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# Cayley Graphs of Finite Groups

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Let $\Gamma(G, T)$ denote the Cayley graph of a finite group $G$ with respect to a normal subset $T$ of $G - \{1\}$.

We compute explicitly the spectrum of $\Gamma(G, T)$ in terms of complex character values. This allows us to determine the number of paths of length $n$ between two arbitrary vertices of $\Gamma(G, T)$ for each $n \in \mathbb{N}$.

Finally, we apply these results to obtain the following theorem. Suppose that $G$ is a finite group which contains a cyclic self-normalizing subgroup $W$ of order $pq$, where $p$ and $q$ are two different odd prime numbers. Define $W_0$ to be the set of all elements of order $pq$ of $W$ and let $T := \bigcup_{g \in G} W_0^g$. Then for any $n \in \mathbb{N}$, the number of paths of length $n$ between two adjacent vertices of the Cayley graph $\Gamma(G, T)$ does not depend on the choice of the two adjacent vertices. Moreover, the rank of $\Gamma(G, T)$ is 4 or 5. 

## 1. THE SPECTRUM OF A CAYLEY GRAPH

Let $G$ be a finite group and $T$ the union of a family of conjugacy classes of $G - \{1\}$. Then by

$$g \rightsquigarrow h :\Leftrightarrow hg^{-1} \in T \qquad (g, h \in G)$$

there is defined a graph $\Gamma(G, T)$ on the elements of $G$. In general, $\Gamma(G, T)$ is directed and it is undirected if and only if $t \in T$ implies $t^{-1} \in T$. $\Gamma(G, T)$ is called the *Cayley graph* of $G$ with respect to $T$.

Let $A$ be the complex group algebra of $G$. Then for each $a \in A$ we define the vectorspace endomorphism $\rho_a$ of $A$ by

$$\rho_a(g) := ga, \qquad (g \in G)$$

and by $\alpha$ we mean the (linear) adjacency map of $\Gamma(G, T)$ which is defined on $A$ by

$$\alpha(g) := \sum_{g \rightsquigarrow h} h. \qquad (g \in G)$$



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Now let

$$z := \alpha(1).$$

Then

$$z = \sum_{t \in T} t. \tag{1}$$

Thus, since $T$ is the union of certain conjugacy classes of $G$, we clearly have $z \in Z(A)$.

Let $A = A_1 \oplus \cdots \oplus A_s$ be the decomposition of $A$ into its minimal ideals and for all $i \in \{1, ..., s\}$ let $e_i$ be the unit of $A_i$. Since $z \in Z(A)$, there are $\lambda_1, ..., \lambda_s \in \mathbb{C}$ such that

$$z = \sum_{i=1}^{s} \lambda_i e_i. \tag{2}$$

In particular, $\{\lambda_1, ..., \lambda_s\}$ is the set of all eigenvalues of $\rho_z$ and the eigenspaces of $\rho_z$ are ideals of $A$.

On the other hand, for each $g \in G$ we have

$$\alpha(g) = \sum_{g \rightsquigarrow h} h = \sum_{1 \rightsquigarrow h} gh = g\alpha(1) = gz = \rho_z(g),$$

thus

$$\alpha = \rho_z \tag{3}$$

and $\{\lambda_1, ..., \lambda_s\}$ is the set of all eigenvalues of $\alpha$ and the eigenspaces of $\alpha$ are ideals of $A$.

Now for each $i \in \{1, ..., s\}$ let $\varphi_i$ be the irreducible representation of $G$ which belongs to $A_i$ and let $\chi_i$ be the character of $\varphi_i$.

Let $j \in \{1, ..., s\}$. Then, by (1) and (2), we have

$$\varphi_j(z) = \varphi_j\left(\sum_{t \in T} t\right) = \sum_{t \in T} \varphi_j(t)$$

and

$$\varphi_j(z) = \varphi_j\left(\sum_{i=1}^{s} \lambda_i e_i\right) = \sum_{i=1}^{s} \lambda_i \varphi_j(e_i).$$

Thus we conclude

$$\sum_{t \in T} \chi_j(t) = \chi_j(z) = \lambda_j \chi_j(1)$$

and have proved the following theorem.

THEOREM 1. *Let G be a finite group and $\Gamma(G, T)$ a Cayley graph of G. Let further $\{\chi_1, ..., \chi_s\}$ be the set of all irreducible complex characters of G and define*

$$\lambda_j := \frac{1}{\chi_j(1)} \sum_{t \in T} \chi_j(t)$$

*for all $j \in \{1, ..., s\}$.*

*Then $\{\lambda_1, ..., \lambda_s\}$ is the set of all values of the spectrum of $\Gamma(G, T)$.*

*Moreover, if $m_j$ is the multiplicity of $\lambda_j$, then*

$$m_j = \sum_{\substack{k=1 \\ \lambda_k = \lambda_j}}^{s} \chi_k(1)^2.$$

Theorem 1 shows how to compute explicitly the spectrum of a Cayley graph of a finite group in terms of complex character values. In particular, we see that the rank of a Cayley graph of a finite group $G$ which by definition is the number of different values of its spectrum does not exceed the number of the irreducible complex representations of $G$.

We now head for a graph theoretic consequence of Theorem 1.

For each $g \in G$ and $n \in \mathbb{N}$ let $c(g, n)$ denote the number of paths of length $n$ from the vertex 1 to the vertex $g$ of the Cayley graph $\Gamma(G, T)$. Then, by induction over $n$, we have

$$\alpha^n(1) = \sum_{g \in G} c(g, n)\, g$$

for all $n \in \mathbb{N}$. This together with (3) yields

$$z^n = \sum_{g \in G} c(g, n)\, g. \tag{4}$$

On the other hand, by (2) and Theorem 1, $z^n$ has the representation

$$z^n = \sum_{i=1}^{s} \frac{1}{\chi_i(1)^n} \left( \sum_{t \in T} \chi_i(t) \right)^n e_i.$$

Furthermore, by [3, Theorem 2.12] we have

$$e_i = \frac{1}{|G|} \sum_{g \in G} \chi_i(1)\, \chi_i(g^{-1})\, g.$$

Thus

$$z^n = \sum_{g \in G} \left[ \frac{1}{|G|} \sum_{i=1}^{s} \frac{\chi_i(g^{-1})}{\chi_i(1)^{n-1}} \left( \sum_{t \in T} \chi_i(t) \right)^n \right] g. \tag{5}$$

Now by (4) and (5) we may deduce the following result which generalizes [2, Theorem 4.2.8] and is similar to [4, Lemma 1].

THEOREM 2. *Let $G$ be a finite group and $\Gamma(G, T)$ a Cayley graph of $G$. Let further $\{\chi_1, ..., \chi_s\}$ be the set of all irreducible complex characters of $G$.*

*For each $g \in G$ and $n \in \mathbb{N}$ let $c(g, n)$ denote the number of paths of length $n$ from the vertex 1 to the vertex $g$ of $\Gamma(G, T)$.*

*Then we have*

$$c(g, n) = \frac{1}{|G|} \sum_{i=1}^{s} \frac{\chi_i(g^{-1})}{\chi_i(1)^{n-1}} \left( \sum_{t \in T} \chi_i(t) \right)^n.$$

Clearly, $G$ acts by left and by right multiplication as a group of graph automorphisms on the points of $\Gamma(G, T)$. Thus Theorem 2 also enables us to compute the number of paths of length $n$, $n \in \mathbb{N}$, between two arbitrary vertices of $\Gamma(G, T)$.

## 2. SOME CAYLEY GRAPHS OF SMALL RANK

Within this section we investigate finite groups $G$ which satisfy the following hypothesis.

*Hypothesis* 3. $G$ possesses a cyclic subgroup $W$ with $w := |W|$ odd. Suppose that $W = W_1 \times W_2$, where $w_i := |W_i|$ and $w_i \neq 1$ for $i \in \{1, 2\}$. Let

$$W_0 := W - W_1 - W_2.$$

For any non-empty subset $A$ of $W_0$ let

$$\mathbf{C}_G(A) = \mathbf{N}_G(A) = W.$$

Let $\kappa_{01}$, $\kappa_{10}$ be faithful irreducible complex characters of $W/W_1$, $W/W_2$, respectively. Define

$$\kappa_{ij} := \kappa_{10}^i \kappa_{01}^j$$

for $i \in \{0, ..., w_1 - 1\}$, $j \in \{0, ..., w_2 - 1\}$.

Finally, let

$$T := \bigcup_{g \in G} W_0^g.$$

The purpose of this section is the application of the results of the first section on the Cayley graph $\Gamma(G, T)$ which now obviously is undirected. The following result is due to Feit and Thompson.

LEMMA 4. *Suppose that G satisfies Hypothesis 3. Then*

(i) *$W_0$ is a TI set in G.*

(ii) *There exists an orthonormal set*

$$\{\bar{\kappa}_{ij} : 0 \leqslant i \leqslant w_1 - 1, 0 \leqslant j \leqslant w_2 - 1\}$$

*of generalized complex characters of G such that the following conditions are true.*

(a) *For all $i \in \{0, ..., w_1 - 1\}$, $j \in \{0, ..., w_2 - 1\}$ the values assumed by $\bar{\kappa}_{ij}$, $\bar{\kappa}_{i0}$, $\bar{\kappa}_{0j}$ lie in $\mathbb{Q}_w$, $\mathbb{Q}_{w_1}$, $\mathbb{Q}_{w_2}$, respectively.*

(b) $\bar{\kappa}_{00} = \mathbf{1}_G$.

(c) *If $t \in W_0$, then $\bar{\kappa}_{ij}(t) = \kappa_{ij}(t)$.*

(d) $\mathbf{1}_G - \bar{\kappa}_{i0} - \bar{\kappa}_{0j} + \bar{\kappa}_{ij} = (\mathbf{1}_W - \kappa_{i0} - \kappa_{0j} + \kappa_{ij})^*$ for all $i \in \{0, ..., w_1 - 1\}$, $j \in \{0, ..., w_2 - 1\}$.

(e) *Every irreducible character of G distinct from all $\pm\bar{\kappa}_{ij}$ vanishes on T.*

*Proof.* See [1, Lemma 13.1]. ∎

Let $\chi$ be a generalized complex character of $G$. We write $\mathbb{Q}(\chi)$ to denote the subfield of $\mathbb{C}$ generated by $\mathbb{Q}$ and the character values $\chi(g)$ for $g \in G$.

Let $e$ be the exponent of $G$. By a theorem of Brauer [3, Theorem 10.3], $\mathbb{Q}_e$ is a splitting field for $G$. Moreover, we have $\mathbb{Q} \subseteq \mathbb{Q}(\chi) \subseteq \mathbb{Q}_e$ and $\mathbb{Q}_e$ is a Galois extension of $\mathbb{Q}$. In particular, each automorphism $\alpha$ of $\mathbb{Q}(\chi)$ is afforded by an automorphism of $\mathbb{Q}_e$. Thus

$$\chi^\alpha : G \to \mathbb{Q}(\chi)$$
$$g \mapsto \chi(g)^\alpha$$

is a character of $G$.

Assume $\chi$ and $\psi$ are two generalized complex characters of $G$. We say that $\chi$ and $\psi$ are *algebraically conjugate* if $\mathbb{Q}(\chi) = \mathbb{Q}(\psi)$ and there exists $\alpha \in \mathrm{Gal}(\mathbb{Q}(\chi) : \mathbb{Q})$ such that $\chi^\alpha = \psi$.

LEMMA 5. *Let $p$, $r \in \{0, ..., w_1 - 1\}$ and $q$, $s \in \{0, ..., w_2 - 1\}$. Then the generalized characters $\bar{\kappa}_{pq}$ and $\bar{\kappa}_{rs}$ are algebraically conjugate iff* $\ker(\kappa_{pq}) = \ker(\kappa_{rs})$.

*Proof.* Let $C(\bar{\kappa}_{pq})$ be the set of all elements of $\{\bar{\kappa}_{ij} : 0 \leqslant i \leqslant w_1 - 1, 0 \leqslant j \leqslant w_2 - 1\}$ which are algebraically conjugate to $\bar{\kappa}_{pq}$ and $D(\bar{\kappa}_{pq})$ the set of the generalized characters $\bar{\kappa}_{ij}$ with $\ker(\kappa_{ij}) = \ker(\kappa_{pq})$.

Let $\bar{\kappa}_{rs} \in C(\bar{\kappa}_{pq})$. Then there exists $\alpha \in \mathrm{Gal}(\mathbb{Q}(\bar{\kappa}_{pq}) : \mathbb{Q})$ such that $\bar{\kappa}_{pq}^\alpha = \bar{\kappa}_{rs}$. Let $t \in G$ such that $\langle t \rangle = W$. Then $t \in W_0$, hence Lemma 4(ii)(c) yields

$\bar{\kappa}_{pq}(t) = \kappa_{pq}(t)$. In particular, if $d := |\ker(\kappa_{pq})|$, then $\bar{\kappa}_{pq}(t)$ is a primitive $wd^{-1}$th root of unity. But then $\kappa_{rs}(t) = \bar{\kappa}_{rs}(t) = \bar{\kappa}^{\alpha}_{pq}(t) = \bar{\kappa}_{pq}(t)^{\alpha}$ is a primitive $wd^{-1}$th root of unity, which implies $|\ker(\kappa_{rs})| = d$. Since $W$ is cyclic, hence $\ker(\kappa_{rs}) = \ker(\kappa_{pq})$. Thus $\bar{\kappa}_{rs} \in D(\bar{\kappa}_{pq})$. Since $\bar{\kappa}_{rs}$ was chosen arbitrarily in $C(\bar{\kappa}_{pq})$, we have

$$C(\bar{\kappa}_{pq}) \subseteq D(\bar{\kappa}_{pq}). \tag{6}$$

Clearly

$$|D(\bar{\kappa}_{pq})| = \varphi(wd^{-1}), \tag{7}$$

where $\varphi$ is the Euler function and $d := |\ker(\kappa_{pq})|$. By [3, Lemma 9.17(c)] we have

$$|C(\bar{\kappa}_{pq})| = |\mathbb{Q}(\bar{\kappa}_{pq}) : \mathbb{Q}|. \tag{8}$$

Finally, since $\bar{\kappa}_{pq}(t)$ is a primitive $wd^{-1}$th root of unity for each $t \in G$ with $\langle t \rangle = W$, hence

$$|\mathbb{Q}(\bar{\kappa}_{pq}) : \mathbb{Q}| \geqslant \varphi(wd^{-1}). \tag{9}$$

Now the conditions (6), (7), (8), and (9) yield the desired conclusion. ∎

The following abbreviation may be convenient. For each $i \in \{0, ..., w_1 - 1\}$ and $j \in \{0, ..., w_2 - 1\}$ let

$$\sigma_{ij} := \begin{cases} (w_1 - 1)(w_2 - 1) & \text{if } i = 0 = j, \\ -(w_1 - 1) & \text{if } i = 0 \neq j, \\ -(w_2 - 1) & \text{if } i \neq 0 = j, \\ 1 & \text{if } i \neq 0 \neq j. \end{cases}$$

LEMMA 6. (i) *For every irreducible character $\chi$ of $G$ distinct from all $\pm\bar{\kappa}_{ij}$ we have*

$$\sum_{t \in T} \chi(t) = 0.$$

(ii) *For all $i \in \{0, ..., w_1 - 1\}$, $j \in \{0, ..., w_2 - 1\}$ we have*

$$\sum_{t \in T} \bar{\kappa}_{ij}(t) = \frac{|G|}{w} \sigma_{ij}.$$

*Proof.* (i) This is an immediate consequence of Lemma 4(ii)(e).

(ii) Let $i \in \{0, ..., w_1 - 1\}$, $j \in \{0, ..., w_2 - 1\}$. Then, by Lemma 4(i) and (ii)(c),

$$\sum_{t \in T} \bar{\kappa}_{ij}(t) = \frac{|G|}{w} \sum_{t \in W_0} \kappa_{ij}(t).$$

Now clearly

$$\sum_{t \in W_0} \bar{\kappa}_{00}(t) = |W_0| = (w_1 - 1)(w_2 - 1).$$

If $(i, j) \neq (0, 0)$, then $\sum_{t \in W} \kappa_{ij}(t) = 0$, thus we have

$$\sum_{t \in W_0} \kappa_{ij}(t) = 1 - \sum_{t \in W_1} \kappa_{ij}(t) - \sum_{t \in W_2} \kappa_{ij}(t). \tag{10}$$

Further $i = 0$ is equivalent to $W_1 \subseteq \ker(\kappa_{ij})$, thus

$$\sum_{t \in W_1} \kappa_{ij}(t) = \begin{cases} w_1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases} \tag{11}$$

Similarly we have

$$\sum_{t \in W_2} \kappa_{ij}(t) = \begin{cases} w_2 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases} \tag{12}$$

Now the full claim follows from (10), (11), and (12). ▮

THEOREM 7. *Suppose that $G$ satisfies Hypothesis 3 and let $\bar{\kappa}_{ij}$ be the generalized complex characters of Lemma 4.*

*Let $\{d_1, ..., d_r\}$ be the set of all divisors of $w$ and for each $k \in \{1, ..., r\}$ choose representatives $i(k) \in \{0, ..., w_1 - 1\}$ and $j(k) \in \{0, ..., w_2 - 1\}$ such that $|\ker(\kappa_{i(k)j(k)})| = d_k$.*

*Finally, define*

$$\lambda_k := \frac{|G| \cdot \sigma_{i(k)j(k)}}{\bar{\kappa}_{i(k)j(k)}(1) \cdot w}$$

*for all $k \in \{1, ..., r\}$.*

*Then $\{\lambda_1, ..., \lambda_r\}$ is the set of all values of the spectrum of the Cayley graph $\Gamma(G, T)$ different from 0.*

*Moreover, if $w$ is square free, then $\lambda_k$ is an integer for all $k \in \{1, ..., r\}$.*

*Proof.* The values of the spectrum are easily obtained from Theorem 1, Lemma 5, and Lemma 6.

Suppose $w$ is square free. Then, by a theorem of Brauer [3, Theorem 8.17] and Lemma 4(ii)(c), $w$ divides $|G|/\bar{\kappa}_{ij}(1)$ for all $i \in \{0, ..., w_1 - 1\}$, $j \in \{0, ..., w_2 - 1\}$. Thus all values of the spectrum of $\Gamma(G, T)$ are integers. ▮

THEOREM 8. *Suppose that $G$ satisfies Hypothesis 3.*

*For each $g \in G$ and $n \in \mathbb{N}$ let $c(g, n)$ denote the number of paths of length $n$ between the vertices 1 and $g$ of the Cayley graph $\Gamma(G, T)$.*

*Let $s$ and $t$ be two elements of $T$ such that the orders of $s$ and $t$ are equal. Then for any $n \in \mathbb{N}$, we have $c(s, n) = c(t, n)$.*

*Proof.* Let $\{d_1, ..., d_r\}$ be the set of all divisors of $w$ and for each $k \in \{1, ..., r\}$ choose representatives $i(k) \in \{0, ..., w_1 - 1\}$ and $j(k) \in \{0, ..., w_2 - 1\}$ such that $|\ker(\kappa_{i(k)j(k)})| = d_k$.

Assume without loss of generality that $t \in W_0$. Then Theorem 2, Lemma 4(ii)(c), (e), Lemma 5, and Lemma 6 yield

$$c(t, n) = \frac{1}{|G|} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \frac{\kappa_{ij}(t^{-1})}{\bar{\kappa}_{ij}(1)^{n-1}} \cdot \frac{|G|^n}{w^n} \sigma_{ij}^n$$
$$= \frac{|G|^{n-1}}{w^n} \sum_{k=1}^{r} \frac{(\sigma_{i(k)j(k)})^n}{(\bar{\kappa}_{i(k)j(k)}(1))^{n-1}} \sum_{\substack{\kappa_{rs} \\ |\ker(\kappa_{rs})| = d_k}} \kappa_{rs}(t^{-1})$$

for all $t \in T$ and $n \in \mathbb{N}$.

Clearly the value on the right-hand side does not change if $t$ is replaced by another element of $W_0$ of the same order. ∎

The following theorem is an immediate consequence of Theorem 7 and Theorem 8.

THEOREM 9. *Suppose that $G$ is a finite group which contains a cyclic self-normalizing subgroup $W$ of order $pq$, where $p$ and $q$ are two different odd prime numbers.*

*Define $W_0$ to be the set of all elements of order $pq$ of $W$ and let $T := \bigcup_{g \in G} W_0^g$.*

*Then for any $n \in \mathbb{N}$, the number of paths of length $n$ between two adjacent vertices of the Cayley graph $\Gamma(G, T)$ does not depend on the choice of the two adjacent vertices. Moreover, the rank of $\Gamma(G, T)$ is 4 or 5.*

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