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Continuous LERW started from interior points

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Abstract

We use the whole-plane Loewner equation to define a family of continuous LERW in finitely connected domains that are started from interior points. These continuous LERW satisfy conformal invariance, preserve some continuous local martingales, and are the scaling limits of the corresponding discrete LERW on the discrete approximation of the domains.

1. Introduction

This paper is a follow-up of [12], in which we defined a family of random curves called continuous LERW in finitely connected plane domains, and proved that they are the scaling limits of the corresponding discrete LERW (loop-erased conditional random walk).

The continuous LERW defined in [12] is a simple curve that grows from a boundary point (or prime end, cf. [1]), say a, of some domain, say D, and aims at a certain target, which could be an interior point, a boundary arc or another boundary point of D. It is an SLE_2-type process that satisfies conformal invariance, which behaves locally like the SLE_2 process in simply connected domains introduced by Oded Schramm [10]. The special cases are when D is a subdomain of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, $a = 0$, and the part of ∂D near a lies on $\mathbb{R}$. In this case, the LERW is the chordal Loewner evolution driven by some semi-martingale, whose martingale part is $\sqrt{2}$ times a Brownian motion, and whose differentiable part contains the information of the domain and the target set. The continuous LERW is first defined in the special cases, and then extended to general cases via conformal maps.

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The corresponding discrete LERW is defined on the graph D^δ, which is the grid approximation of D by $\delta\mathbb{Z}^2$ for some small $\delta > 0$. For the construction, we first start a simple random walk on D^δ from an interior vertex that is closest to a, and stop it when it leaves the domain or hits a vertex that is closest to the target. Then we condition this stopped random walk on the event that it ends at a vertex that is closest to the target. Finally, we erase the loops on this conditional random walk in the order they appear, and get the discrete LERW.

The convergence of the discrete LERW curves to the corresponding continuous LERW curves were proved using the technique introduced by [5]: first use Skorokhod's embedding theorem to prove the convergence of the driving function, and then use the tameness of the discrete LERW curve to prove the convergence of the curves.

This paper will consider the case when the start point a is not a boundary point, but an interior point of D. It is natural to define a discrete LERW that starts from a vertex of D^δ which is closest to a and aims at a given target. The motivation of this paper is to describe the scaling limit of this lattice path. We will uses whole-plane Loewner equation [3] to define a family of random curves, which are still called continuous LERW, and prove that they are the scaling limits of the above discrete LERW.

For the definition of continuous LERW in the domain D started from the interior point $a = 0$ and aimed at another interior point, say z_e, we solve an integral equation as below. For $\xi \in C((-\infty, T))$. Let K_t^ξ and φ_t^ξ, $-\infty < t < T$, be the whole-plane Loewner hulls and maps, respectively, driven by ξ (cf. Section 4.3 of [3] or Section 2.4 of this paper). Suppose $K_t^\xi \subset D \setminus \{z_e\}$ for $-\infty < t < T$. Then for each $t \in (-\infty, T)$, $D \setminus K_t^\xi$ is a finitely connected domain containing z_e. Let

$$X^\xi(t) = (\partial_x \partial_y/\partial_y)[G(D \setminus K_t^\xi, z_e; \cdot) \circ (\varphi_t^\xi)^{-1} \circ \mathrm{e}^{\mathrm{i}} \circ R_{\mathbb{R}}](\xi(t)), \tag{1.1}$$

where $G(D \setminus K_t^\xi, z_e; \cdot)$ is Green's function in $D \setminus K_t^\xi$ with pole at z_e, e^{i} is the map $z \mapsto \mathrm{e}^{\mathrm{i}z}$, and $R_{\mathbb{R}}$ is the conjugate map $z \mapsto \bar{z}$. Let $\kappa = 2$, and $B_{\mathbb{R}}^{(\kappa)}(t)$, $-\infty < t < \infty$, be a driving function for whole-plane SLE_κ (cf. Section 6.6 of [3] or Section 3.2 of this paper). Let $\lambda = 2$, and $\xi(t)$, $-\infty < t < T$, be the solution to the integral equation

$$\xi(t) = B_{\mathbb{R}}^{(\kappa)}(t) + \lambda \int_{-\infty}^{t} X^\xi(t)\mathrm{d}t, \tag{1.2}$$

such that $(-\infty, T)$ is the maximal interval of the solution. It turns out that the solution exists, and is a semi-martingale. So there is a random continuous curve $\beta(t)$, $-\infty \le t < T$, such that $\beta(-\infty) = 0$ and $K_t^\xi = \beta([-\infty, t])$, $-\infty < t < T$. Such β is called the continuous LERW curve in D from 0 to z_e. If the target is a boundary arc or another boundary point, we will use harmonic measure function or Poisson kernel function instead of Green function in (1.1), and keep other formulas in the definition unchanged.

We then prove that these continuous LERW satisfy conformal invariance, and preserve some continuous local martingales generated by generalized Poisson kernels. Finally, we use the technique in [5,12] to show that these continuous LERW are the scaling limits of the corresponding discrete LERW.

The continuous LERW defined in this paper turns out to be locally absolutely continuous w.r.t. the whole-plane SLE_2. In fact, if U is a simply connected subdomain of D that contains the initial point 0, and is bounded away from ∂D and the target, then the continuous LERW stopped at the time τ_U when it exits U has a distribution absolutely continuous w.r.t. the whole-plane

SLE_2 stopped at τ_U. Moreover, there is a local martingale process $M(t)$ such that the above Radon–Nikodym derivative is $M(\tau_U)$. The formula of $M(t)$ will be given in Section 5. So this gives an alternative way to define continuous LERW. First one may use whole-plane SLE_2 and the Radon–Nikodym derivative $M(\tau_U)$ to define a partial continuous LERW (stopped at τ_U), say γ_U, for every U. Using the local martingale property of $M(t)$, one can check that these partial processes are consistent w.r.t. each other: γ_{U_1} stopped at τ_{U_2} has the same distribution as γ_{U_2} stopped at τ_{U_1}. Then one may construct a complete continuous LERW γ such that γ stopped at any τ_U has the distribution of γ_U.

We prefer the definition using the driving function rather than Radon–Nikodym derivative. This is because when we prove the convergence of discrete LERW, the technique in [5] and the Skorokhod's embedding theorem can be easily applied here without major modifications. If one uses the other definition, and tries to prove the convergence, he first has to work out the convergence of a particular discrete LERW to the whole-plane SLE_2, and then show the convergence of the discrete Radon–Nikodym derivative (between discrete LERW) to the continuous Radon–Nikodym derivative $M(\tau_U)$. The first step requires no much less work than the other approach, while the second step seems very difficult to the author.

The Radon–Nikodym derivative approach is useful in other respects. For example, one may use the density functions together with the stochastic coupling technique introduced in [13] to prove the reversibility of continuous LERW without using discrete LERW. One may also use them to show that the continuous LERW is a loop-erasure of a plane Brownian motion restricted in the domain [11].

Unlike the SLE processes started from boundary, there are SLE_κ-type processes started from 0, which are not locally absolutely continuous w.r.t. whole-plane SLE_κ process. One example is the whole-plane Loewner process driven by $\xi(t) = B^{(\kappa)}_{\mathbb{R}}(t) + \sigma t$, where σ is a nonzero real constant. Although this is not the case for continuous LERW, some care is required when dealing with the definition of SLE started from interior points.

We expect that the definition of the continuous LERW started from interior points will shed some light on the definition of some other random curves started from interior points, e.g., the reversal of radial SLE curves, and the scaling limits of self-avoiding walks (SAW) that connect two interior points. In particular, our result implies a description of the reversal of radial SLE_2.

This paper is organized in the following way. In Section 2, we review some basic notation including the radial Loewner equations and whole-plane Loewner equations. We also study the Carathéodory topology restricted to the space of interior hulls. In Section 3, we give the detailed definition of continuous LERW started from interior points, and prove that such LERW satisfies conformal invariance, and preserves a family of continuous local martingales generated by the generalized Poisson kernels. In Section 4, we prove that the solution to (1.2) exists uniquely, and is a semi-martingale. In Section 5, we prove that the continuous LERW started from an interior point is locally absolutely continuous w.r.t. the whole-plane SLE_2 process. In the last section, we introduce a family of discrete LERW defined on the discrete approximation of the domain, and a sketch of a proof is given to show that the scaling limit of this discrete LERW is the continuous LERW defined in this paper.

We will frequently cite notation and theorems from [12]. The readers are suggested have a copy of [12] at hand. We will often use some basic properties of the SLE processes. The reader may refer [9,3] for the background of SLE.

2. Preliminaries

2.1. Some notation

We adopt the notation in Section 2 of [12] about finitely connected domain, conformal closure, prime end, side arc, Green function, generalized Poisson kernel, harmonic measure function, hull and Loewner chain, and etc. But now we call the hull and Loewner chain in [12] the boundary hull and boundary Loewner chain, respectively, to distinguish them from the interior hull and interior Loewner chain that will be defined in this paper.

Throughout this paper, we use the following notation. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote Riemann sphere. Let $\mathbb{H}$ be the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let $\mathbb{D}$ be the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. Let $\mathbb{T}$ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Let $\mathbb{S}_h$ be the strip $\{z \in \mathbb{C} : h > \operatorname{Im} z > 0\}$ for $h > 0$. Let $\mathbb{R}_h$ be the line $\{z \in \mathbb{C} : \operatorname{Im} z = h\}$ for $h \in \mathbb{R}$. Then $\mathbb{S}_h$ is bounded by $\mathbb{R}$ and $\mathbb{R}_h$. Let $\mathbb{A}_h$ be the annulus $\{z \in \mathbb{C} : e^{-h} < |z| < 1\}$ for $h > 0$. We define an almost-$\mathbb{D}$ domain to be a finitely connected subdomain of $\mathbb{D}$ which contains 0 and $\mathbb{A}_h$ for some $h > 0$.

Let e^i be the map $z \mapsto e^{iz}$. Then e^i is the covering map from $\mathbb{H}$ onto $\mathbb{D} \setminus \{0\}$, from $\mathbb{S}_h$ onto $\mathbb{A}_h$, and from $\mathbb{R}$ onto $\mathbb{T}$. Let $R_{\mathbb{R}}(z) = \bar{z}$ be the complex conjugate map. Let $R_{\mathbb{T}}(z) = 1/\bar{z}$ be the reflection about $\mathbb{T}$. Then $e^i \circ R_{\mathbb{R}} = R_{\mathbb{T}} \circ e^i$. For $w \in \mathbb{C}$, let A_w denote the map $z \mapsto w + z$; let M_w denote the map $z \mapsto wz$. Then $e^i \circ A_w = M_{e^i(w)} \circ e^i$. Let $\mathbf{B}(z_0; r)$ be the ball $\{z \in \mathbb{C} : |z - z_0| < r\}$. If σ is a Jordan curve in $\mathbb{C}$, we use $U(\sigma)$ to denote the bounded connected component of $\mathbb{C} \setminus \sigma$, and let $H(\sigma) := \overline{U(\sigma)} = U(\sigma) \cup \sigma$. If I is an interval on $\mathbb{R}$, let $C(I)$ denote the set of real valued continuous functions on I. For $f \in C(I)$, if $[a, b] \subset I$, let $\|f\|_{a,b} = \max\{|f(x)| : x \in [a, b]\}$; if $(-\infty, a] \subset I$, let $\|f\|_a = \sup\{|f(x)| : x \le a\}$.

2.2. Radial Loewner equation

If H is a boundary hull in $\mathbb{D}$ such that $0 \notin H$, then we say that H is a boundary hull in $\mathbb{D}$ w.r.t. 0. For such H, there is a unique map ψ_H that maps $\mathbb{D} \setminus H$ conformally onto $\mathbb{D}$ such that $\psi_H(0) = 0$ and $\psi_H'(0) > 0$. Then $\operatorname{dcap}(H) := \ln(\psi_H'(0)) \ge 0$ is called the capacity of H in $\mathbb{D}$ w.r.t. 0. For example, $\emptyset$ is a boundary hull in $\mathbb{D}$ w.r.t. 0, $\psi_\emptyset = \mathrm{id}_{\mathbb{D}}$, and $\operatorname{dcap}(\emptyset) = 0$. From Schwarz lemma, $|\psi_H(z)| \ge |z|$ for any $z \in \mathbb{D} \setminus H$. If $H_1 \subset H_2$ are boundary hulls in $\mathbb{D}$ w.r.t. 0, define $H_2/H_1 = \psi_{H_1}(H_2 \setminus H_1)$. Then H_2/H_1 is also a boundary hull in $\mathbb{D}$ w.r.t. 0, and we have $\psi_{H_2/H_1} = \psi_{H_2} \circ \psi_{H_1}^{-1}$ and $\operatorname{dcap}(H_1) + \operatorname{dcap}(H_2/H_1) = \operatorname{dcap}(H_2)$. Thus, $|\psi_{H_2}(z)| \ge |\psi_{H_1}(z)|$ for any $z \in \mathbb{D} \setminus H_2$.

The following proposition is the radial version of Lemma 2.8 in [4]. The proof is similar. So we omit the proof.

Proposition 2.1. *Let Ξ be an open neighborhood of $x_0 \in \mathbb{T}$ in $\mathbb{D}$. Suppose W maps Ξ conformally into $\mathbb{D}$ such that, as $z \to \mathbb{T}$ in Ξ, $W(z) \to \mathbb{T}$. Such W extends conformally across $\mathbb{T}$ near x_0 by Schwarz reflection principle. Then we have*

$$\lim_{H \to x_0} \frac{\operatorname{dcap}(W(H))}{\operatorname{dcap}(H)} = |W'(x_0)|^2, \tag{2.1}$$

where $H \to x_0$ means that H is a nonempty hull in $\mathbb{D}$ w.r.t. 0, and $\operatorname{diam}(H \cup \{x_0\}) \to 0$.

Suppose $\xi \in C([0, T))$ for some $T \in (0, +\infty]$. The radial Loewner equation driven by ξ is as follows:

$$\partial_t \psi_t(z) = \psi_t(z) \frac{e^{i\xi(t)} + \psi_t(z)}{e^{i\xi(t)} - \psi_t(z)}, \qquad \psi_0(z) = z. \tag{2.2}$$

For $0 \le t < T$, let L_t be the set of $z \in \mathbb{D}$ such that the solution $\psi_s(z)$ blows up before or at time t. Then L_t is a boundary hull in $\mathbb{D}$ w.r.t. 0, and $\psi_t = \psi_{L_t}$ for each $t \in [0, T)$. We call L_t and ψ_t, $0 \le t < T$, the radial Loewner hulls and maps, respectively, driven by ξ. We have the following proposition.

Proposition 2.2. (a) *Suppose L_t and ψ_t, $0 \le t < T$, are the radial Loewner hulls and maps, respectively, driven by $\xi \in C([0, T))$. Then $(L_t, 0 \le t < T)$ is a boundary Loewner chain in $\mathbb{D}$ avoiding* 0, *and* $\mathrm{dcap}(L_t) = t$ *for any* $0 \le t < T$. *Moreover,*

$$\{e^{i\xi(t)}\} = \bigcap_{\varepsilon \in (0, T-t)} \overline{L_{t+\varepsilon}/L_t}, \quad 0 \le t < T. \tag{2.3}$$

(b) *Suppose L_t, $0 \le t < T$, is a boundary Loewner chain in $\mathbb{D}$ avoiding* 0, *and* $\mathrm{dcap}(L_t) = t$ *for any $0 \le t < T$. Then there is $\xi \in C([0, T))$ such that L_t, $0 \le t < T$, are radial Loewner hulls driven by ξ.*

Proof. This is the main result in [6]. □

The covering radial Loewner equation driven by ξ is:

$$\partial_t \widetilde{\psi}_t(z) = \cot_2(\widetilde{\psi}_t(z) - \xi(t)), \qquad \widetilde{\psi}_0(z) = z. \tag{2.4}$$

In this paper, we use $\cot_2(z)$ to denote the function $\cot(z/2)$. For $0 \le t < T$, let $\widetilde{L}_t$ be the set of $z \in \mathbb{H}$ such that the solution $\widetilde{\psi}_s(z)$ blows up before or at time t. We call $\widetilde{L}_t$ and $\widetilde{\psi}_t$, $0 \le t < T$, the covering radial Loewner hulls and maps, respectively, driven by ξ. Then $\widetilde{\psi}_t$ maps $\mathbb{H} \setminus \widetilde{L}_t$ conformally onto $\mathbb{H}$, and satisfies $\widetilde{\psi}_t(z+2k\pi) = \widetilde{\psi}_t(z)+2k\pi$ for any $k \in \mathbb{Z}$. Since $\mathrm{Im}\cot_2(z) < 0$ for $z \in \mathbb{H}$, so $\mathrm{Im}\,\widetilde{\psi}_t(z)$ decreases in t. Suppose L_t and ψ_t, $0 \le t < T$, are the radial Loewner hulls and maps, respectively, driven by ξ, then for any $t \in [0, T)$, $\widetilde{L}_t = (e^i)^{-1}(L_t)$, and $\psi_t \circ e^i = e^i \circ \widetilde{\psi}_t$.

2.3. Interior hulls and interior Loewner chains

Suppose D is a finitely connected domain. If $\emptyset \ne F \subset D$ is compact and connected, and $D \setminus F$ is also connected, then we say that F is an interior hull in D. If F contains only one point, we say F is degenerate; otherwise, F is non-degenerate. If F is a non-degenerate interior hull in an n-connected domain D, then $D \setminus F$ is an $(n + 1)$-connected domain. If H is another interior hull in D, and $F \subset H$, then $H \setminus F$ is a boundary hull in $D \setminus F$.

Let $T \in (-\infty, +\infty]$. We say the family $F(t)$, $-\infty < t < T$, is an interior Loewner chain in D started from $z_0 \in D$ if (i) for each $t \in (-\infty, T)$, $F(t)$ is a non-degenerate interior hull in D; (ii) $F(t_1) \subsetneq F(t_2)$ for any $t_1 < t_2 < T$; (iii) for any $t_0 \in (-\infty, T)$, $(F(t_0 + t) \setminus F(t_0), 0 \le t < T - t_0)$ is a boundary Loewner chain in $D \setminus F(t_0)$; and (iv) $\bigcap_{-\infty<t<T} F(t) = \{z_0\}$. For any $t_0 < T$, if $(F(t_0 + t) \setminus F(t_0), 0 \le t < T - t_0)$ is started from a prime end $w(t_0)$ of $D \setminus F(t_0)$, then we say that $w(t_0)$ is the prime end determined by $(F(t))$ at time t_0. Suppose u is a continuous (strictly) increasing function on $(-\infty, T)$, and satisfies $u(-\infty) = -\infty$, that is, $\lim_{t\to-\infty} u(t) = -\infty$. Let $u(T) := \lim_{t\to T} u(t)$. Then $F(u^{-1}(t))$, $-\infty < t < u(T)$, is also an interior Loewner chain in D started from z_0. We call it the time-change of $(F(t))$ through u. Suppose $\gamma : [-\infty, T) \to D$ is a simple curve. For $t \in (-\infty, T)$, let $F(t) = \gamma([-\infty, t])$. Then $(F(t))$ is an interior Loewner chain started from $\gamma(-\infty)$. We call such F the interior Loewner chain generated by γ. Then for each $t < T$, $\gamma(t)$ is the prime end determined by $(F(t))$ at time t.

If F is an interior hull in $\widehat{\mathbb{C}}$, and $\infty \notin F$, then we call F a bounded interior hull. For example, if σ is a Jordan curve in $\mathbb{C}$, then $H(\sigma)$ is a bounded interior hull. For any bounded interior hull F, there is a unique function ϕ_F that maps $\widehat{\mathbb{C}} \setminus F$ conformally onto $\widehat{\mathbb{C}} \setminus \overline{r\mathbb{D}}$ for some $r \geq 0$ such that $\phi_F(\infty) = \infty$ and $\phi_F'(\infty) := \lim_{z\to\infty} z/\phi_F(z) = 1$. We call $\mathrm{rad}(F) := r$ the radius of F, and $\mathrm{cap}(F) := \ln(r)$ the capacity of F w.r.t. ∞. Here if F contains only one point, say z_0, then $\phi_F(z) = z - z_0$, so $\mathrm{rad}(F) = 0$ and $\mathrm{cap}(F) = \ln(0) = -\infty$. If F is non-degenerate, then $\mathrm{rad}(F) > 0$ and $\mathrm{cap}(F) \in \mathbb{R}$, and we define $\varphi_F := M_{\mathrm{rad}(F)}^{-1} \circ \phi_F$. Then φ_F maps $\widehat{\mathbb{C}} \setminus F$ conformally onto $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and satisfies $\varphi_F(\infty) = \infty$ and $\varphi_F'(\infty) > 0$. Let $\psi_F = R_{\mathbb{T}} \circ \varphi_F \circ R_{\mathbb{T}}$. Then ψ_F maps $\widehat{\mathbb{C}} \setminus R_{\mathbb{T}}(F)$ conformally onto $\mathbb{D}$, and satisfies $\psi_F(0) = 0$ and $\psi_F'(0) > 0$.

The following results are well known (e.g., cf. [3]). If F is a bounded interior hull, $a, b \in \mathbb{C}$, then $\mathrm{rad}(aF + b) = |a| \,\mathrm{rad}(F)$; $\mathrm{rad}(\overline{\mathbf{B}(z_0; r)}) = r$ for any $z_0 \in \mathbb{C}$ and $r > 0$; $\mathrm{rad}(F) \geq \mathrm{diam}(F)/4$ for any bounded interior hull F, and the equality holds if and only if F is a line segment or a single point. By taking logarithm, we get the corresponding results for $\mathrm{cap}(F)$. Suppose $F_1 \subset F_2$ are two non-degenerate bounded interior hulls. Then $\mathrm{cap}(F_1) \leq \mathrm{cap}(F_2)$, where the equality holds only if $F_1 = F_2$. Let $F_2/F_1 := R_{\mathbb{T}} \circ \varphi_{F_1}(F_2 \setminus F_1)$. Then F_2/F_1 is a boundary hull in $\mathbb{D}$ w.r.t. 0. Moreover, we have

$$\psi_{F_2/F_1} = R_{\mathbb{T}} \circ \varphi_{F_2} \circ \varphi_{F_1}^{-1} \circ R_{\mathbb{T}}, \tag{2.5}$$

and $\mathrm{dcap}(F_2/F_1) = \mathrm{cap}(F_2) - \mathrm{cap}(F_1)$. Since $|\psi_{F_2/F_1}(z)| \geq |z|$ for any $z \in \mathbb{D} \setminus (F_2/F_1)$, so $|\varphi_{F_1}(z)| \geq |\varphi_{F_2}(z)|$ for any $z \in \mathbb{C} \setminus F_2$. If $F_1 \subset F_2 \subset F_3$ are non-degenerate bounded interior hulls, then $F_2/F_1 \subset F_3/F_1$, and $(F_3/F_1)/(F_2/F_1) = F_3/F_2$. Here F_3/F_1 and F_2/F_1 are boundary hulls in $\mathbb{D}$ w.r.t. 0, and the quotient between F_3/F_1 and F_2/F_1 uses the definition in the last subsection.

Let $\mathcal{H}$ denote the set of all bounded interior hulls, and let $\mathcal{H}_0$ denote the set of $H \in \mathcal{H}$ such that $0 \in H$. From Proposition 3.30 in [3], there is an absolute constant $C_{\mathcal{H}} \geq 3$ such that, for any $H \in \mathcal{H}_0$ with $\mathrm{rad}(H) = 1$, and any $z \in \mathbb{C}$ with $|z| > 1$,

$$|\phi_H^{-1}(z) - z| \leq C_{\mathcal{H}}.$$

Suppose $H \in \mathcal{H}_0$ is non-degenerate. Let $H_0 = H/\mathrm{rad}(H) \in \mathcal{H}_0$. Then $\mathrm{rad}(H_0) = 1$. So for any $z \in \mathbb{C}$ with $|z| > \mathrm{rad}(H)$, $\phi_H^{-1}(z) = \mathrm{rad}(H)\phi_{H_0}^{-1}(z/\mathrm{rad}(H))$, which implies

$$|\phi_H^{-1}(z) - z| = |\phi_{H_0}^{-1}(z/\mathrm{rad}(H)) - z/\mathrm{rad}(H)|\,\mathrm{rad}(H) \leq C_{\mathcal{H}}\,\mathrm{rad}(H).$$

If $H \in \mathcal{H}$ is non-degenerate, then there is $z_0 \in H$ with $|z_0| = \mathrm{dist}(0, H)$. Then $H_0 = H - z_0 \in \mathcal{H}_0$, $\mathrm{rad}(H_0) = \mathrm{rad}(H)$, and $\phi_H^{-1} = A_{z_0} \circ \phi_{H_0}^{-1}$. Thus, for any $|z| > \mathrm{rad}(H)$,

$$|\phi_H^{-1}(z) - z| \leq |z_0| + |\phi_{H_0}^{-1}(z) - z| \leq \mathrm{dist}(0, H) + C_{\mathcal{H}}\,\mathrm{rad}(H). \tag{2.6}$$

If $H = \{z_0\}$ is degenerate, (2.6) still holds because $\phi_H^{-1} = A_{z_0}$ and $\mathrm{dist}(0, H) = |z_0|$. For any interior hull H, Since ϕ_H^{-1} maps $\{|z| > \mathrm{rad}(H)\}$ onto $\mathbb{C} \setminus H$, so for any $z \in \mathbb{C} \setminus H$,

$$|\phi_H(z) - z| \leq \mathrm{dist}(0, H) + C_{\mathcal{H}}\,\mathrm{rad}(H). \tag{2.7}$$

2.4. *Whole-plane Loewner equation*

Suppose $F(t)$, $-\infty < t < T$, is an interior Loewner chain (in $\widehat{\mathbb{C}}$) avoiding ∞, that is, $\infty \notin F(t)$ for any $t < T$. If $\mathrm{cap}(F(t)) = t$ for any $t < T$, we say $(F(t))$ is parameterized by capacity. In the general case, $v(t) := \mathrm{cap}(F(t))$ is a continuous increasing function such that $v(-\infty) = -\infty$, and the time-change of $(F(t))$ through v is parameterized by capacity. The following proposition is a combination of a theorem in [3] and its inverse statement.

Proposition 2.3. (i) *Suppose $\xi \in C((-\infty, T))$. Then there is a unique interior Loewner chain K_t, $-\infty < t < T$, which is started from 0, avoids ∞, and is parameterized by capacity, such that the followings hold. For $-\infty < t < T$, let $\varphi_t = \varphi_{K_t}$. Then φ_t satisfies*

$$\partial_t \varphi_t(z) = \varphi_t(z) \frac{e^{i\xi(t)} + \varphi_t(z)}{e^{i\xi(t)} - \varphi_t(z)}; \tag{2.8}$$

and for any $z_0 \in \mathbb{C} \setminus \{0\}$,

$$\lim_{t \to -\infty} e^t \varphi_t(z_0) = z_0. \tag{2.9}$$

(ii) *Suppose K_t, $-\infty < t < T$, is an interior Loewner chain started from 0 avoiding ∞, and is parameterized by capacity. Then there is $\xi \in C((-\infty, T))$ such that for $\varphi_t = \varphi_{K_t}$, (2.8) and (2.9) both hold.*

Proof. (i) This is a special case of Proposition 4.21 in [3], where $\mu_t = \delta_{e^i(\xi(t))}$.
(ii) Fix $t \in (-\infty, T)$. Since $K_t \in \mathcal{H}_0$, $\mathrm{rad}(K_t) = e^t$, and $\phi_{K_t} = M_{e^t} \circ \varphi_t$, so from (2.7),

$$|e^t \varphi_t(z) - z| < C_{\mathcal{H}} e^t, \quad z \in \mathbb{C} \setminus K_t. \tag{2.10}$$

Fix $z_0 \in \mathbb{C} \setminus \{0\}$. Since $\{0\} = \bigcap_{-\infty<t<T} K_t$, so there is $T_0 \in (-\infty, T)$ such that $z_0 \notin K_t$ for $t \le T_0$, which implies that $|e^t \varphi_t(z_0) - z_0| < C_{\mathcal{H}} e^t$ from (2.10). Thus, (2.9) holds.

Fix $b \in (-\infty, T)$. Since $(K_{b+t}/K_b, 0 \le t < T - b)$ is a boundary Loewner chain in $\mathbb{D}$ avoiding 0, and $\mathrm{dcap}(K_{b+t}/K_b) = \mathrm{cap}(K_{b+t}) - \mathrm{cap}(K_b) = t$ for $0 \le t < T - b$, so from Proposition 2.2, there is $\xi_b \in C([0, T - b))$ such that K_{b+t}/K_b, $0 \le t < T - b$, are the radial Loewner hulls driven by ξ_b, and for each $t \in [0, T - b)$,

$$\{e^{i\xi_b(t)}\} = \bigcap_{\varepsilon \in (0, T-b-t)} \overline{(K_{b+t+\varepsilon}/K_b)/(K_{b+t}/K_b)} = \bigcap_{\varepsilon \in (0, T-b-t)} \overline{K_{b+t+\varepsilon}/K_{b+t}}. \tag{2.11}$$

For $t \in (-\infty, T)$, choose $b \in (-\infty, t]$, and let $\chi(t) = e^i(\xi_b(t - b))$. From (2.11), the value of $\chi(t)$ does not depend on the choice of b. Since $\xi_b \in C([0, T - b))$ for each $b < T$, so χ is a $\mathbb{T}$-valued continuous function. Thus, there is $\xi \in C((-\infty, T))$ such that $\chi(t) = e^i(\xi(t))$ for $-\infty < t < T$. Since $e^i(\xi_b(t)) = e^i(\xi(b+t))$ for $0 \le t < T - b$, so K_{b+t}/K_b, $0 \le t < T - b$, are also the radial Loewner hulls driven by $\xi(b + \cdot)$. Let ψ_t^b, $0 \le t < T - b$, be the radial Loewner maps driven by $\xi(b + \cdot)$. Then for $t \in [0, T - b)$, $\psi_t^b = \psi_{K_{b+t}/K_b} = R_{\mathbb{T}} \circ \varphi_{b+t} \circ \varphi_b^{-1} \circ R_{\mathbb{T}}$, and

$$\partial_t \psi_t^b(z) = \psi_t^b(z) \frac{e^{i\xi(b+t)} + \psi_t^b(z)}{e^{i\xi(b+t)} - \circ\psi_t^b(z)}.$$

Since $R_{\mathbb{T}} \circ \psi_t^b = \varphi_{b+t} \circ \varphi_b^{-1} \circ R_{\mathbb{T}}$, and $\varphi_b^{-1} \circ R_{\mathbb{T}}$ maps $\mathbb{D} \setminus (K_{b+t}/K_b)$ onto $\mathbb{C} \setminus K_{b+t}$, so (2.8) holds for $t \in [b, T)$. Since $b \in (-\infty, T)$ could be arbitrary, so (2.8) holds for all $t \in (-\infty, T)$. □

In the above proposition, K_t and φ_t, $-\infty < t < T$, are called the whole-plane Loewner hulls and maps, respectively, driven by ξ. Since $e^i(\xi_b(t)) = e^i(\xi(b + t))$ for $0 \le t < T - b$, and $b \in (-\infty, T)$ is arbitrary, so from (2.11) we get a formula similar to (2.3), which is

$$\{e^{i\xi(t)}\} = \bigcap_{\varepsilon \in (0, T-t)} \overline{K_{t+\varepsilon}/K_t}, \quad -\infty < t < T. \tag{2.12}$$

For $t \in (-\infty, T)$, let $L_t = R_{\mathbb{T}}(K_t)$ and $\psi_t = R_{\mathbb{T}} \circ \varphi_t \circ R_{\mathbb{T}}$. Then $\psi_t = \psi_{K_t}$, $\mathbb{C} \setminus L_t$ is a simply connected domain that contains 0, ψ_t maps $\mathbb{C} \setminus L_t$ conformally onto $\mathbb{D}$, fixes 0, and satisfies

$$\partial_t \psi_t(z) = \psi_t(z) \frac{e^{i\xi(t)} + \psi_t(z)}{e^{i\xi(t)} - \psi_t(z)}. \tag{2.13}$$

We call L_t and ψ_t the inverted whole-plane Loewner hulls and maps, respectively, driven by ξ.

The covering whole-plane Loewner equation is defined as follows. Let $\widetilde{K}_t = (e^i)^{-1}(K_t)$, $-\infty < t < T$. Suppose $\widetilde{\varphi}_t$, $-\infty < t < T$, satisfy that for each t, $\widetilde{\varphi}_t$ maps $\mathbb{C} \setminus \widetilde{K}_I(t)$ conformally onto $-\mathbb{H}$, $e^i \circ \widetilde{\varphi}_t = \varphi_t \circ e^i$, and the following differential equation holds:

$$\partial_t \widetilde{\varphi}_t(z) = \cot_2(\widetilde{\varphi}_t(z) - \xi(t)), \quad -\infty < t < T; \tag{2.14}$$

$$\lim_{t \to -\infty} (\widetilde{\varphi}_t(z) - it) = z, \quad z \in \mathbb{C}. \tag{2.15}$$

Then we call $\widetilde{K}_t$ and $\widetilde{\varphi}_t$ the covering whole-plane Loewner hulls and maps, respectively, driven by ξ. Such family of $\widetilde{\varphi}_t$ exists and is unique. In fact, for each $t \in (-\infty, T)$, we can find some $\widetilde{\varphi}_t$ that maps $\mathbb{C} \setminus \widetilde{K}_I(t)$ conformally onto $-\mathbb{H}$ such that $e^i \circ \widetilde{g}_I(t, \cdot) = g_I(t, \cdot) \circ e^i$. Such $\widetilde{\varphi}_t$ is not unique. Since φ_t is differentiable in t, so one may choose $\widetilde{\varphi}_t$ such that it is also differentiable in t. From (2.8) we conclude that (2.14) must hold. From (2.9) we conclude that $\lim_{t \to -\infty}(\widetilde{\varphi}_t(z) - it) = z + i2n\pi$ for some $n \in \mathbb{Z}$, and such n is the same for every z. Now we replace $\widetilde{\varphi}_t$ by $\widetilde{\varphi}_t - i2n\pi$. Then (2.14) and (2.15) still hold. So we have the existence of $\widetilde{g}_I(t, \cdot)$. The uniqueness follows from the same argument.

For $-\infty < t < T$, let $\widetilde{L}_t = R_{\mathbb{R}}(\widetilde{K}_t)$ and $\widetilde{\psi}_t = R_{\mathbb{R}} \circ \widetilde{\varphi}_t \circ R_{\mathbb{R}}$. Then we have $\widetilde{L}_t = (e^i)^{-1}(L_t)$, $e^i \circ \widetilde{\psi}_t = \psi_t \circ e^i$, $\widetilde{\psi}_t$ maps $\mathbb{C} \setminus \widetilde{L}_t$ conformally onto $\mathbb{H}$, and satisfies

$$\partial_t \widetilde{\psi}_t(z) = \cot_2(\widetilde{\psi}_t(z) - \xi(t)), \quad -\infty < t < T. \tag{2.16}$$

We call $\widetilde{L}_t$ and $\widetilde{\psi}_t$ the inverted covering whole-plane Loewner hulls and maps, respectively, driven by ξ. It is easily seen that for $-\infty < t < T$, the whole-plane Loewner objects driven by ξ at time t, such as $K_t, \varphi_t, L_t, \psi_t, \widetilde{K}_t, \widetilde{\varphi}_t, \widetilde{L}_t, \widetilde{\psi}_t$, are all determined by $e^i(\xi(s))$, $-\infty < s \le t$.

From (2.14) and (2.15), for any $z \in \mathbb{C} \setminus \widetilde{L}_t$, we have

$$\begin{aligned} R_{\mathbb{R}}(\widetilde{\psi}_t(z) - (z - it)) &= \widetilde{\varphi}_t(\overline{z}) - (\overline{z} + it) = \int_{-\infty}^{t} (\cot_2(\widetilde{\varphi}_s(\overline{z}) - \xi(s)) - i) ds \\ &= \int_{-\infty}^{t} i \left(\frac{e^i(\widetilde{\varphi}_s(\overline{z})) + e^i(\xi(s))}{e^i(\widetilde{\varphi}_s(\overline{z})) - e^i(\xi(s))} - 1 \right) ds = \int_{-\infty}^{t} \frac{2ie^i(\xi(s))}{\varphi_s(e^i(\overline{z})) - e^i(\xi(s))} ds. \end{aligned} \tag{2.17}$$

Suppose that $(1 + C_{\mathcal{H}})e^t|e^{iz}| \le 1/2$. Since $|e^i(\overline{z})| = 1/|e^i(z)|$, so for any $s \in (-\infty, t]$,

$$e^{-s}|e^i(\overline{z})| - (1 + C_{\mathcal{H}}) \ge \frac{e^{-s}}{2|e^i(z)|}. \tag{2.18}$$

Note that $e^i(\overline{z}) \in \mathbb{C} \setminus K_t \subset \mathbb{C} \setminus K_s$, $-\infty < s \le t$. From (2.10), for $s \in (-\infty, t]$, we have $|\varphi_s(e^i(\overline{z})) - e^{-s}e^i(\overline{z})| \le C_{\mathcal{H}}$, which together with (2.18) implies that

$$|\varphi_s(e^i(\overline{z})) - e^i(\xi(s))| \ge |\varphi_s(e^i(\overline{z}))| - 1 \ge |e^{-s}e^i(\overline{z})| - C_{\mathcal{H}} - 1 \ge \frac{e^{-s}}{2|e^i(z)|}.$$

From (2.17) and the above formula, we have

$$|\widetilde{\psi}_t(z) - (z - it)| \le 4(1 + C_{\mathcal{H}})e^t|e^{iz}|, \quad \text{if } (1 + C_{\mathcal{H}})e^t|e^{iz}| \le 1/2. \tag{2.19}$$

2.5. Carathéodory topology

The following definition is about the convergence of domains in Carathéodory topology.

Definition 2.1. Suppose D_n is a sequence of domains and D is a domain. We say that (D_n) converges to D, denoted by $D_n \xrightarrow{\text{Cara}} D$, if for every $z \in D$, $\text{dist}^\#(z, \partial^\# D_n) \to \text{dist}^\#(z, \partial^\# D)$. This is equivalent to the followings:

(i) every compact subset of D is contained in all but finitely many D_n's; and
(ii) for every point $z_0 \in \partial^\# D$, $\text{dist}^\#(z_0, \partial^\# D_n) \to 0$ as $n \to \infty$.

A sequence of domains may converge to two different domains. For example, let $D_n = \mathbb{C} \setminus ((-\infty, n])$. Then $D_n \xrightarrow{\text{Cara}} \mathbb{H}$, and $D_n \xrightarrow{\text{Cara}} -\mathbb{H}$ as well. But two different limit domains of the same domain sequence must be disjoint from each other, because if they have nonempty intersection, then one contains some boundary point of the other, which implies a contradiction.

Suppose $D_n \xrightarrow{\text{Cara}} D$, and for each n, f_n is a $\widehat{\mathbb{C}}$ valued function on D_n, and f is a $\widehat{\mathbb{C}}$ valued function on D. We say that f_n converges to f locally uniformly in D, or $f_n \xrightarrow{\text{l.u.}} f$ in D, if for each compact subset F of D, f_n converges to f in the spherical metric uniformly on F. If every f_n is analytic (resp. harmonic), then f is also analytic (resp. harmonic).

Lemma 2.1. *Suppose $D_n \xrightarrow{\text{Cara}} D$, f_n maps D_n conformally onto some domain E_n for each n, and $f_n \xrightarrow{\text{l.u.}} f$ in D. Then either f is constant on D, or f maps D conformally onto some domain E. And in the latter case, $E_n \xrightarrow{\text{Cara}} E$ and $f_n^{-1} \xrightarrow{\text{l.u.}} f^{-1}$ in E.*

This lemma is similar to Theorem 1.8, the Carathéodory kernel theorem, in [7], and the proof is also similar.

Recall that $\mathcal{H}$ is the set of all bounded interior hulls in $\mathbb{C}$. For every sequence (H_n) in $\mathcal{H}$, there is at most one $H \in \mathcal{H}$ such that $\mathbb{C} \setminus H_n \xrightarrow{\text{Cara}} \mathbb{C} \setminus H$ because if we also have $\mathbb{C} \setminus H_n \xrightarrow{\text{Cara}} \mathbb{C} \setminus H'$ for some $H' \in \mathcal{H}$, then from $(\mathbb{C} \setminus H') \cap (\mathbb{C} \setminus H) \neq \emptyset$ we conclude that $\mathbb{C} \setminus H' = \mathbb{C} \setminus H$, and so $H' = H$. We write $H_n \xrightarrow{\mathcal{H}} H$ for $\mathbb{C} \setminus H_n \xrightarrow{\text{Cara}} \mathbb{C} \setminus H$. We will define a metric $d_{\mathcal{H}}$ on $\mathcal{H}$ such that $H_n \to H$ w.r.t. $d_{\mathcal{H}}$ iff $H_n \xrightarrow{\mathcal{H}} H$.

Recall that for each $H \in \mathcal{H}$, ϕ_H maps $\widehat{\mathbb{C}} \setminus H$ conformally onto $\widehat{\mathbb{C}} \setminus \{|z| \le \text{rad}(H)\}$ such that $\phi_H(\infty) = \infty$ and $\phi'_H(\infty) = 1$. So ϕ_H^{-1} is defined on $\{|z| > \text{rad}(H)\}$. For $H_1, H_2 \in \mathcal{H}$, let

$$d_{\mathcal{H}}^{\vee}(H_1, H_2) = |\text{rad}(H_1) - \text{rad}(H_2)| + \sum_{m=1}^{\infty} 2^{-m} \sup\bigg\{ |\phi_{H_1}^{-1}(z) - \phi_{H_2}^{-1}(z)| : |z| \ge (\text{rad}(H_1) \vee \text{rad}(H_2)) + \frac{1}{m} \bigg\}. \tag{2.20}$$

It is clear that $d_{\mathcal{H}}^{\vee}(H_1, H_2) = d_{\mathcal{H}}^{\vee}(H_2, H_1) \ge 0$, and $d_{\mathcal{H}}^{\vee}(H_1, H_2) = 0$ iff $H_1 = H_2$. From (2.6) we have $d_{\mathcal{H}}^{\vee}(H_1, H_2) < \infty$. But $d_{\mathcal{H}}^{\vee}$ may not satisfy the triangle inequality.

We now define a metric $d_{\mathcal{H}}$ from $d_{\mathcal{H}}^{\vee}$ such that for $H_1, H_2 \in \mathcal{H}$,

$$d_{\mathcal{H}}(H_1, H_2) = \inf\bigg\{ \sum_{k=1}^{n} d_{\mathcal{H}}^{\vee}(F_{k-1}, F_k) : F_0 = H_1, F_n = H_2, F_k \in \mathcal{H}, 0 \le k \le n, n \in \mathbb{N} \bigg\}. \tag{2.21}$$

It is clear that $0 \le d_{\mathcal{H}}(H_1, H_2) = d_{\mathcal{H}}(H_2, H_1) \le d^{\vee}_{\mathcal{H}}(H_1, H_2) < \infty$ and $d_{\mathcal{H}}$ satisfies the triangle inequality. We need to check that $d_{\mathcal{H}}(H_1, H_2) = 0$ if and only if $H_1 = H_2$. The "if" part is clear because $d_{\mathcal{H}}(H_1, H_2) \le d^{\vee}_{\mathcal{H}}(H_1, H_2)$. For the "only if" part, we prove by contradiction. Suppose $H_1 \ne H_2$ and $d_{\mathcal{H}}(H_1, H_2) = 0$. If there are $F_k \in \mathcal{H}$, $0 \le k \le n$, such that $F_0 = H_1$ and $F_n = H_2$, then from (2.20) we have

$$\sum_{k=1}^{n} d^{\vee}_{\mathcal{H}}(F_{k-1}, F_k) \ge \sum_{k=1}^{n} |\operatorname{rad}(F_{k-1}) - \operatorname{rad}(F_k)| \ge |\operatorname{rad}(H_1) - \operatorname{rad}(H_2)|.$$

So we have $|\operatorname{rad}(H_1) - \operatorname{rad}(H_2)| \le d_{\mathcal{H}}(H_1, H_2) = 0$, which implies that $\operatorname{rad}(H_1) = \operatorname{rad}(H_2)$. Let $r = \operatorname{rad}(H_1)$. Since $H_1 \ne H_2$, so $\phi^{-1}_{H_1} \ne \phi^{-1}_{H_2}$ on $\{|z| > r\}$. Thus, there is $m \in \mathbb{N}$ such that

$$\sup\left\{|\phi_{H_1}(z) - \phi_{H_2}(z)| : |z| \ge r + \frac{1}{m}\right\} \ge \frac{1}{m}. \tag{2.22}$$

Since $d_{\mathcal{H}}(H_1, H_2) = 0$, so there are $F_k \in \mathcal{H}$, $0 \le k \le n$, such that $F_0 = H_1$ and $F_n = H_2$, and

$$\sum_{k=1}^{n} d^{\vee}_{\mathcal{H}}(F_{k-1}, F_k) < \frac{2^{-2m}}{2m}. \tag{2.23}$$

For any $1 \le j \le n$, since

$$\sum_{k=1}^{j} |\operatorname{rad}(F_{k-1}) - \operatorname{rad}(F_k)| \le \sum_{k=1}^{j} d^{\vee}_{\mathcal{H}}(F_{k-1}, F_k) < \frac{2^{-2m}}{2m} < \frac{1}{2m},$$

so $\operatorname{rad}(F_j) \le \operatorname{rad}(F_0) + \frac{1}{2m} = r + \frac{1}{2m}$. Thus, from (2.20) and (2.22)

$$\begin{aligned}
&\sum_{k=1}^{n} d^{\vee}_{\mathcal{H}}(F_{k-1}, F_k)\\
&\quad\ge \sum_{k=1}^{n} 2^{-2m} \sup\left\{|\phi^{-1}_{F_{k-1}}(z) - \phi^{-1}_{F_k}(z)| : |z| \ge (\operatorname{rad}(F_{k-1}) \vee \operatorname{rad}(F_k)) + \frac{1}{2m}\right\}\\
&\quad\ge 2^{-2m} \sum_{k=1}^{n} \sup\left\{|\phi^{-1}_{F_{k-1}}(z) - \phi^{-1}_{F_k}(z)| : |z| \ge r + \frac{1}{m}\right\}\\
&\quad\ge 2^{-2m} \sup\left\{|\phi^{-1}_{H_1}(z) - \phi^{-1}_{H_2}(z)| : |z| \ge r + \frac{1}{m}\right\} \ge \frac{2^{-2m}}{m},
\end{aligned}$$

which contradicts (2.23). Thus, $d_{\mathcal{H}}$ is a metric on $\mathcal{H}$.

Suppose $H_n \to H$ w.r.t. $d_{\mathcal{H}}$. Then we have $\operatorname{rad}(H_n) \to \operatorname{rad}(H)$ and $\phi^{-1}_{H_n}$ converges to ϕ^{-1}_H uniformly on $\{|z| \ge \operatorname{rad}(H) + \varepsilon\}$ for any $\varepsilon > 0$. Thus, $\{|z| > \operatorname{rad}(H_n)\} \xrightarrow{\text{Cara}} \{|z| > \operatorname{rad}(H)\}$ and $\phi^{-1}_{H_n} \xrightarrow{\text{l.u.}} \phi^{-1}_H$ in $\{|z| > \operatorname{rad}(H)\}$. From Lemma 2.1, we have $\mathbb{C} \setminus H_n = \phi^{-1}_{H_n}(\{|z| > \operatorname{rad}(H_n)\}) \xrightarrow{\text{Cara}} \phi^{-1}_H(\{|z| > \operatorname{rad}(H)\}) = \mathbb{C} \setminus H$, i.e., $H_n \xrightarrow{\mathcal{H}} H$. On the other hand, suppose $\mathbb{C} \setminus H_n \xrightarrow{\text{Cara}} \mathbb{C} \setminus H$. We will show that $H_n \to H$ w.r.t. $d_{\mathcal{H}}$. For this purpose, we will derive a stronger result.

Recall that $\mathcal{H}_0$ is the set of hulls in $\mathcal{H}$ that contains 0. Since $d_{\mathcal{H}}(H_n, H_0) \to 0$ implies $H_n \xrightarrow{\mathcal{H}} H_0$, so $\mathcal{H}_0$ is a closed subset of $(\mathcal{H}, d_{\mathcal{H}})$. For any $F \in \mathcal{H}$, let $\mathcal{H}(F) = \{H \in \mathcal{H} : H \subset F\}$,

$\mathcal{H}_0(F) = \mathcal{H}(F) \cap \mathcal{H}_0$, and $\mathcal{H}_0^x(F) = \{H \in \mathcal{H}_0(F) : \text{cap}(F) \geq x\}$, $x \in \mathbb{R}$. Then $\mathcal{H}_0(F)$ and $\mathcal{H}_0^x(F)$ are closed subsets of $\mathcal{H}(F)$ because rad is continuous w.r.t. $d_{\mathcal{H}}$. The hulls in $\mathcal{H}_0^x(F)$ are non-degenerate because they have finite capacities. If σ is a Jordan curve in $\mathbb{C}$, we write $\mathcal{H}(\sigma)$, $\mathcal{H}_0(\sigma)$, and $\mathcal{H}_0^x(\sigma)$ for $\mathcal{H}(H(\sigma))$, $\mathcal{H}_0(H(\sigma))$, and $\mathcal{H}_0^x(H(\sigma))$, respectively.

Lemma 2.2. *$\mathcal{H}(F)$, $\mathcal{H}_0(F)$ and $\mathcal{H}_0^x(F)$ are all compact subsets of $(\mathcal{H}, d_{\mathcal{H}})$.*

Proof. Let $r_F = \max\{|z| : z \in F\}$. Then for any $H \in \mathcal{H}(F)$, $\text{rad}(H) \leq \text{rad}(F) \leq r_F$. Suppose (H_n) is a sequence in $\mathcal{H}(F)$. By passing to a subsequence, we may assume that $\text{rad}(H_n) \to r_0 \in [0, r_F]$. For each $n \in \mathbb{N}$, let $g_n(z) = \phi_{H_n}^{-1}(z) - z$ for $\widehat{\mathbb{C}} \setminus \overline{\mathbf{B}(0; \text{rad}(H_n))}$. Then g_n is analytic. From (2.6), $|g_n|$ is bounded by $C_F := (1 + C_{\mathcal{H}})r_F$. Since $\text{rad}(H_n) \to r_0$, so $\widehat{\mathbb{C}} \setminus \overline{\mathbf{B}(0; \text{rad}(H_n))} \xrightarrow{\text{Cara}} \widehat{\mathbb{C}} \setminus \overline{\mathbf{B}(0; r_0)}$. Since (g_n) is a normal family, by passing to a subsequence, we may assume that $(g_n) \xrightarrow{\text{l.u.}} g_0$ in $\widehat{\mathbb{C}} \setminus \overline{\mathbf{B}(0; r_0)}$. Then $|g_0|$ is also bounded by C_F on $\widehat{\mathbb{C}} \setminus \overline{\mathbf{B}(0; r_0)}$. Let $f_0(z) = g_0(z) + z$ for $|z| > r_0$. Then $f_0(z) - z$ is bounded, and $\phi_{H_n}^{-1}(z) = g_n(z) + z \to f_0(z)$ uniformly on $\{|z| \geq r\}$ for any $r > r_0$. From Lemma 2.1, f_0 is either constant or a conformal map on $\widehat{\mathbb{C}} \setminus \overline{\mathbf{B}(0; r_0)}$. Since $f_0(z) - z$ is bounded, so f_0 cannot be constant. Thus, f_0 is a conformal map, and $\widehat{\mathbb{C}} \setminus H_n \xrightarrow{\text{Cara}} f_0(\widehat{\mathbb{C}} \setminus \overline{\mathbf{B}(0; r_0)})$. Since $f_0(z) - z$ is bounded, so $\infty = f_0(\infty) \in f_0(\widehat{\mathbb{C}} \setminus \overline{\mathbf{B}(0; r_0)})$ and $f_0'(\infty) = \lim_{n\to\infty}(\phi_{H_n}^{-1})'(\infty) = 1$. Since $f_0(\widehat{\mathbb{C}} \setminus \overline{\mathbf{B}(0; r_0)})$ is simply connected, so its complement in $\widehat{\mathbb{C}}$ is some $H_0 \in \mathcal{H}$. Thus, $f_0 = \phi_{H_0}^{-1}$ and $\text{rad}(H_0) = r_0$.

We now have proved that, by passing to a subsequence, we have $\text{rad}(H_n) \to \text{rad}(H_0)$ and $\phi_{H_n}^{-1} \xrightarrow{\text{l.u.}} \phi_{H_0}^{-1}$ in $\widehat{\mathbb{C}} \setminus \overline{\mathbf{B}(0; r_0)}$. Moreover, for any $|z| > \text{rad}(H_n) \vee \text{rad}(H_0)$,

$$|\phi_{H_n}^{-1}(z) - \phi_{H_0}^{-1}(z)| \leq |\phi_{H_n}^{-1}(z) - z| + |\phi_{H_0}^{-1}(z) - z| \leq 2C_F.$$

Given $\varepsilon > 0$, there is $M \in \mathbb{N}$ such that $2^{-M}(2C_F) < \varepsilon/3$. There is $N \geq M$ such that, for $n \geq N$, $|\text{rad}(H_n) - \text{rad}(H_0)| < (\varepsilon/3) \wedge (1/N)$, and $|\phi_{H_n}^{-1}(z) - \phi_{H_0}^{-1}(z)| \leq \varepsilon/3$ for any $|z| \geq \text{rad}(H_0) + 1/N$. Thus,

$$\begin{aligned} d_{\mathcal{H}}(H_n, H_0) \leq d_{\mathcal{H}}^{\vee}(H_n, H_0) &= |\text{rad}(H_n) - \text{rad}(H_0)| \\ &\quad + \sum_{m=1}^{\infty} 2^{-m} \sup\{|\phi_{H_n}^{-1}(z) - \phi_{H_0}^{-1}(z)| : |z| \geq (\text{rad}(H_n) \vee \text{rad}(H_0)) + 1/m\} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\sum_{m=1}^{N} 2^{-m} + 2C_F \sum_{m=N+1}^{\infty} 2^{-m} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

So we have $d_{\mathcal{H}}(H_n, H_0) \to 0$. Thus, $\mathcal{H}(F)$ is compact. The rest part of the lemma follows from the facts that $\mathcal{H}_0$ and $\{H \in \mathcal{H} : \text{cap}(H) \geq x\}$ are closed. $\square$

Suppose $H_n \xrightarrow{\mathcal{H}} H$. Choose $r_0 \in (0, \infty)$ such that $H \subset \{|z| < r_0\}$. Then $\{|z| = r_0\}$ is a compact subset of $\mathbb{C} \setminus H$. Let $\delta = \text{dist}(H, \{|z| = r_1\}) > 0$. Choose $z_0 \in \partial H = \partial(\mathbb{C} \setminus H)$. Since $\mathbb{C} \setminus H_n \xrightarrow{\text{Cara}} \mathbb{C} \setminus H$, so there is $N \in \mathbb{N}$ such that if $n \geq N$, then $\{|z| = r_0\} \subset \mathbb{C} \setminus H_n$ and $\text{dist}(z_0, \partial(\mathbb{C} \setminus H_n)) < \delta$, which implies that $H_n \cap \{|z| = r_0\} = \emptyset$ and $H_n \cap \{|z| < r_0\} \neq \emptyset$. Since H_n is connected, so $H_n \subset \{|z| < r_0\}$ if $n \geq N$. Thus, $\{H_n : n \geq N\} \subset \mathcal{H}(\{|z| \leq r_0\})$. From Lemma 2.2, $\{H_n : n \in \mathbb{N}\}$ is a pre-compact set. Assume that $H_n \not\to H$ w.r.t. $d_{\mathcal{H}}$. Then there is $\varepsilon > 0$ and a subsequence (H_{n_k}) of (H_n) such that $d_{\mathcal{H}}(H_{n_k}, H) \geq \varepsilon$ for any $k \in \mathbb{N}$. By passing to a subsequence, we may assume that $H_{n_k} \to H'$ w.r.t. $d_{\mathcal{H}}$. Then $H' \neq H$ and $H_{n_k} \xrightarrow{\mathcal{H}} H'$.

Since $H_n \xrightarrow{\mathcal{H}} H$, so the subsequence $H_{n_k} \xrightarrow{\mathcal{H}} H$ as well. Then we must have $H' = H$, which is a contradiction. So $H_n \to H$ w.r.t. $d_{\mathcal{H}}$. So the topology on $\mathcal{H}$ generated by $d_{\mathcal{H}}$ agrees with Carathéodory topology. From (2.20) and (2.21) we see that if $H_n \to H$ w.r.t. $d_{\mathcal{H}}$, then $\mathrm{rad}(H_n) \to \mathrm{rad}(H)$ and $\phi_{H_n}^{-1} \xrightarrow{\text{l.u.}} \phi_H^{-1}$ in $\{|z| > \mathrm{rad}(H)\}$. From Lemma 2.1 we have $\phi_{H_n} \xrightarrow{\text{l.u.}} \phi_H$ in $\mathbb{C} \setminus H$. Recall that for every non-degenerate interior hull H, $\varphi_H = \mathrm{rad}(H)^{-1}\phi_H$ maps $\mathbb{C}\setminus H$ conformally onto $\{|z| > 1\}$ and $\psi_H = R_{\mathbb{T}} \circ \varphi_H \circ R_{\mathbb{T}}$ maps $\mathbb{C}\setminus R_T T(H)$ conformally onto $\mathbb{D}$.

Lemma 2.3. *Let α be a Jordan curve, F be a compact subset of $\mathbb{C}\setminus H(\alpha)$, and $b \in \mathbb{R}$. If $(H_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{H}^b(\alpha)$, then there is $H \in \mathcal{H}^b(\alpha)$ and a subsequence (H_{n_k}) of (H_n) such that $\varphi_{H_{n_k}} \to \varphi_H$ uniformly on F, and $\psi_{H_{n_k}} \to \psi_H$ uniformly on $R_{\mathbb{T}}(F)$.*

Proof. From Lemma 2.2, there is $H \in \mathcal{H}^b(\alpha)$ and a subsequence (H_{n_k}) of (H_n) such that $H_{n_k} \xrightarrow{\mathcal{H}} H$. Then $\mathrm{rad}(H_{n_k}) \to \mathrm{rad}(H)$ and $\phi_{H_{n_k}} \xrightarrow{\text{l.u.}} \phi_H$ in $\mathbb{C} \setminus H$. Since $H \subset H(\alpha)$ and F is a compact subset of $\mathbb{C}\setminus H(\alpha)$, so F is also a compact subset of $\mathbb{C}\setminus H$. Thus, $\phi_{H_{n_k}} \to \phi_H$ uniformly on F. Since $\mathrm{rad}(H_{n_k}) \to \mathrm{rad}(H) \ge e^b$, so $\varphi_{H_{n_k}} \to \varphi_H$ uniformly on F, and $\psi_{H_{n_k}} \to \psi_H$ uniformly on $R_{\mathbb{T}}(F)$. □

3. Continuous LERW

3.1. Continuous boundary LERW

Let Ω be an almost-$\mathbb{D}$ domain, and $p \in \Omega$. Let $\widetilde{\Omega} = (e^i)^{-1}(\Omega)$ and $\widetilde{p} = (e^i)^{-1}(p)$. For $\xi \in C([0,T))$, let ψ_t^{ξ} (resp. $\widetilde{\psi}_t^{\xi}$) and L_t^{ξ} (resp. $\widetilde{L}_t^{\xi}$), $0 \le t < T$, denote the radial (resp. covering radial) Loewner maps and hulls, respectively, driven by ξ. Suppose $L_t^{\xi} \subset \Omega \setminus \{p\}$, that is, $\widetilde{L}_t^{\xi} \subset \widetilde{\Omega} \setminus \widetilde{p}$. Then $\Omega \setminus L_t^{\xi}$ is a finitely connected subdomain of Ω, and contains p. Let $\Omega_t^{\xi} = \psi_t^{\xi}(\Omega \setminus L_t^{\xi})$, $\widetilde{\Omega}_t^{\xi} = (e^i)^{-1}(\Omega_t^{\xi}) = \widetilde{\psi}_t^{\xi}(\widetilde{\Omega} \setminus \widetilde{L}_t^{\xi})$, $p_t^{\xi} = \psi_t^{\xi}(p)$, and $\widetilde{p}_t^{\xi} = \widetilde{\psi}_t^{\xi}(\widetilde{p})$. Then Ω_t^{ξ} is also an almost-$\mathbb{D}$ domain, $p_t^{\xi} \in \Omega_t^{\xi}$, and $\widetilde{p}_t^{\xi} \subset \widetilde{\Omega}_t^{\xi}$. For a finitely connected domain D and $z_0 \in D$, let $G(D, z_0; \cdot)$ denote the Green function in D with the pole at z_0. Let

$$J_t^{\xi} = G(\Omega \setminus L_t^{\xi}, p; \cdot) \circ (\psi_t^{\xi})^{-1} = G(\Omega_t^{\xi}, p_t^{\xi}; \cdot), \tag{3.1}$$

and $\widetilde{J}_t^{\xi} = J_t^{\xi} \circ e^i$. Then $\widetilde{J}_t^{\xi}$ is harmonic on $\widetilde{\Omega}_t^{\xi} \setminus \widetilde{p}_t^{\xi}$, and vanishes on $\mathbb{R}$, so can be extended harmonically across $\mathbb{R}$ by Schwarz reflection principle. Let $X^{\xi}(t) = (\partial_x \partial_y / \partial_y)\widetilde{J}_t^{\xi}(\xi(t))$. The following theorem is similar to Theorem 3.1 in [12]. The difference is that here we use radial Loewner equation. We will prove the theorem in Section 4.1.

Theorem 3.1. (i) *For any $f \in C([0,\infty))$ and $\lambda \in \mathbb{R}$, the equation*

$$\xi(t) = f(t) + \lambda \int_0^t X^{\xi}(s)\mathrm{d}s \tag{3.2}$$

has a solution $\xi(t)$ on $[0,a]$ for some $a > 0$.

(ii) *If for $j = 1, 2$, ξ_j solves (3.2) for $0 \le t < T_j$, and $T_j > 0$, then there is $S > 0$ such that $\xi_1(t) = \xi_2(t)$ for $0 \le t \le S$.*

Remark. The statement of the above theorem is enough for the use of this paper. In fact, the followings are true. Eq. (3.2) has a unique maximal solution $\xi_f(t)(t)$, $0 \le t < T_f$, for some $T_f > 0$. Here we call a solution maximal if it cannot be extended. Moreover, for any $a \ge 0$, $\{f \in C([0,\infty)) : T_f > a\}$ is open w.r.t. $\|\cdot\|_{0,a}$, and $f \mapsto \xi_f$ is $(\|\cdot\|_{0,a}, \|\cdot\|_{0,a})$ continuous on $\{T_f > a\}$. Let $\lambda = 2$ and $f(t) = \sqrt{2}B(t)$, where $B(t)$ is a Brownian motion. Let $\xi(t)$, $0 \le t < T$, be the maximal solution to (3.2). For $0 \le t < T$, let

$$u(t) = \int_0^t \partial_y \widetilde{J}_s^\xi(\xi(s))^2 \mathrm{d}s.$$

One can prove that $(L^\xi_{u^{-1}(t)}, 0 \le t < u(T))$ has the same distribution as the continuous LERW$(\Omega; 1 \to p)$ defined in [12]. The proof is similar to that of Theorem 3.2 in [12]. So the radial Loewner equation plays an equivalent role as chordal Loewner equation in defining a continuous boundary LERW.

3.2. Continuous interior LERW

Let D be a finitely connected domain that contains 0. Fix $z_e \in D \setminus \{0\}$. Let $\Omega = R_{\mathbb{T}}(D)$, $p = R_{\mathbb{T}}(z_e)$, $\widetilde{\Omega} = (\mathrm{e}^{\mathrm{i}})^{-1}(\Omega)$, and $\widetilde{p} = (\mathrm{e}^{\mathrm{i}})^{-1}(p)$. Let $\xi \in C((-\infty, T))$. We use K_t^ξ (resp. L_t^ξ, $\widetilde{L}_t^\xi$) and φ_t^ξ (resp. ψ_t^ξ, $\widetilde{\psi}_t^\xi$), $0 \le t < T$, to denote the whole-plane (resp. inverted whole-plane, inverted covering whole-plane) Loewner hulls and maps, respectively, driven by $\xi \in C((-\infty, T))$. Recall that if $\xi \in C([0, T))$, we use ψ_t^ξ, $\widetilde{\psi}_t^\xi$, L_t^ξ and $\widetilde{L}_t^\xi$ to denote the radial Loewner objects driven by ξ. But this will not cause ambiguity.

If for some $t < T$, $K_t^\xi \subset D \setminus \{z_e\}$, that is, $L_t^\xi \subset \Omega \setminus \{p\}$ or $\widetilde{L}_t^\xi \subset \widetilde{\Omega} \setminus \widetilde{p}$, then let $\Omega_t^\xi = \psi_t^\xi(\Omega \setminus L_t^\xi)$, $p_t^\xi = \psi_t^\xi(p)$, $\widetilde{\Omega}_t^\xi = (\mathrm{e}^{\mathrm{i}})^{-1}(\Omega_t^\xi) = \widetilde{\psi}_t^\xi(\widetilde{\Omega} \setminus \widetilde{L}_t^\xi)$, and $\widetilde{p}_t^\xi = (\mathrm{e}^{\mathrm{i}})^{-1}(p_t^\xi) = \widetilde{\psi}_t^\xi(\widetilde{p})$. Then Ω_t^ξ is an almost-$\mathbb{D}$ domain that contains p_t^ξ.

Let

$$J_t^\xi = G(\Omega \setminus L_t^\xi, p; \cdot) \circ (\psi_t^\xi)^{-1} = G(\Omega_t^\xi, p_t^\xi; \cdot), \tag{3.3}$$

and $\widetilde{J}_t^\xi = J_t^\xi \circ \mathrm{e}^{\mathrm{i}}$. Then $\widetilde{J}_t^\xi$ is a positive harmonic function in $\widetilde{\Omega}_t^\xi \setminus \widetilde{p}_t^\xi$, and vanishes on $\mathbb{R}$. From Schwarz reflection principle, $\widetilde{J}_t^\xi$ extends harmonically across $\mathbb{R}$. Let $X^\xi(t) = (\partial_x \partial_y / \partial_y) \widetilde{J}_t^\xi(\xi(t))$. Recall that $\psi_t^\xi = R_{\mathbb{T}} \circ \varphi_t^\xi \circ R_{\mathbb{T}}$. It is easy to check that the $X^\xi(t)$ here agrees with that in (1.1).

For $a \in \mathbb{R}$, let $\mathcal{T}_a$ denote the topology on $C((-\infty, a])$ generated by $\|\cdot\|_{b,a}$, $b \le a$. For $f_1, f_2 \in C((-\infty, a])$, we write $f_1 \overset{a}{\sim} f_2$ if $\mathrm{e}^{\mathrm{i}}(f_1(t)) = \mathrm{e}^{\mathrm{i}}(f_2(t))$ for any $t \le a$. Let $\mathcal{T}_a^{\mathbb{T}}$ be the set of $S \in \mathcal{T}_a$ such that $\pi_a^{-1}(\pi_a(S)) = S$, where π_a is the projection map from $C((-\infty, a])$ onto $C((-\infty, a])/\overset{a}{\sim}$. Then $\mathcal{T}_a^{\mathbb{T}}$ is also a topology on $C(\mathbb{R})$. Let $\mathcal{F}_a^0$ be the σ-algebra generated by $\mathcal{T}_a^{\mathbb{T}}$. Then $\mathcal{F}_a^0$ agrees with the σ-algebra generated by the functions $f \mapsto \mathrm{e}^{\mathrm{i}}(f(t))$, $t \in (-\infty, a]$. The proposition and theorem below will be proved in Section 4.2.

Proposition 3.1. *If $L_a^\xi \subset \Omega \setminus \{p\}$, then the improper integral $\int_{-\infty}^a X^\xi(t)\mathrm{d}t$ converges.*

Theorem 3.2. *Fix $\lambda \in \mathbb{R}$. For any $f \in C(\mathbb{R})$, the equation*

$$\xi(t) = f(t) + \lambda \int_{-\infty}^t X^\xi(s)\mathrm{d}s \tag{3.4}$$

has a unique maximal solution $\xi_f \in C((-\infty, T_f))$ for some $T_f \in (-\infty, +\infty]$. Moreover,

(i) *for any $a \in \mathbb{R}$, $\{f \in C(\mathbb{R}) : T_f > a\} \in \mathcal{T}_a^{\mathbb{T}}$, and $f \mapsto \xi_f$ is $(\mathcal{T}_a^{\mathbb{T}}, \mathcal{T}_a^{\mathbb{T}})$-continuous on $\{f \in C(\mathbb{R}) : T_f > a\}$;*

(ii) *there does not exist a Jordan curve α such that $\bigcup_{t<T_f} K_t^{\xi_f} \subset H(\alpha) \subset D \setminus \{z_e\}$.*

Let $B_+(t)$ and $B_-(t)$, $0 \le t < \infty$, be two independent Brownian motions. Let $\mathbf{x}$ be a random variable that is uniformly distributed on $[0, 2\pi)$, and independent of $B_\pm(t)$. For $\kappa > 0$ and $t \in \mathbb{R}$, let $B^{(\kappa)}_{\mathbb{R}}(t) = \mathbf{x} + \sqrt{\kappa} B_{\mathrm{sign}(t)}(|t|)$. Then the whole-plane Loewner hulls driven by $B^{(\kappa)}_{\mathbb{R}}(t)$ are called the whole-plane SLE_κ hulls. We will be particularly interested in the case that $\kappa = 2$.

Let $(\mathcal{F}_t)$ be the usual augmentation of $(\mathcal{F}^0_t)$ w.r.t. the distribution of $B^{(2)}_{\mathbb{R}}$. So $(\mathcal{F}_t)$ is right-continuous. Let $\mathcal{F}_\infty = \vee_{t \in \mathbb{R}} \mathcal{F}_t$. Suppose S is a finite $(\mathcal{F}_t)_{t\in\mathbb{R}}$-stopping time. Then for any $t \ge 0$, $S + t$ is an $(\mathcal{F}_t)_{t\in\mathbb{R}}$-stopping time. So we have a filtration $(\mathcal{F}_{S+t})_{t\ge 0}$. For $t \ge 0$, let $B_S(t) := (B^{(2)}_{\mathbb{R}}(S + t) - B^{(2)}_{\mathbb{R}}(S))/\sqrt{2}$. It is well known that $(B_S(t), t \ge 0)$ is an $(\mathcal{F}_{S+t})_{t\ge 0}$-Brownian motion.

Suppose $\xi \in C((-\infty, T))$ is the maximal solution to (3.4) with $f = B^{(2)}_{\mathbb{R}}$ and $\lambda = 2$. Then we call $(K^\xi_t, 0 \le t < T)$ a continuous interior LERW process in D from 0 to z_e, and let it be denoted by $\mathrm{LERW}(D; 0 \to z_e)$. From Theorem 3.2(i), T is an $(\mathcal{F}_t)_{t\in\mathbb{R}}$-stopping time, and $(e^i(\xi(t)))$ is $(\mathcal{F}_t)$-adapted. So for any fixed $a \in \mathbb{R}$, $(\xi(a+t) - \xi(a), 0 \le t < T - a)$ is $(\mathcal{F}_{a+t})_{t\ge 0}$-adapted. Since $K^\xi_t, L^\xi_t, \widetilde{L}^\xi_t, \varphi^\xi_t, \psi^\xi_t, \widetilde{\psi}^\xi_t$ are determined by $e^i \circ \xi(s)$, $-\infty < s \le t$, so they are all $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted. Note that in general $(\xi(t))$ is not $(\mathcal{F}_t)_{t\in\mathbb{R}}$-adapted.

Let $R = \mathrm{dist}(0; \partial D \cup \{z_e\}) > 0$. Fix $r \in (0, R)$. From Theorem 3.2(ii), there is $t_0 \in (-\infty, T)$ such that $K^\xi_{t_0} \not\subset \mathbf{B}(0; r)$. Then $T > t_0 = \mathrm{cap}(K^\xi_{t_0}) \ge \ln(r/4)$. So $T \ge \ln(R/4)$. Fix $a \in (-\infty, \ln(R/4))$. Then $a < T$. Let $T_a = T - a$ and $\xi_a(t) = \xi(a + t) - \xi(a)$ for $0 \le t < T_a$. Then T_a is an $(\mathcal{F}_{a+t})_{t\ge 0}$-stopping time, $(\xi_a(t))$ and $(X^\xi(a + t))$ are $(\mathcal{F}_{a+t})_{t\ge 0}$-adapted. Recall that $B_a(t) = (B^{(2)}_{\mathbb{R}}(a + t) - B^{(2)}_{\mathbb{R}}(a))/\sqrt{2}$ is an $(\mathcal{F}_{a+t})_{t\ge 0}$-Brownian motion, so ξ_a solves the $(\mathcal{F}_{a+t})_{t\ge 0}$-adapted SDE:

$$d\xi_a(t) = \sqrt{2} dB_a(t) + 2X^\xi(a + t)dt, \quad 0 \le t < T_a. \tag{3.5}$$

From Girsanov's theorem [8] and the existence of the radial SLE_2 trace, one can easily show that the interior Loewner chain K^ξ_t, $-\infty < t < T$, is a.s. generated by a simple curve $\beta(t)$, $-\infty \le t < T$, with $\beta(-\infty) = 0$. We call such β an $\mathrm{LERW}(D; 0 \to z_e)$ curve.

Suppose $z_0 \ne z_e \in D$. If $z_0 \in \mathbb{C}$, we define $\mathrm{LERW}(D; z_0 \to z_e)$ to be the image of $\mathrm{LERW}(A^{-1}_{z_0}(D); 0 \to A^{-1}_{z_0}(z_e))$ under the map A_{z_0}. If $z_0 = \infty$, we define $\mathrm{LERW}(D; z_0 \to z_e)$ to be the image of $\mathrm{LERW}(W(D); 0 \to W(z_e))$ under the map $W(z) = 1/z$.

Remark. A continuous $\mathrm{LERW}(\widehat{\mathbb{C}} : 0 \to \infty)$ has the same distribution as a whole-plane SLE_2. This can be seen from the fact that $X^\xi(t) \equiv 0$.

3.3. Conformal invariance

Theorem 3.3. *Let D be a finitely connected domain, and $z_0, z_e \in D$ with $z_0 \ne z_e$. Let $(K_t, -\infty < t < T)$ be an* $\mathrm{LERW}(D; z_0 \to z_e)$ *process. Suppose V maps D conformally onto another finitely connected domain D^*. Then after a time-change, $(V(K_t), -\infty < t < T)$ has the same distribution as $(K^*_t, -\infty < t < T^*)$, which is an* $\mathrm{LERW}(D^*; z^*_0 \to z^*_e)$ *process, where $z^*_0 = V(z_0)$ and $z^*_e = V(z_e)$.*

Proof. WLOG, assume $z_0 = z^*_0 = 0$. Let $\kappa = 2$. From the definition, $K_t = K^\xi_t$ for $-\infty < t < T$, where $\xi(t)$, $-\infty < t < T$, is the maximal solution to the equation

$$\xi(t) = B^{(\kappa)}_{\mathbb{R}}(t) + \left(3 - \frac{\kappa}{2}\right) \int_{-\infty}^t X^\xi_s ds. \tag{3.6}$$

Since $\kappa \le 4$, so from the property of SLE_κ (cf. [9]), a.s. $V^{-1}(\infty) \notin K_t$ for any $t < T$. Since $V(0) = 0$, so $(V(K_t), -\infty < t < T)$ is a.s. an interior Loewner chain started from 0 avoiding ∞. Let $u(t) = \mathrm{cap}(V(K_t))$ for $-\infty < t < T$, and $T^* = u(T)$. Let $v(t) = u^{-1}(t)$ and $K_t^* = V(K_{v(t)})$ for $-\infty < t < T^*$. So (K_t^*) is a time-change of $(V(K_t))$. We will prove that $(K_t^*, -\infty < t < T^*)$ has the same distribution as an $\mathrm{LERW}(D^*; 0 \to z_e^*)$. Since (K_t^*) is parameterized by capacity, so from Proposition 2.3, there is $\xi^* \in C((-\infty, T^*))$ such that $K_t^* = K_t^{\xi^*}$ for $-\infty < t < T^*$. For simplicity, we omit the superscripts ξ, and replace the superscripts ξ^* by $*$ for the whole-plane Loewner objects driven by ξ or ξ^*, respectively, in the rest of this proof.

Recall $\Omega = R_{\mathbb{T}}(D)$, $\widetilde{\Omega} = (\mathrm{e}^{\mathrm{i}})^{-1}(\Omega)$, $p = R_{\mathbb{T}}(z_e)$, $\widetilde{p} = (\mathrm{e}^{\mathrm{i}})^{-1}(p)$, $\Omega_t = \psi_t(\Omega \setminus L_t)$, and $\widetilde{\Omega}_t = (\mathrm{e}^{\mathrm{i}})^{-1}(\Omega_t)$. We can define Ω^*, $\widetilde{\Omega}^*$, p^*, $\widetilde{p}^*$, Ω_t^*, and $\widetilde{\Omega}_t^*$, similarly for D^* and the driving function ξ^*. Let $W = R_{\mathbb{T}} \circ V \circ R_{\mathbb{T}}$. Then W maps Ω conformally onto Ω^*, and $W(p) = p^*$. There is $\widetilde{W}$ that maps $\widetilde{\Omega}$ conformally onto $\widetilde{\Omega}^*$ such that $\mathrm{e}^{\mathrm{i}} \circ \widetilde{W} = W \circ \mathrm{e}^{\mathrm{i}}$. Let

$$W_t = \psi_{u(t)}^* \circ W \circ \psi_t^{-1}, \qquad \widetilde{W}_t = \widetilde{\psi}_{u(t)}^* \circ \widetilde{W} \circ \widetilde{\psi}_t^{-1}, \quad -\infty < t < T. \tag{3.7}$$

Then $\mathrm{e}^{\mathrm{i}} \circ \widetilde{W}_t = W_t \circ \mathrm{e}^{\mathrm{i}}$, and W_t (resp. $\widetilde{W}_t$) maps Ω_t (resp. $\widetilde{\Omega}_t$) conformally onto $\Omega_{u(t)}^*$ (resp. $\widetilde{\Omega}_{u(t)}^*$). Since $W_t(z) \to \mathbb{T}$ as $\Omega_t \ni z \to \mathbb{T}$, and $\widetilde{W}_t(z) \to \mathbb{R}$ as $\widetilde{\Omega}_t \ni z \to \mathbb{R}$, so from Schwarz reflection principle, W_t (resp. $\widetilde{W}_t$) extends conformally across $\mathbb{T}$ (resp. $\mathbb{R}$). Since $W_t(K_{t+\varepsilon}/K_t) = K_{u(t+\varepsilon)}^*/K_{u(t)}^*$ for $-\infty < t < t + \varepsilon < T$. So from (2.12), $W_t(\mathrm{e}^{\mathrm{i}}(\xi(t))) = \mathrm{e}^{\mathrm{i}}(\xi^*(u(t)))$. Since $\mathrm{dcap}(K_{t+\varepsilon}/K_t) = \varepsilon$ and $\mathrm{dcap}(K_{u(t+\varepsilon)}^*/K_{u(t)}^*) = u(t + \varepsilon) - u(t)$, so from (2.1) we have,

$$u'(t) = |W_t'(\mathrm{e}^{\mathrm{i}}(\xi(t)))|^2 = \widetilde{W}_t'(\xi(t))^2, \quad -\infty < t < T. \tag{3.8}$$

Now $\mathrm{e}^{\mathrm{i}} \circ \widetilde{W}_t(\xi(t)) = W_t \circ \mathrm{e}^{\mathrm{i}}(\xi(t)) = \mathrm{e}^{\mathrm{i}}(\xi^*(u(t)))$, so $\widetilde{W}_t(\xi(v(t)))$ is also the driving function of (K_t^*). So we may choose ξ^* such that,

$$\xi^*(u(t)) = \widetilde{W}_t(\xi(t)), \quad -\infty < t < T. \tag{3.9}$$

Differentiate the equality $\widetilde{W}_t \circ \widetilde{\psi}_t(z) = \widetilde{\psi}_{u(t)}^* \circ \widetilde{W}(z)$ w.r.t. t for $t \in (-\infty, T)$ and $z \in \widetilde{\Omega} \setminus \widetilde{L}_t$. From (2.16), (3.8) and (3.9), we have

$$\begin{aligned}\partial_t \widetilde{W}_t(\widetilde{\psi}_t(z)) + \widetilde{W}_t'(\widetilde{\psi}_t(z)) \cot_2(\widetilde{\psi}_t(z) - \xi(t)) &= u'(t) \cot_2(\widetilde{\psi}_{u(t)}^* \circ \widetilde{W}(z) - \xi^*(u(t))) \\ &= \widetilde{W}_t'(\xi(t))^2 \cot_2(\widetilde{W}_t \circ \widetilde{\psi}_t(z) - \widetilde{W}_t(\xi(t))).\end{aligned}$$

Since $\widetilde{\psi}_t$ maps $\widetilde{\Omega} \setminus \widetilde{L}_t$ onto $\widetilde{\Omega}_t$, so for any $w \in \widetilde{\Omega}_t$,

$$\partial_t \widetilde{W}_t(w) = \widetilde{W}_t'(\xi(t))^2 \cot_2(\widetilde{W}_t(w) - \widetilde{W}_t(\xi(t))) - \widetilde{W}_t'(w) \cot_2(w - \xi(t)).$$

Letting $w \to \xi(t)$ in $\widetilde{\Omega}_t$, we get

$$\partial_t \widetilde{W}_t(\xi(v(t))) = -3\widetilde{W}_t''(\xi(v(t))). \tag{3.10}$$

Since W maps $\Omega \setminus L_{v(t)}$ conformally onto $\Omega^* \setminus L_t^*$, and $W(p) = p^*$, so $G(\Omega \setminus L_t, p; \cdot) = G(\Omega^* \setminus L_{u(t)}^*, p^*; \cdot) \circ W$. Thus, $J_t = J_{u(t)}^* \circ W_t$, and so $\widetilde{J}_t = \widetilde{J}_{u(t)}^* \circ \widetilde{W}_t$. Since $X(t) = (\partial_x \partial_y / \partial_y) \widetilde{J}_t(\xi(t))$, $X^*(u(t)) = (\partial_x \partial_y / \partial_y) \widetilde{J}_{u(t)}^*(\xi^*(u(t)))$, so from (3.9),

$$X(t) = \widetilde{W}_t''(\xi(t))/\widetilde{W}_t'(\xi(t)) + \widetilde{W}_t'(\xi(t)) X^*(u(t)), \quad -\infty < t < T. \tag{3.11}$$

We now want to apply Itô's formula. The following non-rigorous argument illustrate the idea of the proof. From (3.6), $\xi(t)$, $-\infty < t < T$, satisfies the SDE

$$\mathrm{d}\xi(t) = \mathrm{d}B_{\mathbb{R}}^{(\kappa)}(t) + \left(3 - \frac{\kappa}{2}\right) X(t)\mathrm{d}t. \tag{3.12}$$

One may think of $B_{\mathbb{R}}^{(\kappa)}(t)$ as $\sqrt{\kappa}B(t)$. From (3.9), (3.10), and Itô's formula, we have

$$\begin{aligned}\mathrm{d}\xi^*(u(t)) &= \widetilde{W}_t'(\xi(t))\mathrm{d}\xi(t) + \partial_t \widetilde{W}_t(\xi(t))\mathrm{d}t + \frac{\kappa}{2}\widetilde{W}_t''(\xi(t))\mathrm{d}t \\ &= \widetilde{W}_t'(\xi(t))dB_{\mathbb{R}}^{(\kappa)}(t) + \left(3 - \frac{\kappa}{2}\right)\left(\widetilde{W}_t'(\xi(t))X(t)\mathrm{d}t - \widetilde{W}_t''(\xi(t))\right)\mathrm{d}t. \end{aligned} \tag{3.13}$$

From (3.11) we then have

$$\mathrm{d}\xi^*(u(t)) = \widetilde{W}_t'(\xi(t))\mathrm{d}B_{\mathbb{R}}^{(\kappa)}(t) + \left(3 - \frac{\kappa}{2}\right)\widetilde{W}_t'(\xi(t))^2 X^*(u(t))\mathrm{d}t.$$

Finally, we use (3.8) to conclude that there is another copy of $B_{\mathbb{R}}^{\kappa}(t)$ such that

$$\mathrm{d}\xi^*(t) = \mathrm{d}B_{\mathbb{R}}^{(\kappa)}(t) + \left(3 - \frac{\kappa}{2}\right) X^*(t)\mathrm{d}t, \quad -\infty < t < T^*.$$

So $\xi^*(t)$ is a driving function for continuous LERW in D^* from 0 to z_e^*. The argument is not rigorous because $B_{\mathbb{R}}^{(\kappa)}(t)$ is not a Brownian motion in the usual sense, and Itô's formula does not directly apply to time-intervals of the form $(-\infty, T)$. We have a way to solve these problems, which is to truncate the time interval.

We will use the filtration $\mathcal{F}_t$, $t \in \mathbb{R}$, in Section 3.2. Suppose that a is a finite $(\mathcal{F}_t)$-stopping time such that $a < T$ always holds. Let $\mathcal{F}_t^a = \mathcal{F}_{a+t}$, $0 \le t < \infty$. Then we have a new filtration $(\mathcal{F}_t^a)_{t\ge 0}$. Let $T_a = T - a > 0$. Then T_a is an $(\mathcal{F}_t^a)$-stopping time. Let $B_a(t) = (B_{\mathbb{R}}^{(\kappa)}(a+t) - B_{\mathbb{R}}^{(\kappa)}(a))/\sqrt{\kappa}$, $0 \le t < \infty$. Then $B_a(t)$ is an $(\mathcal{F}_t^a)$-Brownian motion. Let $\xi_a(t) = \xi(a+t) - \xi(a)$ and $X_a(t) = X(a+t)$. Then (ξ_a) and (X_a) are both $(\mathcal{F}_t^a)$-adapted, and $\xi_a(t)$, $0 \le t < T_a$, satisfies the $(\mathcal{F}_t^a)$-adapted SDE:

$$\mathrm{d}\xi_a(t) = \sqrt{\kappa}\mathrm{d}B_a(t) + \left(3 - \frac{\kappa}{2}\right) X_a(t)\mathrm{d}t. \tag{3.14}$$

Let $u_a(t) = u(a+t) - u(a)$, $0 \le t < T_a$. Then u_a is continuous and increasing on $[0, T_a)$, and $u_a(0) = 0$. Let $\xi_b^*(t) = \xi^*(b+t) - \xi^*(b)$ for $b \in (-\infty, T^*)$ and $t \in [0, T^* - b)$. Let

$$\widetilde{W}_{a,t} = A_{\xi^*(u(a))}^{-1} \circ \widetilde{W}_{a+t} \circ A_{\xi(a)}.$$

Then $(\widetilde{W}_{a,t})$ is also $(\mathcal{F}_t^a)$-adapted. From (3.9), (3.8) and (3.10) we have

$$\xi_{u(a)}^*(u_a(t)) = \widetilde{W}_{a,t}(\xi_a(t)), \tag{3.15}$$

$$u_a'(t) = \widetilde{W}_{a,t}'(\xi_a(t))^2, \tag{3.16}$$

$$\partial_t \widetilde{W}_{a,t}(\xi_a(t)) = -3\widetilde{W}_{a,t}''(\xi_a(t)). \tag{3.17}$$

Now we apply Itô's formula to the $(\mathcal{F}_t^a)$-adapted SDE. From (3.14), (3.15) and (3.17), we have

$$\mathrm{d}\xi_{u(a)}^*(u_a(t)) = \widetilde{W}_{a,t}'(\xi_a(t))\sqrt{\kappa}\mathrm{d}B_a(t) + \left(3 - \frac{\kappa}{2}\right)\left(\widetilde{W}_{a,t}'(\xi_a(t))X_a(t)\mathrm{d}t - \widetilde{W}_{a,t}''(\xi_a(t))\right)\mathrm{d}t.$$

From (3.11) we have

$$\mathrm{d}\xi_{u(a)}^*(u_a(t)) = \widetilde{W}_{a,t}'(\xi_a(t))\sqrt{\kappa}\mathrm{d}B_a(t) + \widetilde{W}_{a,t}'(\xi_a(t))^2\left(3 - \frac{\kappa}{2}\right) X_{u(a)}^*(u_a(t))\mathrm{d}t, \tag{3.18}$$

where $X_b^*(t) = X^*(b+t)$ for $b \in (-\infty, T^*)$ and $t \in [0, T^* - b)$. Now we apply some time-change. Recall that u_a is continuously increasing, and maps $[0, T_a)$ onto $[0, T^* - u(a))$. So its inverse map, say v_a is defined on $[0, T^* - u(a))$. We extend v_a to be defined on $[0, \infty)$ such that if $t > T^* - u(a)$ then $v_a(t) = T_a$. Since (u_a) is $(\mathcal{F}_t^a)$-adapted, so for every $t \in [0, \infty)$, $v_a(t)$ is an $(\mathcal{F}_t^a)$-stopping time. Let $\mathcal{F}_t^{a,v} = \mathcal{F}_{v_a(t)}^a$, $0 \le t < \infty$. Then we have a new filtration $(\mathcal{F}_t^{a,v})_{0 \le t < \infty}$. From (3.16) and (3.18) we see that there is a stopped $(\mathcal{F}_t^{a,v})$-Brownian motion $B_{a,v}(t)$, $0 \le t < T^* - u(a)$, such that $\xi_{u(a)}^*(t)$ satisfies the $(\mathcal{F}_t^{a,v})$-adapted SDE:

$$\mathrm{d}\xi_{u(a)}^*(t) = \sqrt{\kappa}\mathrm{d}B_{a,v}(t) + \left(3 - \frac{\kappa}{2}\right) X_{u(a)}^*(t)\mathrm{d}t, \quad 0 \le t < T^* - u(a). \tag{3.19}$$

Using Proposition 3.1, we may define

$$B^*(t) = \xi^*(t) - \left(3 - \frac{\kappa}{2}\right) \int_{-\infty}^t X^*(s)\mathrm{d}s, \quad -\infty < t < T^*. \tag{3.20}$$

From (3.19) we have

$$\sqrt{\kappa} B_{a,v}(t) = B^*(u(a) + t) - B^*(u(a)), \quad 0 \le t < T^* - u(a). \tag{3.21}$$

From (3.9) and (3.11) we know that $(\mathrm{e}^{\mathrm{i}}(\xi^*(u(t))))$ and $(X^*(u(t)))$ are both $(\mathcal{F}_t)$-adapted. So $(\mathrm{e}^{\mathrm{i}}(B^*(u(t))))$ is also $(\mathcal{F}_t)$-adapted. Especially, $\mathrm{e}^{\mathrm{i}}(B^*(t))$, $-\infty < t \le u(a)$, are $\mathcal{F}_a$-measurable. Since $\mathcal{F}_a = \mathcal{F}_0^a = \mathcal{F}_0^{a,v}$, so from (3.21), $B_{a,v}(t) = (B^*(u(a) + t) - B^*(u(a)))/\sqrt{\kappa}$, $0 \le t < T^* - u(a)$, is a stopped Brownian motion independent of $\mathrm{e}^{\mathrm{i}}(B^*(t))$, $-\infty < t \le u(a)$.

Recall that in the above argument, we need that a is a finite $(\mathcal{F}_t)$-stopping time such that $T > a$ always holds. Let $R = \mathrm{dist}(0, \mathbb{C} \setminus (D^* \setminus \{z_e^*, V(\infty)\}))$. From Theorem 3.2(ii), for any $r \in (0, R)$, there is $t_r < T$ such that $K_{t_r} \not\subset V^{-1}(\mathbf{B}(0; r))$, so $K_{u(t_r)}^* = V(K_{t_r}) \not\subset \mathbf{B}(0; r)$. Thus, $T^* > u(t_r) = \mathrm{cap}(K_{u(t_r)}^*) \ge \ln(r/4)$. So $T^* \ge \ln(R/4)$. Fix any deterministic number $b \in (-\infty, \ln(R/4))$. Then $T^* > b$ always holds. Let $a = u^{-1}(b)$. Then a is a finite stopping time such that $T > a$ always holds, and $u(a) = b$ is a deterministic number. From the last paragraph, we then conclude that $(B^*(b + t) - B^*(b))/\sqrt{\kappa}$, $0 \le t < T^* - b$, is a stopped Brownian motion independent of $\mathrm{e}^{\mathrm{i}}(B^*(t))$, $-\infty < t \le b$. Since this holds for any deterministic number $b \in (-\infty, \ln(R/4))$, so we may extend $B^*(t)$ to be defined on $\mathbb{R}$ such that $(\mathrm{e}^{\mathrm{i}}(B^*(t)))$ has the same distribution as $(\mathrm{e}^{\mathrm{i}}(B_{\mathbb{R}}^{(\kappa)}(t)))$. This means that there is an integer valued random variable $\mathbf{n}$ such that $(B^*(t) - 2\mathbf{n}\pi)$ has the same distribution as $(B_{\mathbb{R}}^{(\kappa)}(t))$. Since $\xi^*(t)$ and $\xi^*(t) - 2\mathbf{n}\pi$ generate the same whole-plane Loewner objects, so by replacing $\xi^*(t)$ by $\xi^*(t) - 2\mathbf{n}\pi$, we may assume that $(B^*(t))$ has the same distribution as $(B_{\mathbb{R}}^{(\kappa)}(t))$. From (3.20), $\xi^*(t)$ solves

$$\xi^*(t) = B^*(t) + \left(3 - \frac{\kappa}{2}\right) \int_{-\infty}^t X^*(s)\mathrm{d}s, \quad -\infty < t < T^*. \tag{3.22}$$

So we can conclude that $K_t^* = V(K_{v(t)})$, $-\infty < t < T^*$, is a stopped LERW process in D^* from 0 to z_e^*. To finish the proof, we need to show that $(-\infty, T^*)$ is a.s. the maximal interval of the solution to (3.22) for the extended function $B^*(t)$, which is now defined on $\mathbb{R}$.

Assume that $(-\infty, T^*)$ is not the maximal interval of the solution. So we have $K_{T^*}^*$, which is an interior hull in $D^* \setminus \{\infty, z_e^*\}$ that contains K_t^* for all $t \in (-\infty, T^*)$. Since $\kappa \le 4$, so a.s. $V(\infty) \notin K_{T^*}^*$. Excluding a null event, we may assume that $K_{T^*}^* \subset D^* \setminus \{\infty, V(\infty), z_e^*\}$. We can find a Jordan curve σ^* in $\mathbb{C}$ such that $K_{T^*}^* \subset H(\sigma^*) \subset D^* \setminus \{\infty, V(\infty), z_e^*\}$. So $V(K_t) = K_{u(t)}^* \subset H(\sigma^*)$ for $-\infty < t < T$. Let $\sigma = V^{-1}(\sigma^*)$. Then σ is a Jordan curve in $\mathbb{C}$, and $H(\sigma) = V^{-1}(H(\sigma^*)) \subset D \setminus \{V^{-1}\infty, \infty, z_e\}$. We have $K_t \subset H(\sigma)$ for $-\infty < t < T$, which contradicts Theorem 3.2(ii). So $(-\infty, T^*)$ is a.s. the maximal interval of the solution, and the proof is finished. □

Remark. The ideas behind (3.8), (3.9) and (3.10) first appeared in [4], which were used there to show that SLE_6 satisfies locality property. From the above proof we see that for any $\kappa \in (0,4]$, the above theorem still holds if $\xi(t)$ is the solution to (3.4) with $f(t) = B^{(\kappa)}_{\mathbb{R}}(t)$ and $\lambda = 3 - \frac{\kappa}{2}$. If $\kappa > 4$, the statement should be modified. We can conclude that after a time-change, $(V(K_t), -\infty < t < S)$ has the same distribution as $(K^*_t, -\infty < t < S^*)$, where $S \in (-\infty, T]$ is the biggest number such that $K_t \subset D \setminus \{V^{-1}(\infty)\}$ for $t \in (-\infty, S)$, and $S^* \in (-\infty, T^*]$ is the biggest number such that $K^*_t \subset D^* \setminus \{V(\infty)\}$ for $t \in (-\infty, S^*)$.

3.4. Local martingales

Let D be a finitely connected domain, $0 \in D$, and $z_e \in D \setminus \{0\}$. Let $p = R_{\mathbb{T}}(z_e)$ and $\Omega = R_{\mathbb{T}}(D)$. For $\xi \in C((-\infty, T))$, let L^ξ_t (resp. $\widetilde{L}^\xi_t$) and ψ^ξ_t (resp. $\widetilde{\psi}^\xi_t$) be the inverted whole-plane (resp. covering whole-plane) Loewner hulls and maps driven by ξ. Suppose $\bigcup_{t<T} L^\xi_t \subset \Omega \setminus \{p\}$. For each $t \in (-\infty, T)$ and $x \in \mathbb{R}$, let $P^\xi(t, x, \cdot)$ be the generalized Poisson kernel in Ω^ξ_t with the pole at e^{ix}, normalized by $P^\xi(t, x, \psi^\xi_t(p)) = 1$, and let $\widetilde{P}^\xi(t, x, \cdot) = P^\xi(t, x, \cdot) \circ e^{i}$. It is standard to check that both P^ξ and $\widetilde{P}^\xi$ are $C^{1,2,h}$ differentiable, where "h" means harmonic.

Lemma 3.1. *For any $t \in (-\infty, T)$ and $z \in \widetilde{\Omega} \setminus \widetilde{L}^\xi_t$, we have $\widetilde{\mathcal{V}}_t(z) = 0$, where*

$$\widetilde{\mathcal{V}}_t(z) = \partial_1 \widetilde{P}^\xi(t, \xi(t), \widetilde{\psi}^\xi_t(z)) + 2\partial_2 \widetilde{P}^\xi(t, \xi(t), \widetilde{\psi}^\xi_t(z)) X^\xi_t + \partial_2^2 \widetilde{P}^\xi(t, \xi(t), \widetilde{\psi}^\xi_t(z)) + 2\,\mathrm{Re}(\partial_{3,z} \widetilde{P}^\xi(t, \xi(t), \widetilde{\psi}^\xi_t(z)) \cot_2(\widetilde{\psi}^\xi_t(z) - \xi(t))).$$

Here ∂_1 and ∂_2 are partial derivatives w.r.t. the first two (real) variables, and $\partial_{3,z} = (\partial_{3,x} - i\partial_{3,y})/2$ is the partial derivative w.r.t. the third (complex) variable.

Proof. For simplicity, we assume that $\partial\Omega$ is smooth, so every boundary point of Ω or Ω^ξ_t corresponds to a prime end. In the general case, we have to work on the conformal closure of Ω. For any $t \in (-\infty, T)$ and $z \in \Omega \setminus L^\xi_t$, let

$$\mathcal{V}_t(z) = \partial_1 P^\xi(t, \xi(t), \psi^\xi_t(z)) + 2\partial_2 P^\xi(t, \xi(t), \psi^\xi_t(z)) X^\xi_t + \partial_2^2 P^\xi(t, \xi(t), \psi^\xi_t(z)) + 2\,\mathrm{Re}\left(\partial_{3,z} P^\xi(t, \xi(t), \psi^\xi_t(z)) \psi^\xi_t(z) \frac{e^{i\xi(t)} + \psi^\xi_t(z)}{e^{i\xi(t)} - \psi^\xi_t(z)}\right). \tag{3.23}$$

It is easy to check that $\mathcal{V}_t \circ e^{i} = \widetilde{\mathcal{V}}_t$. For $t \in (-\infty, T)$, $x \in \mathbb{R}$ and $z \in \partial\Omega$, since $\psi^\xi_t(z) \in \partial\Omega^\xi_t \setminus \mathbb{T}$, so $P^\xi(t, x, \psi^\xi_t(z)) = 0$, which implies that $\partial_2 P^\xi = \partial_2^2 P^\xi = 0$ at $(t, x, \psi^\xi_t(z))$, and

$$\partial_1 P^\xi(t, x, \psi^\xi_t(z)) + 2\,\mathrm{Re}\left(\partial_{3,z} P^\xi(t, \xi(t), \psi^\xi_t(z)) \psi^\xi_t(z) \frac{e^{i\xi(t)} + \psi^\xi_t(z)}{e^{i\xi(t)} - \psi^\xi_t(z)}\right) = 0.$$

Thus, $\mathcal{V}_t$ vanishes on $\partial\Omega$ for $t \in [0, T)$.

Let $\mathcal{W}_t = \mathcal{V}_t \circ (\psi^\xi_t)^{-1}$ and $\widetilde{\mathcal{W}}_t = \mathcal{W}_t \circ e^{i}$. Then $\mathcal{W}_t$ vanishes on $\partial\Omega^\xi_t \setminus \mathbb{T}$ for $t \in (-\infty, T)$, and $\widetilde{\mathcal{W}}_t = \widetilde{\mathcal{V}}_t \circ (\widetilde{\psi}^\xi_t)^{-1}$. Thus, for $t \in (-\infty, T)$ and $w \in \widetilde{\Omega}^\xi_t$,

$$\widetilde{\mathcal{W}}_t(w) = \partial_1 \widetilde{P}^\xi(t, \xi(t), w) + 2\partial_2 \widetilde{P}^\xi(t, \xi(t), w) X^\xi_t + \partial_2^2 \widetilde{P}^\xi(t, \xi(t), w) + 2\,\mathrm{Re}(\partial_{3,z} \widetilde{P}^\xi(t, \xi(t), w) \cot_2(w - \xi(t))). \tag{3.24}$$

Since $\widetilde{P}^\xi(t,\xi(t),\cdot)$ vanishes on $\mathbb{R}\setminus\{\xi(t)+2n\pi : n\in\mathbb{Z}\}$, and $\cot_2(w-\xi(t))$ is real on $\mathbb{R}\setminus\{\xi(t)+2n\pi : n\in\mathbb{Z}\}$, so $\widetilde{\mathcal{W}}_t$ vanishes on $\mathbb{R}\setminus\{\xi(t)+2n\pi : n\in\mathbb{Z}\}$, which implies that $\mathcal{W}_t$ vanishes on $\mathbb{T}\setminus\{e^{i\xi(t)}\}$. So $\mathcal{W}_t$ vanishes on $\partial\Omega_t^\xi\setminus\{e^{i\xi(t)}\}$.

Since $\widetilde{P}^\xi(t,x,\cdot)$ has period 2π, and has simple poles at $x+2n\pi$, $n\in\mathbb{Z}$, so there are $c(t,x)\in\mathbb{R}$ and some analytic function $F(t,x,\cdot)$ defined in some neighborhood of $\mathbb{R}$ such that in that neighborhood, $P^\xi(t,x,w)=\operatorname{Im}(F(t,x,w)+c(t,x)\cot_2(w-x))$. Then we have

$$\partial_1 P^\xi(t,\xi(t),w)=\operatorname{Im}(\partial_1 F(t,\xi(t),w)+\partial_1 c(t,\xi(t))\cot_2(w-\xi(t))),$$

$$\partial_2\widetilde{P}^\xi(t,\xi(t),w) =\operatorname{Im}\left(\partial_2 F(t,\xi(t),w)+\partial_2 c(t,\xi(t))\cot_2(w-\xi(t))+\frac{c(t,\xi(t))}{2\sin_2(w-\xi(t))^2}\right),$$

$$\partial_2^2\widetilde{P}^\xi(t,\xi(t),w) = \operatorname{Im}\Bigg(\partial_2^2 F(t,\xi(t),w)+\partial_2^2 c(t,\xi(t))\cot_2(w-\xi(t)) +\frac{2\partial_2 c(t,\xi(t))}{2\sin_2(w-\xi(t))^2}+\frac{c(t,\xi(t))\cos_2(w-\xi(t))}{2\sin_2(w-\xi(t))^3}\Bigg),$$

$$2\operatorname{Re}(\partial_{3,z}P^\xi(t,\xi(t),w)\cot_2(w-\xi(t))) =\operatorname{Im}\left(2F'(t,\xi(t),w)\cot_2(w-\xi(t))-\frac{c(t,\xi(t))\cos_2(w-\xi(t))}{2\sin_2(w-\xi(t))^3}\right).$$

From (3.24) and the above formulas, $\widetilde{\mathcal{W}}_t(w)$ equals the imaginary part of

$$\begin{aligned}&\partial_1 F(t,\xi(t),w)+\partial_1 c(t,\xi(t))\cot_2(w-\xi(t))+2\Bigg(\partial_2 F(t,\xi(t),w)\\&\quad+\partial_2 c(t,\xi(t))\cot_2(w-\xi(t))+\frac{c(t,\xi(t))}{2\sin_2(w-\xi(t))^2}\Bigg)X_t^\xi+\partial_2^2 F(t,\xi(t),w)\\&\quad+\partial_2^2 c(t,\xi(t))\cot_2(w-\xi(t))+\frac{\partial_2 c(t,\xi(t))}{\sin_2(w-\xi(t))^2}+\frac{c(t,\xi(t))\cos_2(w-\xi(t))}{2\sin_2(w-\xi(t))^3}\\&\quad+2F'(t,\xi(t),w)\cot_2(w-\xi(t))-\frac{c(t,\xi(t))\cos_2(w-\xi(t))}{2\sin_2(w-\xi(t))^3}\\&=G_t(w)+A_1(t)\cot_2(w-\xi(t))+\frac{A_2(t)}{\sin_2(w-\xi(t))^2}\end{aligned}$$

for some function G_t, which is analytic near $\mathbb{R}$, and real valued functions $A_1(t)$ and $A_2(t)$, where $A_2(t)=c(t,\xi(t))X_t^\xi+\partial_2 c(t,\xi(t))$.

Since $J_t^\xi = G(\Omega_t^\xi,\varphi_t^\xi(p);\cdot)$, so for $x\in\mathbb{R}$, $\partial_{\mathbf{n}}J_t^\xi(e^{ix})$ equals the value at $\varphi_t^\xi(p)$ of the (usual) Poisson kernel in Ω_t^ξ with the pole at e^{ix}. Comparing the residues of $\partial_{\mathbf{n}}J_t^\xi(e^{ix})$ and $P^\xi(t,x,\varphi_t^\xi(p))$ at e^{ix}, we conclude that

$$\partial_{\mathbf{n}}J_t^\xi(e^{ix})/(-1/\pi)=P^\xi(t,x,\varphi_t^\xi(p))/(2c(t,x))=1/(2c(t,x)).$$

It is clear that $\partial_{\mathbf{n}}J_t^\xi(e^{ix})=\partial_y\widetilde{J}_t^\xi(x)$. Thus, $c(t,x)\partial_y\widetilde{J}_t^\xi(x)=-1/(2\pi)$ for any $x\in\mathbb{R}$. Differentiating this equality w.r.t. x, we get

$$0=c(t,\xi(t))\partial_x\partial_y\widetilde{J}_t^\xi(\xi(t))+\partial_2 c(t,\xi(t))\partial_y\widetilde{J}_t^\xi(\xi(t))=A_2(t)\partial_y\widetilde{J}_t^\xi(\xi(t)).$$

Thus, $A_2(t)$ vanishes. So $\widetilde{\mathcal{W}}_t(w)$ equals the imaginary part of some analytic function plus $A_1(t)\cot_2(w-\xi(t))$ near $\mathbb{R}$. Hence, $\mathcal{W}_t(w)$ equals the imaginary part of some analytic function plus $-\mathrm{i}A_1(t)\frac{e^{\mathrm{i}\xi(t)}+w}{e^{\mathrm{i}\xi(t)}-w}$ near $\mathbb{T}$. Since $\mathcal{W}_t$ is harmonic in Ω_t^ξ, and vanishes at every prime end of Ω_t^ξ other than $e^{\mathrm{i}\xi(t)}$, so $\mathcal{W}_t = C(t)P^\xi(t,\xi(t),\cdot)$ for some $C(t)\in\mathbb{R}$. Since $P^\xi(t,x,\psi_t^\xi(p)) = 1$ for any $t\in(-\infty,T)$ and $x\in\mathbb{R}$, so from (3.23), we have $\mathcal{V}_t(p) = 0$. Thus, $\mathcal{W}_t(\psi_t^\xi(p)) = 0$. So for $t\in(-\infty,T)$, we have $C(t) = 0$, which implies that $\mathcal{W}_t$ vanishes on Ω_t^ξ, and so $\widetilde{\mathcal{V}}_t = \mathcal{W}_t\circ\psi_t^\xi\circ e^{\mathrm{i}}$ vanishes on $\widetilde{\Omega}\setminus\widetilde{L}_t^\xi$. □

Theorem 3.4. *Let $\beta(t)$, $-\infty\le t<T$, be an* LERW$(D;0\to z_e)$ *curve. For each $t\in(-\infty,T)$, let P_t be the generalized Poisson kernel in $D\setminus\beta([-\infty,t])$ with the pole at $\beta(t)$, normalized by $P_t(z_e) = 1$. Then for any $z\in D\setminus\{0\}$, $(P_t(z))$ is a continuous local martingale.*

Proof. We may assume that the driving function $\xi(t)$, $-\infty<t<T$, is the maximal solution to (3.4) with $f(t) = B_{\mathbb{R}}^{(2)}(t)$ and $\lambda = 2$. Then $\bigcup_{t<T}L_t^\xi\subset\Omega\setminus\{p\}$. Let P^ξ be defined as at the beginning of this subsection. Then $P_t\circ R_{\mathbb{T}}\circ e^{\mathrm{i}}(z) = \widetilde{P}^\xi(t,\xi(t),\widetilde{\psi}_t^\xi(z))$. Let $(\mathcal{F}_t)$ be the filtration generated by $(e^{\mathrm{i}}(B_{\mathbb{R}}^{(2)}(t)))$. Then $(e^{\mathrm{i}}(\xi(t)))$, (ψ_t^ξ), (Ω_t^ξ) and (X_t^ξ) are all $(\mathcal{F}_t)$-adapted. Let $R = \mathrm{dist}(0;\partial D\cup\{z_e\})$. Fix a constant $a\in(-\infty,\ln(R/4))$. Then a is always less than T. Let $T_a = T-a$ and $\xi_a(t) = \xi(a+t)-\xi(a)$ for $0\le t<T_a$. Let $B_a(t) = (B_{\mathbb{R}}^{(2)}(a+t)-B_{\mathbb{R}}^{(2)}(a))/\sqrt{2}$ for $t\ge 0$. Then $B_a(t)$ is an $(\mathcal{F}_{a+t})_{t\ge0}$-Brownian motion, and $\xi_a(t)$ satisfies the $(\mathcal{F}_{a+t})_{t\ge0}$-adapted SDE:

$$\mathrm{d}\xi_a(t) = \sqrt{2}\mathrm{d}B_a(t) + 2X_{a+t}^\xi\mathrm{d}t,\quad 0\le t<T_a. \tag{3.25}$$

For each $t\in[0,T_a)$ and $x\in\mathbb{R}$, let $Q(t,x,\cdot)$ be the generalized Poisson kernel in $M_{e^{\mathrm{i}\xi(a)}}^{-1}(\Omega_{a+t}^\xi)$ with the pole at $e^{\mathrm{i}x}/e^{\mathrm{i}\xi(a)}$, normalized by $Q(t,x,\psi_{a+t}^\xi(p)/e^{\mathrm{i}\xi(a)}) = 1$, and let $\widetilde{Q}(t,x,\cdot) = Q(t,x,\cdot)\circ e^{\mathrm{i}}$. It is clear that $\widetilde{Q}(t,x,z) = \widetilde{P}^\xi(a+t,x+\xi(a),z+\xi(a))$ for $0\le t<T_a$, $x\in\mathbb{R}$ and $z\in A_{e^{\mathrm{i}\xi(a)}}^{-1}(\widetilde{\Omega}_{a+t}^\xi)$. Since $e^{\mathrm{i}\xi(a)}$ is $\mathcal{F}_a$-measurable, and Ω_{a+t}^ξ is $\mathcal{F}_{a+t}$-measurable, so $(Q(t,\cdot,\cdot))$ is $(\mathcal{F}_{a+t})_{t\ge0}$-adapted, and so is $(\widetilde{Q}(t,\cdot,\cdot))$.

For $0\le t<T_a$ and $z\in\widetilde{\Omega}\setminus\widetilde{L}_{a+t}^\xi$, let $\widetilde{g}_t(z) = \widetilde{\psi}_{a+t}^\xi(z)-\xi(a)$. Then $(\widetilde{g}_t)$ is $(\mathcal{F}_{a+t})_{t\ge0}$-adapted, and satisfies $\partial_t\widetilde{g}_t(z) = \cot_2(\widetilde{g}_t(z)-\xi_a(t))$. From Lemma 3.1, we have that

$$\partial_1\widetilde{Q}(t,\xi_a(t),\widetilde{g}_t(z)) + 2\partial_2\widetilde{Q}(t,\xi_a(t),\widetilde{g}_t(z))X_{a+t}^\xi + \partial_2^2\widetilde{Q}(t,\xi_a(t),\widetilde{g}_t(z))$$
$$+ 2\,\mathrm{Re}(\partial_{3,z}\widetilde{Q}(t,\xi_a(t),\widetilde{g}_t(z))\cot_2(\widetilde{g}_t(z)-\xi_a(t))) = 0.$$

Since $P_{a+t}\circ R_{\mathbb{T}}\circ e^{\mathrm{i}}(z) = \widetilde{Q}(t,\xi_a(t),\widetilde{g}_t(z))$, so from Itô's formula, the above formula and that $\partial_t\widetilde{g}_t(z) = \cot_2(\widetilde{g}_t(z)-\xi_a(t))$, we conclude that for any $z\in\widetilde{\Omega}$, $(P_{a+t}\circ R_{\mathbb{T}}\circ e^{\mathrm{i}}(z), 0\le t<T_a)$ is a continuous local martingale. Since $R_{\mathbb{T}}\circ e^{\mathrm{i}}$ maps $\widetilde{\Omega}$ onto $D\setminus\{0\}$, so for any $z\in D\setminus\{0\}$, $(P_t(z), a\le t<T)$ is a continuous local martingale. Since this holds for any $a\in(-\infty,\ln(R/4))$, so the proof is completed. □

Remark. The similar local martingales first appear in [5], which was used to prove the convergence of LERW to radial SLE_2. For the process in the case $(\kappa,\lambda)\ne(2,2)$, so far we do not know any local martingale generated by harmonic functions.

3.5. Other kinds of targets

Suppose D is a finitely connected domain that contains 0, and I_e is a side arc of D. Then $R_{\mathbb{T}}(I_e)$ is a side arc of $\Omega = R_{\mathbb{T}}(D)$. Now we change the definition of J_t^ξ in (3.1) by replacing

$G(\Omega \setminus L_t^\xi, p; \cdot)$ by $H(\Omega \setminus L_t^\xi, R_{\mathbb{T}}(I_e); \cdot)$, which is the harmonic measure of $R_{\mathbb{T}}(I_e)$ in $\Omega \setminus L_t^\xi$, and still let $\widetilde{J}_t^\xi = J_t^\xi \circ \mathrm{e}^{\mathrm{i}}$ and $X_t^\xi = (\partial_x \partial_y / \partial_y) \widetilde{J}_t^\xi (\xi(t))$. Let everything else in Section 3.2 be unchanged. Then Theorem 3.2 still holds. For the new meaning of X_t^ξ, let $\xi \in C((-\infty, T))$ be the maximal solution to (3.4) with $f = B_{\mathbb{R}}^{(2)}$ and $\lambda = 2$. Let K_t^ξ, $-\infty < t < T$, be the whole-plane Loewner hulls driven by ξ. Then we call the interior Loewner chain K_t^ξ, $0 \le t < T$, a continuous interior LERW in D from 0 to I_e. Let it be denoted by LERW$(D; 0 \to I_e)$. Such a Loewner chain is almost surely generated by a random simple curve started from 0, which is called an LERW$(D; 0 \to I_e)$ curve. Through conformal maps, we can then define continuous LERW from any interior point to a side arc. Then we can prove that this kind of continuous LERW is conformally invariant up to a time-change.

Let $\beta(t)$, $0 \le t < T$, denote an LERW$(D; 0 \to I_e)$ curve. For each $t \in [0, T)$, let P_t be the generalized Poisson kernel in $D \setminus \beta([0, t])$ with the pole at $\beta(t)$, normalized by $\int_{I_e} \partial_{\mathbf{n}} P_t(z) \mathrm{d}s(z) = 1$, where $\mathbf{n}$ is the inward unit normal vector, and ds is the measure of length. Then for any fixed $z \in D$, $(P_t(z))$ is a continuous local martingale.

Remark. After a time-change, a continuous LERW$(\mathbb{D}; 0 \to \mathbb{T})$ has the same distribution as a standard disc SLE$_2$ defined in [14].

Now let w_e be a prime ends of D. Then $R_{\mathbb{T}}(w_e)$ is a prime end of Ω. Choose h that maps a neighborhood U of $R_{\mathbb{T}}(w_e)$ in $\widehat{\Omega}$ conformally onto a neighborhood V of 0 in $\overline{\mathbb{H}}$ such that $h(R_{\mathbb{T}}(w_e)) = 0$ and $h(U \cap \widehat{\partial} \Omega) \subset \mathbb{R}$. Here $\widehat{\Omega}$ and $\widehat{\partial} \Omega$ are the conformal closure and conformal boundary, respectively, of Ω as defined in [12]. Change the definition of J_t^ξ by replacing $G(\Omega \setminus L_t^\xi, p; \cdot)$ by $P(\Omega \setminus L_t^\xi, R_{\mathbb{T}}(w_e), h; \cdot)$ in (3.1), where we use $P(\Omega \setminus L_t^\xi, R_{\mathbb{T}}(w_e), h; \cdot)$ to denote the generalized Poisson kernel P in $\Omega \setminus L_t^\xi$ with the pole at $R_{\mathbb{T}}(w_e)$, normalized by $P \circ h^{-1}(z) = -\mathrm{Im}(1/z) + O(1)$, as $z \to 0$ in $\mathbb{H}$. We still let $\widetilde{J}_t^\xi = J_t^\xi \circ \mathrm{e}^{\mathrm{i}}$ and $X_t^\xi = (\partial_x \partial_y / \partial_y) \widetilde{J}_t^\xi (\xi(t))$. For the new meaning of X_t^ξ, let $\xi \in C((-\infty, T))$ be the maximal solution to (3.4) with $f = B_{\mathbb{R}}^{(2)}$ and $\lambda = 2$. Let K_t^ξ, $-\infty < t < T$, be the whole-plane Loewner hulls driven by ξ. Then we call the interior Loewner chain K_t^ξ, $0 \le t < T$, a continuous interior LERW in D from 0 to w_e. Let it be denoted by LERW$(D; 0 \to w_e)$. Such a Loewner chain is almost surely generated by a random simple curve started from 0, which is called an LERW$(D; 0 \to w_e)$ curve. Through conformal maps, we can then define continuous LERW from any interior point to a prime end. Then we can prove that this kind of continuous LERW is conformally invariant up to a time-change.

Let $\beta(t)$, $0 \le t < T$, denote an LERW$(D; 0 \to w_e)$ curve. Fix h that maps a neighborhood U of w_e in $\widehat{D}$ conformally into $\mathbb{H}$ such that $h(w_e) = 0$ and $h(U \cap \widehat{\partial} D) \subset \mathbb{R}$. For each $t \in [0, T)$, let P_t be the generalized Poisson kernel in $D \setminus \beta([0, t])$ with the pole at $\beta(t)$, normalized by $\partial_y (P_t \circ h^{-1})(0) = 1$. Then for any fixed $z \in D$, $(P_t(z))$ is a continuous local martingale.

4. Existence and uniqueness

4.1. The radial equation

In this subsection, we will prove Theorem 3.1. We will use the notation in Section 3.1, and use $\cot_2(z)$, $\sin_2(z)$, $\coth_2(z)$, $\sinh_2(z)$ and $\cosh_2(z)$ to denote the functions $\cot(z/2)$, $\sin(z/2)$, $\coth(z/2)$, $\sinh(z/2)$ and $\cosh(z/2)$, respectively.

Lemma 4.1. *Let* $\xi \in C([0,T))$. *Suppose* $a \in [0,T)$ *and* $H > 0$ *satisfy* $\cosh_2(H) > e^{a/2}$. *Then for any* $z \in \mathbb{C}$ *with* $\operatorname{Im} z \ge H$, $\widetilde{\psi}^\xi_a(z)$ *is meaningful, and* $\cosh_2(\operatorname{Im}\widetilde{\psi}^\xi_a(z)) \ge \cosh_2(H)/e^{a/2}$.

Proof. Let $h > 0$ be the solution of $\cosh_2(h) = \cosh_2(H)/e^{a/2}$. Suppose $z \in \mathbb{C}$ and $\operatorname{Im} z \ge H$. Let $b \in (0,a]$ be the maximal number such that $\widetilde{\psi}^\xi_t(z)$ exists for $t \in [0,b)$. Let $h(t) = \operatorname{Im}\widetilde{\psi}^\xi_t(z)$ for $t \in [0,b)$. From (2.4) we see that there is some real valued function $\theta(t)$ such that

$$h'(t) = \operatorname{Im}\cot_2(\widetilde{\psi}^\xi_t(z) - \xi(t)) = \operatorname{Im}\cot_2(\theta(t) + i h(t)) \ge -\coth_2(h(t)),$$

which implies that $\tanh_2(h(t))h'(t)/2 \ge -1/2$. So for $t \in [0,b)$,

$$\ln\cosh_2(h(t)) - \ln\cosh_2(\operatorname{Im} z) = \ln\cosh_2(h(t)) - \ln\cosh_2(h(0)) \ge -t/2.$$

Thus, $\cosh_2(h(t)) \ge \cosh_2(\operatorname{Im} z)/e^{t/2} \ge \cosh_2(H)/e^{a/2} = \cosh_2(h)$, and so $h(t) \ge h$ for $t \in [0,b)$. Since $h > 0$, so $\widetilde{\psi}^\xi_t(z)$ does not blow up at b. Thus, $b = a$, and $\operatorname{Im}\widetilde{\psi}^\xi_a(z) = \lim_{t\to a^-} h(t) \ge h$. So we have $\cosh_2(\operatorname{Im}\widetilde{\psi}^\xi_a(z)) \ge \cosh_2(H)/e^{a/2}$. □

Lemma 4.2. *Let* $a, h > 0$ *be such that* $\cosh_2(h) > e^{a/2}$. *There is* $C > 0$ *such that, for any* $\eta, \xi \in C([0,a])$, $b \in [0,a]$, *and* $z \in \{\operatorname{Im} z \ge h\}$, $\widetilde{\psi}^\eta_b \circ (\widetilde{\psi}^\xi_b)^{-1}(z)$ *is meaningful, and*

$$|z - \widetilde{\psi}^\eta_b \circ (\widetilde{\psi}^\xi_b)^{-1}(z)| \le C\|\eta - \xi\|_{0,b}. \tag{4.1}$$

Proof. Suppose $\eta, \xi \in C([0,a])$, $b \in [0,a]$, and $\operatorname{Im} z \ge h$. Since $\operatorname{Im}\widetilde{\psi}^\xi_t(w)$ decreases in t, so $\operatorname{Im}(\widetilde{\psi}^\xi_b)^{-1}(z) \ge \operatorname{Im} z \ge h$. From Lemma 4.1, we see that for $0 \le t \le b$, $\widetilde{\psi}^\xi_t \circ (\widetilde{\psi}^\xi_b)^{-1}(z)$ and $\widetilde{\psi}^\eta_t \circ (\widetilde{\psi}^\xi_b)^{-1}(z)$ are meaningful, and $\cosh_2^2(\operatorname{Im}\widetilde{\psi}^\eta_t \circ (\widetilde{\psi}^\xi_b)^{-1}(z)), \cosh_2^2(\operatorname{Im}\widetilde{\psi}^\xi_t \circ (\widetilde{\psi}^\xi_b)^{-1}(z)) \ge \cosh_2^2(h)/e^a$, which implies that

$$\sinh_2^2(\operatorname{Im}\widetilde{\psi}^\eta_t \circ (\widetilde{\psi}^\xi_b)^{-1}(z)), \sinh_2^2(\operatorname{Im}\widetilde{\psi}^\xi_t \circ (\widetilde{\psi}^\xi_b)^{-1}(z)) \ge \cosh_2^2(h)/e^a - 1. \tag{4.2}$$

Since $|\cot_2'(z)| = \frac{1}{2}|\sin_2^{-2}(z)| \le \frac{1}{2}\sinh_2^{-2}(\operatorname{Im} z)$, so if $\operatorname{Im} z_1, \operatorname{Im} z_2 \ge H > 0$, then

$$|\cot_2(z_1) - \cot_2(z_2)| \le \frac{1}{2}\sinh_2^{-2}(H)|z_1 - z_2|. \tag{4.3}$$

Let

$$g(t) = |\widetilde{\psi}^\xi_t \circ (\widetilde{\psi}^\xi_b)^{-1}(z) - \widetilde{\psi}^\eta_t \circ (\widetilde{\psi}^\xi_b)^{-1}(z)|, \quad 0 \le t \le b.$$

From (2.4), (4.2) and (4.3), we see that for any $0 \le t \le b$,

$$\begin{aligned} g(t) &\le \int_0^t \left|\cot_2(\widetilde{\psi}^\xi_s \circ (\widetilde{\psi}^\xi_b)^{-1}(z) - \xi(s)) - \cot_2(\widetilde{\psi}^\eta_s \circ (\widetilde{\psi}^\xi_b)^{-1}(z) - \eta(s))\right| dt \\ &\le \int_0^t C_1(g(s) + |\xi(s) - \eta(s)|)dt \le C_1\int_0^t (g(s) + \|\xi - \eta\|_{0,b})ds, \end{aligned} \tag{4.4}$$

where

$$C_1 = \frac{1/2}{\cosh_2^2(h)/e^a - 1} > 0.$$

Let $C = e^{aC_1} - 1$. Solving (4.4), we get

$$|z - \widetilde{\psi}^\eta_b \circ (\widetilde{\psi}^\xi_b)^{-1}(z)| = g(b) \le (e^{bC_1} - 1)\|\eta - \xi\|_{0,b} \le C\|\eta - \xi\|_{0,b}. \quad \square$$

Lemma 4.3. *There are* $a_0, C > 0$ *such that, for any* $t \in [0, a_0]$, *and* $\zeta, \eta \in C([0, t])$, *we have* $L^\eta{}_t, L_t^\zeta \subset \Omega \setminus \{p\}$, *and* $|X_t^\eta - X_t^\zeta| \le C\|\eta - \zeta\|_{0,t}$.

Proof. There is $H > 0$ such that $\mathbb{S}_H \subset \widetilde{\Omega} \setminus \widetilde{p}$. Choose $a_0 > 0$ such that $e^{a_0} < \cosh_2(H)$. Let $h > 0$ be such that $\cosh_2(h) = \cosh_2(H)/e^{a_0/2}$. Then $\cosh_2(h)^2/e^{a_0} > 1$. Fix $t \in [0, a_0]$. Suppose $\zeta, \eta \in C([0, t])$. From Lemma 4.1, for any $z \in \mathbb{C}$ with $\operatorname{Im} z \ge H$, $\widetilde{\psi}_t^\zeta(z)$ and $\widetilde{\psi}_t^\eta(z)$ are meaningful, and $\operatorname{Im}\widetilde{\psi}_t^\zeta(z), \operatorname{Im}\widetilde{\psi}_t^\eta(z) \ge h$. Thus, $\widetilde{L}_t^\zeta, \widetilde{L^\eta}_t \subset \mathbb{S}_H \subset \widetilde{\Omega} \setminus \widetilde{p}$, and $\mathbb{S}_h \subset \widetilde{\Omega}_t^\zeta \setminus \widetilde{p}_t^\zeta, \widetilde{\Omega}_t^\eta \setminus \widetilde{p}_t^\eta$. So we have $L_t^\zeta, L^\eta{}_t \subset \Omega \setminus \{p\}$.

Choose $h_0 > h_{0.5} > h_1 > h_2 \in (0, h)$ such that $\cosh_2(h_2) > e^{a_0/2}$. Let $C_0 > 0$ be the C given by Lemma 4.2 with $a = a_0$ and $h = h_2$. Let $C_* > 1$ be the number depending only on h and h_0 such that, if f is positive and harmonic in $\mathbb{S}_h$, and has period 2π, then for any $x_1, x_2 \in \mathbb{R}_{h_0}$, $f(x_1) \le C_* f(x_2)$. Let

$$\delta = \min\left\{\frac{h_{0.5} - h_1}{C_0}, \frac{(h_0 - h_{0.5})(h_1 - h_2)}{8h_0C_0C_*}\right\} > 0. \tag{4.5}$$

Suppose first that $\|\eta-\zeta\|_{0,t} < \delta$. Let $m = \inf\{\widetilde{J^\zeta}_t(z) : z \in \mathbb{R}_{h_0}\}$, $M = \sup\{\widetilde{J^\zeta}_t(z) : z \in \mathbb{R}_{h_0}\}$, and $D_\nabla = \sup\{|\nabla\widetilde{J^\zeta}_t(z)| : z \in \mathbb{S}_{h_{0.5}}\}$. Since $\mathbb{S}_h \subset \widetilde{\Omega}_t^\zeta \setminus \widetilde{p}_t^\zeta$, so $\widetilde{J}_t^\zeta$ is positive and harmonic in $\mathbb{S}_h$, and vanishes on $\mathbb{R}$. After a reflection about $\mathbb{R}$, $\widetilde{J}_t^\zeta$ is harmonic in $\{|\operatorname{Im} z| < h\}$, and $|\widetilde{J}_t^\zeta|$ is bounded by M on $\{|\operatorname{Im} z| \le h_0\}$. Moreover, $\widetilde{J}_t^\zeta$ has period 2π. Thus, $M \le C_* m$. From Harnack's inequality, we have

$$D_\nabla \le 2M/(h_0 - h_{0.5}), \tag{4.6}$$

and for any $x \in \mathbb{R}$,

$$\partial_y \widetilde{J}_t^\zeta(x) \ge m/h_0, \qquad |\partial_x\partial_y\widetilde{J^\zeta}_t(x)| \le 4M/h_0^2, \qquad |\partial_x^2\partial_y\widetilde{J^\zeta}_t(x)| \le 12M/h_0^3. \tag{4.7}$$

For $j = 1, 2$, let $\rho_j = (\widetilde{\psi}_t^\eta)^{-1}(\mathbb{R}_{h_j})$. Then ρ_1 and ρ_2 lie in $\widetilde{\Omega} \setminus \widetilde{p} \setminus \widetilde{L^\eta}_t$, and ρ_2 disconnects ρ_1 from $\widetilde{L^\eta}_t$. Since $\cosh_2(h_2) > e^{a_0/2}$ and $t \in [0, a_0]$, so from Lemma 4.2, for any $z \in \mathbb{C}$ with $\operatorname{Im} z \ge h_2$, $\widetilde{\psi}_t^\zeta \circ (\widetilde{\psi}_t^\eta)^{-1}(z)$ is meaningful, so ρ_1 and ρ_2 lie in $\mathbb{H} \setminus \widetilde{L}_t^\zeta$, and ρ_2 disconnects ρ_1 from $\widetilde{L}_t^\zeta$. Thus, ρ_1 and ρ_2 lie in $\widetilde{\Omega} \setminus \widetilde{p} \setminus (\widetilde{L}_t^\zeta \cup \widetilde{L^\eta}_t)$, and ρ_2 disconnects ρ_1 from $\widetilde{L}_t^\zeta \cup \widetilde{L^\eta}_t$. For $\xi \in C([0, t])$, let $G_t^\xi = G(\Omega \setminus L_t^\xi, p; \cdot)$ and $\widetilde{G}_t^\xi = G_t^\xi \circ e^{i}$. Then $\widetilde{J}_t^\xi = \widetilde{G}_t^\xi \circ (\widetilde{\psi}_t^\xi)^{-1}$. For $j = 1, 2$, define

$$N_j = \sup_{z\in\mathbb{R}_{h_j}} \{|\widetilde{J^\eta}_t(z) - \widetilde{J}_t^\zeta(z)|\} = \sup_{z\in\mathbb{R}_{h_j}} \{|\widetilde{G^\eta}_t \circ (\widetilde{\psi}_t^\eta)^{-1}(z) - \widetilde{G}_t^\zeta \circ (\widetilde{\psi^\zeta}_t)^{-1}(z)|\}; \tag{4.8}$$

$$N_j' = \sup_{w\in\rho_j} \{|\widetilde{G}_t^\eta(w) - \widetilde{G}_t^\zeta(w)|\} = \sup_{z\in\mathbb{R}_{h_j}} \{|\widetilde{G}_t^\eta \circ (\widetilde{\psi}_t^\eta)^{-1}(z) - \widetilde{G}_t^\zeta \circ (\widetilde{\psi}_t^\eta)^{-1}(z)|\}. \tag{4.9}$$

Note that $\widetilde{J}_t^\eta - \widetilde{J}_t^\zeta$ is harmonic in $\mathbb{S}_h$, and vanishes on $\mathbb{R}$, since both $\widetilde{J}_t^\eta$ and $\widetilde{J}_t^\zeta$ satisfy these properties. Since the probability that a plane Brownian motion started from a point on $\mathbb{R}_{h_2}$ visits $\mathbb{R}_{h_1}$ before $\mathbb{R}$ is h_2/h_1, so

$$N_2 \le (h_2/h_1)N_1. \tag{4.10}$$

Since every $z \in \widetilde{p}$ is a removable singularity of $\widetilde{G}_t^\eta - \widetilde{G}_t^\zeta$, so after an extension, $\widetilde{G}_t^\eta - \widetilde{G}_t^\zeta$ is harmonic in $\widetilde{\Omega} \setminus (\widetilde{L}_t^\zeta \cup \widetilde{L^\eta}_t)$. Since ρ_1 and ρ_2 lie in $\widetilde{\Omega} \setminus (\widetilde{L}_t^\zeta \cup \widetilde{L^\eta}_t)$, and ρ_2 disconnects ρ_1 from

$\widetilde{L}^\zeta_t \cup \widetilde{L^\eta}_t$, so from the maximum principle, we have

$$N_1' \le N_2'. \tag{4.11}$$

Fix $j \in \{1,2\}$ and $z_0 \in \mathbb{R}_{h_j}$. Since $\operatorname{Im} z_0 \ge h_2$, $\cosh_2(h_2) > e^{a_0/2}$, and $t \in [0, a_0]$, so from Lemma 4.2, the choice of C_0, (4.5), and that $\|\eta - \zeta\|_{0,t} < \delta$, we have

$$|z_0 - \widetilde{\psi}^\zeta_t \circ (\widetilde{\psi}^\eta_t)^{-1}(z_0)| \le C_0\|\eta - \zeta\|_{0,t} < C_0\delta \le h_{0.5} - h_1. \tag{4.12}$$

Thus, $\operatorname{Im} \widetilde{\psi}^\zeta_t \circ (\widetilde{\psi}^\eta_t)^{-1}(z_0) < \operatorname{Im} z_0 + h_{0.5} - h_1 \le h_{0.5}$. On the other hand, since $\widetilde{\psi}^\zeta_t \circ (\widetilde{\psi}^\eta_t)^{-1}(z_0) \in \mathbb{H}$, so $\operatorname{Im} \widetilde{\psi}^\zeta_t \circ (\widetilde{\psi}^\eta_t)^{-1}(z_0) > 0$. Thus, $[z_0, \widetilde{\psi}^\zeta_t \circ (\widetilde{\psi}^\eta_t)^{-1}(z_0)] \subset \mathbb{S}_{h_{0.5}}$. So from (4.6) and (4.12),

$$\begin{aligned} &|\widetilde{G}^\zeta_t \circ (\widetilde{\psi}^\zeta_t)^{-1}(z_0) - \widetilde{G}^\zeta_t \circ (\widetilde{\psi}^\eta_t)^{-1}(z_0)| = |\widetilde{J}^\zeta_t(z_0) - \widetilde{J^\zeta}_t \circ \widetilde{\psi}^\zeta_t \circ (\widetilde{\psi}^\eta_t)^{-1}(z_0)| \\ &\quad \le \sup_{z \in \mathbb{S}_{h_{0.5}}} \{|\nabla \widetilde{J}^\zeta_t(z)|\} \cdot |z_0 - \widetilde{\psi}^\zeta_t \circ (\widetilde{\psi}^\eta_t)^{-1}(z_0)| \\ &\quad \le D_\nabla C_0 \|\eta - \zeta\|_{0,t} \le \frac{2MC_0\|\eta - \zeta\|_{0,t}}{h_0 - h_{0.5}}. \end{aligned} \tag{4.13}$$

Let

$$\Delta = \frac{2MC_0\|\eta - \zeta\|_{0,t}}{h_0 - h_{0.5}}. \tag{4.14}$$

Then from (4.8), (4.9), (4.13) and (4.14), we have

$$|N_j' - N_j| \le \sup_{z \in \mathbb{R}_{h_j}} \{|\widetilde{G}^\zeta_t \circ (\widetilde{\psi}^\zeta_t)^{-1}(z) - \widetilde{G}^\zeta_t \circ (\widetilde{\psi}^\eta_t)^{-1}(z)|\} \le \Delta, \quad j = 1, 2. \tag{4.15}$$

From (4.10), (4.11) and (4.15), we have

$$N_1 \le N_1' + \Delta \le N_2' + \Delta \le N_2 + 2\Delta \le (h_2/h_1)N_1 + 2\Delta.$$

Thus,

$$N_1 \le 2h_1\Delta/(h_1 - h_2). \tag{4.16}$$

From Harnack's inequality, for any $x \in \mathbb{R}$,

$$|\partial_y \widetilde{J}^\eta_t(x) - \partial_y \widetilde{J}^\zeta_t(x)| \le N_1/h_1 \le 2\Delta/(h_1 - h_2); \tag{4.17}$$

$$|\partial_x\partial_y \widetilde{J}^\eta_t(x) - \partial_x\partial_y \widetilde{J}^\zeta_t(x)| \le 4N_1/h_1^2 \le 8\Delta/(h_1(h_1 - h_2)). \tag{4.18}$$

From (4.5), (4.7), (4.14), (4.17), $M \le C_* m$, and that $\|\eta - \zeta\|_{0,t} < \delta$, for any $x \in \mathbb{R}$,

$$\partial_y \widetilde{J}^\eta_t(x) \ge \frac{m}{h_0} - \frac{2\Delta}{h_1 - h_2} > \frac{m}{h_0} - \frac{4MC_0\delta}{(h_0 - h_{0.5})(h_1 - h_2)} \ge \frac{m}{2h_0}. \tag{4.19}$$

From (4.18) and (4.19), we have

$$|\partial_x\partial_y \widetilde{J}^\eta_t(\eta(t))/\partial_y \widetilde{J}^\eta_t(\eta(t)) - \partial_x\partial_y \widetilde{J}^\zeta_t(\eta(t))/\partial_y \widetilde{J}^\eta_t(\eta(t))| \le \frac{16h_0\Delta}{mh_1(h_1 - h_2)}. \tag{4.20}$$

From (4.7), (4.17) and (4.19), we have

$$|\partial_x\partial_y \widetilde{J^\zeta}_t(\eta(t))/\partial_y \widetilde{J}^\eta_t(\eta(t)) - \partial_x\partial_y \widetilde{J^\zeta}_t(\eta(t))/\partial_y \widetilde{J}^\zeta_t(\eta(t))| \le \frac{16M\Delta}{m^2(h_1 - h_2)}. \tag{4.21}$$

From (4.7) and that $M \le C_* m$, for any $x \in \mathbb{R}$,

$$|\partial_x(\partial_x\partial_y/\partial_y)\widetilde{J}_t^\zeta(x)| \le |(\partial_x^2\partial_y/\partial_y)\widetilde{J}_t^\zeta(x)| + |(\partial_x\partial_y/\partial_y)\widetilde{J}_t^\zeta(x)|^2$$
$$\le (12M/h_0^3)/(m/h_0) + (4M/h_0^2)^2/(m/h_0)^2 \le 28C_*^2/h_0^2.$$

Thus,

$$|(\partial_x\partial_y/\partial_y)\widetilde{J^\zeta}_t(\eta(t)) - (\partial_x\partial_y/\partial_y)\widetilde{J}_t^\zeta(\zeta(t))| \le 28C_*^2/h_0^2\|\eta-\zeta\|_{0,t}. \tag{4.22}$$

From (4.14) and (4.20)–(4.22), and that $M \le C_* m$, we get $|X_t^\eta - X_t^\zeta| \le C\|\eta-\zeta\|_{0,t}$, where

$$C := \frac{32C_*C_0h_0/h_1}{(h_0-h_{0.5})(h_1-h_2)} + \frac{32C_*^2C_0}{(h_0-h_{0.5})(h_1-h_2)} + \frac{28C_*^2}{h_0^2} > 0.$$

In the above argument, we assumed that $\|\eta-\zeta\|_{0,t} < \delta$. In the general case, we can find $n \in \mathbb{N}$ such that $\|\eta-\zeta\|_{0,t}/n < \delta$. Let $\zeta_k = \zeta + k(\eta-\zeta)/n$, $0 \le k \le n$. Then

$$|X_t^\eta - X_t^\zeta| \le \sum_{k=1}^n |X_t^{\zeta_k} - X_t^{\zeta_{k-1}}| \le \sum_{k=1}^n C\|\zeta_k-\zeta_{k-1}\|_{0,t} = C\|\eta-\zeta\|_{0,t}. \quad \square$$

Proof of Theorem 3.1. (i) Let $a_0, C > 0$ be given by Lemma 4.3. Use the method of Picard iteration to define a sequence of functions $(\xi_n(t))$ in $C([0,a_0])$ such that $\xi_0(t) = f(t)$, $0 \le t \le a_0$, and for $n \in \mathbb{N}$,

$$\xi_n(t) = f(t) + \lambda\int_0^t X_s^{\xi_{n-1}}\mathrm{d}s, \quad 0 \le t \le a_0. \tag{4.23}$$

Then for $a \in [0,a_0]$, if $0 \le t \le a$, then

$$|\xi_{n+1}(t) - \xi_n(t)| \le |\lambda|\int_0^t |X_s^{\xi_n} - X_s^{\xi_{n-1}}|\mathrm{d}s \le C|\lambda|t\|\xi_n-\xi_{n-1}\|_{0,a}.$$

Thus, $\|\xi_{n+1}-\xi_n\|_{0,a} \le C|\lambda|a\|\xi_n-\xi_{n-1}\|_{0,a}$. Choose $a \in (0,a_0)$ such that $C|\lambda|a < 1/2$. Then (ξ_n) is a Cauchy sequence w.r.t. $\|\cdot\|_{0,a}$. Let $\xi \in C([0,a])$ be the limit of this sequence. Let $n \to \infty$ in (4.23), then ξ solves (3.2) for $0 \le t \le a$.
(ii) Suppose for $j = 1, 2$, ξ_j solves (3.2) for $0 \le t < T_j$ for some $T_j > 0$. Choose $S \in (0, T_1 \wedge T_2 \wedge a_0)$ such that $C|\lambda|S \le 1/2$. Then

$$\|\xi_1-\xi_2\|_{0,S} \le C|\lambda|S\|\xi_1-\xi_2\|_{0,S} \le \|\xi_1-\xi_2\|_{0,S}/2,$$

which implies $\|\xi_1-\xi_2\|_{0,S} = 0$. Thus, $\xi_1(t) = \xi_2(t)$ for $0 \le t \le S$. $\square$

4.2. The whole-plane equation

In this section, we will prove Proposition 3.1 and Theorem 3.2. We use the notation in Section 3.2. Let $R = \mathrm{dist}(0, \partial D \cup \{z_e\}) > 0$ throughout this subsection.

Lemma 4.4. *Suppose $t < \ln(R) - \ln(1 + C_{\mathcal{H}})$. Let $h = \ln(R/\mathrm{e}^t - C_{\mathcal{H}}) > 0$. Then for any $\xi \in C((-\infty,t])$, we have $K_t^\xi \subset D \setminus \{z_e\}$ and $\mathbb{S}_h \subset \widetilde{\Omega}_t^\xi \setminus \widetilde{p}_t^\xi$.*

Proof. Suppose $\xi \in C((-\infty,t])$. From (2.10), if $1 < |z| < \mathrm{e}^h$, then $|(\varphi_t^\xi)^{-1}(z) - \mathrm{e}^t z| \le C_{\mathcal{H}}\mathrm{e}^t$, and so $|(\varphi_t^\xi)^{-1}(z)| < \mathrm{e}^t(|z| + C_{\mathcal{H}}) < R$. Thus, $(\varphi_t^\xi)^{-1}(\{1 < |z| < \mathrm{e}^h\}) \subset \{|z| < R\} \subset D \setminus \{z_e\}$, which implies that $\{1 < |z| < \mathrm{e}^h\} \subset \varphi_t^\xi(D \setminus K_t^\xi \setminus \{z_e\})$, and so $\mathbb{S}_h \subset \widetilde{\Omega}_t^\xi \setminus \widetilde{p}_t^\xi$. Since K_t^ξ is surrounded by $(\varphi_t^\xi)^{-1}(\{1 < |z| < \mathrm{e}^h\})$, so $K_t^\xi \subset \{|z| < R\} \subset D \setminus \{z_e\}$. $\square$

Lemma 4.5. *There are non-increasing functions S_1 and S_2 defined on $(0,\infty)$ with $S_j(h) = O(h\mathrm{e}^{-h})$ as $h \to \infty$, $j = 1,2$, such that for any $h > 0$, if $J(z)$ is positive and harmonic in $\mathbb{S}_h$, vanishes on $\mathbb{R}$, and has period 2π, then for $j = 1,2$,*

$$|(\partial_x^j \partial_y/\partial_y)J(x)| \le S_j(h), \quad \text{for any } x \in \mathbb{R}. \tag{4.24}$$

Proof. There is a positive measure μ on $[0,2\pi)$ such that $J(z) = \int \mathbf{S}_h(z-x)\mathrm{d}\mu(x)$, for any $z \in \mathbb{S}_h$, where

$$\mathbf{S}_h(z) := \operatorname{Im}\left(\frac{z}{h} + \frac{1}{\mathrm{i}}\,\mathrm{P.V.}\sum_{n\in\mathbb{Z},2\nmid n}\frac{\mathrm{e}^{nh}+\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^{nh}-\mathrm{e}^{\mathrm{i}z}}\right). \tag{4.25}$$

Such $\mathbf{S}_h$ is positive and harmonic on $\mathbb{S}_h$, has period 2π, and vanishes on $\mathbb{R}\cup\mathbb{R}_h \setminus \{2m\pi + h\mathrm{i} : m \in \mathbb{Z}\}$. And $2m\pi + h\mathrm{i}$ is a simple pole of $\mathbf{S}_h$ for each $m \in \mathbb{Z}$. In fact, $\mathbb{S}_h \circ \mathrm{e}^{\mathrm{i}}$ is a Poisson kernel in $\mathbb{A}_h$ with the pole at e^{-h}. Let

$$S_j(h) = \sup\{|(\partial_x^j\partial_y/\partial_y)\mathbf{S}_h(x)| : x \in \mathbb{R}\}, \quad j = 1,2.$$

Then for $j = 1,2$, $S_j(h) \in (0,\infty)$, and $|\partial_x^j\partial_y\mathbf{S}_h(x)| \le S_j(h)\partial_y\mathbf{S}_h(x)$ for any $x \in \mathbb{R}$. Since $J(z) = \int \mathbf{S}_h(z-x)\mathrm{d}\mu(x)$, so $|\partial_x^j\partial_y J(x)| \le S_j(h)\partial_y J(x)$ for any $x \in \mathbb{R}$, which implies (4.24). If $h' > h$, applying (4.24) to $J = \mathbf{S}_{h'}$, we find that $S_j(h') \le S_j(h)$. So $S_j(h)$ is non-increasing.

Now for $x \in \mathbb{R}$, $j = 0,1,2$,

$$\partial_x^j\partial_y\mathbf{S}_h(x) = \frac{\mathrm{d}^{j+1}}{\mathrm{d}z^{j+1}}\left(\frac{z}{h} + \frac{1}{\mathrm{i}}\,\mathrm{P.V.}\sum_{n\in\mathbb{Z},2\nmid n}\frac{\mathrm{e}^{nh}+\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^{nh}-\mathrm{e}^{\mathrm{i}z}}\right)\Bigg|_{z=x}.$$

So

$$\begin{aligned}\partial_y\mathbf{S}_h(x) &= \frac{1}{h} + \sum_{n\in\mathbb{Z},2\nmid n}\frac{2\mathrm{e}^{nh}\mathrm{e}^{\mathrm{i}x}}{(\mathrm{e}^{nh}-\mathrm{e}^{\mathrm{i}x})^2} = \frac{1}{h} + \sum_{n\in\mathbb{N},2\nmid n}\left(\frac{2\mathrm{e}^{nh}\mathrm{e}^{\mathrm{i}x}}{(\mathrm{e}^{nh}-\mathrm{e}^{\mathrm{i}x})^2} + \frac{2\mathrm{e}^{-nh}\mathrm{e}^{\mathrm{i}x}}{(\mathrm{e}^{-nh}-\mathrm{e}^{\mathrm{i}x})^2}\right)\\ &= \frac{1}{h} + \operatorname{Re}\sum_{n\in\mathbb{N},2\nmid n}\frac{4\mathrm{e}^{nh}\mathrm{e}^{\mathrm{i}x}}{(\mathrm{e}^{nh}-\mathrm{e}^{\mathrm{i}x})^2} \ge \frac{1}{h} - \sum_{n\in\mathbb{N},2\nmid n}\frac{4\mathrm{e}^{nh}}{(\mathrm{e}^{nh}-1)^2}\\ &\ge \frac{1}{h} - \sum_{n=1}^{\infty}\frac{4\mathrm{e}^{nh}}{(\mathrm{e}^{nh}-1)^2};\end{aligned} \tag{4.26}$$

$$|\partial_x\partial_y\mathbf{S}_h(x)| = \left|\sum_{n\in\mathbb{Z},2\nmid n}\frac{4\mathrm{i}\mathrm{e}^{nh}\mathrm{e}^{\mathrm{i}x}(\mathrm{e}^{nh}+\mathrm{e}^{\mathrm{i}x})}{(\mathrm{e}^{nh}-\mathrm{e}^{\mathrm{i}x})^3}\right| \le \sum_{n\in\mathbb{N},2\nmid n}\frac{8\mathrm{e}^{2nh}}{(\mathrm{e}^{nh}-1)^3} \le \sum_{n=1}^{\infty}\frac{8\mathrm{e}^{2nh}}{(\mathrm{e}^{nh}-1)^3}; \tag{4.27}$$

$$|\partial_x^2\partial_y\mathbf{S}_h(x)| = \left|\sum_{n\in\mathbb{Z},2\nmid n}\frac{-4\mathrm{e}^{nh}\mathrm{e}^{\mathrm{i}x}(\mathrm{e}^{2nh}+4\mathrm{e}^{nh}\mathrm{e}^{\mathrm{i}x}+\mathrm{e}^{\mathrm{i}2x})}{(\mathrm{e}^{nh}-\mathrm{e}^{\mathrm{i}x})^4}\right| \le \sum_{n=1}^{\infty}\frac{24\mathrm{e}^{3nh}}{(\mathrm{e}^{nh}-1)^4}. \tag{4.28}$$

Thus, $1/\partial_y\mathbf{S}_h(x) = O(h)$ and $\partial_x^j\partial_y\mathbf{S}_h(x) = O(\mathrm{e}^{-h})$ for $j = 1,2$, as $h \to \infty$, uniformly in $x \in \mathbb{R}$. So for $j = 1,2$, we have $S_j(h) = O(h\mathrm{e}^{-h})$ as $h \to \infty$. $\square$

For $t \le \ln(R)$, let

$$E_0(t) = e^{t-\ln(R)}, \qquad E_1(t) = (\ln(R) - t)e^{t-\ln(R)}, \qquad E_2(t) = E_0(t) + E_1(t). \quad (4.29)$$

Then $\lim_{t\to-\infty} E_j(t) = 0$, $j = 0, 1, 2$; and $\int_{-\infty}^t E_j(s)ds = E_{j+1}(t)$, $j = 0, 1$.

Lemma 4.6. *There are absolute constants $M, C > 0$ such that, if $t \le \ln(R) - M$ then for any $\xi \in C((-\infty, t])$, $K_t^\xi \subset D \setminus \{z_e\}$ and $|X_t^\xi| \le CE_1(t)$.*

Proof. Since $S_1(x) = O(xe^{-x})$ as $x \to \infty$, so there are $h_0, C_0 > 0$ such that, if $x \ge h_0$ then $S_1(x) \le C_0 xe^{-x}$. Let $M = \ln(2C_{\mathcal{H}} + e^{h_0})$. Suppose $t \le \ln(R) - M$ and $\xi \in C((-\infty, t])$. Let $h = \ln(R/e^t - C_{\mathcal{H}})$. Then $h > h_0$ and $\ln(R) - t \ge h \ge \ln(R/2) - t$. Let $C = 2C_0$. Then $S_1(h) \le C_0 he^{-h} \le CE_1(t)$. From Lemma 4.4, we have $K_t^\xi \subset D \setminus \{z_e\}$ and $\mathbb{S}_h \subset \widetilde{\Omega}_t^\xi \setminus \widetilde{p}_t^\xi$. Since $X_t^\xi = (\partial_x\partial_y/\partial_y)\widetilde{J}_t^\xi(\xi(t))$, and $\widetilde{J}_t^\xi$ is positive and harmonic in $\widetilde{\Omega}_t^\xi \setminus \widetilde{p}_t^\xi$, vanishes on $\mathbb{R}$, and has period 2π, so from Lemma 4.5, we have $|X_t^\xi| \le S_1(h) \le CE_1(t)$. □

Proof of Proposition 3.1. It is easy to check that X_t^ξ is continuous in t. So from the above lemma, the improper integral converges. □

Lemma 4.7. *Suppose $\xi \in C((-\infty, t])$, $z \in \mathbb{C} \setminus \widetilde{L}_t^\xi$, and $s \in (-\infty, t]$. Then*

$$e^s \sinh_2^2(\operatorname{Im} \widetilde{\psi}_s^\xi(z)) \ge e^t \sinh_2^2(\operatorname{Im} \widetilde{\psi}_t^\xi(z)); \quad (4.30)$$

$$\exp(\operatorname{Im} z)/4 \ge e^t \sinh_2^2(\operatorname{Im} \widetilde{\psi}_t^\xi(z)). \quad (4.31)$$

Proof. Let $h(r) = \operatorname{Im} \widetilde{\psi}_r^\xi(z)$ for $r \in (-\infty, t]$. From (2.16), there is a real valued function θ on $(-\infty, t]$ such that for $r \in (-\infty, t]$,

$$h'(r) = \operatorname{Im} \cot_2(\widetilde{\psi}_r^\xi(z) - \xi(r)) = \operatorname{Im} \cot_2(\theta(r) + ih(r)) \le -\tanh_2(h(r)),$$

which implies that $\coth_2(h(r))h'(r) \le -1$. So we have

$$2\ln\sinh_2(h(t)) - 2\ln\sinh_2(h(s)) = \int_s^t \coth_2(h(r))h'(r)dr \le -(t-s).$$

This immediately implies (4.30). Now let t be fixed and let $s \to -\infty$. Since $\widetilde{\psi}_s^\xi(z) - (z - is) \to 0$, so $e^s \sinh_2^2(\operatorname{Im} \widetilde{\psi}_s^\xi(z)) \to \exp(\operatorname{Im} z)/4$, which implies (4.31). □

Lemma 4.8. *Let $h_2 > h_3 > 0$, $s \le t$, and $\zeta, \eta \in C((-\infty, t])$. Let*

$$C_0 = \frac{1}{2\sinh_2^2(h_3)}, \qquad A_0 = \frac{2(1 + C_{\mathcal{H}})e^{s-t}}{\sinh_2^2(h_2)},$$
$$\Delta_0 = e^{C_0}A_0 + (e^{C_0} - 1)\|\eta - \zeta\|_{s,t}. \quad (4.32)$$

Assume that

$$A_0 \le 1, \qquad \Delta_0 \le h_2 - h_3. \quad (4.33)$$

Then for any $z \in \mathbb{C}$ with $\operatorname{Im} z \ge h_2$, $\widetilde{\psi}_t^\eta \circ (\widetilde{\psi}_t^\zeta)^{-1}(z)$ is meaningful, and

$$|\widetilde{\psi}_t^\eta \circ (\widetilde{\psi}_t^\zeta)^{-1}(z) - z| < \Delta_0.$$

Proof. Fix $z \in \mathbb{C}$ with $\operatorname{Im} z \ge h_2$. From (4.31), we have

$$\exp(\operatorname{Im}(\widetilde{\psi}_t^{\zeta})^{-1}(z)) \ge 4\mathrm{e}^t \sinh_2^2(h_2).$$

Then we have $(1 + C_{\mathcal{H}})\mathrm{e}^s \exp(-\operatorname{Im}(\widetilde{\psi}_t^{\zeta})^{-1}(z)) \le A_0/2 \le 1/2$. Thus, from (2.19),

$$|\widetilde{\psi}_s^{\eta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) - ((\widetilde{\psi}_t^{\zeta})^{-1}(z) - \mathrm{i}s)| \le 4(1 + C_{\mathcal{H}})\mathrm{e}^s \exp(-\operatorname{Im}(\widetilde{\psi}_t^{\zeta})^{-1}(z)) \le A_0/2.$$

Similarly, $|\widetilde{\psi}_s^{\zeta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) - ((\widetilde{\psi}_t^{\zeta})^{-1}(z) - \mathrm{i}s)| \le A_0/2$. Thus,

$$|\widetilde{\psi}_s^{\eta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) - \widetilde{\psi}_s^{\zeta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z)| \le A_0. \tag{4.34}$$

Note that $A_0 < \Delta_0 \le h_2 - h_3$. Let t_0 be the maximal number in $(s, t]$ such that, for $r \in [s, t_0)$, $\widetilde{\psi}_r^{\eta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z)$ is meaningful, and

$$g(r) := |\widetilde{\psi}_r^{\eta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) - \widetilde{\psi}_r^{\zeta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z)| < h_2 - h_3.$$

From (4.34), $g(s) \le A_0$. Since $\operatorname{Im} \widetilde{\psi}_r^{\zeta}(w)$ decreases in r, so for $r \le t$,

$$\operatorname{Im} \widetilde{\psi}_r^{\zeta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) \ge \operatorname{Im} \widetilde{\psi}_t^{\zeta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) = \operatorname{Im} z \ge h_2.$$

Thus, for $r \in [s, t_0)$,

$$\operatorname{Im} \widetilde{\psi}_r^{\eta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) \ge \operatorname{Im} \widetilde{\psi}_r^{\zeta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) - (h_2 - h_3) \ge h_3.$$

So $\widetilde{\psi}_r^{\eta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z)$ does not blow up at $r = t_0$, and $\operatorname{Im} \widetilde{\psi}_{t_0}^{\eta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) \ge h_3$. From (4.30), for $r \in [s, t_0]$ and $\xi = \zeta$ or η,

$$\sinh_2^2(\operatorname{Im} \widetilde{\psi}_r^{\xi} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z)) \ge \mathrm{e}^{t_0 - r} \sinh_2^2(\operatorname{Im} \widetilde{\psi}_{t_0}^{\xi} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z)) \ge \mathrm{e}^{t_0 - r} \sinh_2^2(h_3).$$

Thus, for any $r \in [s, t_0]$ and $w \in [\widetilde{\psi}_r^{\zeta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) - \zeta(r), \widetilde{\psi}_r^{\eta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) - \eta(r)]$, we have $\sinh_2^2(\operatorname{Im} w) \ge \mathrm{e}^{t_0 - r} \sinh_2^2(h_3)$. Since $|\cot_2'(w)| \le \frac{1}{2} \sinh_2^{-2}(\operatorname{Im} w)$ for $w \in \mathbb{H}$, so for $r \in [s, t_0]$,

$$\left|\cot_2(\widetilde{\psi}_r^{\eta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) - \eta(r)) - \cot_2(\widetilde{\psi}_r^{\zeta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) - \zeta(r))\right| \le \frac{g(r) + |\eta(r) - \zeta(r)|}{2\mathrm{e}^{t_0 - r} \sinh_2(h_3)^2}.$$

From (2.16) and the above formula, for $r \in [s, t_0]$,

$$\begin{aligned} g(r) &\le g(s) + \int_s^r \frac{g(u) + |\eta(u) - \zeta(u)|}{2\mathrm{e}^{t_0 - u} \sinh_2(h_3)^2} \mathrm{d}u \\ &\le A_0 + C_0 \mathrm{e}^{-t_0} \int_s^r \mathrm{e}^u (g(u) + \|\eta - \zeta\|_{s,t}) \mathrm{d}u. \end{aligned}$$

Solving this inequality, we get

$$g(t_0) \le A_0 \mathrm{e}^{C_0(1 - \mathrm{e}^{s - t_0})} + \|\eta - \zeta\|_{s,t} (\mathrm{e}^{C_0(1 - \mathrm{e}^{s - t_0})} - 1) < \Delta_0 \le h_2 - h_3.$$

From the choice of t_0, we have $t_0 = t$, and so $\widetilde{\psi}_t^{\eta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z)$ is meaningful, and

$$|\widetilde{\psi}_t^{\eta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) - z| = |\widetilde{\psi}_t^{\eta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z) - \widetilde{\psi}_t^{\zeta} \circ (\widetilde{\psi}_t^{\zeta})^{-1}(z)| = g(t) < \Delta_0. \quad \square$$

Suppose f is positive and harmonic in $\{a - H < \operatorname{Im} z < a + H\}$ for some $a \in \mathbb{R}$ and $H > 0$, and has period 2π. From Harnack's inequality, there is $C_* > 0$ depending only on H such that

$$\sup\{f(z) : z \in \mathbb{R}_a\} \le C_* \inf\{f(z) : z \in \mathbb{R}_a\}. \tag{4.35}$$

Let $S_1(h)$ and $S_2(h)$ be given by Lemma 4.5. Let $S_3(h) = S_2(h) + S_1(h)^2$. Then $S_3(h)$ is non-increasing, and $S_3(h) = O(h\mathrm{e}^{-h})$ as $h \to \infty$.

Lemma 4.9. *Let $s \le t \in \mathbb{R}$ and $\zeta,\eta \in C((-\infty,t])$. Let $h > 0$ and $H \in (0, h/8]$. Let $h_\lambda = h - (1+\lambda)H$ for $\lambda = 0, 0.5, 1, 2, 3$. Let C_* be given by (4.35). Let C_0, A_0, Δ_0 be given by (4.32). Suppose $L_t^\zeta \subset \Omega \setminus \{p\}$, $\mathbb{S}_h \subset \widetilde{\Omega}_t^\zeta \setminus \widetilde{p}_t^\zeta$, and*

$$A_0 \le 1, \qquad \Delta_0 \le H^2/(16 C_* h_0). \tag{4.36}$$

Then $L^\eta{}_t \subset \Omega \setminus \{p\}$, and

$$|X_t^\eta - X_t^\zeta| < 192 C_*^2 \Delta_0 / H^2 + S_3(h)|\eta(t) - \zeta(t)|. \tag{4.37}$$

Proof. This lemma is similar to Lemma 4.3. The difference is that this lemma is about the whole-plane Loewner objects, while Lemma 4.3 is about the radial Loewner objects. Recall that $X_t^\zeta = (\partial_x\partial_y/\partial_y)\widetilde{J}_t^\zeta(\zeta(t))$, $\widetilde{J}_t^\zeta$ is positive and harmonic in $\widetilde{\Omega}_t^\zeta \setminus \widetilde{p}_t^\zeta$, and vanishes on $\mathbb{R}$. Since $\mathbb{S}_h \subset \widetilde{\Omega}_t^\zeta \setminus \widetilde{p}_t^\zeta$, so after a reflection, $\widetilde{J^\zeta}_t$ is harmonic in $\{|\operatorname{Im} z| < h\}$. Let $m = \inf\{\widetilde{J^\zeta}_t(z) : z \in \mathbb{R}_{h_0}\}$, $M = \sup\{\widetilde{J^\zeta}_t(z) : z \in \mathbb{R}_{h_0}\}$, and $D_\nabla = \sup\{|\nabla \widetilde{J^\zeta}_t(z)| : z \in \mathbb{S}_{h_{0.5}}\}$. Since $\{h_0 - H < \operatorname{Im} z < h_0 + H\} \subset \mathbb{S}_h$, and $\widetilde{J^\zeta}_t$ has period 2π, so from (4.35), $M \le C_* m$. From Harnack's inequality, we find that (4.6) and (4.7) also hold here. So we have $D_\nabla \le 4M/H$.

From (4.36), we have $\Delta_0 < H = h_2 - h_3$. So (4.33) holds. From Lemma 4.8, we see that for any $z \in \mathbb{C}$ with $\operatorname{Im} z \ge h_2$, both $\widetilde{\psi}_t^\eta \circ (\widetilde{\psi}_t^\zeta)^{-1}(z)$ and $\widetilde{\psi}_t^\zeta \circ (\widetilde{\psi}_t^\eta)^{-1}(z)$ are meaningful, and $|\widetilde{\psi}_t^\eta \circ (\widetilde{\psi}_t^\zeta)^{-1}(z) - z| < \Delta_0$. Fix $w \in \mathbb{C} \setminus (\widetilde{\Omega} \setminus \widetilde{p})$. Let $z = (\widetilde{\psi}_t^\zeta)(w) \in \mathbb{H} \setminus (\widetilde{\Omega}_t^\zeta \setminus \widetilde{p}_t^\zeta)$. Since $\mathbb{S}_h \subset \widetilde{\Omega}_t^\zeta \setminus \widetilde{p}_t^\zeta$, so $\operatorname{Im} z \ge h \ge h_2$. Thus, $|\widetilde{\psi}_t^\eta(w) - z| < \Delta_0$, which implies that $\operatorname{Im}\widetilde{\psi}_t^\eta(w) > \operatorname{Im} z - \Delta_0 > h - H = h_0$. Since this holds for any $w \in \mathbb{C} \setminus (\widetilde{\Omega} \setminus \widetilde{p})$, so $L^\eta{}_t \subset \Omega \setminus \{p\}$ and $\mathbb{S}_{h_0} \subset \widetilde{\Omega}_t^\eta \setminus \widetilde{p}_t^\eta$. On the other hand, since $h_0 < h$, so $\mathbb{S}_{h_0} \subset \mathbb{S}_h \subset \widetilde{\Omega}_t^\zeta \setminus \widetilde{p}_t^\zeta$. Thus, $\widetilde{J}_t^\zeta$ and $\widetilde{J}_t^\eta$ are both harmonic in $\mathbb{S}_{h_0}$.

For $j = 1, 2$, let $\rho_j = (\widetilde{\psi}_t^\eta)^{-1}(\mathbb{R}_{h_j})$. Then ρ_1 and ρ_2 lie in $\widetilde{\Omega} \setminus \widetilde{L^\eta}_t$, and ρ_2 disconnects ρ_1 from $\widetilde{L^\eta}_t$. Since for any $z \in \mathbb{C}$ with $\operatorname{Im} z \ge h_2$, $\widetilde{\psi}_t^\zeta \circ (\widetilde{\psi}_t^\eta)^{-1}(z)$ is meaningful, so ρ_1 and ρ_2 lie in $\mathbb{C} \setminus \widetilde{L}_t^\zeta$, and ρ_2 disconnects ρ_1 from $\widetilde{L}_t^\zeta$. Thus, ρ_1 and ρ_2 lie in $\widetilde{\Omega} \setminus (\widetilde{L}_t^\zeta \cup \widetilde{L^\eta}_t)$, and ρ_2 disconnects ρ_1 from $\widetilde{L}_t^\zeta \cup \widetilde{L^\eta}_t$. For $\xi \in C((-\infty,t])$, let $G_t^\xi = G(\Omega \setminus L_t^\xi, p; \cdot)$ and $\widetilde{G}_t^\xi = G_t^\xi \circ \mathrm{e}^{\mathrm{i}}$. Then $\widetilde{J}_t^\xi = \widetilde{G}_t^\xi \circ (\widetilde{\psi}_t^\xi)^{-1}$. For $j = 1, 2$, define N_j and N_j' by (4.8) and (4.9). Then the same argument can be used to derive (4.10) and (4.11).

Fix $j \in \{1, 2\}$ and $z_0 \in \mathbb{R}_{h_j}$. Since $\operatorname{Im} z_0 \ge h_2$, so from Lemma 4.8, $|z_0 - \widetilde{\psi}_t^\zeta \circ (\widetilde{\psi}_t^\eta)^{-1}(z_0)| < \Delta_0$. Thus, $\operatorname{Im}\widetilde{\psi}_t^\zeta \circ (\widetilde{\psi}_t^\eta)^{-1}(z_0) < \operatorname{Im} z_0 + \Delta_0 \le h_1 + H/2 = h_{0.5}$. On the other hand, we have $\widetilde{\psi}_t^\zeta \circ (\widetilde{\psi}_t^\eta)^{-1}(z_0) \in \mathbb{H}$, so the line segment $[z_0, \widetilde{\psi}_t^\zeta \circ (\widetilde{\psi}_t^\eta)^{-1}(z_0)]$ lies in $\mathbb{S}_{h_{0.5}}$. So

$$\begin{aligned}|\widetilde{G}_t^\zeta \circ (\widetilde{\psi}_t^\zeta)^{-1}(z_0) - \widetilde{G}_t^\zeta \circ (\widetilde{\psi}_t^\eta)^{-1}(z_0)| &= |\widetilde{J}_t^\zeta(z_0) - \widetilde{J^\zeta}_t \circ \widetilde{\psi}_t^\zeta \circ (\widetilde{\psi}_t^\eta)^{-1}(z_0)| \\ &\le \sup_{z \in \mathbb{S}_{h_{0.5}}} \{|\nabla \widetilde{J}_t^\zeta(z)|\} \cdot |z_0 - \widetilde{\psi}_t^\zeta \circ (\widetilde{\psi}_t^\eta)^{-1}(z_0)| < D_\nabla \Delta_0 \le 4M\Delta_0/H.\end{aligned}$$

Let $\Delta = 4M\Delta_0/H$. From (4.8), (4.9), and the above formula, we find that (4.15) also holds here, which together with (4.10) and (4.11) implies (4.16). Thus, (4.17) and (4.18) both hold here. From (4.7), (4.17) and (4.36), $\Delta = 4M\Delta_0/H$, $M \le C_* m$ and $h_1 - h_2 = H$, we find that, for any $x \in \mathbb{R}$,

$$\partial_y \widetilde{J}_t^\eta(x) \ge \frac{m}{h_0} - \frac{2\Delta}{h_1 - h_2} > \frac{m}{h_0} - \frac{8M\Delta_0}{H(h_1 - h_2)} \ge \frac{m}{2h_0}.$$

This is similar to (4.19). Then (4.20) and (4.21) both hold here.

Using (4.20), (4.21), $\Delta = 4M\Delta_0/H$, $M \le C_* m$, $h_1 - h_2 = H$ and $h_0 \le 2h_1$, we get

$$|\partial_x\partial_y \widetilde{J}_t^\eta(\eta(t))/\partial_y \widetilde{J}_t^\eta(\eta(t)) - \partial_x\partial_y \widetilde{J^\zeta}_t(\eta(t))/\partial_y \widetilde{J}_t^\eta(\eta(t))| \le 128 C_* \Delta_0/H^2, \tag{4.38}$$

$$|\partial_x\partial_y \widetilde{J}_t^\zeta(\eta(t))/\partial_y \widetilde{J}_t^\eta(\eta(t)) - \partial_x\partial_y \widetilde{J}_t^\zeta(\eta(t))/\partial_y \widetilde{J^\zeta}_t(\eta(t))| \le 64 C_*^2 \Delta_0/H^2. \tag{4.39}$$

From Lemma 4.5 and the definition of $S_3(h)$, for any $x \in \mathbb{R}$

$$|\partial_x(\partial_x\partial_y/\partial_y)\widetilde{J}_t^\zeta(x)| \le |(\partial_x^2\partial_y/\partial_y)\widetilde{J}_t^\zeta(x)| + |(\partial_x\partial_y/\partial_y)\widetilde{J}_t^\zeta(x)|^2 \le S_3(h).$$

Thus,

$$|(\partial_x\partial_y/\partial_y)\widetilde{J}_t^\zeta(\eta(t)) - (\partial_x\partial_y/\partial_y)\widetilde{J^\zeta}_t(\zeta(t))| \le S_3(h)|\eta(t) - \zeta(t)|. \tag{4.40}$$

Then (4.37) follows from (4.38)–(4.40). □

Lemma 4.10. *For $j = 0, 1, 2$, let $E_j(t)$ be as in (4.29). There are absolute constants $M, C \ge 1$ such that the followings hold.*

(i) *For any $s \le t \le \ln(R) - M$, if $\zeta, \eta \in C((-\infty, t])$ and $\|\eta - \zeta\|_{s,t} \le 1$, then*

$$|X_t^\eta - X_t^\zeta| \le C(E_0(s) + E_1(t)\|\eta - \zeta\|_{s,t}). \tag{4.41}$$

(ii) *For any $t \le \ln(R) - M$ and $\zeta, \eta \in C((-\infty, t])$,*

$$|X_t^\eta - X_t^\zeta| \le C E_1(t)\|\eta - \zeta\|_t. \tag{4.42}$$

Proof. (i) Let $C_* > 0$ be the C_* in (4.35) with $H = 1$. Let

$$C_1 = \max\{20e^3(1 + C_{\mathcal{H}})\exp(5/(2e^4)), 2e^8(\exp(5/(2e^4)) - 1)\} \ge 1. \tag{4.43}$$

Let $h_* > 0$ be such that, if $h \ge h_*$ then $h/e^h \le 1/(32 C_1 C_*)$. Let

$$M = \max\{\ln(e^8 + 20e^3 C_{\mathcal{H}}), \ln(C_{\mathcal{H}} + e^{h_*})\} \ge 1.$$

Suppose $s \le t \le \ln(R) - M$, $\zeta, \eta \in C((-\infty, t])$, and $\|\eta - \zeta\|_{s,t} \le 1$. Let $h = \ln(R/e^t - C_{\mathcal{H}})$. It is straightforward to check that $h \ge \max\{8, h_*, \ln(R/2) - t\}$, and $\ln(R/2) - t \ge 1$. Since $M > \ln(1 + C_{\mathcal{H}})$, from Lemma 4.4, we have $K_t^\zeta, K_t^\eta \subset D \setminus \{z_e\}$ and $\mathbb{S}_h \subset \widetilde{\Omega}_t^\zeta \setminus \widetilde{p}_t^\zeta, \widetilde{\Omega}_t^\eta \setminus \widetilde{p}_t^\eta$.

Let $H = 1$. Then $H \in (0, h/8]$. Let $h_\lambda = h - (1 + \lambda)H$ for $\lambda = 0, 0.5, 1, 2, 3$. Then all $h_\lambda \ge 4$. It is easy to check that $\sinh_2^2(x) \ge e^x/5$ if $x \ge 4$. Let C_0, A_0, Δ_0 be given by (4.32). Then

$$A_0 \le \frac{2(1 + C_{\mathcal{H}})e^{s-t}}{e^{h_2}/5} = \frac{2(1 + C_{\mathcal{H}})e^{s-t}}{e^{h-3}/5} \le \frac{2(1 + C_{\mathcal{H}})e^{s-t}}{e^{-t}R/(10e^3)}$$
$$= 20e^3(1 + C_{\mathcal{H}})E_0(s). \tag{4.44}$$

Since $s \le t \le \ln(R) - M$, so $E_0(s) \le e^{-M} \le 1/(e^8 + 20e^3 C_{\mathcal{H}})$. Thus, $A_0 \le 1$. Since $C_0 \le 5/(2e^{h_3})$, $h_3 = h - 4 \ge 4$, and $h \ge \ln(R/2) - t$, so $C_0 \le 5/(2e^4)$ and $C_0 \le 5e^4 E_0(t)$. Thus,

$$e^{C_0} \le \exp(5/(2e^4)), \qquad e^{C_0} - 1 \le \frac{\exp(5/(2e^4)) - 1}{5/(2e^4)} \cdot 5e^4 E_0(t), \tag{4.45}$$

where the second inequality follows from that $(e^x - 1)/x$ is increasing on $(0, \infty)$. Then from (4.32) and (4.43)–(4.45), we have

$$\Delta_0 \le C_1(E_0(s) + E_0(t)\|\eta - \zeta\|_{s,t}). \tag{4.46}$$

Since $h \ge h_*$, so $h/e^h \le 1/(32C_1C_*)$. Since $h \le \ln(R/e^t)$, so $E_0(t) \le 1/e^h$. From $\|\eta-\zeta\|_{s,t} \le 1$, we have

$$\Delta_0 \le 2C_1E_0(t) \le 2C_1/e^h \le 1/(16C_*h) \le 1/(16C_*h_0).$$

Hence (4.36) holds. From Lemma 4.9, we have

$$|X_t^\eta - X_t^\zeta| \le 192C_*^2\Delta_0 + S_3(h)|\eta(t)-\zeta(t)|. \tag{4.47}$$

Since $S_3(x)$ is non-increasing, and $S_3(x) = O(xe^{-x})$ as $x \to \infty$, so there is an absolute constant $C_S > 0$ such that $S_3(x) \le C_S xe^{-x}$ for any $x \ge 1$. Since $h \ge \ln(R/2) - t \ge 1$, so

$$S_3(h) \le S_3(\ln(R) - \ln(2) - t) \le C_S E_1(\ln(2)+t) \le 2C_S E_1(t). \tag{4.48}$$

Since $\ln(R/2) - t \ge 1$, so $E_0(t) \le E_1(t)$. Let $C = 128C_*^2C_1 + 2C_S \ge 1$. Then (4.41) follows from (4.46)–(4.48).

(ii) If $\|\eta-\zeta\|_t \le 1$, then (4.42) follows from (4.41) by letting $s \to -\infty$. If $\|\eta-\zeta\|_t < \infty$, then there is $n \in \mathbb{N}$ such that $\|\eta-\zeta\|_t < n$. Let $\zeta_k = \zeta + (\eta-\zeta)k/n$, $k = 0, 1, \ldots, n$. Then $\|\zeta_{k-1}-\zeta_k\|_t < 1$ for each k, and $\|\eta-\zeta\|_t = \sum_{k=1}^n \|\zeta_{k-1}-\zeta_k\|_t$. So (4.42) follows from the result in the case $\|\eta-\zeta\|_t \le 1$. If $\|\eta-\zeta\|_t = \infty$, (4.42) always hods. □

Proof of Theorem 3.2. Let M, C be given by Lemma 4.10. Let $a_0 \le \ln(R) - M$ be such that $C|\lambda|E_2(a_0) \le 1/2$. Define a sequence of functions (ξ_n) in $C((-\infty, a_0])$ inductively such that, for any $t \le a_0$ and $n \in \mathbb{N}$, $\xi_0(t) = f(t)$ and

$$\xi_n(t) = f(t) + \lambda\int_{-\infty}^t X_s^{\xi_{n-1}} ds. \tag{4.49}$$

From Proposition 3.1, the above improper integrals converge, and $\|\xi_1 - \xi_0\|_a < \infty$. From Lemma 4.10, for $t \le a_0$, $|X_t^{\xi_{n+1}} - X_t^{\xi_n}| \le CE_1(t)$. So from (4.49), for any $t \le a_0$,

$$|\xi_{n+1}(t) - \xi_n(t)| \le C|\lambda|\int_{-\infty}^t E_1(s)ds\|\xi_n - \xi_{n-1}\|_t \le \|\xi_n - \xi_{n-1}\|_{a_0}/2.$$

Thus, (ξ_n) is a Cauchy sequence w.r.t. $\|\cdot\|_{a_0}$. Let ξ_∞ be the limit. Then ξ_∞ solves (3.4) for $t \in (-\infty, a_0]$.

Let $\mathcal{S}$ be the set of all couples (ξ, T) such that ξ solves (3.4) for $t \in (-\infty, T]$. We have proved that $\mathcal{S}$ is nonempty. Suppose $(\xi, T_0) \in \mathcal{S}$. Let $\mathring{\Omega} = \Omega_{T_0}^\xi$ and $\mathring{p} = p_{T_0}^\xi \in \mathring{\Omega}$. For $\mathring{\xi} \in C([0, S))$ for some $S > 0$, let $\mathring{L}_t^{\mathring{\xi}}$ and $\mathring{\psi}_t^{\mathring{\xi}}$ denote the *radial* Loewner hulls and maps driven by $\mathring{\xi}$. If $\mathring{L}_t^{\mathring{\xi}} \subset \mathring{\Omega} \setminus \{\mathring{p}\}$, let $\mathring{J}_t^{\mathring{\xi}} = G(\mathring{\Omega} \setminus L_t^{\mathring{\xi}}, \mathring{p}; \cdot) \circ (\mathring{\psi}_t^{\mathring{\xi}})^{-1}$, and $\mathring{X}_t^{\mathring{\xi}} = (\partial_x\partial_y/\partial_y)(\mathring{J}_t^{\xi} \circ e^i)(\mathring{\xi}(t))$. From Theorem 3.1(i), the solution to

$$\mathring{\xi}(t) = \xi(T_0) + f(T_0 + t) - f(T_0) + \lambda\int_0^t \mathring{X}_s^{\mathring{\xi}} ds \tag{4.50}$$

exists on $[0, b]$ for some $b > 0$. Let $T_e = T_0 + b > T_0$. Define $\xi_e(t) = \xi(t)$ for $t \le T_0$ and $\xi_e(t) = \mathring{\xi}(t - T_0)$ for $t \in [T_0, T_e]$. It is clear that $\xi_e \in C((-\infty, T_e])$. Since ξ_e agrees with ξ on $(-\infty, T_0]$, so ξ_e solves (3.4) for $t \in (-\infty, T_0]$. For $t \in [0, T_e - T_0]$, we have $\psi_{T_0+t}^{\xi_e} = \mathring{\psi}_t^{\mathring{\xi}} \circ \psi_{T_0}^\xi$ and $L_{T_0+t}^{\xi_e} = L_{T_0}^\xi \cup (\psi_{T_0}^\xi)^{-1}(\mathring{L}_t^{\mathring{\xi}})$, where $\psi_{T_0+t}^{\xi_e}$, $\psi_{T_0}^\xi$ and $L_{T_0+t}^{\xi_e}$, $L_{T_0}^\xi$ are the inverted whole-plane Loewner maps and hulls, while $\mathring{\psi}_t^{\mathring{\xi}}$ and $\mathring{L}_t^{\mathring{\xi}}$ are the radial Loewner maps and hulls. Since $\psi_{T_0}^\xi$

maps p to $\mathring{p}$, and maps $\Omega \setminus L^{\xi_e}_{T_0+t}$ onto $\mathring{\Omega} \setminus \mathring{L}^{\mathring{\xi}}_t$, so

$$\mathring{J}^{\mathring{\xi}}_t = G(\Omega \setminus L^{\xi_e}_{T_0+t}, p; \cdot) \circ (\psi^{\xi}_{T_0})^{-1} \circ (\mathring{\psi}^{\mathring{\xi}}_t)^{-1} = G(\Omega \setminus L^{\xi_e}_{T_0+t}, p; \cdot) \circ (\psi^{\xi_e}_{T_0+t})^{-1}$$
$$= J^{\xi_e}_{T_0+t}.$$

Thus, for $t \in [0, T_e - T_0]$, $\mathring{X}^{\mathring{\xi}}_t = X^{\xi_e}_{T_0+t}$. Since $\xi(T_0) = f(T_0) + \lambda \int_{-\infty}^{T_0} X^{\xi_e}_s \mathrm{d}s$, so from (4.50),

$$\xi_e(T_0 + t) = \mathring{\xi}(t) = \xi(T_0) + f(T_0 + t) - f(T_0) + \lambda \int_{T_0}^{T_0+t} X^{\xi_e}_s \mathrm{d}s$$
$$= f(T_0 + t) + \lambda \int_{-\infty}^{T_0+t} X^{\xi_e}_s \mathrm{d}s, \quad 0 \le t \le T_e - T_0.$$

Thus, $(\xi_e, T_e) \in \mathcal{S}$. So we find that for any $(\xi, T_0) \in \mathcal{S}$, there is $(\xi_e, T_e) \in \mathcal{S}$ such that $T_e > T_0$, and $\xi_e(t) = \xi(t)$ for $t \in (-\infty, T_0]$.

Suppose $(\xi_1, T_1), (\xi_2, T_2) \in \mathcal{S}$. For $j = 1, 2$, as $t \to -\infty$, $\xi_j(t) - f(t) \to 0$, so $\xi_1(t) - \xi_2(t) \to 0$. There is $T < \min\{a_0, T_1, T_2\}$ such that $\|\xi_1 - \xi_2\|_T \le 1$. Then from the argument of the first paragraph, we have $\|\xi_1 - \xi_2\|_T \le \|\xi_1 - \xi_2\|_T/2$. Thus, $\xi_1(t) = \xi_2(t)$ for $-\infty < t \le T$. Let $T_0 \le T_1 \wedge T_2$ be the maximal such that $\xi_1(t) = \xi_2(t)$ for $-\infty < t \le T_0$. Suppose $T_0 < T_1 \wedge T_2$. Let $\mathring{\xi}_1(t) = \xi_1(T_0 + t)$, $\mathring{\xi}_2(t) = \xi_2(T_0 + t)$ for $t \in [0, T_0 - T]$. Then $\mathring{\xi}_1$ and $\mathring{\xi}_2$ both solve Eq. (4.50) for $t \in [0, T_1 \wedge T_2 - T_0]$. From Theorem 3.1(ii), there is $S \in (0, T_1 \wedge T_2 - T_0]$ such that $\mathring{\xi}_1(t) = \mathring{\xi}_2(t)$ for $0 \le t \le S$, which implies that $\xi_1(t) = \xi_2(t)$ for $0 \le t \le T_0 + S$. This contradicts the maximum property of T_0. So $\xi_1(t) = \xi_2(t)$ for $t \in [0, T_1 \wedge T_2]$. Let $T_f = \sup\{T : (\xi, T) \in \mathcal{S}\}$. Define ξ_f on $(-\infty, T_f)$ as follows. For any $t \in (-\infty, T_f)$, choose $(\xi, T) \in \mathcal{S}$ such that $t \le T$, and let $\xi_f(t) = \xi(t)$. Then ξ_f is well defined, and solves (3.4) for $t \in (-\infty, T_f)$. We also have the uniqueness of ξ_f. There is no solution to (3.4) on $(-\infty, T_f]$. Otherwise, there exists some solution on $(-\infty, T_f + \varepsilon]$ for some $\varepsilon > 0$, which contradicts the definition of T_f.

(i) Let M_1, C_1 and M_2, C_2 be the M, C given by Lemmas 4.6 and 4.10, respectively. Let $C = C_1 \vee C_2$ and $M = M_1 \vee M_2$. Choose $a_0 \le \ln(R) - M$ such that $C|\lambda|E_2(a_0) < 1/2$. Then the solution ξ_f exists on $(-\infty, a_0]$ for any $f \in C(\mathbb{R})$.

Fix $a \in \mathbb{R}$. We now prove that $\{f \in C(\mathbb{R}) : T_f > a\} \in \mathcal{T}_a$, and $f \mapsto \xi_f$ is $(\mathcal{T}_a, \mathcal{T}_a)$-continuous on $\{T_f > a\}$. First suppose $a \le a_0$. Then $\{f \in C(\mathbb{R}) : T_f > a\} = C(\mathbb{R}) \in \mathcal{T}_a$. Suppose $\xi_{f_0} \in G \in \mathcal{T}_a$. Then there are $b_0 \le a$ and $\varepsilon \in (0, 1)$ such that $\mathbf{B}_{b_0,a}(\xi_{f_0}, \varepsilon) := \{\xi \in C(\mathbb{R}) : \|\xi - \xi_{f_0}\|_{b_0,a} < \varepsilon\} \subset G$. We may choose $b \le b_0$ and $\delta > 0$ such that $2\delta + 6C|\lambda|E_2(b) < \varepsilon$. Suppose $f \in C(\mathbb{R})$ and $\|f - f_0\|_{b,a} < \delta$. Then

$$|\xi_f(b) - \xi_{f_0}(b)| \le |f(b) - f_0(b)| + |\lambda| \int_{-\infty}^{b} (|X^{\xi_{f_0}}_s| + |X^{\xi_f}_s|)\mathrm{d}s$$
$$\le \|f - f_0\|_{b,a} + |\lambda| \int_{-\infty}^{b} 2C E_1(s)\mathrm{d}s$$
$$= \|f - f_0\|_{b,a} + 2C|\lambda|E_2(b) < \varepsilon.$$

Let $a_1 \in (b, a]$ be the maximal number such that $\|\xi_f - \xi_{f_0}\|_{b,a_1} \le 1$. From Lemmas 4.6 and 4.10, for any $t \in [b, a_1]$,

$$|\xi_f(t) - \xi_{f_0}(t)| \le |f(t) - f_0(t)| + |\lambda| \int_{-\infty}^{b} (|X^{\xi_f}_s| + |X^{\xi_{f_0}}_s|)\mathrm{d}s + |\lambda| \int_{b}^{t} |X^{\xi_f}_s - X^{\xi_{f_0}}_s|\mathrm{d}s$$

$$\leq |f(t) - f_0(t)| + 2C|\lambda| \int_{-\infty}^{b} E_1(s)\mathrm{d}s + C|\lambda| \int_{b}^{t} (E_0(b) + E_1(s)\|\xi_f - \xi_{f_0}\|_{b,s})\mathrm{d}s \leq \|f - f_0\|_{b,a_1} + 3C|\lambda|E_2(b) + C|\lambda|E_2(a_1)\|\xi_f - \xi_{f_0}\|_{b,a_1}.$$

Since $C|\lambda|E_2(a_1) \leq C|\lambda|E_2(a) \leq 1/2$, so

$$\|\xi_{f_0} - \xi_f\|_{b,a_1} \leq \frac{\|f - f_0\|_{b,a_1} + 3C|\lambda|E_1(b)}{1 - C|\lambda|E_2(a_1)} \leq 2\|f - f_0\|_{b,a_1} + 6C|\lambda| < \varepsilon < 1.$$

So we have $a_1 = a$. From the above formula, we have $\|\xi_f - \xi_{f_0}\|_{b_0,a} \leq \|\xi_f - \xi_{f_0}\|_{b,a} < \varepsilon$. Hence $\xi_f \in \mathbf{B}_{b_0,a}(\xi_{f_0}, \varepsilon) \subset G$ if $\|f - f_0\|_{b,a} < \delta$. So $f \to \xi_f$ is $(\mathcal{T}_a, \mathcal{T}_a)$-continuous.

Now consider the case that $a > a_0$. Let $M_0 = \ln(R) - a_0$. Suppose $f_0 \in \{T_f > a_0\}$ and $\xi_{f_0} \in G \in \mathcal{T}_a$. We may choose $h > 0$ such that $\mathbb{S}_h \subset \widetilde{\Omega}_a^{\xi_{f_0}} \setminus \widetilde{p}_a^{\xi_{f_0}}$. Let $H = h/8$ and $h_\lambda = h - (1 + \lambda)H$ for $\lambda = 0, 0.5, 1, 2, 3$. Let $C_* > 0$ be given by (4.35). Recall the definition of $S_3(h)$ before Lemma 4.9. Let

$$\delta_h = h2^{-11}C_*^{-1} \exp\left(-\frac{1}{2}\sinh_2^{-2}(h/2)\right); \tag{4.51}$$

$$M_h = M_0 + \max\left\{0, \ln\left(\frac{2(1 + C_{\mathcal{H}})}{\sinh_2^2(h/2)}\right), \ln\left(\frac{2^{12}(1 + C_{\mathcal{H}})C_*}{h\sinh_2^2(h/2)}\right) + \frac{1}{2\sinh_2^2(h/2)}\right\}; \tag{4.52}$$

$$C_h = 3 \cdot 2^{12}C_*^2 \exp\left(\frac{1}{2}\sinh_2^{-2}(h/2)\right)\left(\frac{2(1 + C_{\mathcal{H}})e^{M_0}}{h^2\sinh_2^2(h/2)} + \frac{1}{h^2}\right) + S_3(h). \tag{4.53}$$

There are $b_0 \leq a_0$ and $\varepsilon \in (0, \delta_h)$ such that $\mathbf{B}_{b_0,a}(\xi_{f_0}, \varepsilon) \subset G$. Let

$$\varepsilon_0 = \min\left\{\delta_h, \frac{\varepsilon}{\exp(C_h|\lambda|(a - a_0))}\right\}. \tag{4.54}$$

There is $b_1 \leq \min\{b_0, \ln(R) - M_h\}$ such that $E_0(b_1) < \varepsilon_0/5$. From the last paragraph, there are $b \leq b_1$ and $\delta \in (0, \varepsilon_0/5)$ such that, if $\|f - f_0\|_{b,a_0} < \delta$ then $\|\xi_f - \xi_{f_0}\|_{b_1,a_0} < \varepsilon_0/5$. Suppose $f \in C(\mathbb{R})$ and $\|f - f_0\|_{b,a} < \delta$. Since $a > a_0$, so $\|\xi_f - \xi_{f_0}\|_{b_1,a_0} < \varepsilon_0/5 < \delta_h$. Let $a_1 \in (a_0, a]$ be the maximal number such that ξ_f is defined on $(-\infty, a_1)$ and $|\xi_f(t) - \xi_{f_0}(t)| < \delta_h$ on $[b_1, a_1)$. Fix $t \in [a_0, a_1)$. Since $t < a$, $\operatorname{Im}\widetilde{\psi}_s^{\xi_{f_0}}(z)$ decreases in s, and $\mathbb{S}_h \subset \widetilde{\Omega}_a^{\xi_{f_0}} \setminus \widetilde{p}_a^{\xi_{f_0}}$, so $\mathbb{S}_h \subset \widetilde{\Omega}_t^{\xi_{f_0}} \setminus \widetilde{p}_t^{\xi_{f_0}}$. Let C_0, A_0, Δ_0 be given by (4.32) with $s = b_1$, $\zeta = \xi_{f_0}$ and $\eta = \xi_f$. Since $t \geq a_0 = \ln(R) - M_0$ and $h_2 \geq h_3 = h/2$, so

$$A_0 \leq \frac{2(1 + C_{\mathcal{H}})e^{M_0}}{\sinh_2^2(h/2)} E_0(b_1). \tag{4.55}$$

Since $C_0 = \frac{1}{2}\sinh_2^{-2}(h/2)$, so from (4.32) and (4.55), we have

$$\Delta_0 \leq \exp\left(\frac{1}{2}\sinh_2^{-2}(h/2)\right)\left(\frac{2(1 + C_{\mathcal{H}})e^{M_0}}{\sinh_2^2(h/2)} E_0(b_1) + \|\xi_f - \xi_{f_0}\|_{b_1,t}\right). \tag{4.56}$$

Using (4.51)–(4.56) and the facts that $b_1 \leq \ln(R) - M_h$, $\|\xi_f - \xi_{f_0}\|_{b_1,t} \leq \delta_h$ and $H = h/8$, one may check that (4.36) holds, i.e., $A_0 \leq 1$ and $\Delta_0 \leq H^2/(16C_*h_0)$. From Lemma 4.9 and (4.56), we have

$$|X_t^{\xi_f} - X_t^{\xi_{f_0}}| \le C_h(E_0(b_1) + \|\xi_f - \xi_{f_0}\|_{b_1,t}), \quad t \in [a_0, a_1). \tag{4.57}$$

Recall that $b \le b_1 \le b_0 \le a_0 < a_1 \le a$. From (3.4) and (4.57), for any $t \in [a_0, a_1)$,

$$\begin{aligned}|\xi_f(t) - \xi_{f_0}(t)| &\le |f(a_0) - f_0(a_0)| + |f(t) - f_0(t)| + |\xi_f(a_0) - \xi_{f_0}(a_0)| \\ &\quad + |\lambda| \int_{a_0}^t |X_s^{\xi_f} - X_s^{\xi_{f_0}}| \mathrm{d}s \\ &\le 2\|f - f_0\|_{b,a} + \|\xi_f - \xi_{f_0}\|_{b_1,a_0} \\ &\quad + C_h|\lambda| \int_{a_0}^t (E_0(b_1) + \|\xi_f - \xi_{f_0}\|_{b_1,s}) \mathrm{d}s.\end{aligned}$$

For $t \in [a_0, a_1)$, let $g(t) = \|\xi_f - \xi_{f_0}\|_{b_1,t}$, then

$$g(t) \le 2\|f - f_0\|_{b,a} + \|\xi_f - \xi_{f_0}\|_{b_1,a_0} + C_h|\lambda| \int_{a_0}^t (E_0(b_1) + g(s)) \mathrm{d}s.$$

Solving this inequality using (4.54) and that $\|f - f_0\|_{b_1,a_0} \le \|f - f_0\|_{b,a_0} < \delta < \varepsilon_0/5$, and $\|\xi_f - \xi_{f_0}\|_{b_1,a_0} < \varepsilon_0/5$, we have that for any $t \in [a_0, a_1)$,

$$\begin{aligned}g(t) &\le \mathrm{e}^{C_h|\lambda|(t-a_0)}(2\|f - f_0\|_{b,a} + \|\xi_f - \xi_{f_0}\|_{b_1,a_0}) + (\mathrm{e}^{C_h|\lambda|(t-a_0)} - 1)E_0(b_1) \\ &< \mathrm{e}^{C_h|\lambda|(a-a_0)}(2\varepsilon_0/5 + \varepsilon_0/5 + \varepsilon_0/5) \le 4\varepsilon/5 < \varepsilon.\end{aligned}$$

So from (4.57) we have $|X_t^{\xi_f} - X_t^{\xi_{f_0}}| < C_h(E_0(b_1) + \varepsilon)$ for any $t \in [a_0, a_1)$. Let

$$S = C_h(E_0(b_1) + \varepsilon) + \sup\{|X_t^{\xi_{f_0}}| : t \in [a_0, a]\} < \infty.$$

Then $|X_t^{\xi_f}| \le S$ for any $t \in [a_0, a_1)$. Since $\xi_f(t) = f(t) + \lambda \int_{-\infty}^t X_s^{\xi_f} \mathrm{d}s$, so $\lim_{t\to a_1} \xi_f(t)$ exists and is finite. By defining $\xi_f(a_1) = \lim_{t\to a_1} \xi_f(t)$, we have ξ_f that solves (3.4) for $-\infty < t \le a_1$. Thus, $T_f > a_1$. Since $\|\xi_f - \xi_{f_0}\|_{b_1,t} = g(t) \le 4\varepsilon/5 < \delta_h$ for all $t \in [a_0, a_1)$, so from the definition of a_1, we have $a_1 = a$. Thus, $T_f > a$ and $\|\xi_f - \xi_{f_0}\|_{b_1,a} = \lim_{t\to a} g(t) \le 4\varepsilon/5 < \varepsilon$. Thus, $f \in \{T_f > a\}$ and $\xi_f \in \mathbf{B}_{b_0,a}(\xi_{f_0}, \varepsilon) \subset G$ if $\|f - f_0\|_{b,a} < \delta$. So $\{T_f > a\} \in \mathcal{T}_a$, and $f \mapsto \xi_f$ is $(\mathcal{T}_a, \mathcal{T}_a)$-continuous on $\{T_f > a\}$.

Let $f_1, f_2 \in C(\mathbb{R})$. Suppose for some $a \in \mathbb{R}$, $T_{f_1} > a$, that is, $\xi_{f_1}(t)$ is defined on $(-\infty, a]$, and $f_1 \overset{a}{\sim} f_2$. Then there is $k \in \mathbb{Z}$ such that $f_2(t) = f_1(t) + 2k\pi$ for $t \le a$. It is clear that $\xi(t) = \xi_{f_1}(t) + 2k\pi$ solves (3.4) with $f = f_2$ for $-\infty < t \le a$. Thus, $T_{f_2} > a$ and $\xi_{f_2}(t) = \xi_{f_1}(t) + 2k\pi$ for $t \le a$, so $\xi_{f_1} \overset{a}{\sim} \xi_{f_2}$. From the results of the last paragraph, we have $\{T_f > a\} \in \mathcal{T}_a^{\mathbb{T}}$, and $f \mapsto \xi_f$ is $(\mathcal{T}_a^{\mathbb{T}}, \mathcal{T}_a^{\mathbb{T}})$-continuous on $\{T_f > a\}$.

(ii) Suppose α is a Jordan curve such that $\bigcup_{t<T_f} K_t^{\xi_f} \subset H(\alpha) \subset D\setminus\{z_e\}$. Then $t = \mathrm{cap}(K_t^{\xi_f}) \le \mathrm{cap}(H(\alpha))$ for any $t < T_f$, so $T_f \le \mathrm{cap}(H(\alpha)) < \infty$. We may choose another Jordan curve α_0 such that $H(\alpha) \subset U(\alpha_0)$ and $H(\alpha_0) \subset D \setminus \{z_e\}$. Let $h = \min\{\ln |\varphi_{H(\alpha)}(z)| : z \in \alpha_0\} > 0$. For any $t < T_f$, since $K_t^{\xi_f} \subset H(\alpha)$, so for any $z \in \alpha_0$, $|\varphi_t^{\xi_f}(z)| = |\varphi_{K_t^{\xi_f}}(z)| \ge |\varphi_{H(\alpha)}(z)| \ge \mathrm{e}^h$. Since α_0 disconnects $K_t^{\xi_f}$ from $\mathbb{C} \setminus (D \setminus \{z_e\})$, so $\{1 < |z| < \mathrm{e}^h\} \subset \varphi_t^{\xi_f}(D \setminus \{z_e\} \setminus K_t^{\xi_f})$. Thus, $\mathbb{S}_h \subset \widetilde{\Omega}_t^{\xi_f} \setminus \widetilde{p}_t^{\xi_f}$ for $t < T_f$. Now $\widetilde{J}_t^{\xi_f}$ is positive and harmonic in $\mathbb{S}_h$, vanishes on $\mathbb{R}$, and has period 2π, so from Lemma 4.5, $|X_t^{\xi_f}| \le S_1(h)$ for $t < T_f$. From (3.4), $\lim_{t\to T_f^-} \xi_f(t)$ exists and is finite. Define $\xi_f(T_f) = \lim_{t\to T_f^-} \xi_f(t)$. Then ξ_f solves (3.4) for $-\infty < t \le T_f$, which is a contradiction. $\square$

5. Partition function

For $\kappa > 0$ and $\lambda \in \mathbb{R}$, let a (κ,λ)-process denote the whole-plane Loewner chain driven by the solution to (3.4) with $f(t) = B^{(\kappa)}_{\mathbb{R}}(t)$. In this section, we will prove that a (κ,λ)-process is locally absolutely continuous w.r.t. the whole-plane SLE_κ processes started from 0. By setting $\kappa = \lambda = 2$, we conclude that the continuous LERW from an interior point to another interior point is locally absolutely continuous w.r.t. the whole-plane SLE_2 process.

Suppose D is a finitely connected domain, $0, z_e \in D$, and $z_e \neq 0$. Let K_t and $\beta(t)$, $-\infty \le t < \infty$, be a whole-plane SLE_κ hulls and trace from 0 to ∞ with the driving function being $\xi(t) = B^{(\kappa)}_{\mathbb{R}}(t)$. Let μ be the distribution of $(\xi(t))$. Let $(\mathcal{F}^0_t)$ be the filtration generated by $(\mathrm{e}^{\mathrm{i}\xi(t)})$. Let $(\mathcal{F}_t)$ be the completion of $(\mathcal{F}^0_t)$ w.r.t. μ. Let ψ_t and $\widetilde{\psi}_t$ be the inverted and covering inverted whole-plane Loewner maps driven by ξ. Let $\varphi_t = \varphi_{K_t}$ and $\phi_t = \phi_{K_t}$. Then $\varphi_t = R_{\mathbb{T}} \circ \psi_t \circ R_{\mathbb{T}}$ and $\phi_t(z) = \mathrm{e}^t \varphi_t(z)$. Let $T \in (-\infty, \infty]$ be the maximal number such that $K_t \subset D \setminus \{z_e\}$ for $-\infty < t < T$. Let J^ξ_t, $\widetilde{J}^\xi_t$, Ω^ξ_t, $\widetilde{\Omega}^\xi_t$, p^ξ_t, and $\widetilde{p}^\xi_t$, $-\infty < t < T$, be as in Section 3.2. For simplicity, we omit the superscripts ξ in this section.

Let $R = \mathrm{dist}(0, \partial D \cup \{z_e\})$. Let $T_R = \ln(R) - \ln(1 + C_{\mathcal{H}})$. Let $h(t) = \ln(R/\mathrm{e}^t - C_{\mathcal{H}}) > 0$ for $t < T_R$. From Lemma 4.4 we have $\mathbb{S}_{h(t)} \subset \widetilde{\Omega}_t \setminus \widetilde{p}_t$ for $t < T_R$. From Lemma 4.5, we conclude that $|(\partial^j_x \partial_y/\partial_y)\widetilde{J}_t(\xi(t))| \le S_j(h(t))$ for $t < T_R$ and $j = 1, 2$, where $S_j(h) = O(h\mathrm{e}^{-h})$ as $h \to \infty$. So for $j = 1, 2$,

$$(\partial^j_x \partial_y/\partial_y)\widetilde{J}_t(\xi(t)) = O(t\mathrm{e}^t), \quad t \to -\infty. \tag{5.1}$$

Now we study the behavior of $\partial_y \widetilde{J}_t(\xi(t))$ as $t \to -\infty$. We have to consider two cases. The first case is that $D = \widehat{\mathbb{C}}$. Then $\Omega_t = \mathbb{D}$ for all $t \in \mathbb{R}$. If $z_e = \infty$ then $p = p_t = 0$ for all $t \in \mathbb{R}$. Thus, $J_t(z) = G(\Omega_t, p_t; z) = -\frac{1}{2\pi}\ln|z|$, and so $\widetilde{J}_t(z) = J_t(\mathrm{e}^{\mathrm{i}z}) = \frac{1}{2\pi}\operatorname{Im} z$. So we have $\partial_y \widetilde{J}_t(\xi(t)) = \frac{1}{2\pi}$ for all $t \in \mathbb{R}$. Now suppose that $D = \widehat{\mathbb{C}}$ and $z_e \notin \{0, \infty\}$. Recall that

$$p_t = \psi_t(p) = R_{\mathbb{T}} \circ \varphi_t \circ R_{\mathbb{T}}(p) = R_{\mathbb{T}} \circ \varphi_t(z_e) = R_{\mathbb{T}}(\mathrm{e}^{-t}\phi_{K_t}(z_e)).$$

From (2.7) we have $|\phi_{K_t}(z_e) - z_e| \le C_{\mathcal{H}}\mathrm{e}^t$. Thus, $p_t = O(\mathrm{e}^t)$ as $t \to -\infty$. We have

$$\widetilde{J}_t(z) = J_t(\mathrm{e}^{\mathrm{i}z}) = G(\Omega_t, p_t; \mathrm{e}^{\mathrm{i}z}) = G(\mathbb{D}, p_t; \mathrm{e}^{\mathrm{i}z}) = -\frac{1}{2\pi}\ln\left|\frac{\mathrm{e}^{\mathrm{i}z} - p_t}{\overline{p}_t\mathrm{e}^{\mathrm{i}z} - 1}\right|.$$

So we have

$$\partial_y \widetilde{J}_t(\xi(t)) = \frac{1}{2\pi}\frac{1 - |p_t|^2}{|1 - p_t\mathrm{e}^{-\mathrm{i}\xi(t)}|^2} = \frac{1}{2\pi} + O(\mathrm{e}^t), \quad t \to -\infty.$$

Thus, when $D = \widehat{\mathbb{C}}$ we always have

$$\partial_y \widetilde{J}_t(\xi(t)) = \frac{1}{2\pi} + O(\mathrm{e}^t), \quad t \to -\infty. \tag{5.2}$$

The second case is that $D \neq \widehat{\mathbb{C}}$. For $t \in (-\infty, T)$, let $G_t(z) = G(D \setminus K_t, z_e; z)$ and $G^\phi_t(z) = G_t(\phi_t^{-1}(z)) = G(\phi_t(D \setminus K_t), \phi_t(z_e); z)$. Since $\Omega_t = R_{\mathbb{T}} \circ \varphi_t(D \setminus K_t) = R_{\mathbb{T}} \circ M^{-1}_{\mathrm{e}^t} \circ \phi_t(D \setminus K_t)$,

$p_t = R_{\mathbb{T}} \circ \varphi_t(z_e) = R_{\mathbb{T}} \circ M_{e^t}^{-1} \circ \phi_t(z_e)$, and $J_t(z) = G(\Omega_t, p_t; z)$, so $J_t = G_t^\phi \circ M_{e^t} \circ R_{\mathbb{T}}$. As t decreases, $D \setminus K_t$ increases, so G_t increases. Let $G_{-\infty}(z) = G(D, z_e; z)$. As $t \to -\infty$, since $K_t \to \{0\}$, so $G_t(z) \to G_{-\infty}(z)$ in $D \setminus \{z_e, 0\}$. Moreover, since $\text{diam}(K_t) \le 4e^t$, so $G_t(z) - G_{-\infty}(z) = O(1/t)$ as $t \to -\infty$, uniformly on any subset of $D \setminus \{z_e, 0\}$ that is bounded away from 0. Using Harnack's inequality, we conclude that $\nabla G_t \to \nabla G_{-\infty}$, as $t \to -\infty$, uniformly on any compact subset of $D \setminus \{z_e, 0\}$.

Let $r = R/2$ and $\delta = R/4$. Let $A = \{r - \delta \le |z| \le r + \delta\}$. Then A is a compact subset of $D \setminus \{z_e, 0\}$. So there are constants $T_A \in (-\infty, T)$ and $M_A \in (0, \infty)$ such that $|\nabla G_t| \le M_A$ on A if $t \le T_A$. From (2.6) we see that $|\phi_t^{-1}(z) - z| \le C_{\mathcal{H}} e^t$ for any $t < \ln|z|$. Let $T_B = T_A \wedge \ln(\delta/C_{\mathcal{H}})$. Suppose $|z| = r$ and $t \le T_B$. Then $|\phi_t^{-1}(z) - z| \le C_{\mathcal{H}} e^t \le \delta$. So $[z, \phi^{-1}(z)] \subset A$. Thus,

$$|G_t^\phi(z) - G_t(z)| = |G_t(\phi_t^{-z}(z)) - G_t(z)| \le M_A |\phi_t^{-1}(z) - z| \le M_A C_{\mathcal{H}} e^t.$$

Since $G_t - G_{-\infty} = O(1/t)$ as $t \to -\infty$, uniformly on $\{|z| = e^t\}$, so $G_t^\phi - G_{-\infty} = O(1/t)$ as $t \to -\infty$, uniformly on $\{|z| = e^t\}$. Since $\widetilde{J}_t = G_t^\phi \circ M_{e^t} \circ R_{\mathbb{T}} \circ e^i$, so as $t \to -\infty$,

$$\widetilde{J}_t(x + i(\ln(r) - t)) = G_t^\phi(re^i x) = G_{-\infty}(re^{ix}) + O(1/t) \tag{5.3}$$

uniformly in $x \in \mathbb{R}$.

Fix $t \in (-\infty, T_B]$. Let $h = \ln(r) - t$. Let $\mathbf{S}_h(z)$ be defined as in (4.25). So $\mathbf{S}_h \circ (e^i)^{-1}$ is a Poisson kernel function in $\{e^{-h} < |z| < 1\}$ with the pole at e^{-h}. Since $\widetilde{J}_t$ is harmonic in $\mathbb{S}_h$, continuous on $\overline{\mathbb{S}_h}$, vanishes on $\mathbb{R}$, and has period 2π, so for any $z \in \mathbb{S}_h$, we have

$$\widetilde{J}_t(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widetilde{J}_t(x + ih) \mathbf{S}_h(z - x) dx.$$

Thus, for any $x_0 \in \mathbb{R}$, $\partial_y \widetilde{J}_t(x_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widetilde{J}_t(x + ih) \partial_y \mathbf{S}_h(x_0 - x) dx$. From (4.25) and the computation in the proof of Lemma 4.5 we have $\partial_y \mathbf{S}_h(x) = \frac{1}{h} + O(e^{-h}) = \frac{1}{h} + O(e^t)$ as $h \to \infty$, uniformly in $x \in \mathbb{R}$. From (5.3) we have $\widetilde{J}_t(x + ih) = G_{-\infty}(re^{ix}) + O(1/t)$ as $t \to -\infty$, uniformly in $x \in \mathbb{R}$. Thus, as $t \to -\infty$, we have

$$\partial_y \widetilde{J}_t(x_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_{-\infty}(re^{ix}) \frac{dx}{h} + O(1/(ht)) = \frac{G_{-\infty}(0)}{\ln(r) - t} + O(1/t^2)$$

uniformly in $x \in \mathbb{R}$, where the second "=" holds because $G_{-\infty}$ is harmonic in $\{|z| \le r\}$. So as $t \to -\infty$, we have

$$-t \partial_y \widetilde{J}_t(\xi(t)) = G_{-\infty}(0) + O(1/t) = G(D, z_e; 0) + O(1/t). \tag{5.4}$$

Next, we study the behavior of $(\partial_t \partial_y/\partial_y)\widetilde{J}_t(\xi(t))$ as $t \to -\infty$. For $t \in (-\infty, T)$, we have $J_t \circ \psi_t \circ R_{\mathbb{T}} = G_t = G(D \setminus K_t, z_e, \cdot)$, which implies that

$$\widetilde{J}_t \circ \widetilde{\psi}_t \circ R_{\mathbb{R}}(z) = G_t(e^{iz}) = G(D \setminus K_t, z_e, e^i(z)). \tag{5.5}$$

Let P_t denote the generalized Poisson kernel in Ω_t with the pole at $e^{i\xi(t)}$, normalized by $P_t(z) = \text{Re} \frac{e^{i\xi(t)} + z}{e^{i\xi(t)} - z} + O(z - e^{i\xi(t)})$ as $z \to e^{i\xi(t)}$. So $P_t \circ \psi_t \circ R_{\mathbb{T}}$ is a generalized Poisson kernel in $D \setminus K_t$ with the pole at $\beta(t)$. If $\delta > 0$ is small, then $K_{t+\delta} \setminus K_t$ is contained in a small ball centered at $\beta(t)$. So it is intuitive that $G_t = G(D \setminus K_t, z_e, \cdot)$ is differentiable in t, and $-\partial_t G_t(z)$ is a generalized Poisson kernel in $D \setminus K_t$ with the pole at $\beta(t)$. This can be proved by expressing $G_t(z)-G_{t+\delta}(z)$ as an integral of Poisson kernels in $D\setminus K_{t+\delta}$ with poles on $K_{t+\delta}\setminus K_t$. We do not go into details here. Thus, there are $C(t)>0$ such that $-\partial_t G_t = C(t)P_t\circ\psi_t\circ R_{\mathbb{T}}$, $-\infty<t<T$. Let $\widetilde{P}_t = P_t\circ \mathrm{e}^{\mathrm{i}}$. Then $\widetilde{P}_t(z) = -\operatorname{Im}\cot_2(z-\xi(t)) + O(z-\xi(t))$ as $z\to\xi(t)$, and we have

$$-\partial_t G_t(\mathrm{e}^{\mathrm{i}z}) = C(t)\widetilde{P}_t\circ\widetilde{\psi}_t(\overline{z}),\quad -\infty<t<T. \tag{5.6}$$

Differentiating (5.5) w.r.t. t and using (2.14) and (5.6), we see that for any $z\in(\mathrm{e}^{\mathrm{i}})^{-1}(D\setminus K_t)$,

$$\begin{aligned}&\partial_t\widetilde{J}_t(\widetilde{\psi}_t(\overline{z})) + \partial_x\widetilde{J}_t(\widetilde{\psi}_t(\overline{z}))\operatorname{Re}\cot_2(\widetilde{\psi}_t(\overline{z})-\xi(t))\\ &\quad + \partial_y\widetilde{J}_t(\widetilde{\psi}_t(\overline{z}))\operatorname{Im}\cot_2(\widetilde{\psi}_t(\overline{z})-\xi(t)) = -C(t)\widetilde{P}_t(\widetilde{\psi}_t(\overline{z})).\end{aligned}$$

Since $\widetilde{\psi}_t\circ R_{\mathbb{R}}$ maps $(\mathrm{e}^{\mathrm{i}})^{-1}(D\setminus K_t)$ onto $\widetilde{\Omega}_t$, so for any $w\in\widetilde{\Omega}_t$,

$$\partial_t\widetilde{J}_t(w) + \partial_x\widetilde{J}_t(w)\operatorname{Re}\cot_2(w-\xi(t)) + \partial_y\widetilde{J}_t(w)\operatorname{Im}\cot_2(w-\xi(t)) = -C(t)\widetilde{P}_t(w). \tag{5.7}$$

Suppose in some neighborhood U of $\xi(t)$, $\widetilde{J}_t = \operatorname{Im}\widetilde{J}_t^{\mathbb{C}}$ and $\widetilde{P}_t = \operatorname{Im}\widetilde{P}_t^{\mathbb{C}}$, where $\widetilde{J}_t^{\mathbb{C}}$ is analytic in U, and $\widetilde{P}_t^{\mathbb{C}}$ is meromorphic with a pole at $\xi(t)$ in U. From (5.7) we have

$$\operatorname{Im}[\partial_t\widetilde{J}_t^{\mathbb{C}}(w)] + \operatorname{Im}[(\widetilde{J}_t^{\mathbb{C}})'(w)\cot_2(w-\xi(t))] = \operatorname{Im}[-C(t)\widetilde{P}_t^{\mathbb{C}}(w)]. \tag{5.8}$$

Comparing the residues at $\xi(t)$ of the two sides, we find that $C(t) = (\widetilde{J}_t^{\mathbb{C}})'(\xi(t)) = \partial_y\widetilde{J}_t(\xi(t))$. Differentiating (5.8) w.r.t. w, we get

$$\begin{aligned}&\partial_t(\widetilde{J}_t^{\mathbb{C}})'(w) + (\widetilde{J}_t^{\mathbb{C}})''(w)\cot_2(w-\xi(t)) + (\widetilde{J}_t^{\mathbb{C}})'(w)\cot_2'(w-\xi(t))\\ &\quad = -(\widetilde{J}_t^{\mathbb{C}})'(\xi(t))(\widetilde{P}_t^{\mathbb{C}})'(w).\end{aligned}$$

Letting $w\to\xi(t)$ in $\widetilde{\Omega}_t$ in the above formula, and comparing the constant term in the power series expansion at $\xi(t)$ of both sides, we get

$$\partial_t(\widetilde{J}_t^{\mathbb{C}})'(\xi(t)) = (\widetilde{J}_t^{\mathbb{C}})'(\xi(t))\lim_{w\to\xi(t)}(-(\widetilde{P}_t^{\mathbb{C}})'(w) - \cot_2'(w)) + \frac{1}{6}(\widetilde{J}_t^{\mathbb{C}})'''(\xi(t)). \tag{5.9}$$

Let $\widetilde{Q}_t = -\widetilde{P}_t - \operatorname{Im}\cot_2$. Then $\widetilde{Q}_t$ is continuous on $\overline{\widetilde{\Omega}_t}$, vanishes on $\mathbb{R}$, equals $-\operatorname{Im}\cot_2$ on $\partial\widetilde{\Omega}_t\setminus\mathbb{R}$, has period 2π, and is harmonic inside $\widetilde{\Omega}_t$. From (5.9) we have

$$(\partial_t\partial_y/\partial_y)\widetilde{J}_t(\xi(t)) = \partial_y\widetilde{Q}_t(\xi(t)) + \frac{1}{6}(\partial_x^2\partial_y/\partial_y)\widetilde{J}_t(\xi(t)). \tag{5.10}$$

For the behavior of $(\partial_t\partial_y/\partial_y)\widetilde{J}_t(\xi(t))$ as $t\to-\infty$, we also need to consider two cases. The first case is $D=\widehat{\mathbb{C}}$. Then $\Omega_t=\mathbb{D}$, so $P_t(z) = \operatorname{Re}\frac{\mathrm{e}^{\mathrm{i}\xi(t)}+z}{\mathrm{e}^{\mathrm{i}\xi(t)}-z}$, which implies that $\widetilde{P}_t(z) = -\operatorname{Im}\cot_2(z)$ and $\widetilde{Q}_t\equiv 0$. From (5.1) and (5.10) we have

$$(\partial_t\partial_y/\partial_y)\widetilde{J}_t(\xi(t)) = O(t\mathrm{e}^t),\quad t\to-\infty. \tag{5.11}$$

The second case is that $D\neq\widehat{\mathbb{C}}$. Let Q_t be continuous on $\overline{\Omega_t}$, harmonic in Ω_t, vanishes on $\mathbb{T}$, and equals to $\operatorname{Re}\frac{\mathrm{e}^{\mathrm{i}\xi(t)}+z}{\mathrm{e}^{\mathrm{i}\xi(t)}-z}$ on $\partial\Omega_t\setminus\mathbb{T}$. Then $\widetilde{Q}_t = Q_t\circ\mathrm{e}^{\mathrm{i}}$. Let $S_t = Q_t\circ R_{\mathbb{T}}\circ\varphi_t$. Then S_t is continuous on $\overline{D\setminus K_t}$, harmonic in $D\setminus K_t$, vanishes on ∂K_t, and $S_t(z) = \operatorname{Re}\frac{\varphi_t(z)+\mathrm{e}^{\mathrm{i}\xi(t)}}{\varphi_t(z)-\mathrm{e}^{\mathrm{i}\xi(t)}}$ on ∂D. Since $\varphi_t(z) = \mathrm{e}^{-t}\phi_t(z)$, so from (2.7) we have $\varphi_t(z) = \mathrm{e}^{-t}z + O(1)$ as $t\to-\infty$, uniformly in $z\in\partial D$. So $S_t = 1+O(\mathrm{e}^t)$ on ∂D as $t\to-\infty$. Since $\operatorname{diam}(K_t)\le 4\mathrm{e}^t$, so $S_t(z) = 1+O(1/t)$

as $t \to -\infty$, uniformly on any compact subset of $D \setminus \{0\}$. The argument that is used to derive (5.4) can be used here to prove that $\partial_y \widetilde{Q}_t(\xi(t)) = -1/t + O(1/t^2)$ as $t \to -\infty$. So from (5.1) and (5.10) we have

$$(\partial_t \partial_y / \partial_y) \widetilde{J}_t(\xi(t)) = -1/t + O(1/t^2), \quad t \to -\infty. \tag{5.12}$$

Let $\alpha = \lambda/\kappa \in \mathbb{R}$. We define $M(t)$ for $t \in (-\infty, T)$. If $D = \widehat{\mathbb{C}}$, let

$$M(t) = (2\pi \partial_y \widetilde{J}_t(\xi(t)))^\alpha \exp\Bigg(-\frac{\kappa}{2}\alpha(\alpha-1)\int_{-\infty}^t \left(\frac{\partial_x \partial_y \widetilde{J}_s(\xi(s))}{\partial_y \widetilde{J}_s(\xi(s))}\right)^2 \mathrm{d}s - \frac{\kappa}{2}\alpha \int_{-\infty}^t \frac{\partial_x^2 \partial_y \widetilde{J}_s(\xi(s))}{\partial_y \widetilde{J}_s(\xi(s))}\mathrm{d}s - \alpha \int_{-\infty}^t \frac{\partial_t \partial_y \widetilde{J}_s(\xi(s))}{\partial_y \widetilde{J}_s(\xi(s))}\mathrm{d}s\Bigg). \tag{5.13}$$

From (5.1) and (5.11) we see that the three improper integrals all converge. From (5.2) we have $\lim_{t\to-\infty} M(t) = 1$. If $D = \widehat{\mathbb{C}}$, let

$$M(t) = \left(\frac{\sqrt{t^2+1}\,\partial_y \widetilde{J}_t(\xi(t))}{G(D, z_e; 0)}\right)^\alpha \exp\Bigg(-\frac{\kappa}{2}\alpha(\alpha-1)\int_{-\infty}^t \left(\frac{\partial_x \partial_y \widetilde{J}_s(\xi(s))}{\partial_y \widetilde{J}_s(\xi(s))}\right)^2 \mathrm{d}s - \frac{\kappa}{2}\alpha \int_{-\infty}^t \frac{\partial_x^2 \partial_y \widetilde{J}_s(\xi(s))}{\partial_y \widetilde{J}_s(\xi(s))}\mathrm{d}s - \alpha \int_{-\infty}^t \left(\frac{\partial_t \partial_y \widetilde{J}_s(\xi(s))}{\partial_y \widetilde{J}_s(\xi(s))} + \frac{s}{\sqrt{s^2+1}}\right)\mathrm{d}s\Bigg). \tag{5.14}$$

From (5.1) and (5.12) we see that the three improper integrals all converge. From (5.4) we have $\lim_{t\to-\infty} M(t) = 1$ in this case.

Lemma 5.1 (*Boundedness*)**.** *Let ρ be a Jordan curve in $\widehat{\mathbb{C}}$ such that $0 \in U(\rho)$ and $H(\rho) \subset D \setminus \{z_e\}$. Let τ_ρ be the first t such that $K_t \cap \rho \neq \emptyset$. Then there is a constant $C \in (0, \infty)$ depending only on ρ, D, and z_e, such that $|\ln(M(t))| \le C$ on $(-\infty, \tau_\rho]$.*

Proof. Let $R_\rho = \mathrm{dist}(0, \rho) > 0$. Then $\ln(R_\rho/4)$ is a lower bound of τ_ρ. From (5.1), (5.2), (5.11) and (5.13), or from (5.1), (5.4), (5.12) and (5.14), we conclude that there is $b \in (-\infty, \ln(R_\rho/4))$ and $C_1 \in (0, \infty)$ depending only on ρ, D, and z_e, such that $|\ln(M(t))| \le C_1$ on $(-\infty, b]$. The boundedness of $|\ln(M(t))|$ on $[b, \tau_\rho]$ follows from Lemma 2.3. $\square$

Now we study the martingale property of $M(t)$. Since $(\widetilde{J}_t)$ is $(\mathcal{F}_t)$-adapted, has period 2π, and $(e^{i\xi(t)})$ is also $(\mathcal{F}_t)$-adapted, so $(\partial_y \widetilde{J}_t(\xi(t)))$ is $(\mathcal{F}_t)$-adapted, and so are $((\partial_x^j \partial_y/\partial_y)\widetilde{J}_t(\xi(t)))$, $j = 1, 2$, and $((\partial_t \partial_y/\partial_y)\widetilde{J}_t(\xi(t)))$. From (5.13) or (5.14) we see that $(M(t))$ is $(\mathcal{F}_t)$-adapted. We will truncate the time interval to apply Itô's formula. Recall that $\ln(R/4)$ is a lower bound of T. Fix $a \in (-\infty, \ln(R/4))$. Let $T_a = T - a > 0$. Let $\mathcal{F}_t^a = \mathcal{F}_{a+t}$, $t \ge 0$. Then T_a is an $(\mathcal{F}_t^a)_{t\ge0}$-stopping time. Let $M_a(t) = M(a+t)$, $0 \le t < T_a$. Then $(M_a(t))$ is $(\mathcal{F}_t^a)$-adapted. From (5.13) or (5.14) we have

$$M_a(t) = M(a)\partial_y \widetilde{J}_a(\xi(a))^{-\alpha} \partial_y \widetilde{J}_{a+t}(\xi(a+t))^\alpha \times \exp\Bigg(-\frac{\kappa}{2}\alpha(\alpha-1)\int_a^{a+t} \left(\frac{\partial_x \partial_y \widetilde{J}_s(\xi(s))}{\partial_y \widetilde{J}_s(\xi(s))}\right)^2 \mathrm{d}s - \frac{\kappa}{2}\alpha \int_a^{a+t} \frac{\partial_x^2 \partial_y \widetilde{J}_s(\xi(s))}{\partial_y \widetilde{J}_s(\xi(s))}\mathrm{d}s - \alpha \int_a^{a+t} \frac{\partial_t \partial_y \widetilde{J}_s(\xi(s))}{\partial_y \widetilde{J}_s(\xi(s))}\mathrm{d}s\Bigg). \tag{5.15}$$

Let $\xi_a(t)=\xi(a+t)-\xi(a)$ and $B_a(t)=\xi_a(t)/\sqrt{\kappa}$, $t\in[0,\infty)$. Then $B_a(t)$ is an $(\mathcal{F}_t^a)$-Brownian motion. Using Itô's formula and the argument in Section 3.3 or Section 3.4, we conclude that $\partial_y\widetilde{J}_{a+t}(\xi(a+t))$, $0\le t<T_a$, satisfies the $(\mathcal{F}_t^a)$-adapted SDE:

$$\begin{aligned}\mathrm{d}\partial_y\widetilde{J}_{a+t}(\xi(a+t)) = {} & \partial_x\partial_y\widetilde{J}_{a+t}(\xi(a+t))\mathrm{d}\xi_a(t)+\frac{\kappa}{2}\partial_x^2\partial_y\widetilde{J}_{a+t}(\xi(a+t))\mathrm{d}t\\ & +\partial_t\partial_y\widetilde{J}_{a+t}(\xi(a+t))\mathrm{d}t.\end{aligned}$$

From Itô's formula, this then implies that

$$\begin{aligned}\frac{\mathrm{d}\partial_y\widetilde{J}_{a+t}(\xi(a+t))^\alpha}{\mathrm{d}\partial_y\widetilde{J}_{a+t}(\xi(a+t))} &= \alpha\frac{\partial_x\partial_y\widetilde{J}_{a+t}(\xi(a+t))}{\partial_y\widetilde{J}_{a+t}(\xi(a+t))}\mathrm{d}\xi_a(t)+\frac{\kappa}{2}\alpha\frac{\partial_x^2\partial_y\widetilde{J}_{a+t}(\xi(a+t))}{\partial_y\widetilde{J}_{a+t}(\xi(a+t))}\mathrm{d}t\\ &\quad+\alpha\frac{\partial_t\partial_y\widetilde{J}_{a+t}(\xi(a+t))}{\partial_y\widetilde{J}_{a+t}(\xi(a+t))}\mathrm{d}t+\frac{\kappa}{2}\alpha(\alpha-1)\left(\frac{\partial_x\partial_y\widetilde{J}_{a+t}(\xi(a+t))}{\partial_y\widetilde{J}_{a+t}(\xi(a+t))}\right)^2\mathrm{d}t.\end{aligned}$$

So from (5.15) we see that $M_a(t)$, $0\le t<T_a$, is a local martingale, and

$$\frac{\mathrm{d}M_a(t)}{M_a(t)}=\alpha\frac{\partial_x\partial_y\widetilde{J}_{a+t}(\xi(a+t))}{\partial_y\widetilde{J}_{a+t}(\xi(a+t))}\mathrm{d}\xi_a(t)=\lambda X(a+t)\frac{\mathrm{d}B_a(t)}{\sqrt{\kappa}}. \tag{5.16}$$

From Lemma 5.1 and that $\lim_{t\to-\infty}M(t)=1$ we conclude that for any ρ as in Lemma 5.1, we have $M(t)$, $-\infty<t\le\tau_\rho$, is a bounded martingale, and so $\mathbf{E}_\mu[M(\tau_\rho)]=M(-\infty)=1$.

Define ν by $\mathrm{d}\nu=M(\tau_\rho)\mathrm{d}\mu$. Then ν is also a probability measure. Now suppose that the distribution of $(\xi(t))$ is ν instead of μ. For $-\infty<t<T$, let

$$\eta(t)=\xi(t)-\lambda\int_{-\infty}^t X^\xi(s)\mathrm{d}s. \tag{5.17}$$

From Proposition 3.1, we see that $\eta(t)$, $-\infty<t<T$, are well defined. Moreover, it is clear that $(\mathrm{e}^{\mathrm{i}\eta(t)})$ is $(\mathcal{F}_t)$-adapted. Fix $a\in(-\infty,\ln(R_\rho/4))$. Then we always have $\tau_\rho>a$. Define $\eta_a(t)=\eta(a+t)-\eta(a)$ for $0\le t<T_a$. Then $(\eta_a(t))$ is $(\mathcal{F}_t^a)$-adapted. And we have

$$\eta_a(t)=\sqrt{\kappa}B_a(t)-\lambda\int_a^{a+t}X^\xi(s)\mathrm{d}s.$$

From (5.16) and Girsanov's theorem, we conclude that, under the measure ν, $(\eta_a(t)/\sqrt{\kappa}, 0\le t\le\tau_\rho-a)$ is a stopped $(\mathcal{F}_t^a)$-adapted Brownian motion, and so is independent of $\mathrm{e}^{\mathrm{i}\eta(t)}$, $-\infty<t\le a$. Since this hold s for any $a\in(-\infty,\ln(R_\rho/4))$, so $(\mathrm{e}^{\mathrm{i}\eta(t)},-\infty<t\le\tau_\rho)$ has the same distribution as $(\mathrm{e}^{\mathrm{i}B_{\mathbb{R}}^{(\kappa)}(t)})$ stopped at some stopping time. Thus, there is a integer valued random variable $\mathbf{n}$ such that $\eta_*(t):=\eta(t)+2\mathbf{n}\pi$, $-\infty<t\le\tau_\rho$, has the same distribution as $B_{\mathbb{R}}^{(\kappa)}(t)$ stopped at some stopping time. From (5.17) we see that $\xi_*(t)=\xi(t)+2\mathbf{n}\pi$, $-\infty<t\le\tau_\rho$, solves the integral equation

$$\xi_*(t)=\eta_*(t)+\int_{-\infty}^t X^{\xi_*}(s)\mathrm{d}s.$$

Here we use the fact that $X^{\xi_*}=X^\xi$. So the whole-plane Loewner chain driven by $\xi_*(t)$, $-\infty<t\le\tau_\rho$, is a (κ,λ) process stopped on hitting ρ. Thus, a (κ,λ) process stopped on hitting ρ has a distribution that is absolutely continuous w.r.t. the whole-plane SLE_κ process stopped on hitting ρ, and the density function is $M(\tau_\rho)$.

The locally absolutely continuity also holds if the target is not an interior point but a boundary arc or a boundary point. In these cases, the process $M(t)$ is defined by (5.14) with $G(D, z_e; 0)$ replaced by a harmonic measure function or a normalized Poisson kernel function valued at 0. Then $M(\tau_\rho)$ is still the density function between the (κ, λ) process and whole-plane SLE_κ process before hitting ρ.

6. Scaling limits of discrete LERW

6.1. Discrete LERW in grid approximation

Let D be a finitely connected domain that contains 0. For $\delta > 0$, let $\delta\mathbb{Z}^2 = \{(j + \mathrm{i}k)\delta : j, k \in \mathbb{Z}\} \subset \mathbb{C}$. We also view $\delta\mathbb{Z}^2$ as a graph whose vertices are $(j + \mathrm{i}k)\delta$, $j, k \in \mathbb{Z}$, and two vertices are adjacent iff the distance between them is δ. We define a graph $\breve{D}^\delta$ that approximates D in $\delta\mathbb{Z}^2$ as follows. The vertex set $V(\breve{D}^\delta)$ is the union of interior vertex set $V_I(\breve{D}^\delta)$ and boundary vertex set $V_\partial(\breve{D}^\delta)$, where $V_I(\breve{D}^\delta) := \delta\mathbb{Z}^2 \cap D$, and $V_\partial(\breve{D}^\delta)$ is the set of ordered pairs $\langle z_1, z_2\rangle$ such that $z_1 \in V_I(\breve{D}^\delta)$, $z_2 \in \partial D$, and there is $z_3 \in \delta\mathbb{Z}^2$ that is adjacent to z_1 in $\delta\mathbb{Z}^2$, such that $[z_1, z_2) \subset [z_1, z_3) \cap D$. Two vertices w_1 and w_2 in $V(\breve{D}^\delta)$ are adjacent iff either $w_1, w_2 \in V_I(\breve{D}^\delta)$, w_1 and w_2 are adjacent in $\delta\mathbb{Z}^2$, and $[w_1, w_2] \subset D$; or for $j = 1$ or 2, $w_j \in V_I(\breve{D}^\delta)$ and $w_{3-j} = \langle w_j, z_3\rangle \in V_\partial(\breve{D}^\delta)$ for some $z_3 \in \partial D$.

Every interior vertex of $\breve{D}^\delta$ has exactly 4 adjacent vertices, and every boundary vertex $w = \langle z_1, z_2\rangle$ has exactly one adjacent vertex, which is the interior vertex z_1. If $\langle z_1, z_2\rangle$ is a boundary vertex, then it determines a boundary point, which is z_2, and a prime end of D, which is the limit in $\widehat{D}$, the conformal closure of D (cf. [1,12]), as $z \to z_2$ along $[z_1, z_2)$.

Let D^δ be the connected component of $\breve{D}^\delta$ that contains 0. Let $V(D^\delta)$ be the set of vertices of D^δ. Let $V_I(D^\delta) := V(D^\delta) \cap V_I(\breve{D}^\delta)$ and $V_\partial(D^\delta) := V(D^\delta) \cap V_\partial(\breve{D}^\delta)$ be the sets of interior vertices and boundary vertices, respectively, of D^δ.

Fix $z_e \in D\setminus\{0, \infty\}$. Let w_e^δ be an interior vertex of D^δ that is closest to z_e. Then $|w_e^\delta - z_e| < \delta$ if δ is small. Let $(q_\delta(0), \ldots, q_\delta(\chi_\delta))$ be the LERW on D^δ started from 0 conditioned to hit w_e^δ before $V_\partial(D^\delta)$. Such LERW is obtained by the following process. First, run a simple random walk on D^δ from 0, stop it on hitting w_e^δ or $V_\partial(D^\delta)$. Second, condition the stopped walk on the event that it hits w_e^δ instead of $V_\partial(D^\delta)$. Finally, erase the loops on the path of this walk, in the order they are created (cf. [2]). Then the obtained simple lattice path is called the LERW on D^δ started from 0 conditioned to hit w_e^δ before $V_\partial(D^\delta)$. So $q_\delta(0) = 0$ and $q_\delta(\chi_\delta) = w_e^\delta$.

Let $E_{-1} = V_\partial(D^\delta)$, $F = \{w_e^\delta\}$, and $E_k = E_{-1} \cup \{q_j : 0 \le j \le k\}$ for $0 \le k \le \chi_\delta - 1$. For each $0 \le k \le \chi_\delta - 1$, let g_k be defined as in Lemma 2.1 in [12] with $A = F$, $B = E_{k-1}$ and $x = q_\delta(k)$. This means that g_k is a function defined on $V(D^\delta)$, which vanishes on $E_k \setminus \{q_\delta(k)\}$, is discrete harmonic on $V_I(D^\delta) \setminus \{q_\delta(0), \ldots, q_\delta(k)\}$, and $g_k(w_e^\delta) = 1$. The following is a special case of Proposition 2.1 in [12].

Proposition 6.1. *For any $v_0 \in V(D^\delta)$, $(g_k(v_0))$ is a martingale up to the first time that $q_\delta(k) \sim w_e^\delta$ or E_k disconnects v_0 from w_e^δ.*

Define q_δ on $[0, \chi_\delta]$ to be the linear interpolation of $q_\delta(k)$, $0 \le k \le \chi_\delta$. Then q_δ is a simple curve in D that connects 0 and w_e^δ. For $0 \le k \le \chi_\delta - 1$, let $D_k = D \setminus q_\delta([0, k])$. When δ is small, the function g_k approximates the generalized Poisson kernel P_k in D_k with the pole at $q_\delta(k)$, normalized by $P_k(z_e) = 1$. Note the resemblance of the discrete martingales preserved by this (discrete) LERW given by Proposition 6.1 and the local martingales preserved by the continuous

LERW given by Theorem 3.4. Suppose $\gamma_0(t)$, $-\infty < t < T_0$, is an LERW$(D; 0 \to z_e)$ curve. We will prove the following theorem about the convergence.

Theorem 6.1. *For any $\varepsilon > 0$, there is $\delta_0 > 0$ such that, if $\delta < \delta_0$, then there are a coupling of q_δ and γ_0, and a continuous increasing function $\widetilde{u}$ that maps $(0, \chi_\delta)$ onto $(-\infty, T_0)$ such that*

$$\mathbf{P}[\sup\{|q_\delta(\widetilde{u}^{-1}(t)) - \gamma_0(t)| : -\infty < t < T_0\} < \varepsilon] > 1 - \varepsilon.$$

6.2. Some estimates

For a non-degenerate interior hull $K \subset D \setminus \{z_e\}$, let ϕ_K, φ_K, and ψ_K be as in the last subsection. So if $K = K_t^\xi$ is a whole-plane Loewner hull at time t driven by some $\xi \in C((-\infty, T))$ with $T > t$, then $\varphi_{K_t^\xi}$ and $\psi_{K_t^\xi}$ agree with the whole-plane Loewner map and inverted whole-plane Loewner map: φ_t^ξ and ψ_t^ξ, respectively. Let $\Omega_K = R_{\mathbb{T}} \circ \varphi_K(D \setminus K)$, $\widetilde{\Omega}_k = (\mathrm{e}^{\mathrm{i}})^{-1}(\Omega_k)$, $p_K = R_{\mathbb{T}} \circ \varphi_K(z_e)$, and $\widetilde{p}_K = (\mathrm{e}^{\mathrm{i}})^{-1}(p_K)$. So Ω_k is a subdomain of $\mathbb{D}$ containing p_K, and $\widetilde{\Omega}_K$ is a periodic subdomain of $\mathbb{H}$. If $K = K_t^\xi$, then $\Omega_{K_t^\xi}$, $\widetilde{\Omega}_{K_t^\xi}$, $p_{K_t^\xi}$, and $\widetilde{p}_{K_t^\xi}$ agree with Ω_t^ξ, $\widetilde{\Omega}_t^\xi$, p_t^ξ, and $\widetilde{p}_t^\xi$, respectively, defined in Section 3.2.

Let α be a Jordan curve in $\mathbb{C}$ such that $0 \in U(\alpha)$ and $H(\alpha) \subset D \setminus \{z_e\}$. Let F be a compact subset of $D \setminus (H(\alpha) \cup \{\infty\})$. Fix $b \in \mathbb{R}$. Throughout this subsection, a constant is called uniform if it depends only on D, z_e, α, F, b. We will frequently apply Lemma 2.3 to $\mathcal{H}^b(\alpha)$ to obtain some uniform constants. We illustrate the idea in the following example. Note that for every $H \in \mathcal{H}^b(\alpha)$, $\varphi_H(F)$ is a compact subset of $\{|z| > 1\}$, so there is $r_H > 0$ such that $|\varphi_H(z)| \ge \mathrm{e}^{r_H}$ for every $z \in F$. From Lemma 2.3, there is a uniform constant $\mathbf{h} > 0$ such that $|\varphi_H(z)| \ge \mathrm{e}^{\mathbf{h}}$ for any $H \in \mathcal{H}^b(\alpha)$ and $z \in F$. Let $F_R = R_{\mathbb{T}}(F)$. Then $|\psi_H(z)| \le \mathrm{e}^{-\mathbf{h}}$ for any $H \in \mathcal{H}^b(\alpha)$ and $z \in F_R$. Suppose $K_a^\xi \subset H(\alpha)$. Then for any $t \in [b, a]$, we have $K_t^\xi \in H^b(\alpha)$, so $|\psi_t^\xi(z)| \le \mathrm{e}^{-\mathbf{h}}$ for any $z \in F_R$. Let $\widetilde{F}_R = (\mathrm{e}^{\mathrm{i}})^{-1}(F_R)$. Since $\psi_t^\xi \circ \mathrm{e}^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}} \circ \widetilde{\psi}_t^\xi$, so $\mathrm{Im}\, \widetilde{\psi}_t^\xi(z) \ge \mathbf{h}$ for any $z \in \widetilde{F}_R$ and $t \in [b, a]$.

The following lemmas are similar to the lemmas in Section 6.1 of [12].

Lemma 6.1. *There are uniform constants $C_1, C_2 > 0$ such that if $K_a^\xi \subset H(\alpha)$, then for any $t_1 \le t_2 \in [b, a]$ and $z \in \widetilde{F}_R$,*

$$|\widetilde{\psi}_{t_2}^\xi(z) - \widetilde{\psi}_{t_1}^\xi(z)| \le C_1|t_2 - t_1|;$$

$$|\widetilde{\psi}_{t_2}^\xi(z) - \widetilde{\psi}_{t_1}^\xi(z) - (t_2 - t_1)\cot_2(\widetilde{\psi}_{t_1}^\xi(z) - \xi(t_1))| \le C_2|t_2 - t_1|(|t_2 - t_1| + \sup_{t \in [t_1, t_2]}\{|\xi(t) - \xi(t_1)|\}).$$

Proof. Suppose $K_a^\xi \subset H(\alpha)$. Then for any $t \in [b, a]$ and $z \in \widetilde{F}_R$, we have $\mathrm{Im}\, \widetilde{\psi}_t^\xi(z) \ge \mathbf{h}$, which implies that $|\cot_2(\widetilde{\psi}_t^\xi(z) - \xi(t))| \le \coth_2(\mathbf{h})$. Since $\varphi_{t_2}^\xi(z) - \varphi_{t_1}^\xi(z) = \int_{t_1}^{t_2} \cot_2(\varphi_t^\xi(z) - \xi(t))\mathrm{d}t$, so $|\widetilde{\psi}_{t_2}^\xi(z) - \widetilde{\psi}_{t_1}^\xi(z)| \le C_1|t_2 - t_1|$ for any $t_1 \le t_2 \in [b, a]$ and $z \in \widetilde{F}_R$, where $C_1 = \coth_2(\mathbf{h}) > 0$. Since $|\cot_2'(w)| \le \frac{1}{2}\sinh_2^{-2}(\mathrm{Im}\, w) \le \frac{1}{2}\sinh_2^{-2}(\mathbf{h})$ for $w \in \mathbb{C}$ with $\mathrm{Im}\, w \ge \mathbf{h}$, and $C_1 \ge 1$, so for $t_1 \le t_2 \in [b, a]$ and $z \in \widetilde{F}_R$,

$$|\cot_2(\widetilde{\psi}_{t_2}^\xi(z) - \xi(t_2)) - \cot_2(\widetilde{\psi}_{t_1}^\xi(z) - \xi(t_1))|$$
$$\le \frac{1}{2}\sinh_2^{-2}(\mathbf{h})(|\varphi_{t_2}^\xi(z) - \varphi_{t_1}^\xi(z)| + |\xi(t_2) - \xi(t_1)|)$$

$$\leq \frac{C_1}{2} \sinh_2^{-2}(\mathbf{h})(|t_2 - t_1| + |\xi(t_2) - \xi(t_1)|).$$

Let $C_2 := \frac{C_1}{2} \sinh_2^{-2}(\mathbf{h}) > 0$. Then for $t_1 \leq t_2 \in [b, a]$ and $z \in \widetilde{F}_R$, we have

$$|\varphi^\xi_{t_2}(z) - \varphi^\xi_{t_1}(z) - (t_2 - t_1)\cot_2(\widetilde{\psi}^\xi_{t_1}(z) - \xi(t_1))|$$
$$= \left| \int_{t_1}^{t_2} \cot_2(\varphi^\xi_t(z) - \xi(t)) - \cot_2(\varphi^\xi_{t_1}(z) - \xi(t_1))\,\mathrm{d}t \right|$$
$$\leq C_2|t_2 - t_1|(|t_2 - t_1| + \sup_{t\in[t_1,t_2]}\{|\xi(t) - \xi(t_1)|\}). \quad \square$$

For $x \in \mathbb{R}$, let $P(K, x, \cdot)$ be the generalized Poisson kernel in Ω_K with the pole at e^{ix}, normalized by $P(K, x, p_K) = 1$. Let $\widetilde{P}(K, x, \cdot) = P(K, x, \cdot) \circ e^{i}$. If $K = K^\xi_t$, then $P(K^\xi_t, x, \cdot)$, and $\widetilde{P}(K^\xi_t, x, \cdot)$ agree with $P^\xi(t, x, \cdot)$ and $\widetilde{P}^\xi(t, x, \cdot)$, respectively, defined in Section 3.4.

Lemma 6.2. *For each $n_1 \in \{0, 1\}$, $n_2, n_3 \in \mathbb{Z}_{\geq 0}$, there is a uniform constant $C > 0$ depending on n_1, n_2, n_3, such that if $K^\xi_a \subset H(\alpha)$, then for any $t \in [b, a]$, $x \in \mathbb{R}$, and $z \in \widetilde{F}_R$, we have*

$$|\partial_1^{n_1}\partial_2^{n_2}\partial_{3,z}^{n_3}\widetilde{P}^\xi(t, x, \widetilde{\psi}^\xi_t(z))| \leq C.$$

Proof. The case $n_1 = 0$ follows from Lemma 2.3 immediately because $K^\xi_t \in \mathcal{H}^b(\alpha)$ for $t \in [b, a]$, and if (H_n) is a sequence in $\mathcal{H}^b(\alpha)$, and $H_n \xrightarrow{\mathcal{H}} H$, then

$$\partial_2^{n_2}\partial_{3,z}^{n_3}\widetilde{P}(H_n, x, \widetilde{\psi}_{H_n}(z)) \to \partial_2^{n_2}\partial_{3,z}^{n_3}\widetilde{P}(H, x, \widetilde{\psi}_H(z))$$

uniformly in $x \in \mathbb{R}$ and $z \in \widetilde{F}_R$, for any $n_2, n_3 \in \mathbb{Z}_{\geq 0}$.

Now we consider the case $n_1 = 1$. First suppose that ∂D is analytic, i.e., ∂D is the disjoint union of analytic Jordan curves. Let $K \in \mathcal{H}^b(\alpha)$. Then $\Omega_K = R_{\mathbb{T}} \circ \varphi_K(D \setminus K)$ also have analytic boundary. Since $P(K, x, \cdot)$ vanishes on $\partial\Omega_K$ except at e^{ix}, so $P(K, x, \cdot)$ extends harmonically across $\partial\Omega_k \setminus \{e^{ix}\}$. Thus, $\widetilde{P}(K, x, \cdot)$ extends harmonically across $\partial\widetilde{\Omega}_K \setminus \{x + 2n\pi : n \in \mathbb{Z}\}$. For $x, y \in \mathbb{R}$, let $Q_y(K, x, \cdot)$ be a continuous function on $\overline{\Omega_K} \setminus \{e^{ix}\}$ such that $Q_y(K, x, \cdot)$ is harmonic in Ω_K; vanishes on $\mathbb{T} \setminus \{x\}$; behaves like $c\,\mathrm{Re}\,\frac{e^{ix}+z}{e^{ix}-z} + O(1)$ near e^{ix} for some $c \in \mathbb{R}$; and

$$Q_y(K, x, z) = -2\,\mathrm{Re}\left(\partial_{3,z}P(K, x, z)z\frac{e^{iy} + z}{e^{iy} - z}\right), \quad z \in (\partial\Omega_K \setminus \mathbb{T}) \cup \varphi_K(p).$$

Such $Q_y(K, x, \cdot)$ exists uniquely. Let $\widetilde{Q}_y(K, x, \cdot) = Q_y(K, x, \cdot) \circ e^{i}$. From (2.13) and the values of $P^\xi(t, x, \cdot)$ at $\partial\Omega^\xi_t \setminus \mathbb{T} = \psi^\xi_t(\partial\Omega)$ and $p^\xi_t = \psi^\xi_t(p)$, it is easy to check that $\partial_1 P^\xi(t, x, z) = Q_{\xi(t)}(K^\xi_t, x, z)$, and so $\partial_1\widetilde{P}^\xi(t, x, z) = \widetilde{Q}_{\xi(t)}(K^\xi_t, x, z)$. Using Lemma 2.3, we can conclude that for any $n_2, n_3 \in \mathbb{Z}_{\geq 0}$, $\partial_2^{n_2}\partial_{3,z}^{n_3}\widetilde{Q}_y(K, x, \psi_K(z))$ is uniformly bounded in $x, y \in \mathbb{R}$ and $z \in \widetilde{F}_R$. So the proof in the case that $n_1 = 1$ and ∂D is analytic is finished.

Now we consider the case that $n_1 = 1$ but ∂D may not be analytic. We may find V that maps D conformally onto D_0 with analytic boundary, such that $V(0) = 0$. Moreover, suppose $F_0 := V(F)$, $\alpha_0 := V(\alpha)$, and $V(H(\alpha))$ do not contain ∞, and $z_0 := V(z_e) \neq \infty$. Then α_0 is a Jordan curve in $\mathbb{C}$ such that $0 \in U(\alpha_0)$, $H(\alpha_0) = V(H(\alpha)) \subset D_0 \setminus \{z_0\}$, and F_0 is a compact subset of $D_0 \setminus (H(\alpha_0) \cup \{\infty\})$. Let $W = R_{\mathbb{T}} \circ V \circ W$, $\Omega_0 = R_{\mathbb{T}}(D_0)$, and $p_0 = R_{\mathbb{T}}(z_0)$. Then W maps Ω conformally onto Ω_0. Let $\widetilde{\Omega}_0 = (e^{i})^{-1}(\Omega_0)$. Choose $\widetilde{W}$ that maps $\widetilde{\Omega}$ conformally onto

$\widetilde{\Omega}_0$ such that $W \circ e^i = e^i \circ \widetilde{W}$. There is $b_0 \in \mathbb{R}$ such that if H is an interior hull in D with $0 \in H$ and $\mathrm{cap}(H) \geq b$, then $\mathrm{cap}(V(H)) \geq b_0$.

Suppose $K_a^\xi \subset \mathcal{H}(\alpha)$. Using the argument in Section 3.3, we conclude that there are $a_0 \in \mathbb{R}$, $\xi_0 \in C((-\infty, a_0])$, and a continuous increasing function u that maps $(-\infty, a]$ onto $(-\infty, a_0]$ such that $V(K_t^\xi) = K_{u(t)}^{\xi_0}$ for $-\infty < t \leq a$. Then we have $K_s^{\xi_0} \in \mathcal{H}(\alpha_0)$ for $-\infty < s \leq a_0$, and $u(b) \geq b_0$. Let

$$W_t = \psi_{u(t)}^{\xi_0} \circ W \circ (\psi_t^\xi)^{-1}, \qquad \widetilde{W}_t = \widetilde{\psi}_{u(t)}^{\xi_0} \circ \widetilde{W} \circ (\widetilde{\psi}_t^\xi)^{-1}, \quad -\infty < t \leq a. \tag{6.1}$$

Using the argument in Section 3.3, we can conclude that $W_t \circ e^i = e^i \circ \widetilde{W}_t$, $u'(t) = \widetilde{W}_t'(\xi(t))^2$, $\xi_0(u(t)) = \widetilde{W}_t(\xi(t))$, and for any $w \in \widetilde{\Omega}_t$,

$$\partial_t \widetilde{W}_t(w) = \widetilde{W}_t'(\xi(t))^2 \cot_2(\widetilde{W}_t(w) - \widetilde{W}_t(\xi(t))) - \widetilde{W}_t'(w) \cot_2(w - \xi(t)).$$

So we have that for any $z \in \Omega_t$,

$$\partial_t W_t(z) = |W_t'(e^{i\xi(t)})|^2 W_t(z) \frac{W_t(e^{i\xi(t)}) + W_t(z)}{W_t(e^{i\xi(t)}) - W_t(z)} - W_t'(z) z \frac{e^{i\xi(t)} + z}{e^{i\xi(t)} - z}. \tag{6.2}$$

For $-\infty < t \leq a_0$ and $x \in \mathbb{R}$, let $P_0^{\xi_0}(t, x, \cdot)$ be the generalized Poisson kernel in $\psi_t^{\xi_0}(\Omega_0 \setminus L_t^{\xi_0})$ with the pole at e^{ix}, normalized by $P_0^{\xi_0}(t, x, \psi_t^{\xi_0}(p_0)) = 1$; and let $\widetilde{P}_0^{\xi_0}(t, x, \cdot) = P_0^{\xi_0}(t, x, \cdot) \circ e^i$. Then we have

$$\widetilde{P}^\xi(t, x, z) = \widetilde{P}_0^{\xi_0}(u(t), \widetilde{W}_t(x), \widetilde{W}_t(z)), \quad -\infty < t \leq a. \tag{6.3}$$

Let $\widetilde{F}_{0,R} = (e^i)^{-1}(R_{\mathbb{T}}(F_0))$. Since D_0 has analytic boundary, so for any $n_1 \in \{0, 1\}$, $n_2, n_3 \in \mathbb{Z}_{\geq 0}$, there is a uniform constant C depending on n_1, n_2, and n_3 such that for $t \in [b_0, a_0]$ and $z \in \widetilde{F}_{0,R}$,

$$|\partial_1^{n_1} \partial_2^{n_2} \partial_{3,z}^{n_3} \widetilde{P}_0^{\xi_0}(t, x, \widetilde{\psi}_t^{\xi_0}(z))| \leq C.$$

From (6.3) and that $u([b, a]) \subset [b_0, a_0]$ and $u'(t) = \widetilde{W}_t'(\xi(t))^2$, we suffice to prove that for any $n_1 \in \{0, 1\}$ and $n_2 \in \mathbb{Z}_{\geq 0}$ with $n_1 + n_2 \geq 1$, there is a uniform constant C depending on n_1 and n_2 such that $|\partial_t^{n_1} \partial_z^{n_2} \widetilde{W}_t(z)| \leq C$ for any $t \in [b, a]$ and $z \in \mathbb{R} \cup \widetilde{\psi}_t^\xi(\widetilde{F}_R)$. Since $e^i \circ \widetilde{W}_t = W_t \circ e^i$, so we suffice to prove that there is a uniform constant $\delta_0 > 0$ such that $|W_t(z)| > \delta_0$ for $t \in [b, a]$ and $z \in \mathbb{T} \cup \psi_t^\xi(F_R)$; and for any $n_1 \in \{0, 1\}$ and $n_2 \in \mathbb{Z}_{\geq 0}$ with $n_1 + n_2 \geq 1$, there is a uniform constant C_0 depending on n_1 and n_2 such that $|\partial_t^{n_1} \partial_z^{n_2} W_t(z)| \leq C_0$ for any $t \in [b, a]$ and $z \in \mathbb{T} \cup \psi_t^\xi(F_R)$.

For the existence of δ_0, we consider two cases. The first case is $z \in \mathbb{T}$. This is trivial because $|W_t(z)| = 1$ on $\mathbb{T}$. The second case is $z \in \psi_t^\xi(F_R)$. From (6.1) and that $\psi_t^{\xi_0} = R_{\mathbb{T}} \circ \varphi_t^\xi \circ R_{\mathbb{T}}$, the inequality in this case is equivalent to that $|\varphi_{u(t)}^{\xi_0}(z)| \leq 1/\delta_0$ for any $t \in [a, b]$ and $z \in F_0$. This can be proved by applying Lemma 2.3 to $\mathcal{H}^{b_0}(\alpha_0)$ and using the facts that $\varphi_{u(t)}^{\xi_0} = \varphi_{K_{u(t)}^{\xi_0}}$, $K_{u(t)}^{\xi_0} \in \mathcal{H}^{b_0}(\alpha_0)$ for $t \in [b, a]$, and $\infty \notin \varphi_H(F_0)$ for every $H \in \mathcal{H}^{b_0}(\alpha_0)$.

Next we consider the existence of C_0. We first consider the case $n_1 = 0$. For any $n_2 \in \mathbb{Z}_{\geq 0}$, the uniform boundedness of $\partial_z^{n_2} W_t(z)$ on $\psi_t^\xi(F_R)$ follows immediately from Lemma 2.3 applied to $\mathcal{H}^b(\alpha)$ and $\mathcal{H}^{b_0}(\alpha_0)$. Using Lemma 2.3 we may also obtain uniform numbers $r \in (0, 1)$ and $M \in (0, \infty)$ such that for $t \in [b, a]$, we have $\{r \leq |z| < 1\} \subset \Omega_t^\xi$, and $|W_t(z)| \leq M$ on $\{|z| = r\}$. Then the uniform boundedness of $\partial_z^{n_2} W_t(z)$ on $\mathbb{T}$ follows from Cauchy's integral formula. A similar argument together with (6.2) proves the case $n_1 = 1$. The two fractions in (6.2) do not cause any problem because they are uniformly bounded as long as z and $W_t(z)$ are uniformly bounded away from $\mathbb{T}$, which are true for $z \in \psi_t^\xi(F_R)$ and $z \in \{|z| = r\}$. $\square$

Lemma 6.3. *There is a uniform constant $C > 0$ such that if $K_a^\xi \subset H(\alpha)$, then for any $t, t' \in [b, a]$, $|X^\xi(t)| \le C$ and $|X^\xi(t) - X^\xi(t')| \le C(|t - t'| + |\xi(t) - \xi(t')|)$.*

Proof. Suppose $K_a^\xi \subset H(\alpha)$. Write $\widetilde{J}^\xi(t, x)$ for $\widetilde{J}_t^\xi(x)$. Note that $X^\xi(t) = (\partial_{2,z}^2/\partial_{2,z})\widetilde{J}^\xi(t, \xi(t))$. So it suffices to prove that there is a uniform constant $C > 0$ such that for any $t \in [b, a]$ and $x \in \mathbb{R}$, $|\partial_1^{n_1}\partial_{2,z}^{n_2}(\partial_{2,z}^2/\partial_{2,z})\widetilde{J}^\xi(t, x)| \le C$ for $n_1, n_2 \in \{0, 1\}$. We need to show that $|\partial_{2,z}\widetilde{J}^\xi(t, x)|$ is bounded from below by a positive uniform constant, and $|\partial_1^{n_1}\partial_{2,z}^{n_2+1}\widetilde{J}^\xi(t, x)|$ is bounded from above by a positive uniform constant. The proof is similar to that of the above lemma. $\square$

Lemma 6.4. *There is a uniform constant $C > 0$ such that if $K_a^\xi \subset H(\alpha)$, then for any $t_1 \le t_2 \in [b, a]$ and $z \in \widetilde{F}_R$, we have*

$$|\partial_1\widetilde{P}^\xi(t_2, \xi(t_2), \widetilde{\psi}_{t_2}^\xi(z)) - \partial_1 P^\xi(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^\xi(z))| \le C(|t_2 - t_1| + |\xi(t_2) - \xi(t_1)|).$$

Proof. This follows from Lemma 3.1, and the above three lemmas. $\square$

Lemma 6.5. *There is a uniform constant $d_1 > 0$ such that, if $K_a^\xi \subset H(\alpha)$, then for any $z \in \widetilde{F}_R$, and any $t_1 < t_2 \in [b, a]$ that satisfy $|t_2 - t_1| \le d_1$, we have*

$$\begin{aligned}
&\widetilde{P}^\xi(t_2, \xi(t_2), \widetilde{\psi}_{t_2}^\xi(z)) - \widetilde{P}^\xi(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^\xi(z)) \\
&\quad = \partial_2\widetilde{P}^\xi(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^\xi(z)) \cdot [(\xi(t_2) - \xi(t_1)) - (t_2 - t_1)X_{t_1}^\xi] \\
&\qquad + \frac{1}{2}\partial_2^2\widetilde{P}^\xi(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^\xi(z)) \cdot [(\xi(t_2) - \xi(t_1))^2 - 2(t_2 - t_1)] \\
&\qquad + O(A^2) + O(AB) + O(AB^2) + O(B^3),
\end{aligned}$$

where $A := |t_2 - t_1|$, $B := \sup_{s,t\in[t_1,t_2]}\{|\xi(s) - \xi(t)|\}$, and $O(X)$ is some number whose absolute value is bounded by $C|X|$ for some uniform constant $C > 0$.

Proof. We may choose a compact subset F' of $\mathbb{D} \setminus H(\rho)$ such that F is contained in the interior of F'. Let $F'_R = R_{\mathbb{T}}(F)$ and $\widetilde{F}'_R = (e^i)^{-1}(F'_R)$. So F_R and $\widetilde{F}_R$ are contained in the interiors of F'_R and $\widetilde{F}'_R$, respectively. Applying Lemma 2.3 to $\mathcal{H}^b(\alpha)$, we obtain a uniform constant $d_0 > 0$ such that for any $K \in \mathcal{H}^b(\alpha)$, we have $\operatorname{dist}(\psi_K(F_R), \partial\psi_K(F'_R)) \ge d_0$. So there is a uniform constant $\widetilde{d}_0$ such that $\operatorname{dist}(\widetilde{\psi}_K(\widetilde{F}_R), \partial\widetilde{\psi}_K(\widetilde{F}'_R)) \ge \widetilde{d}_0$ for any $K \in \mathcal{H}^b(\alpha)$. Suppose $K_a^\xi \subset H(\alpha)$. From Lemma 6.1 and the existence of $\widetilde{d}_0$, we get a uniform constant $d_1 > 0$ such that if $s, t \in [b, a]$ satisfy $|s - t| \le d_1$ then for any $z \in \widetilde{F}_R$, $[\widetilde{\psi}_s^\xi(z), \widetilde{\psi}_t^\xi(z)] \subset \widetilde{\psi}_s^\xi(\widetilde{F}'_R)$.

Fix $z \in \widetilde{F}_R$ and $t_1 < t_2 \in [0, a]$ with $|t_2 - t_1| \le d_1$. Let $P_1 = \widetilde{P}^\xi(t_2, \xi(t_2), \widetilde{\psi}_{t_2}^\xi(z))$, $P_2 = \widetilde{P}^\xi(t_1, \xi(t_2), \widetilde{\psi}_{t_2}^\xi(z))$, $P_3 = \widetilde{P}^\xi(t_1, \xi(t_1), \widetilde{\psi}_{t_2}^\xi(z))$, $P_4 = \widetilde{P}^\xi(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^\xi(z))$. Then

$$\widetilde{P}^\xi(t_2, \xi(t_2), \widetilde{\psi}_{t_2}^\xi(z)) - \widetilde{P}^\xi(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^\xi(z)) = (P_1 - P_2) + (P_2 - P_3) + (P_3 - P_4). \quad (6.4)$$

Now $P_1 - P_2 = \int_{t_1}^{t_2}\partial_1\widetilde{P}^\xi(t, \xi(t_2), \widetilde{\psi}_{t_2}^\xi(z))\mathrm{d}t$. Fix any $t \in [t_1, t_2]$. Applying Lemmas 6.1 and 6.2 to $\widetilde{F}'_R$ and using $[\widetilde{\psi}_t^\xi(z), \widetilde{\psi}_{t_2}^\xi(z)] \subset \widetilde{\psi}_t^\xi(\widetilde{F}'_R)$, we have

$$\partial_1\widetilde{P}^\xi(t, \xi(t_2), \widetilde{\psi}_{t_2}^\xi(z)) - \partial_1\widetilde{P}^\xi(t, \xi(t), \widetilde{\psi}_t^\xi(z)) = O(A) + O(B).$$

Applying Lemma 6.4 to $\widetilde{F}_R$, we have

$$\partial_1 \widetilde{P}^{\xi}(t, \xi(t), \widetilde{\psi}_t^{\xi}(z)) - \partial_1 \widetilde{P}^{\xi}(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^{\xi}(z)) = O(A) + O(B).$$

So we get

$$P_1 - P_2 = \partial_1 \widetilde{P}^{\xi}(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^{\xi}(z))(t_2 - t_1) + O(A^2) + O(AB).$$

Applying Lemma 6.2 to $\widetilde{F}'_R$, since $\widetilde{\psi}_{t_2}^{\xi}(z) \in \widetilde{\psi}_{t_1}^{\xi}(\widetilde{F}'_R)$, so we have

$$\begin{aligned} P_2 - P_3 &= \partial_2 \widetilde{P}^{\xi}(t_1, \xi(t_1), \widetilde{\psi}_{t_2}^{\xi}(z))(\xi(t_2) - \xi(t_1)) \\ &\quad + \frac{1}{2}\partial_2^2 \widetilde{P}^{\xi}(t_1, \xi(t_1), \widetilde{\psi}_{t_2}^{\xi}(z))(\xi(t_2) - \xi(t_1))^2 + O(B^3). \end{aligned}$$

Applying Lemmas 6.1 and 6.2 to $\widetilde{F}'_R$, since $[\widetilde{\psi}_{t_1}^{\xi}(z), \widetilde{\psi}_{t_2}^{\xi}(z)] \subset \widetilde{\psi}_{t_1}^{\xi}(\widetilde{F}'_R)$, so we have

$$\partial_2^j \widetilde{P}^{\xi}(t_1, \xi(t_1), \widetilde{\psi}_{t_2}^{\xi}(z)) - \partial_2^j \widetilde{P}^{\xi}(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^{\xi}(z)) = O(A),$$

for $j = 1, 2$. Thus

$$\begin{aligned} P_2 - P_3 &= \partial_2 \widetilde{P}^{\xi}(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^{\xi}(z))(\xi(t_2) - \xi(t_1)) \\ &\quad + \frac{1}{2}\partial_2^2 \widetilde{P}^{\xi}(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^{\xi}(z))(\xi(t_2) - \xi(t_1))^2 + O(AB) + O(AB^2) + O(B^3). \end{aligned}$$

Applying Lemmas 6.1 and 6.2 to $\widetilde{F}'_R$, since $[\widetilde{\psi}_{t_1}^{\xi}(z), \widetilde{\psi}_{t_2}^{\xi}(z)] \subset \widetilde{\psi}_{t_1}^{\xi}(\widetilde{F}'_R)$, so we have

$$\begin{aligned} P_3 - P_4 &= 2\,\mathrm{Re}(\partial_{3,z} \widetilde{P}^{\xi}(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^{\xi}(z))(\widetilde{\psi}_{t_2}^{\xi}(z) - \widetilde{\psi}_{t_1}^{\xi}(z))) + O(A^2) \\ &= 2\,\mathrm{Re}(\partial_{3,z} \widetilde{P}^{\xi}(t_1, \xi(t_1), \widetilde{\psi}_{t_1}^{\xi}(z))(t_2 - t_1)\cot_2(\widetilde{\psi}_{t_1}^{\xi}(z) - \xi(t_1))) \\ &\quad + O(AB) + O(A^2). \end{aligned}$$

The conclusion then follows from (6.4) and Lemma 3.1. □

6.3. Convergence of driving functions

We may choose mutually disjoint Jordan curves α_j, $j = 0, 1, 2$, in $\mathbb{C}$ such that $0 \in U(\alpha_0) \subset U(\alpha_1) \subset U(\alpha_2)$ and $H(\alpha_2) \subset D \setminus \{z_e\}$. Fix $b \in \mathbb{R}$ such that $b < \ln(d_0/4) - 1$, where $d_0 = \mathrm{dist}(0, \alpha_0)$. So any $H \in \mathcal{H}_0$ with $\mathrm{cap}(H) \le b$ must satisfy $H \subset U(\alpha_0)$. Let F be a compact subset of $D \setminus H(\alpha_2)$ whose interior is not empty. From now on, a uniform constant is a number that depends only on $D, z_e, \alpha_0, \alpha_1, \alpha_2, F, b$, and some other variables we will specify. Let $O(X)$ denote some number whose absolute value is bounded by $C|X|$ for some uniform constant $C > 0$.

Let L^{δ} denote the set of simple lattice paths $X = (X(0), \ldots, X(s))$, $s \in \mathbb{N}$, on D^{δ}, such that $X(0) = 0$, $X(k) \in D$ for $0 \le k \le s$, and $\bigcup_{k=0}^{s}(X(k-1), X(k)] \subset H(\alpha_1)$. Let $\mathrm{Set}(X) = \{X(0), \ldots, X(s)\}$, $\mathrm{Tip}(X) = X(s)$, $H_X = \bigcup_{k=1}^{s}[X(k-1), X(k)]$, and $D_X = D \setminus H_X$. Let P_X be the generalized Poisson kernel in D_X with the pole at $\mathrm{Tip}(X)$, normalized by $P_X(z_e) = 1$, and g_X be defined on $V(D^{\delta})$ such that $g_X \equiv 0$ on $V_{\partial}(D^{\delta}) \cup \mathrm{Set}(X) \setminus \{\mathrm{Tip}(X)\}$, $\Delta_{D^{\delta}} g_X \equiv 0$ on $V_I(D^{\delta}) \setminus \mathrm{Set}(X)$, and $g_X(w_e^{\delta}) = 1$. Let L_b^{δ} be the set of $X \in L^{\delta}$ such that $\mathrm{cap}(H_X) \ge b$. Then we have the following proposition about the convergence of g_X to P_X.

Proposition 6.2. *For any $\varepsilon > 0$, there is a uniform constant $\delta_0 > 0$ depending on ε such that, if $0 < \delta < \delta_0$, then for any $X \in L_b^\delta$, and any $w \in V(D^\delta) \cap (D \setminus H(\alpha_2))$, we have $|g_X(w) - P_X(w)| < \varepsilon$.*

Sketch of the Proof. This proposition is similar to Proposition 6.1 in [12]. So we only give a sketch of the proof. Suppose that the proposition is not true. Then there are $\varepsilon_0 > 0$, a sequence $\delta_n \to 0^+$, a sequence of lattice paths $X_n \in L_b^{\delta_n}$, and a sequence of lattice points $w_n \in V(D^\delta) \cap (D \setminus H(\alpha_2))$, such that $|g_{X_n}(w_n) - P_{X_n}(w_n)| \geq \varepsilon_0$. By passing to a subsequence, one may assume that $w_n \to w_0$, and $D_{X_n} \stackrel{\text{Cara}}{\longrightarrow} D_0$. Then P_{X_n} tends to a generalized Poisson kernel function in D_0. Using linear interpolation to extend each g_{X_n} to a continuous function defined in the unions of lattice squares inside D_{X_n}. Since each g_{X_n} is a positive harmonic function, so from Harnack's inequality, we can conclude that the extended $\{g_{X_n}\}$ is uniform Lipschitz on any compact subset of D_0. Applying Arzelà–Ascoli theorem, by passing to a subsequence, we conclude that $g_{X_n} \to g_0$ locally uniformly in D_0. Then one can check that g_0 is a positive harmonic function. With a little more work, one can prove that g_0 is also a generalized Poisson kernel, and in fact, $g_0 = P_0$. So if $w_0 = \lim w_n \in D_0$, we immediately get a contradiction. If $w_0 \notin D_0$, then $w_0 \in \partial D$. From $w_n \to \partial D$ we get $g_{X_n}(w_n) \to 0$ and $P_{X_n}(w_n) \to 0$, which also gives a contradiction. □

Let the LERW curve q_δ on $[0, \chi_\delta]$ be defined as in Section 6.1. For $0 \leq t \leq \chi_\delta$, let $v_\delta(t) = \text{cap}(q_\delta([0,t]))$, and $T_\delta = v_\delta(\chi_\delta)$. Then v_δ is an increasing function, and maps $[0, \chi_\delta]$ onto $[-\infty, T_\delta]$. Let $\beta_\delta(t) = q_\delta(v_\delta^{-1}(t))$, $-\infty \leq t \leq T_\delta$. From Proposition 2.3, there is some $\xi_\delta \in C((-\infty, T_\delta])$ such that $\beta_\delta([-\infty, t]) = K_t^{\xi_\delta}$ for $-\infty < t \leq T_\delta$. Let n_∞ be the first n such that $(q_\delta(n-1), q_\delta(n)]$ intersects α_0. We may choose $\delta < \text{dist}(\alpha_0, \alpha_1)$. Then $q_\delta([0, n_\infty]) \subset U(\alpha_1)$. Let $T_{\alpha_0}^\delta = v_\delta(n_\infty)$. Let n_0 be the first n such that $v_\delta(n) \geq b$. Pick any $d > 0$. Define a sequence (n_j) by the following. For $j \geq 1$, let n_{j+1} be the first $n \geq n_j$ such that $n = n_\infty$, or $v_\delta(n) - v_\delta(n_j) \geq d^2$, or $|\xi_\delta(n) - \xi_\delta(n_j)| \geq d$, whichever comes first. Let $(\mathcal{F}_n)$ be the filtration generated by $(q_\delta(n))$. Let $\mathcal{F}'_j = \mathcal{F}_{n_j}$, $0 \leq j < \infty$. Then we may derive the following proposition, which is similar to Proposition 6.2 in [12]. Since the proofs of these two propositions are almost identical, so we omit the proof here.

Proposition 6.3. *There are a uniform constant $d_0 > 0$ and a uniform constant $\delta_0(d) > 0$ that depends only on d such that, if $d < d_0$ and $\delta < \delta_0(d)$, then for all $j \geq 0$,*

$$\mathbf{E}\left[(\xi_\delta(v_\delta(n_{j+1})) - \xi_\delta(v_\delta(n_j))) - \int_{v_\delta(n_j)}^{v_\delta(n_{j+1})} X_t^{\xi_\delta} \mathrm{d}t | \mathcal{F}'_j \right] = O(d^3);$$

$$\mathbf{E}[(\xi_\delta(v_\delta(n_{j+1})) - \xi_\delta(v_\delta(n_j)))^2 - 2(v_\delta(n_{j+1}) - v_\delta(n_j)) | \mathcal{F}'_j] = O(d^3).$$

Let $\xi_0(t)$, $-\infty < t < T_0$, be the maximal solution to

$$\xi_0(t) = B_{\mathbb{R}}^{(2)}(t) + 2\int_{-\infty}^{t} X_s^{\xi_0} \mathrm{d}s,$$

where $B_{\mathbb{R}}^{(2)}(t)$, $t \in \mathbb{R}$, is defined in Section 3.2. Let $\beta_0(t)$, $-\infty < t < T_0$, be the whole-plane Loewner curve driven by ξ_0. Then β_0 is a continuous LERW$(D; 0 \to z_e)$ curve.

If α is a Jordan curve in $\mathbb{C}$ with $0 \in U(\alpha)$, and β defined on $[-\infty, T)$ is a curve in $\mathbb{C}$ with $\beta(-\infty) = 0$, let $T_\alpha(\beta)$ be the first t such that $\beta(t) \in \alpha$, if such t exists; otherwise let $T_\alpha(\beta) = T$.

Since $q_\delta([0, T^\delta_{\alpha_0}])$ intersects α_0, so $T_{\alpha_0}(q_\delta) \le T^\delta_{\alpha_0}$. Using the above proposition, we are able to derive the following theorem, which is similar to Theorem 6.2 in [12]. The proof uses Skorokhod Embedding Theorem, the method in the proof of Theorem 3.7 in [5], and the Markov property of $(\mathrm{e}^{\mathrm{i}}(B^{(2)}_{\mathbb{R}}(t)))$. Again, we omit the proof here.

Theorem 6.2. *Suppose α is a Jordan curve in $\mathbb{C}$ with $0 \in U(\alpha)$, and $H(\alpha) \subset D \setminus \{z_e\}$. For every $b \in \mathbb{R}$ and $\varepsilon > 0$, there is $\delta_0 > 0$ such that if $\delta < \delta_0$ then there is a coupling of the processes $(\xi_\delta(t))$ and $(\xi_0(t))$ such that*

$$\mathbf{P}[\sup\{|\mathrm{e}^{\mathrm{i}}(\xi_\delta(t)) - \mathrm{e}^{\mathrm{i}}(\xi_0(t))| : t \in [b, T_\alpha(\beta_\delta) \vee T_\alpha(\beta_0)]\} < \varepsilon] > 1 - \varepsilon.$$

Here if ξ_δ or ξ_0 is not defined on $[b, T_\alpha(\beta_\delta) \vee T_\alpha(\beta_0)]$, we set the value of sup *to be $+\infty$.*

6.4. *Convergence of the curves*

So far, we have derived the convergence of the driving functions. Using the above theorem, Lemma 2.3, and the regularity of discrete LERW path (cf. Lemma 3.4 in [10] and Lemma 7.2 in [12]), we may derive the following theorem, which is similar to Theorem 7.1 in [12]. It is about the local convergence of the curves. Here we omit its proof.

Theorem 6.3. *Let α be as in the above theorem. For every $\varepsilon > 0$, there is $\delta_0 > 0$ such that if $\delta < \delta_0$ then there is a coupling of the processes $(\beta_\delta(t))$ and $(\beta_0(t))$ such that*

$$\mathbf{P}[\sup\{|\beta_\delta(t) - \beta_0(t)| : t \in [-\infty, T_\alpha(\beta_\delta) \vee T_\alpha(\beta_0)]\} < \varepsilon] > 1 - \varepsilon.$$

Finally, we may lift the local convergence to the global convergence, and so finish the proof of Theorem 6.1. The argument used here is almost identical to that in Section 7.2 of [12]. A slight difference is that now $\mathcal{A}$ is the set of Jordan curves α such that $0 \in U(\alpha)$ and $H(\alpha) \subset D \setminus \{z_e\}$; and $\mathcal{B}$ is the set of continuous curves $\beta : [-\infty, T) \to D$ for some $T \in \mathbb{R}$, with $\beta(-\infty) = 0$.

6.5. *Other kinds of targets*

Let D be a finitely connected domain that contains 0. Suppose w_e is a prime end of D that satisfies $w_e \in \delta_e\mathbb{Z}^2$ for some $\delta_e > 0$, and ∂D is flat near w_e, which means that there is $r > 0$ such that $D \cap \{z \in \mathbb{C} : |z - w_e| < r\} = (w_e + a\mathbb{H}) \cap \{z \in \mathbb{C} : |z - w_e| < r\}$ for some $a \in \{\pm 1, \pm \mathrm{i}\}$. For $\delta > 0$, let $w_e^\delta = w_e + \mathrm{i}a\delta$.

Let $\mathcal{M}$ be the set of $\delta > 0$ such that $w_e \in \delta\mathbb{Z}^2$. If $\delta \in \mathcal{M}$ is small enough, then $\langle w_e^\delta, w_e\rangle$ is a boundary vertex of $\breve{D}^\delta$, which determines the boundary point and prime end w_e, and there is a lattice path on D^δ that connects 0 with w_e without passing through any other boundary vertex. Here we do not distinguish w_e from the boundary vertex $\langle w_e^\delta, w_e\rangle$. Let $F = \{w_e\}$, $E_{-1} = V_\partial(D^\delta) \setminus F$, and $E_k = E_{-1} \cup \{q_j : 0 \le j \le k\}$ for $0 \le k \le \chi_\delta - 1$. Let $(q_\delta(0), \dots, q_\delta(\chi_\delta))$ be the LERW on D^δ started from 0 conditioned to hit F before E_{-1}. So $q_\delta(0) = 0$ and $q_\delta(\chi_\delta) = w_e$. Extend q_δ to be defined on $[0, \chi_\delta]$ such that q_δ is linear on $[k-1, k]$ for each $1 \le k \le \chi_\delta$. Then q_δ is a simple curve in $D \cup \{w_e\}$ that connects 0 and w_e.

For each $0 \le k \le \chi_\delta - 1$, let h_k be defined as in Lemma 2.1 in [12] with $A = F$, $B = E_{k-1}$ and $x = q_\delta(k)$. This means that h_k is a function defined on $V(D^\delta)$, which vanishes on $F \cup E_k \setminus \{q_\delta(k)\}$, is discrete harmonic on $V_I(D^\delta) \setminus \{q_\delta(0), \dots, q_\delta(k)\}$, and $h_k(w_e^\delta) - h_k(w_e) = 1$. Then for any fixed vertex v_0 on D^δ, $(h_k(v_0))$ is a martingale up to the time when $q_\delta(k) = w_e^\delta$ or E_k disconnects v_0 from w_e. Let $D_k = D \setminus q_\delta([-1, k])$. Then $q_\delta(k)$ is a prime end of D_k. Note that h_k vanishes on $q_\delta(0), \dots, q_\delta(k-1)$ and all boundary vertices of D^δ, is discrete harmonic at all interior vertices of D^δ except $q_\delta(0), \dots, q_\delta(k)$, and $h_k(w_e^\delta) = 1$. So when δ is small, $\delta \cdot h_k$ is close to the generalized Poisson kernel P_k in D_k with the pole at $q_\delta(k)$ normalized by $\partial_{\mathbf{n}} P_k(w_e) = 1$. Suppose $\beta_0(t)$, $0 \le t < S$, is an LERW$(D; 0_+ \to w_e)$ curve. Then we can prove that Theorem 6.1 still holds for q_δ and β_0 defined here if we replace "$\delta < \delta_0$" by "$\delta \in \mathcal{M}$ and $\delta < \delta_0$".

Now suppose I_e is a side arc of D that is bounded away from 0_+. Let I_e^δ be the set of boundary vertices of D^δ which determine prime ends that lie on I_e. If δ is small enough, I_e^δ is nonempty, and there is a lattice path on D^δ that connecting δ with I_e^δ without passing through any boundary vertex not in I_e^δ. Then we let $F = I_e^\delta$, $E_{-1} = V_\partial(D^\delta) \setminus F$, and $E_k = E_{-1} \cup \{q_j : 0 \le j \le k\}$ for $0 \le k \le \chi_\delta - 1$. Let $(q_\delta(0), \dots, q_\delta(\chi_\delta))$ be the LERW on D^δ started from δ conditioned to hit F before E_{-1}. So $q_\delta(0) = 0$ and $q_\delta(\chi_\delta) \in I_e$.

For each $0 \le k \le \chi_\delta - 1$, let h_k be defined as in Lemma 2.1 in [12] with $A = F$, $B = E_{k-1}$ and $x = q_\delta(k)$. This means that h_k is a function defined on $V(D^\delta)$, which vanishes on $F \cup E_k \setminus \{q_\delta(k)\}$, is discrete harmonic on $V_I(D^\delta) \setminus \{q_\delta(0), \dots, q_\delta(k)\}$, and

$$\sum_{w_1 \sim w_2, w_2 \in I_e^\delta} (h_k(w_1) - h_k(w_2)) = 1.$$

When δ is small, the function h_k seems to be close to the generalized Poisson kernel P_k in D_k with the pole at $q_\delta(k)$ normalized by $\int_{I_e} \partial_{\mathbf{n}} P_k(z) \mathrm{d}s(z) = 1$. Let β_0 be an LERW$(D; 0 \to I_e)$ curve.

If I_e is a whole side of D, then we can prove that Theorem 6.1 still holds for q_δ and β_0 defined here. If I_e is not a whole side, for the purpose of convergence, we may need some additional boundary conditions. Suppose the two ends of I_e correspond to $w_e^1, w_e^2 \in \partial D$, near which ∂D is flat, and $w_e^1, w_e^2 \in \delta_e \mathbb{Z}^2$ for some $\delta_e > 0$. Let $\mathcal{M}$ be the set of $\delta > 0$ such that $w_e^1, w_e^2 \in \delta \mathbb{Z}^2$. Then Theorem 6.1 still holds for q_δ and γ_0 defined here if we replace "$\delta < \delta_0$" by "$\delta \in \mathcal{M}$ and $\delta < \delta_0$".

6.6. Restriction and reversibility

Using Theorem 6.1 and the properties of the discrete LERW, we may derive the restriction and reversibility properties of the continuous LERW defined in this paper.

Corollary 6.1. *Let D be a finitely connected domain, $z_0 \in D$, and I_e is a side arc of D. Let $\beta(t)$, $0 \le t < T$, be an* LERW$(D; z_0 \to I_e)$ *curve. Then a.s. $\widehat{\lim}_{t \to S} \beta(t)$, the limit of $\beta(t)$ in $\widehat{D}$, as $t \to T^-$, exists and lies on I_e. Moreover, the distribution of $\widehat{\lim}_{t \to S} \beta(t)$ is proportional to the harmonic measure in D viewed from z_0 restricted to I_e. If $J_e \subset I_e$ is another side arc of D, then after a time-change, $\beta(t)$ conditioned on the event that $\widehat{\lim}_{t \to S} \beta(t) \in J_e$ has the same distribution as an* LERW$(D; z_0 \to J_e)$ *curve. This is still true when J_e shrinks to a single boundary point, say z_e, in which case, the conditioned curve $\beta(t)$ has the same distribution as an* LERW$(D; z_0 \to z_e)$ *curve, after a time-change.*

As pointed out by [10,5], LERW is closely related with UST (uniform spanning tree) by Wilson's algorithm. This is also true for the LERW we considered here. The LERW started from an interior vertex w_0 of D^δ conditioned to exit D at the given boundary point w_e can be reconstructed as follows. Let T be an UST with wired boundary condition, i.e., all boundary vertices of D^δ are identified as a single vertex. In that case, there is only one lattice path that connects w_0 with ∂D^δ. Now we condition that this path ends ∂D^δ at w_e. Then this path is the above LERW. In fact, the reversal of such a path is the LERW started from w_e conditioned to hit

w_0 before exiting D, as considered in [12]. The LERW from one interior vertex w_0 to another interior vertex w_e could be constructed as follows. Divide ∂D^δ into two sets: S_0 and S_e. Identify $S_0 \cup \{w_0\}$ as a single vertex: w_0^*; identify $S_e \cup \{w_e\}$ as another single vertex: w_e^*. Let T be the UST on this quotient graph conditioned on the event that the two end points of the lattice path on T connecting w_0^* and w_e^* are w_0 and w_e. Then the lattice path on T connecting w_0^* and w_e^* is the LERW from w_0 to w_e. Here the distribution of T does not depend on the choice of S_0 and S_e. So it is clear that the reversal of this LERW is the LERW from w_e to w_0. From Theorem 6.1, we have the following two corollaries.

Corollary 6.2. *Let D be a finitely connected domain, and $z_1 \neq z_2 \in D$. Let $\beta(t)$, $0 \le t < T$, be an* LERW$(D; z_1 \to z_2)$ *curve. Then after a time-change, the reversal of β has the same distribution as an* LERW$(D; z_2 \to z_1)$ *curve. Especially, if $\beta(t)$, $-\infty < t < \infty$, is a whole-plane* SLE$_2$ *curve, then $(W(\beta(-t)))$ has the same distribution as $(\beta(t))$, where $W(z) = 1/\overline{z}$. So we get the reversibility of the whole-plane* SLE$_2$ *curve.*

Corollary 6.3. *Let D be a finitely connected domain, $z_0 \in D$, and w_0 is a prime end of D. Let $\beta(t)$, $0 \le t < T$, be an interior* LERW$(D; z_0 \to w_0)$ *curve. Then after a time-change, the reversal of β has the same distribution as a boundary* LERW$(D; w_0 \to z_0)$ *curve, which is defined in* [12].

Remarks.

(i) Using the stochastic coupling technique in [13] and the partition function given in Section 5, we may give analytic proofs of Corollaries 6.2, 6.3 and 6.1 without using the approximation of discrete LERW.

(ii) For the discrete LERW connecting two interior points, one may let T be the UST on the discrete approximation with free boundary condition, and let LERW be the only curve on this UST connecting w_0^* and w_e^*. This discrete LERW converges to the continuous LERW with free boundary condition. It is defined similarly as the continuous LERW defined here, except that in (1.1) we must use a Green function with Neumann boundary condition on ∂D.

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