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Regularity of the Free Boundary in Singular Stochastic Control

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1. INTRODUCTION

This paper studies the regularity of the free boundary which arises from a stationary problem of singular stochastic control in which the state space has dimension greater than one. The optimal cost function u will be shown to satisfy a variational inequality of the form

$$Lu \leqslant f, \qquad \nabla u + c \geqslant 0,$$
$$(Lu - f) \prod_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} + c_i \right) = 0,$$

where L is a second-order linear elliptic operator with constant coefficients, f is a given function, and $c = (c_1, ..., c_n)$ is a given constant vector. It is well known that such a variational inequality gives rise to a free-boundary problem.

In one dimension, this type of singular control problem has been investigated by many authors, including Bather and Chernoff [BC], Benes *et al.* [BSW], Karatzas [Kar], Menaldi and Robin [MR], and Chow *et al.* [CMR]. One result shown in these papers is that the optimal control is a diffusion process with reflection at the free boundary (one or two points in the one-dimensional case). In the higher-dimensional case, a similar optimal policy has not been constructed (except in the [SS] paper described below) due to the lack of information about the regularity of the associated free boundary. This regularity question has been a long-standing open question and a serious obstacle to the development of a satisfactory theory of singular stochastic control in higher dimensions.

In the present paper, the regularity question will be partially answered. We will show that under certain assumptions the free boundary is smooth

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away from some "corner points" (see Theorem 4.11). The method used is to show that the optimal cost function u is smooth enough to apply the known results of Caffarelli [Caf] and Kinderlehrer and Nirenberg [KN] which then guarantee the required degree of smoothness of the free boundary. In a closely related work, Soner and Shreve [SS] used this same method to prove the regularity of the free boundary for the singular stochastic control problem they studied. (In their problem it is possible to exert control in any direction, while in the problem considered here control can be exerted only in the positive coordinate directions. Largely as a result of this, their free boundary is bounded and has no "corner points," while in this paper the free boundary is unbounded and points can exist having less then C^1 regularity.) Their paper is limited to two dimensions while this paper is not. On the other hand, their paper constructs an optimal control process (as a diffusion with reflection at the free boundary) while this paper does not. (In [MT] an optimal control is constructed in a higher-dimensional setting by use of probabilistic methods which do not require precise knowledge about the regularity of the free boundary.)

This paper is organized as follows. Section 2 introduces the singular control problem to be studied and some important notation. Section 3 proves some preliminary results about the smoothness of the optimal cost function and studies certain other functions that approximate it. The smoothness of the free boundary is proved in Section 4, with the main result being Theorem 4.11.

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2. Preliminaries and Notation

Let $y(t) = (y_1(t), ..., y_n(t))$ denote the state at time t of a controlled system governed for $t \geqslant 0$ by the following Itô equations

$$y_i(t) = x_i + v_i(t) + \int_0^t g_i(y(s))\, ds + \sum_{j=1}^{n} \int_0^t \sigma_{ij}(y(s))\, dw_j(s), \qquad i = 1, ..., n, \tag{2.1}$$

where $x = (x_1, ..., x_n)$ is the initial state, $v = (v_1, ..., v_n)$ is the control vector, $g = (g_1, ..., g_n)$ is the drift vector, $\sigma = [\sigma_{ij}]_{i,j=1}^{n}$ is the diffusion matrix, and $w(t) = (w_1(t), ..., w_n(t))$ is a standard Wiener process in $\mathbf{R}^n$. The control vector $\{v(t); t \geqslant 0\}$ is assumed to be a progressively measurable random process whose components are non-negative, right continuous, and nondecreasing and have finite moments of all orders for every $t \geq 0$ (see [MR, CMR]). The set of all such controls v will be denoted by V.

The associated optimal control problem is to minimize an expected cost function defined by

$$J_x(v) = E\left\{\int_0^\infty f(y_x(t))\, e^{-\alpha t}\, dt + \sum_{i=1}^n c_i \int_0^\infty e^{-\alpha t}\, dv_i(t)\right\}, \tag{2.2}$$

where y_x is used in place of y to emphasize the dependence on the initial state x, $f(x)$ and $c_i \geq 0$, $i = 1, 2, ..., n$, represent the unit costs for operating and controlling the system, respectively, and $\alpha > 0$ is the discount factor. (A good way to think of this is as an inventory problem, with $y_i(t)$ the stock level at time t of the ith product. Interpreting $v_i(t)$ as the cumulative amount of the ith product ordered up to time t, it is natural that v_i be non-negative and nondecreasing.)

The value function u is the optimal cost given by

$$u(x) = \inf\{J_x(v);\, v \in V\}, \qquad x \in \mathbf{R}^n, \tag{2.3}$$

where the infimum is over the admissible set V of singular controls v. To reduce the difficulties of dealing with singular controls, related problems with classical control will now be introduced (which will be seen later to be penalized problems). For each $\varepsilon > 0$, let V_ε denote the set of all controls $v \in V$ such that v is Lipschitz continuous with probability one and

$$0 \leq \frac{dv_i}{dt}(t) \leq \frac{1}{\varepsilon} \qquad \text{a.e. for} \quad t \geq 0,\ i = 1, 2, ..., n, \text{ almost surely.} \tag{2.4}$$

The corresponding optimal cost function u^ε is given by

$$u^\varepsilon(x) = \inf\{J_x(v);\, v \in V_\varepsilon\}. \tag{2.5}$$

In the subsequent analysis it is assumed that the following conditions hold:

$$\left.\begin{array}{ll} \text{(i)} & \text{the drift vector } g = (g_1, ..., g_n) \text{ is constant} \\ \text{(ii)} & \text{the diffusion matrix } \sigma = [\sigma_{ij}]_{i,j=1}^n \text{ is constant, with} \\ & \sigma\sigma^{\mathrm{T}} \text{ positive definite } (\sigma^{\mathrm{T}} \text{ denotes the transpose of } \sigma). \end{array}\right\} \tag{2.6}$$

Let L be the linear elliptic operator defined by

$$Lu \equiv -\sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j} - \sum_{i=1}^n g_i \frac{\partial u}{\partial x_i} + \alpha u, \tag{2.7}$$

where $a_{ij}=\frac{1}{2}\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk}$. Then for problem (2.5) an application of the dynamic programming principle yields the following Hamilton–Jacobi–Bellman equation (see [MR] or [CMR]) for the value function u^{ε}:

$$Lu^{\varepsilon}+\frac{1}{\varepsilon}\sum_{i=1}^{n}\left(\frac{\partial u^{\varepsilon}}{\partial x_i}+c_i\right)^{-}=f, \qquad x\in\mathbf{R}^n. \tag{2.8}$$

(Throughout this paper, for any $t\in\mathbf{R}$, let $t^{+}=\max\{t,0\}$ and $t^{-}=\max\{-t,0\}$ be the positive and negative parts of t as usual.) As $\varepsilon\to 0+$, one deduces from (2.8) that the solution u of the original problem (2.3) satisfies the variational inequality

$$\left.\begin{aligned} &Lu\leqslant f, \frac{\partial u}{\partial x_i}+c_i\geqslant 0, \qquad i=1,...,n,\\ &(Lu-f)\prod_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}+c_i\right)=0\end{aligned}\right\} \text{a.e. for } \quad x\in\mathbf{R}^n, \tag{2.9}$$

which involves a free-boundary problem [KS].

In other terminology and notation, for any open $\Omega\subset\mathbf{R}^n$, let $(\cdot,\cdot)$ denote the usual inner product in $L^2(\Omega)$. Let $W^{m,p}(\Omega)$ denote the usual Sobolev space of real-valued functions on Ω whose generalized derivatives of order less than or equal to m are in $L^p(\Omega)$, $1\leqslant p\leqslant\infty$. Let $H^m(\Omega)=W^{m,2}(\Omega)$ for $m=0,1,2,...$, and let $H_0^1(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. A bilinear form $a(u,v)$ is *coercive on* $H_0^1(\Omega)$ if there is a constant $\alpha_0>0$ such that

$$a(u,u)\geqslant\alpha_0\,\|u\|^2 \qquad \text{for every} \quad u\in H_0^1(\Omega), \tag{2.10}$$

where $\|\cdot\|$ is the norm in $H^1(\Omega)$ (and also the norm in $H_0^1(\Omega)$ when restricted to that space). The bilinear form $a(u,v)$ we will consider is that associated with the operator L of (2.7), namely,

$$a(u,v)\equiv\int_\Omega\left\{\sum_{i,j=1}^{n}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}-\sum_{i=1}^{n}g_i\frac{\partial u}{\partial x_i}v+\alpha uv\right\}dx, \tag{2.11}$$

where the open set $\Omega\subset\mathbf{R}^n$ will be chosen later. Let $W^{2,\infty}_{\mathrm{loc}}$ denote the Sobolev space of functions on $\mathbf{R}^n$ whose restrictions to any bounded open $\Omega\subset\mathbf{R}^n$ are in $W^{2,\infty}(\Omega)$.

3. Preliminary Results on the Smoothness of u and Study of Certain Approximating Functions u_ε

To obtain some a priori estimates for the value function, we assume that there exist constants $K\geqslant k>0$ and $m\geqslant 1$ such that the unit-cost function

$f\colon \mathbf{R}^n \to \mathbf{R}$, the unit-cost vector $c \in \mathbf{R}^n$ for control, and the discount factor $\alpha \in \mathbf{R}$ satisfy the following conditions:

$$\left.\begin{array}{ll}
\text{(i)} & k\,|x^+|^m - K \leqslant f(x) \leqslant K(1+|x|^m), \qquad \forall x \in \mathbf{R}^n,\\
\text{(ii)} & |f(x)-f(x')| \leqslant K(1+|x|^{m-1}+|x'|^{m-1})\,|x-x'|,\\
& \forall x, x' \in \mathbf{R}^n,\\
\text{(iii)} & f \in C^3(\mathbf{R}^n) \text{ and } f \text{ is convex, with}\\
& 0 \leqslant \dfrac{\partial^2 f}{\partial z^2}(x) \leqslant K(1+|x|^q) \qquad \forall x \in \mathbf{R}^n,\\
& q=(m-2)^+, \text{ and for any second order}\\
& \text{directional derivative } \partial^2/\partial z^2,\\
\text{(iv)} & \alpha > 0 \quad \text{and} \quad c_i \geqslant 0 \quad \text{for} \quad i=1, ..., n,
\end{array}\right\} \tag{3.1}$$

where $x^+ = (x_1^+, ..., x_n^+)$.

THEOREM 3.1. *Suppose that the conditions* (2.6) *and* (3.1) *hold. Then the optimal cost function u defined by* (2.3) *is a continuous function such that, for the same* $m \geqslant 1$ *and* $q=(m-2)^+$ *and for some other constants* $K \geqslant k > 0$ *(independent of x and x'), the following properties are satisfied*:

$$\left.\begin{array}{ll}
\text{(i)} & k\,|x^+|^m - K \leqslant u(x) \leqslant K(1+|x|^m), \qquad \forall x \in \mathbf{R}^n\\
\text{(ii)} & |u(x)-u(x')| \leqslant K(1+|x|^{m-1}+|x'|^{m-1})\,|x-x'|,\\
& \forall x, x' \in \mathbf{R}^n,\\
\text{(iii)} & u \textit{ belongs to } W^{2,\infty}_{\mathrm{loc}} \textit{ and is convex, with}\\
& 0 \leqslant \dfrac{\partial^2 u}{\partial z^2}(x) \leqslant K(1+|x|^q), \qquad \textit{a.e. for} \quad x \in \mathbf{R}^n,\\
& \textit{for any second order directional derivative } \partial^2/\partial z^2.
\end{array}\right\} \tag{3.2}$$

Proof. Under the conditions (2.6) and (3.1), it follows from a known estimate (see (2.15) in [MR] with $p=m$, $T \to \infty$ and λ large enough so that $\alpha_p^\lambda < \alpha/2$) that the solution y_x^0 of (2.1) with $v=0$ satisfies

$$E\int_0^\infty |y_x^0(t)|^m\, e^{-\alpha t}\, dt \leqslant K(1+|x|^m), \qquad \forall x \in \mathbf{R}^n, \tag{3.3}$$

for some constant $K>0$. Using (2.3), (2.2), and (3.1-i), we easily obtain

$$u(x) \leqslant J_x(0) \leqslant K(1+|x|^m), \qquad \forall x \in \mathbf{R}^n,$$

where $K>0$ is some other constant. (In what follows, for convenience, K and k will denote "generic" positive constants which may denote different constants in different estimates.) Thus the upper bound of (3.2-i) is proved.

For each fixed $x \in \mathbf{R}^n$, let

$$V_x \equiv \{v \in V; J_x(v) \leqslant J_x(0)\}.$$

Using (2.2) and the lower bound from (3.1-i), we obtain for some $K > 0$ that

$$E \int_0^\infty |[y_x(t)]^+|^m e^{-\alpha t} \, dt \leqslant K(1 + |x|^m), \qquad \forall x \in \mathbf{R}^n, \forall v \in V_x. \tag{3.4}$$

Because of assumption (2.6), $y_x^0 = y_x - v$, so (3.3) gives

$$E \int_0^\infty |y_x(t) - v(t)|^m e^{-\alpha t} \, dt \leqslant K(1 + |x|^m), \qquad \forall x \in \mathbf{R}^n. \tag{3.5}$$

Since each $v_i \geqslant 0$, $|y_x(t)| \leqslant |[y_x(t)]^+| + |y_x(t) - v|$, so (3.4) and (3.5) imply that there is a constant $K > 0$ (independent of x and v) such that

$$E \int_0^\infty |y_x(t)|^m e^{-\alpha t} \, dt \leqslant K(1 + |x|^m), \qquad \forall x \in \mathbf{R}^n, \forall v \in V_x. \tag{3.6}$$

On the other hand, defining $\xi(t) = tg + \sigma w(t)$ (σw is a matrix product with w considered a column vector here), (2.1) gives $y_x(t) = x + v + \xi(t)$, so $|[y_x(t)]^+| \geqslant |(y_x(t) - v)^+| = |(x + \xi)^+|$, and $|\xi| = |x - (x + \xi)| \geqslant |x^+ - (x + \xi)^+| \geqslant |x^+| - |(x + \xi)^+|$, so that

$$|[y_x(t)]^+| \geqslant |x^+| - |\xi(t)|.$$

Thus for some constants $K \geqslant k > 0$ we have the lower bound

$$E \int_0^\infty |[y_x(t)]^+|^m e^{-\alpha t} \, dt \geqslant k \, |x^+|^m - K, \qquad \forall x \in \mathbf{R}^n. \tag{3.7}$$

In view of the fact that

$$J_x(v) \geqslant E \int_0^\infty f(y_x(t)) \, e^{-\alpha t} \, dt, \qquad \forall v \in V,$$

(2.3), (3.1-i), and (3.7) easily give the lower bound in (3.2-i).

For any $x, x' \in \mathbf{R}^n$, it is easy to check that

$$|u(x) - u(x')| \leqslant \sup\{|J_x(v) - J_{x'}(v)|; v \in V_x \cup V_{x'}\}. \tag{3.8}$$

But

$$|J_x(v) - J_{x'}(v)| \leqslant E \int_0^\infty |f(y_x(t)) - f(y_{x'}(t))| \, e^{-\alpha t} \, dt.$$

Property (3.2-ii) for u follows from this by using (3.1-ii), the fact that $y_x(t) - y_{x'}(t) = x - x'$ (from (2.1) and (2.6)), and the fact that there is a positive constant K such that for any $v \in V_x \cup V_{x'}$ we have

$$E \int_0^\infty |y_x(t)|^{m-1} e^{-\alpha t}\, dt \leqslant K(1 + |x|^{m-1} + |x'|^{m-1}), \tag{3.9}$$

with the corresponding fact also true for $y_{x'}(t)$. In fact, if $v \in V_x$, (3.9) follows immediately from (3.6) by using the Hölder inequality. On the other hand, if $v \in V_{x'}$, (3.9) follows from the corresponding fact for $y_{x'}(t)$ and the estimate

$$\begin{aligned} |y_x|^{m-1} &\leqslant [|y_{x'}| + |x - x'|]^{m-1} \\ &\leqslant 2^{m-1}[|y_{x'}|^{m-1} + 2^{m-1}(|x|^{m-1} + |x'|^{m-1})]. \end{aligned}$$

For $i = 1, ..., n$, let $\Delta_i x$ be the row n-vector with Δx_i as ith entry and all other entries zero. For $i = 1, ..., n$ and for any function $F\colon \mathbf{R}^n \to \mathbf{R}$, define the second difference of F in the x_i direction by

$$\delta_i^2 F(x) = F(x + \Delta_i x) - 2F(x) + F(x - \Delta_i x), \qquad \forall x \in \mathbf{R}^n. \tag{3.10}$$

It is easy to check the fact that

$$\delta_i^2 u(x) \leqslant \sup\{\delta_i^2 J_x(v);\, v \in V_x\}. \tag{3.11}$$

Since $f \in C^2(\mathbf{R}^n)$, we clearly have for $i = 1, ..., n$ and $x \in \mathbf{R}^n$ that

$$\delta_i^2 f(x) = (\Delta x_i)^2 \int_0^1 \int_{-\lambda}^{\lambda} \frac{\partial^2 f}{\partial x_i^2}(x_1, ..., x_i + \mu \Delta x_i, ..., x_n)\, d\mu\, d\lambda. \tag{3.12}$$

Since $y_{x \pm \Delta_i x}(t) = y_x(t) \pm \Delta_i x$, the results (3.11), (2.2), (3.12), condition (3.1-iii), and the Hölder inequality applied to (3.6) imply the upper bound on the following:

$$0 \leqslant \delta_i^2 u(x) \leqslant K(1 + |x|^q)(\Delta x_i)^2, \qquad 1 \leqslant i \leqslant n,\ x \in \mathbf{R}^n,\ |\Delta x_i| \leqslant 1. \tag{3.13}$$

To prove the lower bound of (3.13), it clearly suffices to prove the convexity of u. In view of the definition of u in (2.3), to show the convexity of u it clearly suffices to prove the joint convexity of $J_x(v)$ in (x, v), that is, that

$$J_{\theta x + (1-\theta) x'}(\theta v + (1 - \theta) v') \leqslant \theta J_x(v) + (1 - \theta) J_{x'}(v'),$$

for any $x, x' \in \mathbf{R}^n$, any $v, v' \in V$, and any $\theta \in [0, 1]$. But convexity of $J_x(v)$ in (x, v) clearly follows from the fact that $y_x(t, v)$ depends linearly on (x, v) and from the fact that the set V and the function f are both convex.

It remains only to prove that $u \in W^{2,\infty}_{\mathrm{loc}}$. Let B be any open ball and let $\varphi \in C_0^\infty(\mathbf{R}^n)$ be any test function with support contained in B. Since $(\Delta x_i)^{-2}\,\delta_i^2 u(x)$ is bounded on B for $|\Delta x_i| \leqslant 1$ (by (3.13)), there is a sequence $\eta_k \to 0+$ as $k \to \infty$ such that, denoting by g_k the result of replacing Δx_i by η_k in $(\Delta x_i)^{-2}\,\delta_i^2 u(x)$, we have $g_k \to Q$ weakly in $L^p(B)$ for some p with $1 < p < \infty$. It is then easy to show that

$$\int_{\mathbf{R}^n} \varphi(x)\, Q(x)\, dx = \int_{\mathbf{R}^n} \frac{\partial^2 \varphi}{\partial x_i^2}\, u(x)\, dx, \qquad \forall \varphi \in C_0^\infty(B),$$

so that $Q = \partial^2 u/\partial x_i^2$ is a generalized derivative. Existence and local boundedness of mixed second-order generalized derivatives can now be proved easily as follows. For $k = 1, \ldots, n$, let e_k denote the unit vector in the direction of the positive x_k axis. For any fixed $i \neq j$ with $1 \leqslant i, j \leqslant n$, let y be a new coordinate whose axis points in the $(e_i + e_j)/\sqrt{2}$ direction. Then $\partial^2 u/\partial x_i \partial x_j = \partial^2 u/\partial y^2 - (\partial^2 u/\partial x_i^2 + \partial^2 u/\partial x_j^2)/2$. ∎

Recall that Eq. (2.8) is

$$Lu^\varepsilon + \frac{1}{\varepsilon} \sum_{i=1}^{n} \left(\frac{\partial u^\varepsilon}{\partial x_i} + c_i\right)^- = f, \qquad x \in \mathbf{R}^n. \tag{3.14}$$

In what follows, we will study the related equation

$$Lu_\varepsilon + \frac{1}{\varepsilon} \sum_{i=1}^{n} \beta\left(\frac{\partial u_\varepsilon}{\partial x_i} + c_i\right) = f, \qquad x \in \mathbf{R}^n, \tag{3.15}$$

in which the nonsmooth function λ^- has been replaced by a smooth $\beta(\lambda)$, with $\beta \in C^\infty(\mathbf{R})$, β convex and nonincreasing, and

$$\beta(\lambda) = \begin{cases} 0 & \text{if } \lambda \geqslant 0, \\ -2\lambda - 1 & \text{if } \lambda \leqslant -1, \\ \text{positive} & \text{if } \lambda < 0. \end{cases} \tag{3.16}$$

Note that such a β can easily be constructed by mollification of a function with similar properties which is only piecewise smooth.

We will now show that (3.15) is the Hamilton–Jacobi–Bellman equation of a control problem. For any $\varepsilon > 0$, let U_ε denote the set of all progressively measurable random processes (η, ξ) from $[0, \infty)$ into $\mathbf{R}^n \times \mathbf{R}^n$ whose components η_i and ξ_i are nonnegative and satisfy for $1 \leqslant i \leqslant n$, $t \geqslant 0$, and all $s \in \mathbf{R}$ that

$$-s\eta_i(t) - \frac{1}{\varepsilon}\beta(s) \leqslant \xi_i(t) \leqslant \frac{1}{\varepsilon}.$$

Note that for $s=-1$ this gives $\eta_i(t)\leqslant 2/\varepsilon$. Let

$$J_x(\eta,\xi)=J_x(v)+E\int_0^\infty \sum_{i=1}^n \xi_i(t)\,e^{-\alpha t}\,dt, \qquad \text{with} \quad v=\int_0^t \eta(s)\,ds. \tag{3.17}$$

Define

$$u_\varepsilon(x)=\inf\{J_x(\eta,\xi);\,(\eta,\xi)\in U_\varepsilon\}, \qquad x\in\mathbf{R}^n. \tag{3.18}$$

The Hamilton–Jacobi–Bellman equation for this problem is fairly easily seen to be (3.15). (In checking this it is useful to keep in mind that for each fixed t and i, the condition $-s\eta_i(t)-\beta(s)/\varepsilon\leqslant\xi_i(t)$ for all $s\in\mathbf{R}$ in the definition of U_ε is equivalent to the line $y=-\xi_i(t)-s\eta_i(t)$ in the sy-plane being below $y=\beta(s)/\varepsilon$, the graph of β/ε. Clearly, the convex function β/ε is the supremum of all such linear functions.)

THEOREM 3.2. *With the same assumptions as in Theorem* 3.1, *there exist positive constants K, k, and ε_0 such that for all ε with $0<\varepsilon<\varepsilon_0$, the optimal cost u_ε given by* (3.18) *satisfies the following*:

$$\left.\begin{array}{ll}
\text{(i)} & k\,|x^+|^m-K\leqslant u_\varepsilon(x)\leqslant K(1+|x|^m)\\[6pt]
\text{(ii)} & \left|\dfrac{\partial u_\varepsilon}{\partial x_i}(x)\right|\leqslant K(1+|x|^{m-1})\\[6pt]
\text{(iii)} & u_\varepsilon\in W^{2,\infty}_{\mathrm{loc}},\ u_\varepsilon \textit{ is convex, and}\\
& 0\leqslant\dfrac{\partial^2 u_\varepsilon}{\partial z^2}(x)\leqslant K(1+|x|^q)\\
& \textit{with } q=(m-2)^+, \textit{for every } x\in\mathbf{R}^n\\
& \textit{and every second order directional derivative}\\
& \partial^2/\partial z^2.
\end{array}\right\} \tag{3.19}$$

Moreover, for each $x\in\mathbf{R}^n$, $u_\varepsilon(x)\to u(x)$ as $\varepsilon\to 0+$.

Proof. The properties (3.19) can be proved in virtually the same way as the properties (3.2) were proved in Theorem 3.1. In place of V_x in that proof, use

$$V_{x,\varepsilon}=\{(\eta,\xi)\in U_\varepsilon;\,J_x(\eta,\xi)\leqslant J_x(0,0)\}.$$

To show the pointwise convergence of u_ε to u, let V_0 denote the set of all controls in V such that $v(t)$ is uniformly Lipschitz continuous for $t\geqslant 0$. It was proved in [CMR] that the optimal cost u can alternatively be defined by

$$u(x)=\inf\{J_x(v);\,v\in V_0\}, \qquad \forall x\in\mathbf{R}^n. \tag{3.20}$$

It is obvious that $u_\varepsilon(x) \geqslant u(x)$ for every $\varepsilon > 0$ and every $x \in \mathbf{R}^n$. For any $\delta > 0$ and any $x \in \mathbf{R}^n$, because of (3.20) we can find a $v \in V_0$ such that $J_x(v) \leqslant u(x) + \delta/2$. Taking $\xi_i(t) \equiv \delta\alpha(2n)^{-1}$ for $1 \leqslant i \leqslant n$ and $t \geqslant 0$ and η so that $v = \int_0^t \eta(s)\, ds$, for $\varepsilon > 0$ small enough, $(\eta, \xi) \in U_\varepsilon$, so (3.17) gives $J_x(\eta, \xi) \leqslant u(x) + \delta$. ∎

The following six lemmas and all the related definitions are used only for the proof of the next theorem (Theorem 3.9). Define

$$A_0 = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} - \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}, \tag{3.21}$$

i.e., $A_0 u \equiv Lu - \alpha u$. Define $\psi(x) = (\Lambda + |x|^2)^{-P}$ for each $x \in \mathbf{R}^n$, where $\Lambda > 0$ and $P > 0$ are constants to be chosen later. Define

$$H = \{\varphi;\, \varphi\psi^{1/2} \in L^2(\mathbf{R}^n)\} \qquad \text{with the norm } \quad |\varphi|_H = |\varphi\psi^{1/2}|_{L^2(\mathbf{R}^n)} \tag{3.22}$$

$$V = \left\{\varphi \in H;\, \text{for } i = 1, ..., n,\, \frac{\partial\varphi}{\partial x_i} \text{ exists and } \frac{\partial\varphi}{\partial x_i}\psi^{1/2} \in L^2(\mathbf{R}^n)\right\}, \tag{3.23}$$

where $\partial\varphi/\partial x_i$ denotes the generalized derivative. In V use the norm

$$|\varphi|_V = \left[\, |\varphi|_H^2 + \sum_{i=1}^{n} \left|\frac{\partial\varphi}{\partial x_i}\psi^{1/2}\right|_{L^2(\mathbf{R}^n)}^2\right]^{1/2}. \tag{3.24}$$

Clearly H is a real Hilbert space with inner product

$$\langle w, z\rangle = \int_{\mathbf{R}^n} w(x)\, z(x)\, \psi(x)\, dx \qquad \forall w, z \in H. \tag{3.25}$$

Let V' be the space dual to V. We consider $V \subset H = H' \subset V'$. For $v' \in V'$ and $v \in V$, denote the value of v' on v by $\langle v', v\rangle$. (This can be done so as not to conflict with the use of $\langle\ ,\ \rangle$ in (3.25).) For V', use the usual norm

$$|v'|_{V'} = \sup_{v \in V,\, |v|_V \leqslant 1} \langle v', v\rangle, \forall v' \in V'. \tag{3.26}$$

For any $y \in \mathbf{R}^n$, define $B(y) = \sum_{i=1}^n \beta(y_i + c_i)$. For any $\alpha^* > 0$ and $F \in L^\infty_{\mathrm{loc}}(\mathbf{R}^n)$ we say that $U \in W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}^n)$ is a *weak solution* of

$$A_0 U + \frac{1}{\varepsilon} B(\nabla U) + \alpha^* U = F \tag{3.27}$$

if and only if for every test function $v \in C^1(\mathbf{R}^n)$ of compact support we have

$$\int_{\mathbf{R}^n} \left[\sum_{i,j=1}^{n} a_{ij} \frac{\partial U}{\partial x_i} \frac{\partial v}{\partial x_j} - \sum_{i=1}^{n} g_i \frac{\partial U}{\partial x_i} v + \frac{1}{\varepsilon} B(\nabla U)\, v + \alpha^* U v - F v\right] dx = 0. \tag{3.28}$$

Lemma 3.3. *There is a large enough constant $\alpha_0 > \alpha$ such that for every $g \in V'$ there exsits a unique weak solution $u \in V$ to the equation*

$$A_0 u + \frac{1}{\varepsilon} B(\nabla u) + \alpha_0 u = g. \tag{3.29}$$

Moreover, the solution $u \in V$ depends continuously on $g \in V'$.

Proof. Use Corollary 1.8 in Chap. III of [KS] with their $\mathbf{K} = X$ equal to our V and their Au equal to our $A_0 u + (1/\varepsilon) B(\nabla u) + \alpha_0 u - g$. It is easy to show that A is continuous from V into V'. It is straightforward to show that for a large enough $\alpha_0 > \alpha$ there is a constant $\nu_0 > 0$ such that

$$\langle Au - Av, u - v\rangle \geqslant \nu_0 \, |u - v|_V^2 \qquad \forall u, v \in V. \tag{3.30}$$

This shows that A is monotone and coercive. Thus Coro. 1.8 of [KS] guarantees the existence of a weak solution $u \in V$ for every $g \in V'$.

From (3.30) we also easily obtain uniqueness and continuous dependence. ∎

For any $\mu > 0$, $q > 0$, and $f: \mathbf{R}^n \to \mathbf{R}$, define

$$\|f\|_{\mu, q} = |f(x)(\mu + |x|^2)^{-q}|_{L^\infty(\mathbf{R}^n)}. \tag{3.31}$$

For any $q > 0$, let Z_q be the set of all continuous functions $f: \mathbf{R}^n \to \mathbf{R}$ such that $f(x)(1 + |x|^2)^{-q} \to 0$ as $|x| \to \infty$.

Lemma 3.4. *If $q > 0$ and $P > n/2 + 2q$, then*

(a) *$f \in Z_q \Rightarrow f \in V'$ and*

(b) *if $f \in Z_q$, $f_k \in Z_q$ for $k = 1, 2, \ldots$, and for some $\mu > 0$ we have $\|f_k - f\|_{\mu, q} \to 0$ as $k \to \infty$, then $f_k \to f$ in V' as $k \to \infty$.*

Proof. (a) It is easy to see that if $f \in Z_q$ with $P > n/2 + 2q$, then $f \in H \subset V'$.

(b) Again using $P > n/2 + 2q$, it is easy to use the Cauchy-Schwarz inequality to show that $|f_k - f|_{V'} \leqslant K \, \|f_k - f\|_{\mu, q}$ for some constant K. ∎

Let φ be a mollification kernel (fixed in what follows), i.e., $\varphi \in C^\infty(\mathbf{R}^n)$, $\varphi(x) \geqslant 0$ for all $x \in \mathbf{R}^n$, $\varphi(x) \equiv 0$ for $|x| \geqslant 1$, and $\int_{\mathbf{R}^n} \varphi(x)\, dx = 1$. For any $f \in Z_q$, $k = 1, 2, \ldots$, and $x \in \mathbf{R}^n$, define

$$F_k(x) = \begin{cases} f(x) & \text{if } |x| \leqslant k \\ 0 & \text{if } |x| > k \end{cases} \qquad f_k(x) = \int_{\mathbf{R}^n} k^n \varphi(k(x - y))\, F_k(y)\, dy.$$

LEMMA 3.5. *Let $q>0$. Let $f\in Z_q$ and let $f_1, f_2, \ldots$ be defined as above. Then*

(a) *$f_k\in C_0^\infty(\mathbf{R}^n)$ for $k=1,2,\ldots$,*

(b) *for every $x\in\mathbf{R}^n$, $\lim_{k\to\infty} f_k(x)=f(x)$, the convergence being uniform on any compact set,*

(c) *for every constant $\lambda>0$, $\|f_k-f\|_{\lambda,q}\to 0$ as $k\to\infty$, and*

(d) *for every constant $\lambda>0$, $\lim_{k\to\infty}\|f_k\|_{\lambda,q}=\|f\|_{\lambda,q}$.*

Proof. Properties (a) and (b) follow immediately from Theorems 1.5 and 1.7 of [Ag]. The proof of (c) is straightforward. Once (c) has been proved, (d) follows immediately. ∎

LEMMA 3.6. *Let $q>0$. For $i=1,2$, let $F_i\in Z_q$ and $u_i\in C^1(\mathbf{R}^n)\cap Z_q$. Let ε and α^* be positive constants. For $i=1,2$, let u_i be a weak solution of*

$$A_0u_i+\frac{1}{\varepsilon}B(\nabla u_i)+\alpha^*u_i=F_i.$$

Then for every η with $0<\eta<\alpha^*$ there is a $\Lambda_0>0$ such that for $\lambda\geqslant\Lambda_0$,

$$(\alpha^*-\eta)\,\|u_1-u_2\|_{\lambda,q}\leqslant\|F_1-F_2\|_{\lambda,q}.$$

Λ_0 here depends only on n, the coefficients of A_0, q, ε, the Lipschitz constant of B, and η.

Proof. This follows from the method of proof of Thm. 2.14 in [MT]. Theorem 8.19 of [GT] is also used. ∎

LEMMA 3.7. *Let u be the weak solution guaranteed by Lemma 3.3 of (3.29), where $g\in V'\cap C(\mathbf{R}^n)$. Then for any $\mu\in(0,1)$, $u\in C^{1,\mu}_{\mathrm{loc}}(\mathbf{R}^n)$. If also $g\in C^{0,\mu}_{\mathrm{loc}}(\mathbf{R}^n)$, then $u\in C^{2,\mu}_{\mathrm{loc}}(\mathbf{R}^n)$.*

Proof. A bootstrap argument repeatedly using Theorems 8.3 and 8.9 and Lemma 9.16 of [GT] and Theorem 5.4 in [Ad] shows that $u\in C^{1,\mu}_{\mathrm{loc}}(\mathbf{R}^n)$ for any $\mu\in(0,1)$. If also $g\in C^{0,\mu}_{\mathrm{loc}}(\mathbf{R}^n)$, by the Schauder theory (e.g., Lemma 6.10 of [GT]) we have $u\in C^{2,\mu}_{\mathrm{loc}}(\mathbf{R}^n)$. ∎

LEMMA 3.8. *Let $\varepsilon>0$, $P>0$, and $q>0$ be constants, with $P>n/2+2q$. Then there is a $\Lambda_1>0$ such that for $\Lambda\geqslant\Lambda_1$, if $f\in Z_q$ and if u is the unique weak solution guaranteed by Lemmas 3.3 and 3.4 to*

$$A_0u+\frac{1}{\varepsilon}B(\nabla u)+\alpha_0u=f,$$

then $\|f\|_{\Lambda,q}\geqslant(\alpha_0/2)\,\|u\|_{\Lambda,q}$.

Proof. Let f_1, $f_2, \ldots$ be defined as they were immediately before Lemma 3.5. For $k = 1, 2, \ldots$, let u_k be the unique solution in $C^2(\mathbf{R}^n) \cap W_1^\infty(\mathbf{R}^n)$ of $A_0 u_k + (1/\varepsilon)\, B(\nabla u_k) + \alpha_0 u_k = f_k$ guaranteed by Propositions 3.2 and 3.3 of [Bur]. Apply Lemma 3.6 with $\alpha^* = \alpha_0$, $F_1 = f_k$, and $F_2 = 0$. Thus for $\eta = \alpha_0/2$ there is a $\Lambda_1 > 0$ such that for $\Lambda \geqslant \Lambda_1$ we have

$$\frac{\alpha_0}{2} \|u_k\|_{\Lambda, q} \leqslant \|f_k\|_{\Lambda, q}. \tag{3.32}$$

By Lemma 3.5(c) we have $\|f_k - f\|_{\Lambda, q} \to 0$ as $k \to \infty$. Thus by Lemma 3.4(b) we have $f_k \to f$ in V'. By the continuity part of the statement of Lemma 3.3, $u_k \to u$ in V. Thus some subsequence of $\{u_k\}_{k=1}^\infty$ converges almost everywhere to u. Taking $k \to \infty$ in (3.32) for this subsequence, using Lemma 3.5(d), we get the stated result. ∎

THEOREM 3.9. *Let $\varepsilon > 0$. Let $\alpha > 0$ be our discount factor. Let m and f be as in* (3.1). *Let $q > m/2$. Then* (3.15) *has a weak solution $u_\varepsilon \in Z_q$. This weak solution is unique among all continuous functions of at most polynomial growth* (*i.e., functions in $Z_{q'}$ for some $q' > 0$*). *Moreover, for every $\mu \in (0, 1)$, $u_\varepsilon \in C^{2,\mu}_{\mathrm{loc}}(\mathbf{R}^n)$.*

Proof. We assume as usual that $P > n/2 + 2q$. Choose q' with $m/2 < q' < q$. Note that $f \in Z_{q'}$. For any $u \in Z_{q'}$, define $U = T_f u$ to be the weak solution of

$$A_0 U + \frac{1}{\varepsilon} B(\nabla U) + \alpha_0 U = (\alpha_0 - \alpha)\, u + f$$

guaranteed by Lemmas 3.3 and 3.4. For u_1 and u_2 in $Z_{q'}$, let $U_1 = T_f u_1$ and $U_2 = T_f u_2$. Apply Lemma 3.6 with $F_1 = (\alpha_0 - \alpha)\, u_1 + f$ and $F_2 = (\alpha_0 - \alpha)\, u_2 + f$. By Lemma 3.8 (with q replaced by q'), $\|U_1\|_{1, q'}$ and $\|U_2\|_{1, q'}$ are finite, so $U_1 - U_2 \in Z_q$. Thus for $\eta = \alpha/2$, for large enough Λ_0 we have

$$\left(\alpha_0 - \frac{\alpha}{2}\right) \|T_f u_1 - T_f u_2\|_{\Lambda_0, q} \leqslant \|(\alpha_0 - \alpha)(u_1 - u_2)\|_{\Lambda_0, q}. \tag{3.33}$$

From the last part of the statement of Lemma 3.6, this same Λ_0 works for all q' with $m/2 < q' < q$. Note that (3.33) shows that T_f is a contraction map in $\|\cdot\|_{\Lambda_0, q}$ norm with contraction constant $(\alpha_0 - \alpha)(\alpha_0 - \alpha/2)^{-1} < 1$. Since any weak solution of (3.15) in some $Z_{q'}$ space is a fixed point of T_f, this proves the uniqueness part of the theorem. Having proved this, we may assume hereafter that $P = n + m$. Since $Z_r \subset Z_s$ for $0 < r < s$, it suffices to continue our proof assuming that $q < m/2 + n/4$ (so that $P = n + m > n/2 + 2q$).

We will now prove that T_f is a contraction map of Z_q into itself. Assume now that $u_1, u_2 \in Z_q$. We wish to prove that (3.33) remains true. Using Lemma 3.5, we can find sequences $\{u_{1,k}\}_{k=1}^{\infty}$ and $\{u_{2,k}\}_{k=1}^{\infty}$ in $C_0^\infty(\mathbf{R}^n)$ which converge in $\|\cdot\|_{\Lambda_{0,q}}$ norm to u_1 and u_2 respectively. Equation (3.33) is clearly true when u_1 and u_2 are replaced by $u_{1,k}$ and $u_{2,k}$, respectively. Since, for $i=1,2$, $(\alpha_0-\alpha)\,u_{i,k}+f \to (\alpha_0-\alpha)\,u_i+f$ in $\|\cdot\|_{\Lambda_{0,q}}$ norm as $k\to\infty$, by Lemma 3.4 this convergence also occurs in V'; thus by Lemma 3.3, $T_f u_{i,k} \to T_f u_i$ in V, so some subsequence of $\{T_f u_{i,k}\}_{k=1}^{\infty}$ converges almost everywhere to $T_f u_i$, so

$$\|T_f u_1 - T_f u_2\|_{\Lambda_{0,q}} \leqslant \frac{\alpha_0-\alpha}{\alpha_0-\alpha/2}\,\|u_1-u_2\|_{\Lambda_{0,q}} \qquad \forall u_1, u_2 \in Z_q.$$

(The above argument shows that $T_f u_1 - T_f u_2 \in Z_q$. A similar argument shows that $T_f u_1$ and $T_f u_2$ are individually in Z_q.)

Let u be the unique fixed point of T_f in Z_q. Clearly then u is a weak solution of (3.15), so $u=u_\varepsilon$ in the sense of the statement of this theorem. (We will see in the next theorem that this does not conflict with our previous definition of u_ε in (3.18).) By Lemma 3.7, the assertions about the smoothness of $u=u_\varepsilon$ follow immediately. ∎

Remark. The previous theorem proved existence and uniqueness among all functions with at most polynomial growth as $|x|\to\infty$. That these are the appropriate functions to study is seen by considering the corresponding linear problem (the above problem with $B\equiv 0$). See p. 226 of [Mir] for a brief discussion and a reference to a paper which solves the linear problem in such spaces.

THEOREM 3.10. *Make the same assumptions as in Theorem* 3.2. *Then the optimal cost* u_ε *given by* (3.18) *is the solution* u_ε *of the Hamilton–Jacobi–Bellman equation* (3.15). *Moreover, for every* $\mu\in(0,1)$, $u_\varepsilon \in C^{2,\mu}_{\mathrm{loc}}(\mathbf{R}^n)$.

Proof. Fix $\varepsilon>0$ and $q>m/2$. Let u_ε be the unique solution of (3.15) in $C^{2,\mu}_{\mathrm{loc}}(\mathbf{R}^n)\cap Z_q$, for every $\mu\in(0,1)$, guaranteed by Theorem 3.9. We will prove that

$$u_\varepsilon(x) = \inf\{J_x(\eta,\xi);\,(\eta,\xi)\in U_\varepsilon\}, \qquad x\in\mathbf{R}^n. \tag{3.34}$$

Indeed, for any $(\eta,\xi)\in U_\varepsilon$ we can use Itô's formula to get

$$u_\varepsilon(x) = E\left\{\int_0^T (Lu_\varepsilon)(y_x(t))\,e^{-\alpha t}\,dt - \int_0^T \eta(t)\cdot\nabla u_\varepsilon(y_x(t))\,e^{-\alpha t}\,dt \right.$$
$$\left. + u_\varepsilon(y_x(T))\,e^{-\alpha T}\right\}$$

for any $T>0$. Because $u_\varepsilon \in Z_q$, we may let T go to infinity and use (3.15) to deduce that

$$u_\varepsilon(x) = E\left\{\int_0^\infty f(y_x(t))\, e^{-\alpha t}\, dt\right\}$$
$$+\sum_{i=1}^n E\left\{\int_0^\infty \left[-\frac{1}{\varepsilon}\beta\left(\frac{\partial u_\varepsilon}{\partial x_i}(y_x(t))+c_i\right)-\eta_i(t)\frac{\partial u_\varepsilon}{\partial x_i}(y_x(t))\right] e^{-\alpha t}\, dt\right\}. \tag{3.35}$$

By the definition of U_ε, we obtain from (3.35) the inequality

$$u_\varepsilon(x) \leqslant J_x(\eta, \xi) \qquad \forall(\eta, \xi)\in U_\varepsilon, \forall x \in \mathbf{R}^n. \tag{3.36}$$

Now define $\hat\eta(y) = (\hat\eta_1(y), ..., \hat\eta_n(y))$ and $\hat\xi(y) = (\hat\xi_1(y), ..., \hat\xi_n(y))$ by

$$\left.\begin{aligned}
&\hat\eta_i(y) = -\frac{1}{\varepsilon}\beta'\left(\frac{\partial u_\varepsilon}{\partial x_i}(y)+c_i\right)\\
&\hat\xi_i(y) = \frac{1}{\varepsilon}\beta'\left(\frac{\partial u_\varepsilon}{\partial x_i}(y)+c_i\right)\left[\frac{\partial u_\varepsilon}{\partial x_i}(y)+c_i\right]-\frac{1}{\varepsilon}\beta\left(\frac{\partial u_\varepsilon}{\partial x_i}(y)+c_i\right)\\
&\qquad i=1, ..., n,\ y\in\mathbf{R}^n,
\end{aligned}\right\} \tag{3.37}$$

which produces an optimal feedback law for the penalized problem. That is, we solve the stochastic differential equation (see [BL, Thm. 3.5 in Chap. 2])

$$\begin{cases} d\hat y_x(t) = [g+\hat\eta(\hat y_x(t))]\, dt + \sigma dw(t), & t>0,\\ \hat y_x(0) = x, \end{cases}$$

and for $\hat\xi(t) \equiv \hat\xi(\hat y_x(t))$ and $\hat\eta(t) \equiv \hat\eta(\hat y_x(t))$, we have

$$u_\varepsilon(x) = J_x(\hat\eta, \hat\xi) \qquad \text{and} \qquad (\hat\eta(t), \hat\xi(t)) \in U_\varepsilon. \tag{3.38}$$

Clearly (3.36) and (3.38) together prove (3.34). ∎

THEOREM 3.11. *Make the assumptions of Theorem 3.2. Fix p with $n < p < \infty$. Let $\Omega \subset \mathbf{R}^n$ be an open ball. Then there is a sequence $\{\varepsilon_k\}_{k=1}^\infty$ with $\varepsilon_k \to 0+$ as $k\to\infty$ such that for $1 \leqslant i, j \leqslant n$*

$$u_{\varepsilon_k} \to u \qquad \textit{and} \qquad \frac{\partial u_{\varepsilon_k}}{\partial x_i} \to \frac{\partial u}{\partial x_i} \textit{ uniformly on } \bar\Omega$$

and

$$\frac{\partial^2 u_{\varepsilon_k}}{\partial x_i\,\partial x_j} \to \frac{\partial^2 u}{\partial x_i\,\partial x_j} \textit{ weakly in } L^p(\Omega), \qquad \textit{as} \quad k\to\infty.$$

Proof. By the proof of Theorem 3.2, there is a $K_1 > 0$ such that

$$|u_\varepsilon| \leqslant K_1, \qquad |\partial u_\varepsilon/\partial x_i| \leqslant K_1, \qquad \text{and} \qquad |\partial^2 u_\varepsilon/\partial x_i\, \partial x_j| \leqslant K_1$$
$$\text{on} \quad \Omega \text{ for } 1 \leqslant i, j \leqslant n,\ 0 < \varepsilon < \varepsilon_0.$$

Since $W^{2,p}(\Omega)$ is reflexive (see, for example, [Ad, p. 46]), there is a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ with $\varepsilon_k \to 0+$ as $k \to \infty$ such that u_{ε_k} converges weakly in $W^{2,p}(\Omega)$. Since $u_{\varepsilon_k} \to u$ pointwise (by Theorem 3.2) and since weak limits are unique, $u_{\varepsilon_k} \to u$ weakly in $W^{2,p}(\Omega)$ as $k \to \infty$. Since $p > n$, by the Rellich–Kondrachov Theorem (Thm. 6.2 in [Ad]) the imbedding map $W^{2,p}(\Omega) \to C^1(\bar{\Omega})$ is compact. Thus $u_{\varepsilon_k} \to u$ and $\partial u_{\varepsilon_k}/\partial x_i \to \partial u/\partial x_i$ (for $i = 1, ..., n$) uniformly on $\bar{\Omega}$ as $k \to \infty$. ∎

THEOREM 3.12. *Make the same assumptions as in Theorem 3.2. Then for every $\mu \in (0, 1)$, $u_\varepsilon \in C^{4,\mu}_{\text{loc}}(\mathbf{R}^n)$.*

Proof. Since u_ε satisfies (3.15), we have

$$Lu_\varepsilon = f - \frac{1}{\varepsilon} \sum_{i=1}^{n} \beta\left(\frac{\partial u_\varepsilon}{\partial x_i} + c_i\right), \qquad x \in \mathbf{R}^n. \tag{3.39}$$

Let $\mu \in (0, 1)$. Since $u_\varepsilon \in C^{2,\mu}_{\text{loc}}(\mathbf{R}^n)$ (by Theorem 3.10), $f \in C^3(\mathbf{R}^n)$, and $\beta \in C^\infty(\mathbf{R})$, the R.H.S. of (3.39) is in $C^{1,\mu}_{\text{loc}}(\mathbf{R}^n)$. Thus by Theorem 36,V of [Mir], $u_\varepsilon \in C^{3,\mu}_{\text{loc}}(\mathbf{R}^n)$. Thus the R.H.S. of (3.39) is in $C^{2,\mu}_{\text{loc}}(\mathbf{R}^n)$, so again applying Theorem 36,V of [Mir] we have $u_\varepsilon \in C^{4,\mu}_{\text{loc}}(\mathbf{R}^n)$. ∎

4. REGULARITY OF THE FREE BOUNDARY AWAY FROM "CORNER POINTS"

THEOREM 4.1. *Let the assumptions of Theorem 3.1 be satisfied. Then for $i = 1, ..., n$ there exists a real-valued function $\psi_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, such that*

$$\frac{\partial u}{\partial x_i}(x) + c_i = 0 \qquad \text{if} \quad x_i \leqslant \psi_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$$

and

$$\frac{\partial u}{\partial x_i}(x) + c_i > 0 \qquad \text{if} \quad x_i > \psi_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$$

for each $x = (x_1, ..., x_n) \in \mathbf{R}^n$.

Proof. The proof is essentially the same as that given for Theorem 4.1 in [MR]. ∎

DEFINITION 4.2. For any i with $i = 1, ..., n$, define

$$\mathscr{S}_i = \left\{ x \in \mathbf{R}^n; \frac{\partial u}{\partial x_j}(x) + c_j > 0 \text{ for all } j \neq i \right\} \tag{4.1}$$

and

$$\mathscr{F}_i = \mathscr{S}_i \cap \{x = (x_1, ..., x_n) \in \mathbf{R}^n; x_i = \psi_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)\}. \tag{4.2}$$

The free boundary is $\overline{\mathscr{S}_1 \cap \cdots \cap \mathscr{S}_n} \sim (\mathscr{S}_1 \cap \cdots \cap \mathscr{S}_n)$. We will show that each portion $\mathscr{F}_i$, $i = 1, ..., n$, of this is regular. All other free boundary points will be called *corner points*. By symmetry, it clearly suffices to study the regularity of $\mathscr{F}_n$. This is what will be done below.

If $n = 3$ with $\psi_1(x_2, x_3) \equiv \psi_2(x_1, x_3) \equiv \psi_3(x_1, x_2) \equiv 0$, $\mathscr{S}_1 \cap \mathscr{S}_2 \cap \mathscr{S}_3$ is the principal octant. $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$ are quarter planes, and the corner points are points on the nonnegative coordinate axes. The reader should be warned that this paper does not prove that the corner points always have this simple type of structure (although the authors believe that to be true).

LEMMA 4.3. *Assume* (2.6) *and assume that* $\alpha > 0$ *is constant. Let* $\Omega \subset \mathbf{R}^n$ *be an open ball. Let* $a(u, v)$ *be defined by* (2.11). *Then* $a(u, v)$ *is coercive on* $H_0^1(\Omega)$.

Proof. For any $u \in H_0^1(\Omega)$ it is easy to prove that

$$\int_\Omega u \frac{\partial u}{\partial x_i} \, dx = 0 \qquad \text{for} \quad i = 1, ..., n.$$

From the positive definiteness of (a_{ij}) and the fact that $\alpha > 0$, the coercivity of $a(u, v)$ on $H_0^1(\Omega)$ follows easily. ∎

DEFINITION 4.4. (Compare with problem 5 on pp. 30, 31 of [Fr].) We say that w is a *local solution of*

$$a(w, v - w) \geqslant (F, v - w) \qquad \text{for every} \quad v \in K, \tag{4.3}$$

where

$$K = \{v \in H^1(\Omega); v \geqslant 0 \text{ a.e. in } \Omega\}, \tag{4.4}$$

if and only if $w \in K$ and for every $\eta \in C_0^\infty(\Omega)$ with $\eta \geqslant 0$ we have

$$a(w, \eta(v - w)) \geqslant \int_\Omega F\eta(v - w) \, dx \qquad \text{for every} \quad v \in K. \tag{4.5}$$

THEOREM 4.5. *Let the assumptions of Theorem* 3.2 *be satisfied. Let* Ω *be an open ball with* $\bar{\Omega} \subset \mathscr{S}_n$. *Then* $w \equiv \partial u / \partial x_n + c_n$ *is a local solution of* (4.3) *with* K *given by* (4.4) *and* $F \equiv \partial f / \partial x_n + \alpha c_n$.

Proof. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be the sequence in Theorem 3.11 and let $u_k = u_{\varepsilon_k}$ for $k = 1, 2, \ldots$. Since $\partial u/\partial x_1 + c_1 > 0, \ldots, \partial u/\partial x_{n-1} + c_{n-1} > 0$ on $\bar{\Omega}$, since $u \in C^1(\mathbf{R}^n)$ (by Theorem 3.1), and since $\partial u_k/\partial x_i \to \partial u/\partial x_i$ for $1 \leqslant i \leqslant n$ uniformly on $\bar{\Omega}$ as $k \to \infty$ (by Theorem 3.11), there is a K_2 such that for $k \geqslant K_2$ we have $\partial u_k/\partial x_1 + c_1 > 0, \ldots, \partial u_k/\partial x_{n-1} + c_{n-1} > 0$ on $\bar{\Omega}$. Thus $\beta(\partial u_k/\partial x_1 + c_1) \equiv \cdots \equiv \beta(\partial u_k/\partial x_{n-1} + c_{n-1}) \equiv 0$ on Ω for $k \geqslant K_2$ so that (3.15), which u_k satisfies because of Theorem 3.10, becomes

$$Lu_k + \frac{1}{\varepsilon_k}\beta\left(\frac{\partial u_k}{\partial x_n} + c_n\right) = f, \qquad x \in \Omega, k \geqslant K_2.$$

Fix $\delta \in (0, 1)$. By Theorem 3.10, $u_k \in C^{2,\delta}(\bar{\Omega})$ for $k = 1, 2, \ldots$. Thus $f - (1/\varepsilon_k)\,\beta(\partial u_k/\partial x_n + c_n) \in C^{1,\delta}(\bar{\Omega})$, so that the Schauder theory (see Theorem 6.17 of [GT]) shows that $u_k \in C^{3,\delta}(\bar{\Omega})$, so that

$$L\frac{\partial u_k}{\partial x_n} + \frac{1}{\varepsilon_k}\beta'\left(\frac{\partial u_k}{\partial x_n} + c_n\right)\frac{\partial^2 u_k}{\partial x_n^2} = \frac{\partial f}{\partial x_n} \qquad \text{on } \Omega \text{ for } k \geqslant K_2.$$

Let K be given by (4.4), let $\eta \in C_0^{\infty}(\Omega)$ with $\eta \geqslant 0$, and let $v \in K$. Defining $w_k \equiv \partial u_k/\partial x_n + c_n$ for $k = 1, 2, \ldots$, we clearly have for $k \geqslant K_2$ that

$$(Lw_k, \eta(v - w_k)) + \left(\frac{1}{\varepsilon_k}\beta'(w_k)\frac{\partial^2 u_k}{\partial x_n^2}, \eta(v - w_k)\right) = \left(\frac{\partial f}{\partial x_n} + \alpha c_n, \eta(v - w_k)\right).$$

The first term is clearly equal to $a(w_k, \eta(v - w_k))$. At points where $w_k \geqslant v \geqslant 0$, $\beta'(w_k) = 0$, so the integrand of the second term is zero. But at points where $w_k < v$, the integrand of the second term is nonpositive, since $\varepsilon_k > 0$, $\beta'(w_k) \leqslant 0$, $\partial^2 u_k/\partial x_n^2 \geqslant 0$ (by Theorem 3.2), $\eta \geqslant 0$, and $v - w_k > 0$. Thus

$$a(w_k, \eta(v - w_k)) \geqslant \left(\frac{\partial f}{\partial x_n} + \alpha c_n, \eta(v - w_k)\right) \qquad \text{for} \quad k \geqslant K_2.$$

Taking the limit as $k \to \infty$, using the full strength of the convergence of u_k to u described in Theorem 3.11, we now wish to obtain

$$a(w, \eta(v - w)) \geqslant \left(\frac{\partial f}{\partial x_n} + \alpha c_n, \eta(v - w)\right).$$

This does not come trivially, since $a(w_k, \eta(v - w_k))$ involves the term

$$\int_{\Omega} -\sum_{i,j=1}^{n} a_{ij}\eta \frac{\partial w_k}{\partial x_i}\frac{\partial w_k}{\partial x_j}\,dx,$$

which does not necessarily converge to

$$\int_{\Omega} -\sum_{i,j=1}^{n} a_{ij}\eta \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\,dx \qquad \text{as} \quad k \to \infty.$$

However, all the other terms of $a(w_k, \eta(v-w_k))$ converge to their expected limits, while

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\eta \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\,dx \leqslant \liminf_{k\to\infty} \int_\Omega \sum_{i,j=1}^n a_{ij}\eta \frac{\partial w_k}{\partial x_i}\frac{\partial w_k}{\partial x_j}\,dx.$$

(To see this, apply Lemma II.3.27 of [DS] to $L^2(D)\oplus \cdots \oplus L^2(D)$ [with n terms in this sum] with

$$\|(f_1, ..., f_n)\| \equiv \sqrt{\int_D \sum_{i,j=1}^n a_{ij}\eta f_i f_j}, \qquad \text{where} \quad D=\{x\in \mathbf{R}^n; \eta(x)>0\}.\Big)$$

The desired result now follows with no problem. Note that $w\geqslant 0$ a.e. in Ω because of (2.9). ∎

THEOREM 4.6. *Let the assumptions and notations be the same as in Theorem* 4.5. *Then* $w=\partial u/\partial x_n + c_n \in W^{2,\infty}(\Omega)$ *and* w *satisfies*

$$Lw\geqslant F, \qquad w\geqslant 0, \qquad (Lw-F)\,w=0 \text{ a.e. in } \Omega. \tag{4.6}$$

Proof. Let B be an open ball with $\bar{\Omega}\subset B\subset \bar{B}\subset \mathscr{S}_n$. By [Fr, problem 5, pp. 30, 31], the fact that $w=\partial u/\partial x_n + c_n$ is a local solution as described in Definition 4.4 (with Ω replaced by B) proves that $w\in W^{2,p}(\Omega)$ for every p with $1<p<\infty$. (The authors actually used Thm. I.1 on p. 7 of [Br] to prove problem 5 of [Fr] instead of using problem 1 on p. 29 of [Fr].)

Using [Fr, problem 1, p. 44], we then get $w=\partial u/\partial x_n + c_n \in W^{2,\infty}(\Omega)$. The method of proof of the special case of [Fr, problem 1, p. 44] sufficient for our needs involves showing that (3.18) on p. 26 of [Fr] holds with A, f, u, Ω, g, and φ replaced by L, F^*, γw, B, 0, and 0, respectively. Here $\gamma\in C_0^\infty(B)$ with $0\leqslant\gamma\leqslant 1$ on B and $\gamma\equiv 1$ on Ω, while

$$F^* \equiv \gamma F - \sum_{i,j=1}^n \left\{a_{ij} w \frac{\partial^2 \gamma}{\partial x_i\,\partial x_j} + 2a_{ij}\frac{\partial w}{\partial x_i}\frac{\partial \gamma}{\partial x_j}\right\} - \sum_{i=1}^n g_i \frac{\partial \gamma}{\partial x_i} w.$$

(To understand F^*, see the hint for problem 5 on pp. 30, 31 of [Fr].) Then Theorem 4.1 of [Fr] can be applied with the same replacements as above. Since $\gamma\equiv 1$ on Ω, an easy consequence of this proof is that (4.6) holds. ∎

LEMMA 4.7. *As in problem* 6 *on p.* 203 *of* [Fr], *for* $x_n>0$ *define* $\theta=\theta(x)=\cos^{-1}(x_n/|x|)$. *Let* $z=\cos\theta=x_n/r$, *where* $r=|x|$ *as usual. If for some constant* λ, $u=r^\lambda g(z)$ *on some open subset of* $\{x\in\mathbf{R}^n; x_n>0\}$, *where* $g\in C^2(\mathbf{R})$, *then*

$$\Delta u = r^{\lambda-2}[g''(z)(1-z^2)+g'(z)(1-n)\,z+\lambda(\lambda+n-2)\,g(z)].$$

Proof. This can be shown by straightforward (but tedious) computation. ∎

To obtain a crucial technical result (that $F = \partial f/\partial x_n + \alpha c_n < 0$ on $\mathscr{F}_n$), we need a generalization of Lemma 7.3 on p. 195 of [Fr]. This generalization may be of interest in its own right. Except for φ (which we will take to be 0 in our application) and Ω (which we will take to be an open ball with $\bar{\Omega} \subset \mathscr{S}_n$), the notation chosen below shows how we will apply the theorem.

THEOREM 4.8. *Let Ω be a domain in $\mathbf{R}^n$ and let w be a solution of the obstacle problem*

$$Lw - F \geqslant 0, \qquad w \geqslant \varphi, \qquad (Lw - F)(w - \varphi) = 0 \text{ a.e. in } \Omega,$$

$$|w|_{C^{1,1}(\Omega)} \leqslant M < \infty, \qquad \varphi \in C^3(\Omega).$$

Here L is given by (2.7). *We assume that* (2.6) *holds, that $(a_{ij}) = \frac{1}{2}\sigma\sigma^{\mathrm{T}}$, and that $\alpha \geqslant 0$.*

Assume also that $F \in C^1(\Omega)$ and that $-F + L\varphi$ and $\nabla(-F + L\varphi)$ do not vanish simultaneously in Ω. Then $-F + L\varphi > 0$ on the free boundary of w in Ω.

Proof. Let the coincidence set Λ of w in Ω be defined by $\Lambda = \{x \in \Omega;\ w(x) = \varphi(x)\}$. Let the free boundary of w in Ω be denoted by Γ, where $\Gamma = \partial\Lambda \cap \Omega$. We will first show that $-F + L\varphi \geqslant 0$ on Γ. Assume for contradiction that there were an $x_0 \in \Gamma$ at which $(-F + L\varphi)(x_0) < 0$. Then $v \equiv w - \varphi$ satisfies $Lv = Lw - L\varphi \geqslant F - L\varphi > 0$ and $v \geqslant 0$ in a neighborhood of x_0, with the minimum 0 of v being attained at the interior point x_0. By the strong maximum principle (e.g., Thm. 8.19 of [GT] with their L and u replaced by our $-L$ and $-v$, respectively) we have $v \equiv 0$ on that neighborhood. Thus $w \equiv \varphi$ on that neighborhood, contradicting our assumption that $x_0 \in \Gamma$.

Making the nonsingular linear change of variables $y = 2^{-1/2}\sigma^{-1}x$ converts the above problem into a similar one in which (a_{ij}) is replaced by the identity matrix. Thus, without loss of generality, we will assume from now on that (a_{ij}) is the identity matrix, so that $Lw = -\Delta w - \sum_{i=1}^n g_i\, \partial w/\partial x_i + \alpha w$.

Using problem 6 on p. 203 of [Fr] there is a ψ with $0 < \psi < \pi/2$ and a λ with $1 < \lambda < 2$ such that the function $v = |x|^\lambda f_\lambda(\theta)$ is harmonic and positive on the cone $K_\psi = \{x;\, x_n > 0,\ \cos^{-1}(x_n/|x|) < \psi\}$, with $v = 0$ on ∂K_ψ. (The method of construction of v in [Fr] guarantees that v will also be harmonic on a slightly larger cone K_{ψ^*} introduced below.) Taking $z = \cos\theta$ and $g(z) = f_\lambda(\theta)$, we have $g(\cos\psi) = f_\lambda(\psi) = 0$. Clearly $g'(\cos\psi) \neq 0$ (since otherwise $g(z)$ would be the zero solution of its ordinary differential equation). Since $g(z) > 0$ for $\cos\psi < z \leqslant 1$ (i.e., for

$\psi > \theta \geqslant 0$), we clearly must have $g'(\cos \psi) > 0$. Thus there is a ψ^* with $\psi < \psi^* < \pi/2$ such that $f_\lambda(\theta) = g(\cos \theta) < 0$ for $\psi < \theta \leqslant \psi^*$. This ψ^* gives us the opening size we will use for a new cone

$$K_{\psi^*} = \{x; x_n > 0, \cos^{-1}(x_n/|x|) < \psi^*\}.$$

Let $\Psi \equiv -F + L\varphi$. We have already proved that $\Psi \geqslant 0$ on Γ. What we have to prove is that $\Psi > 0$ on Γ. Thus assume for contradiction that for some point $x_0 \in \Gamma$ we have $\Psi(x_0) = 0$. Since, by assumption, $\Psi(x_0)$ and $\nabla\Psi(x_0)$ cannot both be zero, we must have $\nabla\Psi(x_0) \neq 0$. We can assume, without loss of generality (by translating and rotating our coordinate system if necessary), that our origin is at x_0 and that the positive x_n-axis points opposite to the direction of $\nabla\Psi(x_0)$. For a small enough $R > 0$ we then have

$$\Psi < 0 \qquad \text{in} \quad K_{\psi^*} \cap B_R(x_0),$$

where $B_R(x_0)$ is the open ball of radius R centered at x_0. The function $V \equiv w - \varphi$ has $LV = Lw - L\varphi \geqslant F - L\varphi = -\Psi$, so

$$LV > 0 \qquad \text{in} \quad K_{\psi^*} \cap B_R(x_0).$$

Since $V \geqslant 0$, the strong maximum principle (e.g., Thm. 8.19 of [GT]; note that $\alpha \geqslant 0$ is used here) gives

$$V > 0 \qquad \text{in} \quad K_{\psi^*} \cap B_R(x_0).$$

Fix an $\varepsilon > 0$ such that $\lambda + \varepsilon < 2$. Let $r = |x|$ as usual. As we will prove below, there is an r_0 with $0 < r_0 < R$ such that

$$L(r^\lambda f_\lambda(\theta) + r^{\lambda+\varepsilon}) < 0 \qquad \text{in} \quad K_{\psi^*} \cap B_{r_0}(x_0). \tag{4.7}$$

Since $f_\lambda(\psi^*) < 0$ and $\varepsilon > 0$, there is an r_1 with $0 < r_1 \leqslant r_0$ such that $r^\lambda f_\lambda(\psi^*) + r^{\lambda+\varepsilon} \leqslant 0$ for $0 \leqslant r \leqslant r_1$. Thus for any K with $0 < K < 1$ we have

$$V \geqslant 0 \geqslant K(r^\lambda f_\lambda(\theta) + r^{\lambda+\varepsilon}) \qquad \text{in} \quad \partial K_{\psi^*} \cap \overline{B_{r_1}(x_0)}.$$

But on the other portion of the boundary of $K_{\psi^*} \cap B_{r_1}(x_0)$ (i.e., on $K_{\psi^*} \cap \partial B_{r_1}(x_0)$), since $V > 0$ on $K_{\psi^*} \cap \partial B_{r_1}(x_0)$, we can easily find a constant K with $0 < K < 1$ such that

$$V \geqslant K(r^\lambda f_\lambda(\theta) + r^{\lambda+\varepsilon}) \qquad \text{on} \quad K_{\psi^*} \cap \partial B_{r_1}(x_0).$$

Thus (as soon as we have proved that (4.7) holds for some $0 < r_0 < R$), we have

$$V - K(r^\lambda f_\lambda(\theta) + r^{\lambda+\varepsilon}) \geqslant 0 \qquad \text{on the boundary of} \quad K_{\psi^*} \cap B_{r_1}(x_0)$$

and

$$L(V - K(r^{\lambda} f_{\lambda}(\theta) + r^{\lambda+\varepsilon}))$$
$$= LV - KL(r^{\lambda} f_{\lambda}(\theta) + r^{\lambda+\varepsilon}) > 0 \quad \text{on} \quad K_{\psi^*} \cap B_{r_1}(x_0),$$

so that by the maximum principle (e.g., Thm. 8.1 of [GT]) we have

$$V - K(r^{\lambda} f_{\lambda}(\theta) + r^{\lambda+\varepsilon}) \geqslant 0 \quad \text{on} \quad K_{\psi^*} \cap B_{r_1}(x_0).$$

But on the coincidence set Λ, $w \equiv \varphi$ so that $V \equiv 0$ and $\nabla V \equiv 0$. Since $x_0 \in \partial \Lambda$, $V(x_0) = 0$ and $\nabla V(x_0) = 0$. Since w and φ are both in $C^{1,1}(\Omega)$, so is $V = w - \varphi$. Thus for some $M_1 > 0$ and some neighborhood N of x_0,

$$V(x) \leqslant M_1 |x - x_0|^2 \quad \text{for} \quad x \in N,$$

so that $V(x)$ can grow no faster than $M_1 r^2$ going away from x_0. But on the positive x_n-axis (with $\theta = 0$) we have

$$V(x) \geqslant K r^{\lambda} f_{\lambda}(0) + K r^{\lambda+\varepsilon} \quad \text{for} \quad 0 \leqslant r \leqslant r_1,$$

so (since $1 < \lambda < \lambda + \varepsilon < 2$, $K > 0$, and $f_{\lambda}(0) > 0$) $V(x)$ is growing faster than $M_1 r^2$, which gives our contradiction.

Thus it remains only to prove that there is an r_0 with $0 < r_0 < R$ such that (4.7) holds. Note that $r^{\lambda} f_{\lambda}(\theta)$ and $r^{\lambda+\varepsilon}$ are both of the form considered in Lemma 4.7 (with $g(\cos\theta) \equiv f_{\lambda}(\theta)$ in the first case and $g(z) \equiv 1$ in the second). Using the fact that $|g(\cos\theta)|$ and $|g'(\cos\theta)|$ are bounded for $0 \leqslant \theta \leqslant \psi^*$, while $\Delta(r^{\lambda} f_{\lambda}(\theta)) = 0$ and $\Delta r^{\lambda+\varepsilon} = r^{\lambda+\varepsilon-2}(\lambda+\varepsilon)(\lambda+\varepsilon+n-2)$, straightforward calculations and estimates give the result without too much difficulty. ∎

In addition to the above assumptions on f, we will also assume that

for $i = 1, ..., n$, $(\partial f/\partial x_i) + \alpha c_i$ and $\nabla(\partial f/\partial x_i)$ never vanish simultaneously. (4.8)

COROLLARY 4.9. *Let the assumptions and notation be the same as in Theorem 4.5. Assume that* (4.8) *holds. Then* $\partial f/\partial x_n + \alpha c_n < 0$ *on* $\mathscr{F}_n$.

Proof. Let x_0 be any point of $\mathscr{F}_n$. Let Ω be an open ball centered at x_0 with $\bar{\Omega} \subset \mathscr{S}_n$. Let $\varphi \equiv 0$. Let $w = \partial u/\partial x_n + c_n$ and $F = \partial f/\partial x_n + \alpha c_n$ as usual. Then the hypotheses of Theorem 4.8 are satisfied because of Theorem 4.6 and (4.8). ($w \in C^{1,1}(\Omega)$ follows from $w \in W^{2,\infty}(\Omega)$.) The conclusion is that $-F + L\varphi = -\partial f/\partial x_n - \alpha c_n > 0$ on the free boundary of w in Ω. Because of the result of Theorem 4.1, $w(x) = 0$ and $w(x) > 0$ both happen at points x arbitrarily close to x_0, so x_0 is in the free boundary of w in Ω. Thus $\partial f/\partial x_n + \alpha c_n < 0$ at x_0. ∎

One more technical result must be proved before the main result can be stated and proved. The proof of the following theorem is a modified form of the proof in [Ath], which itself derived from the original idea in [Alt].

THEOREM 4.10. *Let the assumptions and notation be the same as in Theorem 4.5. Assume that* (4.8) *holds. Then any point* $\bar{x} \in \mathscr{F}_n$ *is a point of positive Lebesgue density for the coincidence set.*

Proof. Let $\bar{x} = (\bar{x}_1, ..., \bar{x}_n) \in \mathscr{F}_n$. Let Ω_0 be an open ball of radius $2R$ (with $R > 0$) centered at $\bar{x}$ with $\bar{\Omega}_0 \subset \mathscr{S}_n$. By Corollary 4.9 we may take R small enough so that $\partial f/\partial x_n + \alpha c_n < 0$ on Ω_0. Since $w(\bar{x}_1, ..., \bar{x}_{n-1}, \bar{x}_n + R) > 0$, we may take r with $0 < r < R$ so that $w(x) > 0$ whenever $x \in \mathbf{R}^n$ is no more than r units of distance away from $(\bar{x}_1, ..., \bar{x}_{n-1}, \bar{x}_n + R)$. Let $\rho = \rho(x)$ be the function which assigns to any $x \in \mathbf{R}^n$ its distance to the "vertical" line through $\bar{x}$, i.e.,

$$\rho(x) = \rho(x_1, ..., x_n) = [(x_1 - \bar{x}_1)^2 + \cdots + (x_{n-1} - \bar{x}_{n-1})^2]^{1/2}.$$

Now define the set

$$D = \{(x_1, ..., x_n) \in \mathbf{R}^n; \rho(x_1, ..., x_n) < r \text{ and } \psi_n(x_1, ..., x_{n-1}) < x_n < \bar{x}_n + R\}.$$

Note that since $\partial f/\partial x_n + \alpha c_n < 0$ on $\bar{D} \cap \{x_n = \bar{x}_n + R\}$, the fact that $\partial^2 f/\partial x_n^2 \geqslant 0$ on $\mathbf{R}^n$ implies that $\partial f/\partial x_n + \alpha c_n < 0$ on $\bar{D}$. Also define

$$\eta = \eta(x_1, ..., x_n) = \left[\left(\rho(x_1, ..., x_n) - \frac{r}{2}\right)^+\right]^4.$$

Note that $\eta \geqslant 0$, $\eta \in C^2(\mathbf{R}^n)$, and that when $\rho \leqslant r/2$ we have $\eta \equiv 0$. For a ("large") $M > 0$ and for ("small") $\delta > 0$ and $\varepsilon > 0$ to be chosen later, for any $\xi = (\xi_1, ..., \xi_{n-1}) \in \mathbf{R}^{n-1}$ with $|\xi| < \delta$, and for any $x \in \bar{D}$, define

$$W(x) \equiv M \frac{\partial w}{\partial x_n}(x) + \sum_{k=1}^{n-1} \xi_k \frac{\partial w}{\partial x_k}(x) - w(x) + \varepsilon\eta(x).$$

We will apply the maximum principle (e.g., Thm. 6 in Chap. 2 of [PW]) to the function $-W$, the operator $-L$, and the set D.

Before we do this, let us make the (trivial) modification of Theorem 4.5 and our other results which allows Ω to be a ball which has been (linearly) stretched in the x_n-direction. Since the set $\bar{D}$ might be extremely long in the x_n-direction, we may need such a set in order to have $\bar{D} \subset \Omega \subset \bar{\Omega} \subset \mathscr{S}_n$. (It may be that no ball Ω can satisfy these inclusions.) We will assume that Theorem 4.5 and our other results have been modified in this way and that $\bar{D} \subset \Omega \subset \bar{\Omega} \subset \mathscr{S}_n$. To achieve this last, it is crucial to know that $\bar{x} = (\bar{x}_1, ..., \bar{x}_n) \in \mathscr{F}_n$ implies that $\hat{x} = (\bar{x}_1, ..., \bar{x}_{n-1}, x_n) \in \mathscr{S}_n$ for every $x_n < \bar{x}_n$.

(To see this, assume for contradiction that $\partial u/\partial x_j + c_j = 0$ at $\hat{x}$ for some $j = 1, ..., n-1$. Define $\Delta = \bar{x}_n - x_n$. Choose $\tau > 0$ such that every point no more than τ units from $\bar{x}$ is in $\mathscr{S}_n$. Let $\bar{x}^*$ and $\hat{x}^*$ be τ units in the negative j-coordinate direction from $\bar{x}$ and $\hat{x}$, respectively. Since $\partial u/\partial x_j \equiv -c_j$ on the segment from $\hat{x}^*$ to $\hat{x}$ and $\partial u/\partial x_n \equiv -c_n$ on the segment from $\hat{x}$ to $\bar{x}$, $u(\bar{x}) - u(\hat{x}^*) = -c_j\tau - c_n\Delta$. Since $\partial u/\partial x_n \geqslant -c_n$ on the segment from $\hat{x}^*$ to $\bar{x}^*$ and $\partial u/\partial x_j > -c_j$ on the segment from $\bar{x}^*$ to $\bar{x}$, $u(\bar{x}) - u(\hat{x}^*) > -c_j\tau - c_n\Delta$. This contradiction proves the result.)

Returning to the problem of applying the maximum principle, since $Lw \equiv \partial f/\partial x_n + \alpha c_n$ on D (by Theorem 4.6),

$$LW = M\frac{\partial^2 f}{\partial x_n^2} + \sum_{k=1}^{n-1} \xi_k \frac{\partial^2 f}{\partial x_k\, \partial x_n} - \left(\frac{\partial f}{\partial x_n} + \alpha c_n\right) + \varepsilon L\eta \qquad \text{on } D.$$

Since $\partial f/\partial x_n + \alpha c_n < 0$ on $\bar{D}$ while $L\eta$ and all the $\partial^2 f/\partial x_k\, \partial x_n$ are bounded on $\bar{D}$ and since $\partial^2 f/\partial x_n^2 \geqslant 0$, it is clearly possible to choose $\varepsilon > 0$ and $\delta > 0$ small enough so that $LW \geqslant 0$ on D whenever $|\xi| < \delta$. Thus either $W > 0$ on $\bar{D}$ or W attains its minimum on $\bar{D}$ at some point of ∂D. (The continuity of W on $\bar{D}$ comes from Theorem 4.6.) Our goal (as we shall see) is to show that $W \geqslant 0$ on $\bar{D}$, so if $W > 0$ on $\bar{D}$, we are done. Thus it suffices to show that $W \geqslant 0$ on ∂D. We will do this by proving that $W \geqslant 0$ on each of the following subsets of ∂D:

(a) First consider $\partial D \cap \{w = 0\}$. At any x in this set, $\partial w/\partial x_k = 0$ for $1 \leqslant k \leqslant n$. (To see this, consider the line l through x in the x_k-coordinate direction. Restrict w to l and consider the result a function of the single variable x_k. This function has $\partial w/\partial x_k$ restricted to l as its derivative [by Theorem 4.6] and attains its minimum value [zero] at the x_k-value corresponding to x.) Thus on this set $W(x) \equiv \varepsilon\eta(x) \geqslant 0$.

(b) Next, consider

$$N_\beta = \{x \in \partial D;\, w(x) > 0 \text{ and } \operatorname{dist}(x, \partial D \cap \{w = 0\}) < \beta\},$$

where $\beta > 0$ is chosen small enough so N_β contains no point x with $x_n = \bar{x}_n + R$. Thus for $x \in N_\beta$ we must have $\rho(x) = r$. Since $\partial w/\partial x_n = \partial^2 u/\partial x_n^2 \geqslant 0$, while w, $\partial w/\partial x_1, ..., \partial w/\partial x_{n-1}$ are Lipschitz continuous on $\bar{D}$ (by Theorem 4.6) and are 0 on $\partial D \cap \{w = 0\}$ (see (a) above), we may clearly choose $\beta > 0$ smaller if necessary so that $W > 0$ on N_β.

(c) Finally, consider the remaining set $R = \{x \in \partial D;\, w(x) > 0$ and $x \notin N_\beta\}$. At any $x \in R$, $\partial^2 u/\partial x_n^2 > 0$. (To see this, assume for contradiction that there is an $x_0 \in R$ with $\partial^2 u/\partial x_n^2 = 0$ there. Since $\partial^2 u/\partial x_n^2 \geqslant 0$ on $\Omega \cap \{w > 0\}$, $\partial^2 u/\partial x_n^2$ takes an interior minimum on $\Omega \cap \{w > 0\}$ at x_0. But on $\Omega \cap \{w > 0\}$, by Theorem 4.6 we have $L(\partial^2 u/\partial x_n^2) = \partial^2 f/\partial x_n^2 \geqslant 0$. Thus by the maximum principle $\partial^2 u/\partial x_n^2 \equiv 0$ on $\Omega \cap \{w > 0\}$, from which we can prove that $w(x_0)=0$, which gives a contradiction.) Therefore $\partial^2 u/\partial x_n^2 \geqslant m>0$ on R for some constant m. Thus for large enough M we have $W \geqslant 0$ on this portion of the boundary.

Thus $W \geqslant 0$ on ∂D, so $W \geqslant 0$ on $\bar{D}$ by the maximum principle. For $\rho \leqslant r/2$ we have $\eta \equiv 0$, so on $D \cap \{\rho \leqslant r/2\}$ we therefore have that

$$M \frac{\partial w}{\partial x_n} + \sum_{k=1}^{n-1} \xi_k \frac{\partial w}{\partial x_k} \geqslant w > 0.$$

Note that the L.H.S. is the directional derivative of w in the direction $(\xi_1, ..., \xi_{n-1}, M)$. It is easy to see that each point x of the region

$$\{x \in \mathbf{R}^n;\ \rho(x) < r/2 \text{ and } \bar{x} - x \text{ is a positive multiple of } (\xi_1, ..., \xi_{n-1}, M)$$
$$\text{for some } \quad \xi \in \mathbf{R}^{n-1} \text{ with } |\xi| < \delta\}$$

(which coincides with a cone in a neighborhood of $\bar{x}$) must be in the coincidence set. (Otherwise $x \in D$, $w(x)>0$, and going from x in the direction $(\xi_1, ..., \xi_{n-1}, M)$ increases w, so we stay in D. This contradicts the fact that we eventually come to $\bar{x}$ with $w(\bar{x})=0$.) Since this region is in the coincidence set, it follows trivially that $\bar{x}$ is a point of positive Lebesgue density for the coincidence set. ▮

THEOREM 4.11. *Let the assumptions and notation be the same as in Theorem* 4.5. *Assume that* (4.8) *holds. Then in some neighborhood of any point* $x_0 \in \mathscr{F}_n$,

(1) $\mathscr{F}_n$ *is a* C^1 *hypersurface and, in the* $w>0$ *region,* $\partial^2 w/\partial x_i\, \partial x_j$ (*for any* $i, j = 1, 2, ..., n$) *is continuous up to* $\mathscr{F}_n$.

(2) $\mathscr{F}_n$ *is a* $C^{1,\alpha}$ *hypersurface for every positive* $\alpha < 1$.

(3) *If* $f \in C^{k,\mu}$ *with* k *an integer,* $k \geqslant 2$, *and* $0<\mu<1$, *then* $\mathscr{F}_n$ *is a* $C^{k,\mu}$ *hypersurface.*

(4) *If* f *is real analytic, then* $\mathscr{F}_n$ *is a real analytic hypersurface.*

Proof. Assertion (1) comes from applying Thm. 3 of [Caf]. All but one of Caffarelli's main hypotheses are stated in 1.2 on p. 157. Let $x_0 \in \mathscr{F}_n$. Then x_0 is a point of positive Lebesgue density for the coincidence set by Theorem 4.10. Let Ω be an open ball centered at x_0 with $\bar{\Omega} \subset \mathscr{S}_n$. Caffarelli's W is that portion of our Ω for which $w>0$. His elliptic operator A is our $\sum_{i,j=1}^n a_{ij} \partial^2/\partial x_i\, \partial x_j$. His v is our w. From Theorem 4.6 we have

$w \in W^{2,\infty}(\Omega)$ so that $w \in C^{1,1}(\Omega)$. From (4.6) we have $w \geqslant 0$. His f (defined by $A(v) = f$ on W) is our

$$G \equiv -\sum_{i=1}^{n} g_i \frac{\partial w}{\partial x_i} + \alpha w - \frac{\partial f}{\partial x_n} - \alpha c_n$$

$$\left(\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 w}{\partial x_i \, \partial x_j} \equiv G \text{ on } \{x \in \Omega;\, w(x) > 0\} \right).$$

In Corollary 4.9 we proved that $\partial f/\partial x_n + \alpha c_n < 0$ at x_0. Since $w \in C^{1,1}(\Omega)$, clearly $w(x_0) = 0$ and $\nabla w(x_0) = 0$, and clearly by choosing our ball Ω small enough there is a constant λ_0 such that $G \geqslant \lambda_0 > 0$ on a neighborhood of $\bar{\Omega}$. Thus we may take G to be the f^* of [Caf]. (Note that $f^* \in C^{0,1/2}$ is guaranteed, since $G \in C^{0,1}$.) The $\partial_1 W$ of [Caf] is our $\mathscr{F}_n \cap \Omega$. As mentioned above, w and ∇w are zero on this set. The only remaining hypothesis that needs to be checked is that x_0 is a point of positive Lebesgue density for the coincidence set (see Theorem 2). That is assured by Theorem 4.10.

Assertions (2), (3), and (4) of our theorem then follow from Theorem 1′ of [KN]. Their u is our w, their Ω is our $\{x \in \Omega;\, w(x) > 0\}$, their equation $F(x, u, Du, D^2u) = 0$ is our $Lw - \partial f/\partial x_n - \alpha c_n = 0$. Our w has zero Cauchy data on $\mathscr{F}_n$ since w and ∇w are zero there. With our $x_0 \in \mathscr{F}_n$ as "origin," the condition $F(0, 0, 0, 0) \neq 0$ becomes $\partial f/\partial x_n + \alpha c_n \neq 0$ at x_0, which was proved in Corollary 4.9. Conditions (I) and (II) hold because of assertion (1) of our theorem, proved above. Thus the conclusions of Theorem 1′ hold in our case. If we assume that $f \in C^{k,\mu}$ with $k \geqslant 2$, $0 < \mu < 1$, then $F(x, w, Dw, D^2w) = Lw - \partial f/\partial x_n - \alpha c_n$ is of class $C^{k-1,\mu}$ as a function of its arguments, so (with our $k-1$ taken as the m of [KN]) the free boundary Γ (our $\mathscr{F}_n \cap \Omega$) is of class $C^{k,\mu}$. If f is assumed to be real analytic, then $\mathscr{F}_n$ is a real analytic hypersurface. ∎

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