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Blockwise estimate of the fundamental matrix of linear singularly perturbed differential systems with small delay and its application to uniform asymptotic solution

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Abstract

A singularly perturbed system of linear differential equations with a small delay is considered. Estimates of blocks of the fundamental matrix solution to this system uniformly valid for all sufficiently small values of the parameter of singular perturbations are obtained in the cases of time-independent and time-dependent coefficients of the system. In the first case the system is considered on an infinite time-interval, while in the second case it is considered on a finite one. These estimates are applied to justify a uniform asymptotic solution of an initial-value problem for this system in both cases.

1. Introduction

The notion of the fundamental matrix solution is a basic one in the theory of linear differential systems without as well as with delay. Majority of results in this theory was obtained by application of this notion. Therefore, it is very important to study various properties of the fundamental matrix solution. In the present paper, we derive a blockwise estimate of the fundamental matrix solution (or, simply, the fundamental matrix) to the following system:

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$$\frac{dz(t)}{dt} = \int_{-h}^{0} \left[d_\eta A_\varepsilon(t,\eta)\right] z(t+\varepsilon\eta), \quad t \geqslant 0, \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter ($\varepsilon \ll 1$), $h > 0$ is a given constant independent of ε, the matrix $A_\varepsilon(t,\eta)$ and the vector z have the block form

$$A_\varepsilon(t,\eta) = \begin{pmatrix} A_1(t,\eta) & A_2(t,\eta) \\ (1/\varepsilon)A_3(t,\eta) & (1/\varepsilon)A_4(t,\eta) \end{pmatrix}, \qquad z = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1.2}$$

where the blocks $A_1(t,\eta)$ and $A_4(t,\eta)$ are of the dimensions $n \times n$ and $m \times m$, respectively, the blocks x and y are of the dimensions n and m, respectively.

Such a type of singularly perturbed systems with the small delay has a considerable interest in the theory as well as in applications. Theoretically, it is interesting because the system contains simultaneously two types of perturbations, associated with the small multiplier for a part of the derivatives and with the small delay. Practically, such systems are interesting because they can serve as mathematical models in engineering problems. Some examples of such problems and the corresponding references can be found in [9].

Estimates of the fundamental matrix for various particular cases of system (1.1)–(1.2) were obtained in a number of works in the open literature. Thus in [12], the system, containing only the "fast" variable y (the "fast" system) and a single pointwise delay, was considered. An exponent-type estimate of the fundamental matrix was obtained on a finite interval of the time. Further, this result was extended to the "fast" system with single pointwise and distributed delays [6], and to the "fast" system with the general type of delay [1]. In [5,10], a blockwise estimate of the fundamental matrix of the system, containing both ("slow" x and "fast" y) variables, was derived on a finite time-interval. In [5], the case of single pointwise and distributed delays was considered, while in [10], the case of multiple pointwise and distributed delays was studied. The case of time-independent system on an infinite time-interval was analysed in [11]. A blockwise estimate of the fundamental matrix was obtained for the system containing single pointwise and distributed delays. Various applications of the above mentioned results on the estimates of the fundamental matrix can be found in [1,5,6,9–12]. Note that estimates of the fundamental matrix for some singularly perturbed differential systems with nonsmall delay were obtained in [2,3,14]. Since singularly perturbed differential systems with small and nonsmall delay much differ each other, methods for analysis of these two types of singularly perturbed systems are essentially different.

In the present paper, we obtain blockwise estimates of the fundamental matrix of system (1.1)–(1.2) in the following two cases: (1) the matrix A_ε is time-independent and $t \in [0, +\infty)$ (Section 2); (2) the matrix A_ε is time-dependent and $t \in [0, T]$, where $T > 0$ is a given constant independent of ε (Section 3). In Section 4, a uniform asymptotic solution of system (1.1)–(1.2) with a given initial condition is constructed and justified in both cases. The justification of this asymptotic solution is based on the blockwise estimate of the fundamental matrix obtained in the previous sections.

The following main notations are applied in this paper:

(1) E^n is the n-dimensional real Euclidean space;
(2) $\|\cdot\|$ denotes the Euclidean norm of either a vector or a matrix with complex, in general, elements;
(3) I_n is the n-dimensional identity matrix;
(4) $\mathcal{C}$ denotes the set of complex numbers;
(5) $\mathrm{Re}\,\lambda$ denotes the real part of a complex number λ;
(6) $\mathrm{Im}\,\lambda$ denotes the imaginary part of a complex number λ;
(7) i denotes the imaginary unit, i.e., $i = \sqrt{-1}$;
(8) $\mathrm{Var}_{[a_1,a_2]} A(\eta)$ denotes the variation of a matrix-function $A(\eta)$ on the interval $\eta \in [a_1, a_2]$;
(9) $\mathrm{col}(x, y)$, where $x \in E^n$, $y \in E^m$, denotes the block vector with the upper block x and the lower block y.

2. The case of the time-independent matrix A_ε

We shall assume:

A1. The matrix-functions $A_j(\eta)$ $(j = 1, \ldots, 4)$ are given for $\eta \in (-\infty, +\infty)$ and satisfy the following conditions:

(a) $A_j(\eta) = 0$, $\forall \eta \geqslant 0$;
(b) $A_j(\eta) = A_j(-h)$, $\forall \eta \leqslant -h$;
(c) $A_j(\eta)$ is continuous from the left for $\eta \in (-h, 0)$;
(d) $A_j(\eta)$ has bounded variation on the interval $\eta \in [-h, 0]$.

Consider two systems associated with the original system (1.1)–(1.2). The first system is the reduced-order one

$$\frac{d\bar{x}(t)}{dt} = \bar{A}\bar{x}(t), \quad t \geqslant 0,\ \bar{x} \in E^n, \tag{2.1}$$

where

$$\bar{A} = \bar{A}_1 - \bar{A}_2\bar{A}_4^{-1}\bar{A}_3, \quad \bar{A}_j = \int_{-h}^{0} dA_j(\eta) \quad (j = 1, \ldots, 4). \tag{2.2}$$

Here, we assume that the matrix $\bar{A}_4$ is invertible.

The second system is the boundary-layer one

$$\frac{d\tilde{y}(\xi)}{d\xi} = \int_{-h}^{0} \left[dA_4(\eta)\right] \tilde{y}(\xi + \eta), \quad \xi \geqslant 0,\ \tilde{y} \in E^m. \tag{2.3}$$

We shall assume:

A2. All eigenvalues of the matrix $\bar{A}$ lie inside the left-hand half-plane.

A3. All roots μ of the equation

$$\det \tilde{\Delta}(\mu) = 0, \quad \tilde{\Delta}(\mu) = \int_{-h}^{0} \exp(\mu\eta)\, dA_4(\eta) - \mu I_m,$$

satisfy the inequality $\operatorname{Re}\mu < -\gamma$, where $\gamma > 0$ is some constant.

Obtaining the estimate of the fundamental matrix to system (1.1)–(1.2) in the case of the time-independent coefficients is based on the asymptotic analysis (as $\varepsilon \to +0$) of the set of roots of the characteristic equation of this system

$$\det \Delta(\lambda, \varepsilon) = 0, \quad \Delta(\lambda, \varepsilon) = \int_{-h}^{0} \exp(\varepsilon\lambda\eta)\, dA_\varepsilon(\eta) - \lambda I_{n+m}. \tag{2.4}$$

2.1. Asymptotic analysis of the set of roots of the characteristic equation

Consider the following equation for μ:

$$\det \hat{\Delta}(\mu, \varepsilon) = 0, \quad \hat{\Delta}(\mu, \varepsilon) = \varepsilon \int_{-h}^{0} \exp(\mu\eta)\, dA_\varepsilon(\eta) - \mu I_{n+m}. \tag{2.5}$$

It is clear that if, for any $\varepsilon > 0$, $\lambda(\varepsilon)$ is a root of Eq. (2.4), then $\mu(\varepsilon) = \varepsilon\lambda(\varepsilon)$ is a root of Eq. (2.5). Similarly, if $\mu(\varepsilon)$ is a root of (2.5), then $\lambda(\varepsilon) = \mu(\varepsilon)/\varepsilon$ is a root of (2.4).

Setting $\varepsilon = 0$ in (2.5), one obtains the equation

$$\mu^n \det \tilde{\Delta}(\mu) = 0. \tag{2.6}$$

Note that the set of roots of Eq. (2.6) consists of $\mu = 0$ and the set of roots of the equation $\det \tilde{\Delta}(\mu) = 0$.

Lemma 2.1. *Under assumptions* A1 *and* A3, *let* $\{\varepsilon_k\}$ *and* $\{\mu_k\}$ *be two sequences such that*:

(1) $\{\varepsilon_k\}$ *is positive and convergent to zero*;
(2) $\operatorname{Re}\mu_k \geqslant -\gamma$;
(3) *Eq.* (2.5) *is satisfied for all* $(\mu, \varepsilon) = (\mu_k, \varepsilon_k)$ $(k = 1, 2, \ldots)$.

Then the sequence $\{\mu_k\}$ *converges to zero.*

Proof (by contradiction). Assume that the statement of the lemma is wrong. Then there exists a number $\delta > 0$ and a subsequence of the sequence $\{\mu_k\}$, such that this subsequence lies outside of the closed circle which center is at the origin, and the radius equals δ. For the sake of simplicity (but without loss of generality), we assume that this subsequence coincides with the sequence $\{\mu_k\}$. Thus, we have

$$|\mu_k| > \delta \quad (k = 1, 2, \ldots). \tag{2.7}$$

The sequence $\{\mu_k\}$ can be either bounded or not. First, let us consider the case of the bounded sequence $\{\mu_k\}$. In this case, there exists a convergent subsequence of $\{\mu_k\}$. For the sake of simplicity (but without loss of generality), we assume that this subsequence coincides with the sequence $\{\mu_k\}$. Let $\bar{\mu} = \lim_{k\to+\infty} \mu_k$. Then due to the second condition of the lemma ($\operatorname{Re}\mu_k \geqslant -\gamma$) and Eq. (2.7), we obtain that $\operatorname{Re}\bar{\mu} \geqslant -\gamma$ and $|\bar{\mu}| \geqslant \delta$. Substituting $\mu = \mu_k$ and $\varepsilon = \varepsilon_k$ into Eq. (2.5), and calculating the limit as $k \to +\infty$ of the resulting equation, one has that $\bar{\mu}$ is a root of Eq. (2.6). However, the latter is impossible because $\bar{\mu} \neq 0$ and it cannot be a root of the equation $\det \tilde{\Delta}(\mu) = 0$ (see assumption A3).

Now, let us proceed to the case of the unbounded sequence $\{\mu_k\}$. In this case, there exists a subsequence of $\{\mu_k\}$ which tends to infinity. For the sake of simplicity (but without loss of generality), we assume that this subsequence coincides with the sequence $\{\mu_k\}$. Then $\lim_{k\to+\infty} |\mu_k| = +\infty$. Taking into account (1.2), one can rewrite Eq. (2.5) with $(\mu, \varepsilon) = (\mu_k, \varepsilon_k)$ as follows:

$$(-1)^{n+m}\mu_k^{n+m} + \mu_k^{n+m-1} f_1(\mu_k, \varepsilon_k) + \cdots + f_{n+m}(\mu_k, \varepsilon_k) = 0, \tag{2.8}$$

where $f_j(\mu_k, \varepsilon_k)$ $(j = 1, 2, \ldots, n+m)$ are polinoms of ε_k with coefficients depending on μ_k. Due to the second condition of the lemma, these coefficients are bounded uniformly in k. Hence, $\{f_j(\mu_k, \varepsilon_k)\}$ is bounded uniformly in j and k. Dividing both parts of Eq. (2.8) by μ_k^{n+m} and calculating the limit of the resulting equation as $k \to +\infty$, we obtain the contradiction $(-1)^{n+m} = 0$. This contradiction and the one in the case of the bounded sequence $\{\mu_k\}$ imply that the statement of the lemma is true. □

Let $\bar{\lambda}_s$ $(s = 1, \ldots, q \leqslant n)$ be all different eigenvalues of the matrix $\bar{A}$.

Lemma 2.2. *Under assumptions* A1 *and* A3, *let* $\{\varepsilon_k\}$ *and* $\{\lambda_k\}$ *be two sequences such that*:

(1) $\{\varepsilon_k\}$ *is positive and convergent to zero*;
(2) $\lim_{k\to+\infty} \varepsilon_k\lambda_k = 0$;
(3) *Eq.* (2.4) *is satisfied for all* $(\lambda, \varepsilon) = (\lambda_k, \varepsilon_k)$ $(k = 1, 2, \ldots)$.

Then the sequence $\{\lambda_k\}$ *can be partitioned into a finite number* (*no more than* q) *of different subsequences each of which converges to one of the numbers* $\bar{\lambda}_s$ $(s = 1, \ldots, q)$.

Proof (by contradiction). Assume that the statement of the lemma is wrong. Then there exists a number $\delta > 0$ and a subsequence of the sequence $\{\lambda_k\}$, such that this subsequence lies outside of all the closed circles with the centers at the points $\bar{\lambda}_s$ $(s = 1, \ldots, q)$ and with the same radius δ. For the sake of simplicity (but without loss of generality), we assume that this subsequence coincides with the sequence $\{\lambda_k\}$. Thus, we have

$$|\lambda_k - \bar{\lambda}_s| > \delta \quad (k = 1, 2, \ldots,\ s = 1, \ldots, q). \tag{2.9}$$

Due to the second condition of the lemma and to assumption A3, one has $|\det \tilde{\Delta}(\varepsilon_k\lambda_k)| \geqslant a$ for all sufficiently large k, where $a > 0$ is some constant independent of k. Hence, applying the formula for the determinant of a block matrix, one can rewrite the equation $\det \Delta(\lambda_k, \varepsilon_k) = 0$ in the equivalent form for all sufficiently large k:

$$\det\left\{\int_{-h}^{0}\exp(\varepsilon_k\lambda_k\eta)\,dA_1(\eta)-\lambda_k I_n - \left(\int_{-h}^{0}\exp(\varepsilon_k\lambda_k\eta)\,dA_2(\eta)\right)\left(\tilde{\Delta}(\varepsilon_k\lambda_k)\right)^{-1}\left(\int_{-h}^{0}\exp(\varepsilon_k\lambda_k\eta)\,dA_3(\eta)\right)\right\}=0. \tag{2.10}$$

The sequence $\{\lambda_k\}$ can be either bounded or not. First, we consider the case of the bounded sequence $\{\lambda_k\}$. In this case there exists a convergent subsequence of $\{\lambda_k\}$. For the sake of simplicity (but without loss of generality), we assume that this subsequence coincides with the sequence $\{\lambda_k\}$. Let $\bar{\lambda}=\lim_{k\to+\infty}\lambda_k$. Due to (2.9), $|\bar{\lambda}-\bar{\lambda}_s|\geqslant\delta$ $(s=1,\ldots,q)$. Calculating the limit of Eq. (2.10) as $k\to+\infty$, one has $\det(\bar{A}-\bar{\lambda}I_n)=0$. The latter contradicts to the assumption that $\bar{\lambda}_s$ $(s=1,\ldots,q)$ are all different eigenvalues of the matrix $\bar{A}$.

Now, let us proceed to the case of the unbounded sequence $\{\lambda_k\}$. This case is analysed similarly to the case of the unbounded sequence $\{\mu_k\}$ in the proof of Lemma 2.1, and it yields the contradiction $(-1)^n=0$. This contradiction and the one in the case of the bounded sequence $\{\lambda_k\}$ show that the statement of the lemma is true. □

Remark 2.1. Note that Lemmas 2.1 and 2.2 are an extension of results of [7].

Let $\sigma_1>\sigma_2>0$ and $\rho_1<\rho_2$ are numbers, such that

$$-\sigma_1<\operatorname{Re}\bar{\lambda}_s<-\sigma_2,\quad \rho_1<\operatorname{Im}\bar{\lambda}_s<\rho_2\quad (s=1,\ldots,q). \tag{2.11}$$

Consider the domains $\mathcal{D}_1=\{\lambda\in\mathcal{C}\colon -\sigma_1<\operatorname{Re}\lambda<-\sigma_2,\ \rho_1<\operatorname{Im}\lambda<\rho_2\}$ and $\mathcal{D}_2(\varepsilon)=\{\lambda\in\mathcal{C}\colon \operatorname{Re}\lambda<-\gamma/\varepsilon\}$.

Theorem 2.1. *Under assumptions* A1–A3, *for all sufficiently small* $\varepsilon>0$, *any root of the characteristic equation* (2.4) *belongs either to the domain* $\mathcal{D}_1$ *or to the domain* $\mathcal{D}_2(\varepsilon)$.

Proof (by contradiction). Assume that the statement of the theorem is wrong. Then there exist sequences $\{\varepsilon_k\}$ and $\{\lambda_k\}$, such that:

(a) $\{\varepsilon_k\}$ is positive and convergent to zero;
(b) $\operatorname{Re}\lambda_k\geqslant-\gamma/\varepsilon_k$ $(k=1,2,\ldots)$;
(c) $\{\lambda_k\}$ does not belong to $\mathcal{D}_1$;
(d) Eq. (2.4) is satisfied for all $(\lambda,\varepsilon)=(\lambda_k,\varepsilon_k)$ $(k=1,2,\ldots)$.

Consider the sequence $\{\mu_k\}$, such that $\mu_k=\varepsilon_k\lambda_k$ $(k=1,2,\ldots)$. It is easy to see that the sequences $\{\varepsilon_k\}$ and $\{\mu_k\}$ satisfy the conditions of Lemma 2.1. Hence, $\lim_{k\to+\infty}\mu_k=0$. The latter implies that the sequences $\{\varepsilon_k\}$ and $\{\lambda_k\}$ satisfy the conditions of Lemma 2.2. Consequently, using (2.11), one has that $\lambda_k\in\mathcal{D}_1$ for all sufficiently large k. The latter contradicts to the condition (c), which proves the theorem. □

Corollary 2.1. *Under assumptions* A1–A3, *there exists a positive number* α, *such that the following inequality is satisfied for all sufficiently small* $\varepsilon > 0$: $\operatorname{Re}\lambda(\varepsilon) < -\alpha$, *where* $\lambda(\varepsilon)$ *is any root of Eq.* (2.4).

Proof. The corollary is an immediate consequence of Theorem 2.1. The number α can be taken as $0 < \alpha \leqslant \sigma_2$. □

2.2. Estimation of some integrals

Let $\Psi(t,\varepsilon)$ be the fundamental matrix of system (1.1)–(1.2) in the case of the time-independent coefficients. Based on the well-known result of the representation of the fundamental matrix to a linear autonomous differential system with delay [13], and using results of Section 2.1 (Corollary 2.1), one has for any sufficiently small $\varepsilon > 0$

$$\Psi(t,\varepsilon) = \frac{1}{2\pi i}\lim_{\beta\to+\infty}\int_{-\alpha-i\beta}^{-\alpha+i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda,$$
$$\Omega(\lambda,t,\varepsilon) = \exp(\lambda t)\Delta^{-1}(\lambda,\varepsilon), \quad t > 0, \tag{2.12}$$

where $0 < \alpha \leqslant \sigma_2$, and the curve of the integration is the straight-line segment connecting the initial and terminal points.

Along with the integral in (2.12), let us consider the following integrals:

$$\Phi_1(t,\varepsilon) = \int_{\partial\mathcal{D}_1}\Omega(\lambda,t,\varepsilon)\,d\lambda, \qquad \Phi_2(t,\varepsilon) = \lim_{\beta\to+\infty}\int_{-\gamma/\varepsilon-i\beta}^{-\gamma/\varepsilon+i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda, \tag{2.13}$$

$$\Phi_3(t,\varepsilon) = \lim_{\beta\to+\infty}\int_{-\nu_1+i\beta}^{-\nu_2+i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda, \qquad \Phi_4(t,\varepsilon) = \lim_{\beta\to+\infty}\int_{-\nu_1-i\beta}^{-\nu_2-i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda, \tag{2.14}$$

where $\partial\mathcal{D}_1$ is the boundary of the domain $\mathcal{D}_1$ with any direction of the motion along it, ν_1 and ν_2 are any real numbers satisfying the inequality $\gamma/\varepsilon \geqslant \nu_1 \geqslant \nu_2 \geqslant \sigma_2$. In the second integral in (2.13) and in both integrals in (2.14), the curve of the integration is the straight-line segment connecting the initial and terminal points.

In this section, we derive estimates of the integrals in (2.13) and (2.14), which will be used in the sequel.

Lemma 2.3. *Under assumptions* A1–A3, *the following inequality is satisfied for all* $t > 0$ *and sufficiently small* $\varepsilon > 0$: $\|\Phi_1(t,\varepsilon)\| \leqslant a\exp(-\sigma_2 t)$, *where* $a > 0$ *is some constant independent of* ε.

Proof. First, let us note that for all sufficiently small $\varepsilon > 0$

$$\|\Omega(\lambda,t,\varepsilon)\| \leqslant \exp(-\sigma_2 t)\|\Delta^{-1}(\lambda,\varepsilon)\| \quad \forall t > 0,\ \lambda \in \partial\mathcal{D}_1. \tag{2.15}$$

Thus, in order to prove the lemma, one has to estimate properly the norm of $\Delta^{-1}(\lambda,\varepsilon)$ on the boundary of $\mathcal{D}_1$.

Since $\lim_{\varepsilon\to+0}\varepsilon\lambda = 0$ uniformly in $\lambda\in\partial\mathcal{D}_1$, we obtain that $\lim_{\varepsilon\to+0}\tilde{\Delta}(\varepsilon\lambda) = \bar{A}_4$ uniformly in $\lambda\in\partial\mathcal{D}_1$. Hence, $\tilde{\Delta}(\varepsilon\lambda)$ is invertible for all sufficiently small $\varepsilon>0$ and $\lambda\in\partial\mathcal{D}_1$, and

$$\left\|\tilde{\Delta}^{-1}(\varepsilon\lambda)\right\| \leqslant a_1, \quad \lambda\in\partial\mathcal{D}_1, \tag{2.16}$$

where $a_1>0$ is some constant independent of ε.

Let us denote

$$\Delta_1(\lambda,\varepsilon) = \int_{-h}^{0}\exp(\varepsilon\lambda\eta)\,dA_1(\eta) - \lambda I_n, \qquad \Delta_2(\mu) = \int_{-h}^{0}\exp(\mu\eta)\,dA_2(\eta),$$

$$\Delta_3(\mu) = \int_{-h}^{0}\exp(\mu\eta)\,dA_3(\eta). \tag{2.17}$$

Using (2.17), the matrix $\Delta(\lambda,\varepsilon)$ can be rewritten in the form

$$\Delta(\lambda,\varepsilon) = \begin{pmatrix} \Delta_1(\lambda,\varepsilon) & \Delta_2(\varepsilon\lambda) \\ (1/\varepsilon)\Delta_3(\varepsilon\lambda) & (1/\varepsilon)\tilde{\Delta}(\varepsilon\lambda) \end{pmatrix}. \tag{2.18}$$

Now, applying the Frobenius formula [4] to the matrix $\Delta(\lambda,\varepsilon)$, one can conclude that if the matrix $H(\lambda,\varepsilon) = \Delta_1(\lambda,\varepsilon) - \Delta_2(\varepsilon\lambda)\tilde{\Delta}^{-1}(\varepsilon\lambda)\Delta_3(\varepsilon\lambda)$ is invertible, then the matrix $\Delta^{-1}(\lambda,\varepsilon)$ exists and has the form

$$\Delta^{-1}(\lambda,\varepsilon) = \begin{pmatrix} H^{-1}(\lambda,\varepsilon) & -\varepsilon H^{-1}(\lambda,\varepsilon)\Delta_2(\varepsilon\lambda)\tilde{\Delta}^{-1}(\varepsilon\lambda) \\ -\tilde{\Delta}^{-1}(\varepsilon\lambda)\Delta_3(\varepsilon\lambda)H^{-1}(\lambda,\varepsilon) & \varepsilon\tilde{\Delta}^{-1}(\varepsilon\lambda)H_1(\lambda,\varepsilon) \end{pmatrix}, \tag{2.19}$$

where

$$H_1(\lambda,\varepsilon) = I_m + \Delta_3(\varepsilon\lambda)H^{-1}(\lambda,\varepsilon)\Delta_2(\varepsilon\lambda)\tilde{\Delta}^{-1}(\varepsilon\lambda). \tag{2.20}$$

Since $\lim_{\varepsilon\to+0}H(\lambda,\varepsilon) = \bar{A} - \lambda I_n$ uniformly in $\lambda\in\partial\mathcal{D}_1$, and all the eigenvalues of $\bar{A}$ lie inside the domain $\mathcal{D}_1$, the matrix $H(\lambda,\varepsilon)$ is invertible for all sufficiently small $\varepsilon>0$ and $\lambda\in\partial\mathcal{D}_1$, and

$$\left\|H^{-1}(\lambda,\varepsilon)\right\| \leqslant a_2, \quad \lambda\in\partial\mathcal{D}_1, \tag{2.21}$$

where $a_2>0$ is some constant independent of ε.

Thus, the matrix $\Delta^{-1}(\lambda,\varepsilon)$ exists for all sufficiently small $\varepsilon>0$ and $\lambda\in\partial\mathcal{D}_1$. Moreover, Eqs. (2.19), (2.20) and inequalities (2.16), (2.21) yield for all sufficiently small $\varepsilon>0$

$$\left\|\Delta^{-1}(\lambda,\varepsilon)\right\| \leqslant a_3, \quad \lambda\in\partial\mathcal{D}_1, \tag{2.22}$$

where $a_3>0$ is some constant independent of ε. Now, estimating the matrix-function $\Phi_1(t,\varepsilon)$ by application of inequalities (2.15) and (2.22), one immediately obtains the statement of the lemma. □

Lemma 2.4. *Under assumptions* A1–A3*, the following inequality is satisfied for all* $t > 0$ *and sufficiently small* $\varepsilon > 0$: $\|\Phi_2(t,\varepsilon)\| \leqslant a\exp(-\gamma t/\varepsilon)$*, where* $a > 0$ *is some constant independent of* ε.

Proof. First, assuming that $\Delta^{-1}(\lambda,\varepsilon)$ exists, let us rewrite it in the form

$$\Delta^{-1}(\lambda,\varepsilon) = N(\lambda,\varepsilon) - \frac{1}{\lambda}I_{n+m}, \quad N(\lambda,\varepsilon) = \frac{1}{\lambda}\Delta^{-1}(\lambda,\varepsilon)\int_{-h}^{0}\exp(\varepsilon\lambda\eta)\,dA_\varepsilon(\eta), \tag{2.23}$$

yielding

$$\int_{-\gamma/\varepsilon-i\beta}^{-\gamma/\varepsilon+i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda = \int_{-\gamma/\varepsilon-i\beta}^{-\gamma/\varepsilon+i\beta}\exp(\lambda t)N(\lambda,\varepsilon)\,d\lambda - \int_{-\gamma/\varepsilon-i\beta}^{-\gamma/\varepsilon+i\beta}\frac{\exp(\lambda t)}{\lambda}I_{n+m}\,d\lambda. \tag{2.24}$$

Since

$$\lim_{\beta\to+\infty}\int_{-\gamma/\varepsilon-i\beta}^{-\gamma/\varepsilon+i\beta}\frac{\exp(\lambda t)}{\lambda}\,d\lambda = 0, \quad t > 0,\ \varepsilon > 0, \tag{2.25}$$

one has to estimate (as $\beta \to +\infty$) only the first integral in the right-hand part of (2.24).

We have

$$\left\|\exp(\lambda t)N(\lambda,\varepsilon)\right\| = \exp\left(\frac{-\gamma t}{\varepsilon}\right)\left\|N(\lambda,\varepsilon)\right\| \quad \forall t > 0,\ \lambda \in \mathcal{L}(\varepsilon), \tag{2.26}$$

where $\mathcal{L}(\varepsilon) = \{\lambda \in \mathcal{C}\colon\ \lambda = -\gamma/\varepsilon + i\beta,\ \beta \in (-\infty,+\infty)\}$.

Thus, the proof of the lemma is reduced to a proper estimation of the norm of the matrix $N(\lambda,\varepsilon)$ on the straight line $\mathcal{L}(\varepsilon)$.

Begin with a block-form representation for the matrix $\Delta^{-1}(\lambda,\varepsilon)$. Applying the Frobenius formula [4] to the matrix $\Delta(\lambda,\varepsilon)$, given by its block form (2.18), one can conclude that if the matrices $\Delta_1(\lambda,\varepsilon)$ and $G(\lambda,\varepsilon) = \tilde{\Delta}(\varepsilon\lambda) - \Delta_3(\varepsilon\lambda)\Delta_1^{-1}(\lambda,\varepsilon)\Delta_2(\varepsilon\lambda)$ are invertible, then the matrix $\Delta^{-1}(\lambda,\varepsilon)$ exists and has the form

$$\Delta^{-1}(\lambda,\varepsilon) = \begin{pmatrix} \Delta_1^{-1}(\lambda,\varepsilon)G_1(\lambda,\varepsilon) & -\varepsilon\Delta_1^{-1}(\lambda,\varepsilon)\Delta_2(\varepsilon\lambda)G^{-1}(\lambda,\varepsilon) \\ -G^{-1}(\lambda,\varepsilon)\Delta_3(\varepsilon\lambda)\Delta_1^{-1}(\lambda,\varepsilon) & \varepsilon G^{-1}(\lambda,\varepsilon) \end{pmatrix}, \tag{2.27}$$

where

$$G_1(\lambda,\varepsilon) = I_n + \Delta_2(\varepsilon\lambda)G^{-1}(\lambda,\varepsilon)\Delta_3(\varepsilon\lambda)\Delta_1^{-1}(\lambda,\varepsilon). \tag{2.28}$$

Using Eq. (2.17), we can rewrite the matrix $\Delta_1(\lambda,\varepsilon)$ as follows:

$$\Delta_1(\lambda,\varepsilon)=\lambda M(\lambda,\varepsilon),\quad M(\lambda,\varepsilon)=\frac{1}{\lambda}\int_{-h}^{0}\exp(\varepsilon\lambda\eta)\,dA_1(\eta)-I_n,\quad \lambda\in\mathcal{L}(\varepsilon). \tag{2.29}$$

Since $|1/\lambda|\leqslant\varepsilon/\gamma$, $\lambda\in\mathcal{L}(\varepsilon)$, we obtain that $\lim_{\varepsilon\to+0}M(-\gamma/\varepsilon+i\beta,\varepsilon)=-I_n$ uniformly in $\beta\in(-\infty,+\infty)$. Hence, $M(\lambda,\varepsilon)$ (and, consequently, $\Delta_1(\lambda,\varepsilon)$) is invertible for all sufficiently small $\varepsilon>0$ and $\lambda\in\mathcal{L}(\varepsilon)$, and

$$\left\|\Delta_1^{-1}(\lambda,\varepsilon)\right\|\leqslant\frac{a_1}{[(\gamma/\varepsilon)^2+(\operatorname{Im}\lambda)^2]^{1/2}},\quad \lambda\in\mathcal{L}(\varepsilon), \tag{2.30}$$

where $a_1>0$ is some constant independent of ε. Note that inequality (2.30) implies that $\|\Delta_1^{-1}(\lambda,\varepsilon)\|\leqslant\varepsilon a_1/\gamma$ for all sufficiently small $\varepsilon>0$ and $\lambda\in\mathcal{L}(\varepsilon)$.

Now, let us proceed to the matrix $G(\lambda,\varepsilon)$. In order to prove the existence of its inverse matrix, first we shall show the existence of the matrix $\tilde{\Delta}^{-1}(\mu)$ for $\mu\in\mathcal{M}=\{\mu\in\mathcal{C}\colon\ \mu=-\gamma+i\beta,\ \beta\in(-\infty,+\infty)\}$. The latter follows directly from assumption A3. Moreover, this assumption yields

$$\left\|\tilde{\Delta}^{-1}(\varepsilon\lambda)\right\|\leqslant\frac{a_2/\varepsilon}{[(\gamma/\varepsilon)^2+(\operatorname{Im}\lambda)^2]^{1/2}},\quad \varepsilon>0,\ \lambda\in\mathcal{L}(\varepsilon), \tag{2.31}$$

where $a_2>0$ is some constant independent of ε. Note that inequality (2.31) implies the uniform boundness of the matrix $\tilde{\Delta}^{-1}(\varepsilon\lambda)$ for all $\varepsilon>0$ and $\lambda\in\mathcal{L}(\varepsilon)$.

Rewriting the matrix $G(\lambda,\varepsilon)$ in the form

$$G(\lambda,\varepsilon)=\tilde{\Delta}(\varepsilon\lambda)\left[I_m-\tilde{\Delta}^{-1}(\varepsilon\lambda)\Delta_3(\varepsilon\lambda)\Delta_1^{-1}(\lambda,\varepsilon)\Delta_2(\varepsilon\lambda)\right]$$

and taking into account inequalities (2.30) and (2.31), one directly obtains that the matrix $G(\lambda,\varepsilon)$ is invertible for all sufficiently small $\varepsilon>0$ and $\lambda\in\mathcal{L}(\varepsilon)$, and

$$\left\|G^{-1}(\lambda,\varepsilon)\right\|\leqslant\frac{a_3/\varepsilon}{[(\gamma/\varepsilon)^2+(\operatorname{Im}\lambda)^2]^{1/2}},\quad \lambda\in\mathcal{L}(\varepsilon), \tag{2.32}$$

where $a_3>0$ is some constant independent of ε. Note that (2.32) implies the uniform boundness of $G^{-1}(\lambda,\varepsilon)$ for all sufficiently small $\varepsilon>0$ and $\lambda\in\mathcal{L}(\varepsilon)$. The latter along with the inequality (2.30) yields the uniform boundness of the matrix $G_1(\lambda,\varepsilon)$ for all sufficiently small $\varepsilon>0$ and $\lambda\in\mathcal{L}(\varepsilon)$.

Thus, the matrix $\Delta^{-1}(\lambda,\varepsilon)$ exists for all sufficiently small $\varepsilon>0$ and $\lambda\in\mathcal{L}(\varepsilon)$, and has the form (2.27)–(2.28).

Now, by using (2.27) and the block form of the matrix $A_\varepsilon(\eta)$ (see Eq. (1.2)), the matrix $N(\lambda,\varepsilon)$ becomes

$$N(\lambda,\varepsilon)=\frac{1}{\lambda}\begin{pmatrix}N_1(\lambda,\varepsilon) & N_2(\lambda,\varepsilon)\\ N_3(\lambda,\varepsilon) & N_4(\lambda,\varepsilon)\end{pmatrix}, \tag{2.33}$$

where

$$N_j(\lambda,\varepsilon)=\Delta_1^{-1}(\lambda,\varepsilon)\left[G_1(\lambda,\varepsilon)\mathcal{A}_j(\varepsilon\lambda)-\Delta_2(\varepsilon\lambda)G^{-1}(\lambda,\varepsilon)\mathcal{A}_{j+2}(\varepsilon\lambda)\right]\quad (j=1,2), \tag{2.34}$$

$$N_k(\lambda,\varepsilon)=G^{-1}(\lambda,\varepsilon)\left[\mathcal{A}_k(\varepsilon\lambda)-\Delta_3(\varepsilon\lambda)\Delta_1^{-1}(\lambda,\varepsilon)\mathcal{A}_{k-2}(\varepsilon\lambda)\right]\quad (k=3,4), \tag{2.35}$$

$$\mathcal{A}_l(\mu) = \int_{-h}^{0} \exp(\mu\eta)\, dA_l(\eta) \quad (l = 1, \ldots, 4). \tag{2.36}$$

Equations (2.33)–(2.36) and inequalities (2.30), (2.32) yield for all sufficiently small $\varepsilon > 0$

$$\|N(\lambda, \varepsilon)\| \leqslant \frac{a_4/\varepsilon}{[(\gamma/\varepsilon)^2 + (\operatorname{Im}\lambda)^2]}, \quad \lambda \in \mathcal{L}(\varepsilon), \tag{2.37}$$

where $a_4 > 0$ is some constant independent of ε.

Now, Eqs. (2.24)–(2.26) and inequality (2.37) leads directly to the statement of the lemma. □

Lemma 2.5. *Under assumption* A1, *the following equations hold*: $\Phi_k(t, \varepsilon) = 0$ $(k = 3, 4)$, $t > 0$, $\varepsilon > 0$.

Proof. We have

$$\|\Omega(\lambda, t, \varepsilon)\| \leqslant \|\Delta^{-1}(\lambda, \varepsilon)\|, \quad \lambda \in \mathcal{N}(\beta, \varepsilon),\ t > 0,\ \varepsilon > 0, \tag{2.38}$$

where $\mathcal{N}(\beta, \varepsilon) = \{\lambda \in \mathcal{C}\colon \lambda = -\nu + i\beta,\ \sigma_2 \leqslant \nu \leqslant \gamma/\varepsilon\}$.

Transforming the matrix $\Delta(\lambda, \varepsilon)$ to the form

$$\Delta(\lambda, \varepsilon) = \lambda \left[\frac{1}{\lambda} \int_{-h}^{0} \exp(\varepsilon\lambda\eta)\, dA_\varepsilon(\eta) - I_{n+m} \right],$$

one directly obtains that, for any $\varepsilon > 0$, there exist two constants $b(\varepsilon) > 0$ and $c(\varepsilon) > 0$, such that $\Delta(\lambda, \varepsilon)$ is invertible for all $\lambda \in \mathcal{N}(\beta, \varepsilon)$, $|\beta| > b(\varepsilon)$, and

$$\|\Delta^{-1}(\lambda, \varepsilon)\| \leqslant \frac{c(\varepsilon)}{|\beta|}, \quad \lambda \in \mathcal{N}(\beta, \varepsilon),\ |\beta| > b(\varepsilon). \tag{2.39}$$

Inequalities (2.38) and (2.39) directly yield the statement of the lemma. □

2.3. *Another representation for the fundamental matrix $\Psi(t, \varepsilon)$*

In this section, using the integral representation (2.12) of the fundamental matrix $\Psi(t, \varepsilon)$, we obtain another integral representation for this matrix, which is based on the structure of the set of roots of the characteristic equation (2.4) studied in Section 2.1. Further in this section, the new integral representation for $\Psi(t, \varepsilon)$ is applied to obtain a preliminary estimate of this matrix.

Theorem 2.2. *Under assumptions* A1–A3, *for any sufficiently small $\varepsilon > 0$, the fundamental matrix $\Psi(t, \varepsilon)$ has the form*

$$\Psi(t,\varepsilon)=\frac{1}{2\pi i}\left\{\int\limits_{\partial\mathcal{D}_1}\Omega(\lambda,t,\varepsilon)\,d\lambda+\lim_{\beta\to+\infty}\int\limits_{-\gamma/\varepsilon-i\beta}^{-\gamma/\varepsilon+i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda\right\},\quad t>0,$$

where the direction of the motion along $\partial\mathcal{D}_1$ is opposite to the clockwise one.

Proof. Let $\varepsilon>0$ be any such small that Theorem 2.1 is valid, and t be any positive.

For any $\beta>\max\{|\rho_1|,|\rho_2|\}$, consider the domains

$$\mathcal{D}_3=\{\lambda\in\mathcal{C}\colon -\sigma_1<\operatorname{Re}\lambda<-\sigma_2,\ -\beta<\operatorname{Im}\lambda<\rho_1\},$$
$$\mathcal{D}_4=\{\lambda\in\mathcal{C}\colon -\sigma_1<\operatorname{Re}\lambda<-\sigma_2,\ \rho_2<\operatorname{Im}\lambda<\beta\},$$

and

$$\mathcal{D}_5=\{\lambda\in\mathcal{C}\colon -\gamma/\varepsilon<\operatorname{Re}\lambda<-\sigma_1,\ -\beta<\operatorname{Im}\lambda<\beta\},$$

where σ_k and ρ_k $(k=1,2)$ are given in (2.11). Then, taking into account Theorem 2.1, one has due to the Cauchy theorem

$$\int\limits_{\partial\mathcal{D}_l}\Omega(\lambda,t,\varepsilon)\,d\lambda=0\quad(l=3,4,5),\tag{2.40}$$

where $\partial\mathcal{D}_l$ is the boundary of the domain $\mathcal{D}_l$ $(l=3,4,5)$ with the clockwise direction of the motion along it.

Using (2.40) yields the following for any $\beta>\max\{|\rho_1|,|\rho_2|\}$:

$$\begin{aligned}\int\limits_{-\sigma_2-i\beta}^{-\sigma_2+i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda&=\int\limits_{-\sigma_2-i\beta}^{-\sigma_2+i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda+\sum_{l=3}^{5}\int\limits_{\partial\mathcal{D}_l}\Omega(\lambda,t,\varepsilon)\,d\lambda\\&=\int\limits_{\partial\mathcal{D}_1}\Omega(\lambda,t,\varepsilon)\,d\lambda+\int\limits_{-\gamma/\varepsilon-i\beta}^{-\gamma/\varepsilon+i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda\\&\quad+\int\limits_{-\sigma_2-i\beta}^{-\gamma/\varepsilon-i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda+\int\limits_{-\gamma/\varepsilon+i\beta}^{-\sigma_2+i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda.\end{aligned}\tag{2.41}$$

Applying Lemma 2.5, one obtains from (2.41)

$$\lim_{\beta\to+\infty}\int\limits_{-\sigma_2-i\beta}^{-\sigma_2+i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda=\int\limits_{\partial\mathcal{D}_1}\Omega(\lambda,t,\varepsilon)\,d\lambda+\lim_{\beta\to+\infty}\int\limits_{-\gamma/\varepsilon-i\beta}^{-\gamma/\varepsilon+i\beta}\Omega(\lambda,t,\varepsilon)\,d\lambda.\tag{2.42}$$

Now, the statement of the theorem follows directly from Eqs. (2.12) and (2.42). □

Lemma 2.6. *Under assumptions* A1–A3*, the fundamental matrix $\Psi(t,\varepsilon)$ satisfies the following inequality for all sufficiently small $\varepsilon>0$: $\|\Psi(t,\varepsilon)\|\leqslant a\exp(-\sigma_2 t)$, $t>0$, where $a>0$ and $\sigma_2>0$ are some constants independent of ε (σ_2 is given in* (2.11)).

Proof. The statement of the lemma follows immediately from Theorem 2.2 and Lemmas 2.3 and 2.4. □

2.4. Main result

Let $\Psi_1(t,\varepsilon)$, $\Psi_2(t,\varepsilon)$, $\Psi_3(t,\varepsilon)$, and $\Psi_4(t,\varepsilon)$ be the upper left-hand, upper right-hand, lower left-hand, and lower right-hand blocks of the fundamental matrix $\Psi(t,\varepsilon)$ of the dimensions $n\times n$, $n\times m$, $m\times n$, and $m\times m$, respectively.

Theorem 2.3. *Under assumptions* A1–A3, *the following inequalities are satisfied for all* $t>0$ *and sufficiently small* $\varepsilon>0$:

$$\left\|\Psi_k(t,\varepsilon)\right\| \leqslant a\exp(-\sigma_2 t) \quad (k=1,3), \qquad \left\|\Psi_2(t,\varepsilon)\right\| \leqslant a\varepsilon\exp(-\sigma_2 t),$$
$$\left\|\Psi_4(t,\varepsilon)\right\| \leqslant a\left[\varepsilon\exp(-\sigma_2 t)+\exp\left(\frac{-\gamma t}{\varepsilon}\right)\right],$$

where $a>0$, $\sigma_2>0$, *and* $\gamma>0$ *are some constants independent of* ε (σ_2 *and* γ *are given in* (2.11) *and assumption* A3, *respectively*).

Proof. The inequalities for $\Psi_1(t,\varepsilon)$ and $\Psi_3(t,\varepsilon)$, claimed in the theorem, follow directly from Lemma 2.6. Let us prove the inequalities for $\Psi_2(t,\varepsilon)$ and $\Psi_4(t,\varepsilon)$. Denoting

$$\Gamma(t,\varepsilon)=\begin{pmatrix}\Psi_2(t,\varepsilon)\\ \Psi_4(t,\varepsilon)\end{pmatrix} \tag{2.43}$$

and taking into account that $\Psi_2(t,\varepsilon)$ and $\Psi_4(t,\varepsilon)$ are the corresponding blocks of the fundamental matrix to system (1.1)–(1.2) in the case of the time-independent coefficients, we obtain the equation for $\Gamma(t,\varepsilon)$

$$\frac{d\Gamma(t,\varepsilon)}{dt}=\int_{-h}^{0}\left[dA_\varepsilon(\eta)\right]\Gamma(t+\varepsilon\eta,\varepsilon), \quad t>0, \tag{2.44}$$

and the initial conditions

$$\Gamma(0,\varepsilon)=\begin{pmatrix}0\\ I_m\end{pmatrix}, \qquad \Gamma(t,\varepsilon)=0, \quad t<0. \tag{2.45}$$

Let the $(m\times m)$-matrix $\Theta(t,\varepsilon)$ be the fundamental matrix of the equation

$$\varepsilon\frac{dy(t)}{dt}=\int_{-h}^{0}\left[dA_4(\eta)\right]y(t+\varepsilon\eta), \quad t>0,\ y\in E^m. \tag{2.46}$$

Assumption A3 and results of [13] directly yield for all $\varepsilon>0$

$$\left\|\Theta(t,\varepsilon)\right\| \leqslant a\exp\left(\frac{-\gamma t}{\varepsilon}\right), \quad t>0, \tag{2.47}$$

where $a>0$ is some constant independent of ε.

Transforming the variable Γ in (2.44) and (2.45) as

$$\Gamma(t,\varepsilon) = \tilde{\Gamma}(t,\varepsilon) + \hat{\Gamma}(t,\varepsilon), \quad \hat{\Gamma}(t,\varepsilon) = \begin{pmatrix} 0 \\ \Theta(t,\varepsilon) \end{pmatrix}, \tag{2.48}$$

we obtain the problem

$$\frac{d\tilde{\Gamma}(t,\varepsilon)}{dt} = \int_{-h}^{0} \big[dA_\varepsilon(\eta)\big]\,\tilde{\Gamma}(t+\varepsilon\eta,\varepsilon) + F(t,\varepsilon), \quad t > 0,$$

$$\tilde{\Gamma}(t,\varepsilon) = 0, \quad t \leqslant 0, \tag{2.49}$$

where

$$F(t,\varepsilon) = \begin{pmatrix} \int_{-h}^{0} [dA_2(\eta)]\,\Theta(t+\varepsilon\eta,\varepsilon) \\ 0 \end{pmatrix}.$$

From (2.47), one has for all $\varepsilon > 0$

$$\big\|F(t,\varepsilon)\big\| \leqslant a \exp\left(\frac{-\gamma t}{\varepsilon}\right), \quad t > 0, \tag{2.50}$$

where $a > 0$ is some constant independent of ε.

Rewriting problem (2.49) in the equivalent integral form (by application of the variation-of-constant formula [13])

$$\tilde{\Gamma}(t,\varepsilon) = \int_{0}^{t} \Psi(t-s,\varepsilon)F(s,\varepsilon)\,ds, \quad t > 0, \tag{2.51}$$

and using Lemma 2.6 and inequality (2.50), one obtains from (2.51) the following inequality for all sufficiently small $\varepsilon > 0$

$$\big\|\tilde{\Gamma}(t,\varepsilon)\big\| \leqslant a\varepsilon \exp(-\sigma_2 t), \quad t > 0, \tag{2.52}$$

where $a > 0$ is some constant independent of ε.

Now, Eq. (2.48) and inequalities (2.47) and (2.52) yield the inequalities for $\Psi_2(t,\varepsilon)$ and $\Psi_4(t,\varepsilon)$ claimed in the theorem. □

3. The case of the time-dependent matrix A_ε

We shall assume:

A4. The matrix-functions $A_j(t,\eta)$ $(j = 1,\dots,4)$ are given for $(t,\eta) \in [0,T] \times (-\infty,+\infty)$ and satisfy the following conditions:

(a) $A_j(t,\eta) = 0$, $\forall t \in [0,T]$, $\eta \geqslant 0$;
(b) $A_j(t,\eta) = A_j(t,-h)$, $\forall t \in [0,T]$, $\eta \leqslant -h$;
(c) $A_j(t,\eta)$ is continuous in $t \in [0,T]$ uniformly in $\eta \in (-\infty,+\infty)$;

(d) $A_j(t,\eta)$ is continuous from the left in $\eta \in (-h,0)$ for each $t \in [0,T]$;
(e) $A_j(t,\eta)$ has bounded variation in η on the interval $[-h,0]$ for each $t \in [0,T]$ and $\mathrm{Var}_{[-h,0]} A_j(t,\cdot) \leqslant d$, where $d > 0$ is some constant.

A5. All roots $\lambda(t)$ of the equation

$$\det\left[\int_{-h}^{0} \exp(\lambda\eta)\, d_\eta A_4(t,\eta) - \lambda I_m\right] = 0$$

satisfy the inequality $\mathrm{Re}[\lambda(t)] \leqslant -2\chi$ for all $t \in [0,T]$, where $\chi > 0$ is some constant.

Let $\Psi(t,s,\varepsilon)$ be the fundamental matrix of system (1.1)–(1.2). Let $\Psi_1(t,s,\varepsilon)$, $\Psi_2(t,s,\varepsilon)$, $\Psi_3(t,s,\varepsilon)$, and $\Psi_4(t,s,\varepsilon)$ be the upper left-hand, upper right-hand, lower left-hand, and lower right-hand blocks of the matrix $\Psi(t,s,\varepsilon)$ of the dimensions $n \times n$, $n \times m$, $m \times n$, and $m \times m$, respectively.

Theorem 3.1. *Under assumptions* A4 *and* A5, *the following inequalities are satisfied for all sufficiently small* $\varepsilon > 0$ *and* $0 \leqslant s \leqslant t \leqslant T$:

$$\left\|\Psi_k(t,s,\varepsilon)\right\| \leqslant a \quad (k = 1,3), \qquad \left\|\Psi_2(t,s,\varepsilon)\right\| \leqslant a\varepsilon,$$
$$\left\|\Psi_4(t,s,\varepsilon)\right\| \leqslant a\left[\varepsilon + \exp\left(-\frac{\chi(t-s)}{\varepsilon}\right)\right],$$

where $a > 0$ *is some constant independent of* ε.

Proof. Let us prove the inequalities for $\Psi_l(t,s,\varepsilon)$ $(l = 2,4)$. The two other inequalities are proved similarly.

Taking into account that $\Psi_2(t,s,\varepsilon)$ and $\Psi_4(t,s,\varepsilon)$ are the corresponding blocks of the fundamental matrix to system (1.1)–(1.2), we obtain that they satisfy the equations

$$\frac{\partial \Psi_2(t,s,\varepsilon)}{\partial t} = \int_{-h}^{0} \left[d_\eta A_1(t,\eta)\right] \Psi_2(t+\varepsilon\eta,s,\varepsilon) + \int_{-h}^{0} \left[d_\eta A_2(t,\eta)\right] \Psi_4(t+\varepsilon\eta,s,\varepsilon), \quad t > s, \tag{3.1}$$

$$\varepsilon\frac{\partial \Psi_4(t,s,\varepsilon)}{\partial t} = \int_{-h}^{0} \left[d_\eta A_3(t,\eta)\right] \Psi_2(t+\varepsilon\eta,s,\varepsilon) + \int_{-h}^{0} \left[d_\eta A_4(t,\eta)\right] \Psi_4(t+\varepsilon\eta,s,\varepsilon), \quad t > s, \tag{3.2}$$

and the initial conditions

$$\Psi_2(t,s,\varepsilon) = 0, \quad t \leqslant s, \tag{3.3}$$

$$\Psi_4(s,s,\varepsilon) = I_m, \qquad \Psi_4(t,s,\varepsilon) = 0, \quad t < s. \tag{3.4}$$

Let $X(t,s,\varepsilon)$ and $Y(t,s,\varepsilon)$ be the fundamental matrices of the equations

$$\frac{dx(t)}{dt} = \int_{-h}^{0} \big[d_\eta A_1(t,\eta)\big] x(t+\varepsilon\eta), \quad t>0,\ x \in E^n,$$

and

$$\varepsilon \frac{dy(t)}{dt} = \int_{-h}^{0} \big[d_\eta A_4(t,\eta)\big] y(t+\varepsilon\eta), \quad t>0,\ y \in E^m,$$

respectively.

From assumptions A4, A5, and results of [1], one directly has for all sufficiently small $\varepsilon > 0$

$$\big\|X(t,s,\varepsilon)\big\| \leqslant a, \quad 0 \leqslant s \leqslant t \leqslant T, \tag{3.5}$$

$$\big\|Y(t,s,\varepsilon)\big\| \leqslant a \exp\left[-\frac{\chi(t-s)}{\varepsilon}\right], \quad 0 \leqslant s \leqslant t \leqslant T, \tag{3.6}$$

where $a > 0$ is some constant independent of ε.

Using the variation-of-constant formula [13], we can rewrite Eqs. (3.1)–(3.4) in the equivalent form

$$\Psi_2(t,s,\varepsilon) = \int_s^t X(t,\tau,\varepsilon)\left\{\int_{-h}^{0}\big[d_\eta A_2(\tau,\eta)\big]\Psi_4(\tau+\varepsilon\eta,s,\varepsilon)\right\}d\tau, \quad t \geqslant s, \tag{3.7}$$

$$\Psi_4(t,s,\varepsilon) = Y(t,s,\varepsilon) + \frac{1}{\varepsilon}\int_s^t Y(t,\tau,\varepsilon)\left\{\int_{-h}^{0}\big[d_\eta A_3(\tau,\eta)\big]\Psi_2(\tau+\varepsilon\eta,s,\varepsilon)\right\}d\tau, \quad t \geqslant s. \tag{3.8}$$

Substituting (3.8) into (3.7), changing the order of the integration (by the Fubini's theorem), and taking into account (3.3), (3.4), and that $Y(t,s,\varepsilon) = 0$, $t < s$, one obtains

$$\Psi_2(t,s,\varepsilon) = F_1(t,s,\varepsilon) + \int_s^t d\tau \int_{-h}^{0}\big[d_\eta F_2(t,\tau,\eta,\varepsilon)\big]\Psi_2(\tau+\varepsilon\eta,s,\varepsilon), \quad t \geqslant s, \tag{3.9}$$

where

$$F_1(t,s,\varepsilon) = \int_s^t X(t,\omega,\varepsilon)\left\{\int_{-h}^{0}\big[d_\zeta A_2(\omega,\zeta)\big]Y(\omega+\varepsilon\zeta,s,\varepsilon)\right\}d\omega, \tag{3.10}$$

$$F_2(t,\tau,\eta,\varepsilon) = \frac{1}{\varepsilon}F_1(t,\tau,\varepsilon)A_3(\tau,\eta). \tag{3.11}$$

Using (3.5) and (3.6) yields for all sufficiently small $\varepsilon > 0$

$$\|F_1(t,s,\varepsilon)\| \leqslant a\varepsilon, \quad 0 \leqslant s \leqslant t \leqslant T, \tag{3.12}$$

where $a > 0$ is some constant independent of ε.

Now, applying the method of successive approximations to Eq. (3.9) and taking into account inequality (3.12) and assumption A4, one directly obtains, for all sufficiently small $\varepsilon > 0$, the inequality for $\Psi_2(t,s,\varepsilon)$ claimed in the theorem. Then, the inequality for $\Psi_4(t,s,\varepsilon)$, claimed in the theorem, directly follows from Eq. (3.8) using (3.6) and the inequality for $\Psi_2(t,s,\varepsilon)$. □

4. Asymptotic solution

Consider the problem

$$\frac{dz(t)}{dt} = \int_{-h}^{0} \left[d_\eta A_\varepsilon(t,\eta)\right] z(t+\varepsilon\eta) + f_\varepsilon(t), \quad t \geqslant 0, \tag{4.1}$$

$$z(\tau) = \varphi(\tau), \quad \tau \in [-\varepsilon h, 0], \tag{4.2}$$

where the matrix $A_\varepsilon(t,\eta)$ and the vector z have the block form given in (1.2), and the vectors $f_\varepsilon(t)$ and $\varphi(\tau)$ have the block form

$$f_\varepsilon(t) = \operatorname{col}\left\{f_1(t), \frac{1}{\varepsilon} f_2(t)\right\}, \qquad \varphi(\tau) = \operatorname{col}\left\{\varphi_1(\tau), \varphi_2(\tau)\right\}, \tag{4.3}$$

the blocks f_1 and φ_1 are of the dimension n, while the blocks f_2 and φ_2 are of the dimension m.

In this section, a uniform asymptotic solution of problem (4.1)–(4.2) is constructed and justified in both, time-independent and time-dependent matrix A_ε, cases. The justification of the asymptotic solution is based on the estimate of the fundamental matrix obtained in the previous sections. We shall begin with the case of the time-dependent matrix A_ε. Then, the case of the time-independent matrix A_ε will be considered.

4.1. Asymptotic solution of (4.1)–(4.2) (the time-dependent case)

In order to save the space, we restrict the consideration by the first-order asymptotic solution. Such a restriction does not lead to the loss of generality because the formal construction and the justification of the first-order asymptotic solution contain all the peculiarities arising in the obtaining asymptotic solution of arbitrary order.

In this section, in addition to assumptions A4 and A5, we shall assume:

A6. The matrices $A_j(t,\eta)$ $(j = 1,\dots,4)$ satisfy the following conditions:

(a) $A_j(t,\eta)$ is twice continuously differentiable with respect to $t \in [0,T]$ uniformly in $\eta \in (-\infty,+\infty)$;

(b) the first- and second-order partial derivatives of $A_j(t,\eta)$ with respect to t have bounded variations in η on the interval $[-h,0]$ for each $t \in [0,T]$ and

$$\operatorname*{Var}_{[-h,0]} \frac{\partial^k A_j(t,\cdot)}{\partial t^k} \leqslant d \quad (k = 1,2),$$

where $d > 0$ is some constant.

A7. The vector-functions $f_j(t)$ $(j = 1,2)$ are twice continuously differentiable for $t \in [0,T]$.

A8. There exists a number $\varepsilon_0 > 0$, such that the vector-functions $\varphi_j(\tau)$ $(j = 1,2)$ are twice continuously differentiable for $\tau \in [-\varepsilon_0 h, 0]$.

4.1.1. Formal representation of the first-order asymptotic solution

We seek the first-order asymptotic solution $z_1(t,\varepsilon) = \operatorname{col}\{x_1(t,\varepsilon), y_1(t,\varepsilon)\}$ of (4.1)–(4.2) on the interval $t \in [0,T]$ in the form

$$x_1(t,\varepsilon) = \bar{x}_0(t) + \varepsilon \bar{x}_1(t) + x_0^b(\xi) + \varepsilon x_1^b(\xi), \quad \xi = \frac{t}{\varepsilon}, \tag{4.4}$$

$$y_1(t,\varepsilon) = \bar{y}_0(t) + \varepsilon \bar{y}_1(t) + y_0^b(\xi) + \varepsilon y_1^b(\xi). \tag{4.5}$$

In (4.4)–(4.5), terms with an overbar form the outer asymptotic solution, while the terms with the superscript b form the boundary correction in a neighborhood of $t = 0$. The boundary correction terms are considerable only for nonlarge values of ξ, and they vanish as $\xi \to +\infty$. Such an approach and its modifications (the boundary function method) were applied in the open literature to obtain an asymptotic solution of singularly perturbed differential equations without delays as well as to differential-difference equations (including neutral type ones) with a small delay but without the small multiplier for a part of the derivatives (see [16–19]). Its modification was also applied to a boundary-value problem for a class of singularly perturbed functional-differential equations with small deviations of the argument (see [8]). In [5], the zeroth-order asymptotic solution of the form similar to (4.4)–(4.5) was constructed and justified for a particular case of problem (4.1)–(4.2). An asymptotic solution for singularly perturbed linear time-independent systems with nonsmall delay was obtained in [15]. Although the structure of this solution (the outer solution plus the boundary-layer correction) is similar to that in (4.4)–(4.5), the obtaining the asymptotic solution as well as its justification substantially differ from those in the case of the small delay.

Equations and initial conditions for the terms of the asymptotic solution (4.4)–(4.5) are obtained substituting $z_1(\cdot,\varepsilon)$ into the problem (4.1)–(4.2) instead of $z(\cdot)$ and equating terms of the same power of ε on both sides of the equations, separately depending on t and ξ.

4.1.2. Equations for $\bar{x}_0(t)$ and $\bar{y}_0(t)$

These equations have the form

$$\frac{d\bar{x}_0(t)}{dt} = \bar{A}_1(t)\bar{x}_0(t) + \bar{A}_2(t)\bar{y}_0(t) + f_1(t), \quad t \in [0,T], \tag{4.6}$$

$$0 = \bar{A}_3(t)\bar{x}_0(t) + \bar{A}_4(t)\bar{y}_0(t) + f_2(t), \quad t \in [0, T], \tag{4.7}$$

where $\bar{A}_j(t) = \int_{-h}^{0} d_\eta A_j(t, \eta)$ $(j = 1, \ldots, 4)$.

Due to assumption A5, the matrix $\bar{A}_4(t)$ is nonsingular for $t \in [0, T]$. Hence, Eq. (4.7) can be uniquely resolved with respect to $\bar{y}_0(t)$ as follows:

$$\bar{y}_0(t) = -\bar{A}_4^{-1}(t)\big[\bar{A}_3(t)\bar{x}_0(t) + f_2(t)\big], \quad t \in [0, T]. \tag{4.8}$$

Substituting (4.8) into Eq. (4.6) yields the differential equation for $\bar{x}_0(t)$

$$\frac{d\bar{x}_0(t)}{dt} = \big[\bar{A}_1(t) - \bar{A}_2(t)\bar{A}_4^{-1}(t)\bar{A}_3(t)\big]\bar{x}_0(t) + f_1(t) - \bar{A}_2(t)\bar{A}_4^{-1}(t)f_2(t),$$
$$t \in [0, T]. \tag{4.9}$$

The initial condition for this equation will be obtained below.

Completing this section, let us note that, due to assumption A6, the matrix-functions $\bar{A}_j(t)$ $(j = 1, \ldots, 4)$ are twice continuously differentiable on the interval $t \in [0, T]$. The latter along with assumption A7 provides the twice continuous differentiability of $\bar{x}_0(t)$ and $\bar{y}_0(t)$ on the interval $t \in [0, T]$.

4.1.3. Equations for $x_0^b(\xi)$ and $y_0^b(\xi)$

These equations have the form

$$\frac{dx_0^b(\xi)}{d\xi} = 0, \quad \xi \geqslant 0, \tag{4.10}$$

$$\frac{dy_0^b(\xi)}{d\xi} = \int_{-h}^{0} \big[d_\eta A_3(0, \eta)\big] x_0^b(\xi + \eta) + \int_{-h}^{0} \big[d_\eta A_4(0, \eta)\big] y_0^b(\xi + \eta), \quad \xi \geqslant 0. \tag{4.11}$$

According to the above-mentioned property of the boundary correction terms, we have to require that $x_0^b(\xi) \to 0$ as $\xi \to +\infty$. Due to this requirement, one has from (4.10)

$$x_0^b(\xi) = 0, \quad \forall \xi \in [0, +\infty). \tag{4.12}$$

4.1.4. Initial conditions for obtaining the zeroth-order terms

These conditions have the form

$$\bar{x}_0(0) + x_0^b(\zeta) = \varphi_1(0), \tag{4.13}$$

$$\bar{y}_0(0) + y_0^b(\zeta) = \varphi_2(0), \tag{4.14}$$

where $\zeta = \tau/\varepsilon$, $\zeta \in [-h, 0]$.

Due to (4.12), one has from (4.13) the initial condition for Eq. (4.9)

$$\bar{x}_0(0) = \varphi_1(0), \tag{4.15}$$

yielding a unique solution $\bar{x}_0(t)$ of problem (4.9), (4.15) on the interval $t \in [0, T]$. Once this solution is known, $\bar{y}_0(t)$ is obtained directly from Eq. (4.8) for $t \in [0, T]$.

Remark 4.1. Since the solution $z(t,\varepsilon)=\{x(t,\varepsilon),y(t,\varepsilon)\}$ of problem (4.1)–(4.2) is determined not only on the interval $[0,T]$ but also on the interval $[-\varepsilon h,0)$, one has to determine the outer zeroth-order terms $\bar{x}_0$ and $\bar{y}_0$ for negative values of their argument. We shall do this as follows: $\bar{x}_0(\tau)=\psi_1^0(\tau)$, $\bar{y}_0(\tau)=\psi_2^0(\tau)$, $\tau<0$, where $\psi_j^0(\tau)$ $(j=1,2)$ are any twice continuously differentiable functions for $\tau\leqslant 0$ satisfying the conditions:

$$\frac{d^k\psi_1^0(\tau)}{d\tau^k}\bigg|_{\tau=0}=\frac{d^k\bar{x}_0(t)}{dt^k}\bigg|_{t=0},\qquad \frac{d^k\psi_2^0(\tau)}{d\tau^k}\bigg|_{\tau=0}=\frac{d^k\bar{y}_0(t)}{dt^k}\bigg|_{t=0},\qquad k=0,1,2.$$

Now, Eqs. (4.13) and (4.14) yield

$$x_0^b(\zeta)=0,\quad \zeta\in[-h,0],\tag{4.16}$$

$$y_0^b(\zeta)=\varphi_2(0)-\bar{y}_0(0),\quad \zeta\in[-h,0].\tag{4.17}$$

Using (4.12), (4.16), and assumption A5, and taking into account results of [13], one directly obtains that problem (4.11), (4.17) has a unique solution satisfying the inequality

$$\|y_0^b(\xi)\|\leqslant a\exp(-\kappa\xi),\quad \forall\xi\geqslant 0,\tag{4.18}$$

where $a>0$ and $\kappa>0$ are some constants.

4.1.5. Equations for $\bar{x}_1(t)$ and $\bar{y}_1(t)$

These equations have the form

$$\frac{d\bar{x}_1(t)}{dt}=\bar{A}_1(t)\bar{x}_1(t)+\bar{A}_2(t)\bar{y}_1(t)+\int_{-h}^{0}\eta\,d_\eta A_1(t,\eta)\,\frac{d\bar{x}_0(t)}{dt}+\int_{-h}^{0}\eta\,d_\eta A_2(t,\eta)\,\frac{d\bar{y}_0(t)}{dt},\tag{4.19}$$

$$\frac{d\bar{y}_0(t)}{dt}=\bar{A}_3(t)\bar{x}_1(t)+\bar{A}_4(t)\bar{y}_1(t)+\int_{-h}^{0}\eta\,d_\eta A_3(t,\eta)\,\frac{d\bar{x}_0(t)}{dt}+\int_{-h}^{0}\eta\,d_\eta A_4(t,\eta)\,\frac{d\bar{y}_0(t)}{dt}.\tag{4.20}$$

Similarly to Section 4.1.2, this set of equations can be rewritten in an equivalent form as follows:

$$\frac{d\bar{x}_1(t)}{dt}=\big[\bar{A}_1(t)-\bar{A}_2(t)\bar{A}_4^{-1}(t)\bar{A}_3(t)\big]\bar{x}_1(t)+\bar{A}_2(t)\bar{A}_4^{-1}(t)\frac{d\bar{y}_0(t)}{dt}+\left[\int_{-h}^{0}\eta\,d_\eta A_1(t,\eta)-\bar{A}_2(t)\bar{A}_4^{-1}(t)\int_{-h}^{0}\eta\,d_\eta A_3(t,\eta)\right]\frac{d\bar{x}_0(t)}{dt}$$

$$+\left[\int_{-h}^{0}\eta\, d_\eta A_2(t,\eta)-\bar{A}_2(t)\bar{A}_4^{-1}(t)\int_{-h}^{0}\eta\, d_\eta A_4(t,\eta)\right]\frac{d\bar{y}_0(t)}{dt},$$

$$t\in[0,T], \tag{4.21}$$

$$\bar{y}_1(t)=\bar{A}_4^{-1}(t)\left\{\frac{d\bar{y}_0(t)}{dt}-\bar{A}_3(t)\bar{x}_1(t)-\int_{-h}^{0}\eta\, d_\eta A_3(t,\eta)\frac{d\bar{x}_0(t)}{dt}\right.$$
$$\left.-\int_{-h}^{0}\eta\, d_\eta A_4(t,\eta)\frac{d\bar{y}_0(t)}{dt}\right\},\quad t\in[0,T]. \tag{4.22}$$

The initial condition for Eq. (4.21) will be obtained below.

Taking into account that $\bar{x}_0(t)$ and $\bar{y}_0(t)$ are twice continuously differentiable for $t\in[0,T]$, and using assumption A6, one directly has that $d\bar{x}_1(t)/dt$ is continuous, as well as $d\bar{y}_1(t)/dt$ exists and is continuous, for $t\in[0,T]$.

4.1.6. Equations for $x_1^b(\xi)$ and $y_1^b(\xi)$

These equations have the form

$$\frac{dx_1^b(\xi)}{d\xi}=\int_{-h}^{0}\big[d_\eta A_2(0,\eta)\big]y_0^b(\xi+\eta),\quad \xi\geqslant 0, \tag{4.23}$$

$$\frac{dy_1^b(\xi)}{d\xi}=\int_{-h}^{0}\big[d_\eta A_3(0,\eta)\big]x_1^b(\xi+\eta)+\int_{-h}^{0}\big[d_\eta A_4(0,\eta)\big]y_1^b(\xi+\eta)$$
$$+\int_{-h}^{0}\big[dA_{41}(\eta)\big]\xi y_0^b(\xi+\eta),\quad \xi\geqslant 0, \tag{4.24}$$

where $A_{41}(\eta)=\partial A_4(t,\eta)/\partial t|_{t=0}$.

Integrating Eq. (4.23) from $\xi=0$ to an arbitrary ξ yields

$$x_1^b(\xi)=x_1^b(0)+\int_0^{\xi}\left\{\int_{-h}^{0}\big[d_\eta A_2(0,\eta)\big]y_0^b(s+\eta)\right\}ds,\quad \xi\geqslant 0. \tag{4.25}$$

Requiring that $x_1^b(\xi)\to 0$ as $\xi\to+\infty$, one directly has from (4.25)

$$x_1^b(0)=-\int_0^{+\infty}\left\{\int_{-h}^{0}\big[d_\eta A_2(0,\eta)\big]y_0^b(s+\eta)\right\}ds. \tag{4.26}$$

Note that the convergence of the integral in (4.26) directly follows from (4.18).

Using (4.25) and (4.26), we obtain

$$x_1^b(\xi) = -\int_{\xi}^{+\infty}\left\{\int_{-h}^{0}\left[d_\eta A_2(0,\eta)\right] y_0^b(s+\eta)\right\} ds, \quad \xi \geqslant 0, \tag{4.27}$$

yielding by (4.18)

$$\left\| x_1^b(\xi)\right\| \leqslant a\exp(-\kappa\xi), \quad \xi \geqslant 0, \tag{4.28}$$

where $a > 0$ and $\kappa > 0$ are some constants.

4.1.7. Initial conditions for obtaining the first-order terms

These conditions have the form

$$\zeta\left[\frac{d\bar{x}_0(t)}{dt}\bigg|_{t=0}\right] + \bar{x}_1(0) + x_1^b(\zeta) = \zeta\left[\frac{d\varphi_1(\tau)}{d\tau}\bigg|_{\tau=0}\right], \quad \zeta \in [-h, 0], \tag{4.29}$$

$$\zeta\left[\frac{d\bar{y}_0(t)}{dt}\bigg|_{t=0}\right] + \bar{y}_1(0) + y_1^b(\zeta) = \zeta\left[\frac{d\varphi_2(\tau)}{d\tau}\bigg|_{\tau=0}\right], \quad \zeta \in [-h, 0]. \tag{4.30}$$

Setting $\zeta = 0$ in Eq. (4.29) and using Eq. (4.26), one obtains

$$\bar{x}_1(0) = \int_{0}^{+\infty}\left\{\int_{-h}^{0}\left[d_\eta A_2(0,\eta)\right] y_0^b(s+\eta)\right\} ds, \tag{4.31}$$

yielding a single solution $\bar{x}_1(t)$ of problem (4.21), (4.31) on the interval $t \in [0, T]$. Once this solution is known, $\bar{y}_1(t)$ is obtained directly from Eq. (4.22) on the interval $t \in [0, T]$.

Remark 4.2. Similarly to Remark 4.1, we have to extend the outer first-order terms $\bar{x}_1$ and $\bar{y}_1$ to the domain of negative values of their argument. We shall do this extension as follows: $\bar{x}_1(\tau) = \psi_1^1(\tau)$, $\bar{y}_1(\tau) = \psi_2^1(\tau)$, $\tau < 0$, where $\psi_j^1(\tau)$ $(j = 1, 2)$ are any continuously differentiable functions for $\tau \leqslant 0$ satisfying the conditions:

$$\frac{d^k\psi_1^1(\tau)}{d\tau^k}\bigg|_{\tau=0} = \frac{d^k\bar{x}_1(t)}{dt^k}\bigg|_{t=0}, \quad \frac{d^k\psi_2^1(\tau)}{d\tau^k}\bigg|_{\tau=0} = \frac{d^k\bar{y}_1(t)}{dt^k}\bigg|_{t=0}, \quad k = 0, 1.$$

From Eqs. (4.29) and (4.30), one directly has the initial conditions for the first-order boundary corrections

$$x_1^b(\zeta) = \zeta\left[\frac{d\varphi_1(\tau)}{d\tau}\bigg|_{\tau=0} - \frac{d\bar{x}_0(t)}{dt}\bigg|_{t=0}\right] - \bar{x}_1(0), \quad \zeta \in [-h, 0], \tag{4.32}$$

$$y_1^b(\zeta) = \zeta\left[\frac{d\varphi_2(\tau)}{d\tau}\bigg|_{\tau=0} - \frac{d\bar{y}_0(t)}{dt}\bigg|_{t=0}\right] - \bar{y}_1(0), \quad \zeta \in [-h, 0]. \tag{4.33}$$

Using (4.17), (4.18), (4.28), (4.32), and assumption A5, and taking into account results of [13], one directly obtains that problem (4.24), (4.33) has a unique solution satisfying the inequality

$$\left\| y_1^b(\xi)\right\| \leqslant a\exp(-\kappa\xi), \quad \xi \geqslant 0, \tag{4.34}$$

where $a > 0$ and $\kappa > 0$ are some constants.

Thus, we have completed the formal construction of the first-order uniform asymptotic solution to problem (4.1)–(4.2) in the case of the time-dependent matrix A_ε.

4.1.8. Justification of the first-order asymptotic solution

Theorem 4.1. *Under assumptions* A4–A8*, the unique solution of problem* (4.1)–(4.2) $z(t,\varepsilon) = \mathrm{col}\{x(t,\varepsilon), y(t,\varepsilon)\}$ *satisfies the inequality* $\|z(t,\varepsilon) - z_1(t,\varepsilon)\| \leqslant a\varepsilon^2$ *for* $t \in [0,T]$ *and all sufficiently small* $\varepsilon > 0$*, where* $z_1(t,\varepsilon) = \mathrm{col}\{x_1(t,\varepsilon), y_1(t,\varepsilon)\}$ *is the first-order asymptotic solution obtained in Sections* 4.1.1–4.1.7*, and* $a > 0$ *is some constant independent of* ε.

Proof. First of all, let us note that the existence and uniqueness of solution of problem (4.1)–(4.2) directly follows from results of [13].

Let us make the following transformation of variables in (4.1)–(4.2):

$$z(t,\varepsilon) = z_1(t,\varepsilon) + v(t,\varepsilon). \tag{4.35}$$

Substituting (4.35) into (4.1)–(4.2) and applying results of Sections 4.1.1–4.1.7, one obtains after some rearrangement the following problem for the new variable $v(t,\varepsilon)$:

$$\frac{dv(t,\varepsilon)}{dt} = \int_{-h}^{0} \bigl[d_\eta A_\varepsilon(t,\eta)\bigr] v(t+\varepsilon\eta,\varepsilon) + g(t,\varepsilon), \quad t \in [0,T], \tag{4.36}$$

$$v(\tau,\varepsilon) = \phi(\tau,\varepsilon), \quad \tau \in [-\varepsilon h, 0], \tag{4.37}$$

where

$$g(t,\varepsilon) = \mathrm{col}\bigl\{g_1(t,\varepsilon), g_2(t,\varepsilon)\bigr\}, \qquad \phi(\tau,\varepsilon) = \mathrm{col}\bigl\{\phi_1(\tau,\varepsilon), \phi_2(\tau,\varepsilon)\bigr\}, \tag{4.38}$$

$g_1 \in E^n$, $g_2 \in E^m$, $\phi_1 \in E^n$, $\phi_2 \in E^m$, the vector-functions $g_j(t,\varepsilon)$, $\phi_j(\tau,\varepsilon)$ $(j = 1,2)$ are expressed in a known way by $z_1(t,\varepsilon)$, they are continuous in $t \in [0,T]$ and $\tau \in [-\varepsilon h, 0]$, respectively, and satisfy the following inequalities for all sufficiently small $\varepsilon > 0$:

$$\bigl\|g_1(t,\varepsilon)\bigr\| \leqslant a\varepsilon\left[\varepsilon + \exp\left(-\frac{\kappa t}{\varepsilon}\right)\right], \quad \bigl\|g_2(t,\varepsilon)\bigr\| \leqslant a\varepsilon, \quad t \in [0,T], \tag{4.39}$$

$$\bigl\|\phi_j(\tau,\varepsilon)\bigr\| \leqslant a\varepsilon^2, \quad j = 1,2, \ \tau \in [-\varepsilon h, 0], \tag{4.40}$$

where $a > 0$ and $\kappa > 0$ are some constants independent of ε.

Using the variation of constant formula [13], one directly has from (4.36) and (4.37)

$$v(t,\varepsilon) = \Psi(t,0,\varepsilon)\phi(0,\varepsilon) + \int_0^{\varepsilon h} \Psi(t,\omega,\varepsilon)\Lambda_\varepsilon \left\{ \int_{-h}^{-\omega/\varepsilon} \bigl[d_\eta A(t,\eta)\bigr]\phi(\omega+\varepsilon\eta) \right\} d\omega$$
$$+ \int_0^t \Psi(t,s,\varepsilon) g(s,\varepsilon)\, ds, \quad t \in [0,T], \tag{4.41}$$

where $\Psi(t,s,\varepsilon)$ is the fundamental matrix of system (1.1)–(1.2), and

$$\Lambda_\varepsilon = \begin{pmatrix} I_n & 0 \\ 0 & (1/\varepsilon)I_m \end{pmatrix}, \qquad A(t,\eta) = \begin{pmatrix} A_1(t,\eta) & A_2(t,\eta) \\ A_3(t,\eta) & A_4(t,\eta) \end{pmatrix}. \tag{4.42}$$

Using Theorem 3.1 and inequalities (4.39), (4.40), one directly obtains from Eq. (4.41) for all sufficiently small $\varepsilon > 0$

$$\|v(t,\varepsilon)\| \leqslant a\varepsilon^2, \quad t \in [0,T]. \tag{4.43}$$

Now, the statement of the theorem directly follows from Eq. (4.35) and inequality (4.43). □

4.2. Asymptotic solution of (4.1)–(4.2) (the time-independent case)

In this section, in addition to assumptions A1–A3 and A8, we shall assume:

A9. The vector-functions $f_j(t)$ $(j = 1,2)$ are twice continuously differentiable for $t \in [0,+\infty)$, and $d^k f_j(t)/dt^k$ $(j = 1,2,\ k = 0,1,2)$ are bounded for $t \in [0,+\infty)$.

The formal representation of the first-order asymptotic solution $z_1(t,\varepsilon) = \mathrm{col}\{x_1(t,\varepsilon), y_1(t,\varepsilon)\}$ of (4.1)–(4.2) on the interval $t \in [0,+\infty)$ in the case of the time-independent matrix A_ε is the same as (4.4)–(4.5). The algorithm of obtaining the terms of the asymptotic expansion is the same as presented in Sections 4.1.2–4.1.7 with obvious simplification owing to the time-independent character of A_ε. Assumption A9 along with assumptions A1 and A2 provide the existence and boundness of $d^k\bar{x}_0(t)/dt^k$, $d^k\bar{y}_0(t)/dt^k$ $(k = 0,1,2)$ and $d^k\bar{x}_1(t)/dt^k$, $d^k\bar{y}_1(t)/dt^k$ $(k = 0,1)$ for $t \in [0,+\infty)$. The justification of the first-order asymptotic solution is carried out very similarly to the proof of Theorem 4.1 using Theorem 2.3 instead of Theorem 3.1. Thus, we have the following proposition.

Theorem 4.2. *Let the matrix A_ε be time independent. Then, under assumptions* A1–A3, A8, *and* A9, *the unique solution of problem* (4.1)–(4.2) $z(t,\varepsilon) = \mathrm{col}\{x(t,\varepsilon), y(t,\varepsilon)\}$ *satisfies the inequality $\|z(t,\varepsilon) - z_1(t,\varepsilon)\| \leqslant a\varepsilon^2$ for $t \in [0,+\infty)$ and all sufficiently small $\varepsilon > 0$, where $z_1(t,\varepsilon)$ is the first-order asymptotic solution, and $a > 0$ is some constant independent of ε.*

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