapping, provided we can get a starting point. This is given by  $q = 3$ , using Theorem 3.1(ii). Indeed, repeating the argument used for inequality (5.8) with  $p = 3$  we get

$$\begin{aligned} & \|z\|_{W^{m+1,2}(\Omega_1)}^2 + \|v\|_{W^{m+1,2}(\Omega_1)}^2 \\ & \leq C(\|z\|_{W^{m-1,14}(\Omega_1)}^{14} + \|v\|_{W^{m-1,14}(\Omega_1)}^{14} + \|z\|_{W^{m,7/3}(\Omega_1)}^{7/3} + \|v\|_{W^{m,7/3}(\Omega_1)}^{7/3} + 1), \end{aligned}$$

where at the right-hand side the first two terms are bounded for  $m = 2$  by Proposition 5.7, and the other two terms can be reabsorbed using the interpolation given by Lemma 5.8. By (5.9), bootstrapping on  $q$  we then obtain the desired result.  $\square$

### 6. Intrinsic regularity for vanishing viscosity solutions

This section is devoted to the proof of regularity for vanishing viscosity solutions introduced in Definition 1.1, which provides a foliation result for intrinsic minimal graphs, i.e. solutions to Eq. (1.3). Regularity estimates are obtained by making use of the estimates proved in the previous section, since they were uniform in  $\epsilon$ . In particular we will obtain Hölder continuity, that in definitive will be sufficient for  $C^\infty$  regularity, in the horizontal direction, while no regularity is expected in the transverse direction, in accordance with the geometry outlined by the foliation.

To come back to the notation of Definition 1.1, we will indicate by  $u$  a vanishing viscosity solution of (1.3) and by  $(u_n)$  its approximating sequence referring to the vanishing positive real sequence  $(\epsilon_n)$ . The corresponding nonlinear vector fields will be denoted by  $X_{1,n} = \sigma_1^j(x_1, x_2, u_n(x_1, x_2))\partial_{x_j}$ ,  $X_{2,n} = \epsilon_n \sigma_2^j(x_1, x_2, u_n(x_1, x_2))\partial_{x_j}$  and  $Y_n = \frac{1}{\epsilon_n} X_{2,n}$ . Accordingly, we will use  $\nabla_{\epsilon_n}$  and  $W_{\epsilon_n}^{k,p}$  for the natural gradient and Sobolev spaces. For the limit equation we will use the notations  $X = X_{1,\infty} = \sigma_1^j(x_1, x_2, u(x_1, x_2))\partial_{x_j}$ ,  $\nabla_0 = (X, 0)$ ,  $W_0^{k,p}$ , while  $\nabla_E$  and  $W_E^{k,p}$  still stand for the usual Euclidean gradient and Sobolev spaces.

**Theorem 6.1.** *Let  $u$  be a vanishing viscosity solution of Eq. (1.3). Then*

$$Xu \in W_{E,\text{loc}}^{1,p}(\Omega) \quad \text{for all } p > 1. \tag{6.1}$$

**Proof.** To prove (6.1) it suffices to obtain the following estimate uniform in  $\epsilon$ : for every ball  $B \Subset \Omega$  and all  $p > 1$  there exists a positive constant  $C$  such that

$$\|\nabla_{\epsilon_n} u_n\|_{W_E^{1,p}(B)} \leq C \tag{6.2}$$

for all  $n$ , and this is a direct consequence of (5.2) and Theorem 5.1 with  $m = 2$ .  $\square$

**Proposition 6.2.** *Let  $u$  be a vanishing viscosity solution of Eq. (1.3). Then*

$$X_{1,n}u_n \rightarrow Xu, \quad X_{2,n}u_n \rightarrow 0 \tag{6.3}$$

as  $n \rightarrow +\infty$  weakly in  $W_{E,\text{loc}}^{1,2}(\Omega)$ . Moreover Eq. (1.3) can be represented as

$$X^2u = 0$$

and is satisfied weakly in the Sobolev sense, and hence, pointwise a.e. in  $\Omega$ , i.e.

$$\int_{\Omega} XuX^\dagger\phi = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

**Proof.** Since the approximating solution  $u_n$  is of class  $C^\infty$ , we can use the nondivergence form of Eq. (1.3)

$$\sum_{i,j=1}^2 a_{ij}(\nabla_{\epsilon_n} u_n) X_{i,n} X_{j,n} u_n = 0.$$

Here

$$a_{ij}(\nabla_{\epsilon_n} u_n) \rightarrow a_{ij}(\nabla_0 u) = \delta_{i1} \delta_{j1} \quad \text{in } L^p,$$

while (6.2) and the definition of vanishing viscosity solution imply

$$X_{1,n} u_n \rightarrow Xu, \quad X_{2,n} u_n \rightarrow 0$$

as  $n \rightarrow +\infty$  weakly in  $W_{E,\text{loc}}^{1,2}(\Omega)$ . Hence letting  $n$  go to  $\infty$  in the nondivergence form equation we conclude

$$X^2 u = 0$$

in the Sobolev sense.  $\square$

**Proof of Theorem 1.2.** By Theorem 5.1 with  $m = 3$  and inequality (5.2) we have that  $X^2 u \in W_{E,\text{loc}}^{1,p}(\Omega)$  for all  $p > 1$ . Then, by Sobolev–Morrey embedding in  $\mathbb{R}^2$ ,  $X^2 u$  is Hölder continuous, and this implies that the equation  $X^2 u = 0$ , satisfied weakly by the previous proposition, is satisfied pointwise everywhere in  $\Omega$ .  $\square$

We can now give a new pointwise definition of derivative in the direction of vector fields  $X_1$  and  $X_2$ .

**Definition 6.3.** Let  $V$  be a Lipschitz vector field on  $\Omega$  and let  $\xi_0 \in \Omega$  and  $\gamma(s)$  be a solution to problem  $\gamma' = V(\gamma)$ ,  $\gamma(0) = \xi_0$ . We say that a function  $f \in C_{\text{loc}}^\alpha(\Omega)$ , with  $\alpha \in ]0, 1[$ , has Lie derivative in the direction of the vector field  $V$  in  $\xi_0$  if there exists

$$\left. \frac{d}{ds}(f \circ \gamma) \right|_{s=0},$$

and we will denote its value by  $Vf(\xi_0)$ .

If the weak derivative of a function  $f$  is sufficiently regular, then the two notions of derivatives coincide. For the proof of the following result see [16, Proposition 5.2].

**Proposition 6.4.** *If  $f \in C_{\text{loc}}^\alpha(\Omega)$  for some  $\alpha \in ]0, 1[$  and its weak derivatives satisfy  $X_{1,u} f \in C_{\text{loc}}^\alpha(\Omega)$ ,  $Y_u f \in L_{\text{loc}}^p(\Omega)$  with  $p > 1/\alpha$ , then for all  $\xi \in \Omega$  the Lie derivative  $X_{1,u} f(\xi)$  exists and coincides with the weak one.*

We are now ready to prove the main result concerning the foliation:

**Proof of Corollary 1.3.** By Theorem 1.2 we have  $X^2 u = 0$ , and the previous proposition ensures that  $X(Xu)$  is a Lie derivative, so by Definition 6.3 we have

$$\frac{d^2}{ds^2} u(\gamma(s)) = X^2 u = 0. \quad \square$$

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