Linear Algebra and its Applications 437 (2012) 242–270

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Two commuting operators associated with a tridiagonal pair

Sarah Bockting-Conrad

Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706-1388, USA

ARTICLE INFO

Article history:
Received 20 October 2011
Accepted 8 February 2012
Available online 17 March 2012

Submitted by R. Brualdi

AMS classification:
Primary: 15A21
Secondary: 05E30

Keywords:
Tridiagonal pair
Leonard pair
q-Serre relations

ABSTRACT

Let $\mathbb{K}$ denote a field and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. We consider an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions: (i) Each of A, A^* is diagonalizable; (ii) there exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that $A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leqslant i \leqslant d$, where $V_{-1} = 0$ and $V_{d+1} = 0$; (iii) there exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ for $0 \leqslant i \leqslant \delta$, where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$; (iv) there does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$. We call such a pair a *tridiagonal pair* on V. It is known that $d = \delta$; to avoid trivialities assume $d \geqslant 1$. We show that there exists a unique linear transformation $\Delta : V \to V$ such that $(\Delta - I)V_i^* \subseteq V_0^* + V_1^* + \cdots + V_{i-1}^*$ and $\Delta(V_i + V_{i+1} + \cdots + V_d) = V_0 + V_1 + \cdots + V_{d-i}$ for $0 \leqslant i \leqslant d$. We show that there exists a unique linear transformation $\Psi : V \to V$ such that $\Psi V_i \subseteq V_{i-1} + V_i + V_{i+1}$ and $(\Psi - \Lambda)V_i^* \subseteq V_0^* + V_1^* + \cdots + V_{i-2}^*$ for $0 \leqslant i \leqslant d$, where $\Lambda = (\Delta - I)(\theta_0 - \theta_d)^{-1}$ and θ_0 (resp. θ_d) denotes the eigenvalue of A associated with V_0 (resp. V_d). We characterize Δ, Ψ in several ways. There are two well-known decompositions of V called the first and second split decomposition. We discuss how Δ, Ψ act on these decompositions. We also show how Δ, Ψ relate to each other. Along this line we have two main results. Our first main result is that Δ, Ψ commute. In the literature on TD pairs, there is a scalar β used to describe the eigenvalues. Our second main result is that each of $\Delta^{\pm 1}$ is a polynomial of degree d in Ψ, under a minor assumption on β.

E-mail address: bockting@math.wisc.edu

0024-3795/$ - see front matter
http://dx.doi.org/10.1016/j.laa.2012.02.007

1. Introduction

Throughout this paper, $\mathbb{K}$ denotes a field and $\overline{\mathbb{K}}$ denotes the algebraic closure of $\mathbb{K}$.

We begin by recalling the notion of a tridiagonal pair. We will use the following terms. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. For a linear transformation $A : V \to V$ and a subspace $W \subseteq V$, we say that W is an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that $W = \{v \in V | Av = \theta v\}$. In this case, θ is called the *eigenvalue* of A associated with W. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1 [2, Definition 1.1]**.** Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on V we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions:

(i) Each of A, A^* is diagonalizable.
(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leqslant i \leqslant d), \tag{1}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.
(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leqslant i \leqslant \delta), \tag{2}$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.
(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We say the pair A, A^* is *over* $\mathbb{K}$.

Note 1.2. According to a common notational convention A^* denotes the conjugate-transpose of A. We are not using this convention. In a TD pair A, A^* the linear transformations A and A^* are arbitrary subject to (i)–(iv) above.

Referring to the TD pair in Definition 1.1, by [2, Lemma 4.5] the scalars d and δ are equal. We call this common value the *diameter* of A, A^*. To avoid trivialities, throughout this paper we assume that the diameter is at least one.

We now give some background on TD pairs. The concept of a TD pair originated in the theory of Q-polynomial distance-regular graphs [17]. Since that beginning the TD pairs have been investigated in a systematic way; for notable papers along this line see [1–10,12,15,18]. Several of these papers focus on a class of TD pair said to be sharp. These papers ultimately led to the classification of sharp TD pairs [1]. In spite of this classification, there are still some intriguing aspects of TD pairs which have not yet been fully studied. In this paper, we investigate one of those aspects.

We now summarize the present paper. Given a TD pair A, A^* on V we introduce two linear transformations $\Delta : V \to V$ and $\Psi : V \to V$ that we find attractive. We characterize Δ, Ψ in several ways. There are two well-known decompositions of V called the first and second split decomposition [2, Section 4]. We discuss how Δ, Ψ act on these decompositions. We also show how Δ, Ψ relate to each other.

We now describe Δ, Ψ in more detail. For the rest of this section, fix an ordering $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) of the eigenspaces of A (resp. A^*) which satisfies (1) (resp. (2)). For $0 \leqslant i \leqslant d$ let θ_i (resp.

θ^*_i) denote the eigenvalue of A (resp. A^*) corresponding to V_i (resp. V^*_i). We show that there exists a unique linear transformation $\Delta : V \to V$ such that both

$$\begin{aligned}&(\Delta - I)V^*_i \subseteq V^*_0 + V^*_1 + \cdots + V^*_{i-1},\\ &\Delta(V_i + V_{i+1} + \cdots + V_d) = V_0 + V_1 + \cdots + V_{d-i}\end{aligned}$$

for $0 \leqslant i \leqslant d$. We show that there exists a unique linear transformation $\Psi : V \to V$ such that both

$$\begin{aligned}&\Psi V_i \subseteq V_{i-1} + V_i + V_{i+1},\\ &\left(\Psi - \frac{\Delta - I}{\theta_0 - \theta_d}\right)V^*_i \subseteq V^*_0 + V^*_1 + \cdots + V^*_{i-2}\end{aligned}$$

for $0 \leqslant i \leqslant d$. By construction,

$$\Psi V^*_i \subseteq V^*_0 + V^*_1 + \cdots + V^*_{i-1} \quad (0 \leqslant i \leqslant d).$$

Before discussing Δ and Ψ further, we recall the split decompositions of V. For $0 \leqslant i \leqslant d$ define

$$\begin{aligned}U_i &= (V^*_0 + V^*_1 + \cdots + V^*_i) \cap (V_i + V_{i+1} + \cdots + V_d),\\ U^{\Downarrow}_i &= (V^*_0 + V^*_1 + \cdots + V^*_i) \cap (V_0 + V_1 + \cdots + V_{d-i}).\end{aligned}$$

By [2, Theorem 4.6], both the sums $V = \sum_{i=0}^d U_i$ and $V = \sum_{i=0}^d U^{\Downarrow}_i$ are direct. We call $\{U_i\}_{i=0}^d$ (resp. $\{U^{\Downarrow}_i\}_{i=0}^d$) the *first split decomposition* (resp. *second split decomposition*) of V. By [2, Theorem 4.6], the maps A, A^* act on the split decompositions in the following way:

$$\begin{aligned}&(A - \theta_i I)U_i \subseteq U_{i+1} \quad (0 \leqslant i \leqslant d-1), && (A - \theta_d I)U_d = 0,\\ &(A^* - \theta^*_i I)U_i \subseteq U_{i-1} \quad (1 \leqslant i \leqslant d), && (A^* - \theta^*_0 I)U_0 = 0,\\ &(A - \theta_{d-i} I)U^{\Downarrow}_i \subseteq U^{\Downarrow}_{i+1} \quad (0 \leqslant i \leqslant d-1), && (A - \theta_0 I)U^{\Downarrow}_d = 0,\\ &(A^* - \theta^*_i I)U^{\Downarrow}_i \subseteq U^{\Downarrow}_{i-1} \quad (1 \leqslant i \leqslant d), && (A^* - \theta^*_0 I)U^{\Downarrow}_0 = 0.\end{aligned}$$

We now summarize how Δ, Ψ act on the split decompositions of V. We show that

$$\begin{aligned}&\Delta U_i = U^{\Downarrow}_i,\\ &(\Delta - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1},\\ &(\Delta - I)U^{\Downarrow}_i \subseteq U^{\Downarrow}_0 + U^{\Downarrow}_1 + \cdots + U^{\Downarrow}_{i-1}\end{aligned}$$

for $0 \leqslant i \leqslant d$. We also show that

$$\begin{aligned}&\Psi U_i \subseteq U_{i-1} \quad (1 \leqslant i \leqslant d), && \Psi U_0 = 0,\\ &\Psi U^{\Downarrow}_i \subseteq U^{\Downarrow}_{i-1} \quad (1 \leqslant i \leqslant d), && \Psi U^{\Downarrow}_0 = 0.\end{aligned}$$

We now discuss how Δ, Ψ relate to each other. Along this line we have two main results. Our first main result is that Δ, Ψ commute. In order to state the second result, we define

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (1 \leqslant i \leqslant d).$$

Our second main result is that both

$$\Delta = I + \frac{\eta_1(\theta_0)}{\vartheta_1}\Psi + \frac{\eta_2(\theta_0)}{\vartheta_1\vartheta_2}\Psi^2 + \cdots + \frac{\eta_d(\theta_0)}{\vartheta_1\vartheta_2\cdots\vartheta_d}\Psi^d,$$
$$\Delta^{-1} = I + \frac{\tau_1(\theta_d)}{\vartheta_1}\Psi + \frac{\tau_2(\theta_d)}{\vartheta_1\vartheta_2}\Psi^2 + \cdots + \frac{\tau_d(\theta_d)}{\vartheta_1\vartheta_2\cdots\vartheta_d}\Psi^d$$

provided that each of $\vartheta_1, \vartheta_2, \ldots, \vartheta_d$ is nonzero. Here τ_i, η_i are the polynomials

$$\tau_i = (x-\theta_0)(x-\theta_1)\cdots(x-\theta_{i-1}),$$
$$\eta_i = (x-\theta_d)(x-\theta_{d-1})\cdots(x-\theta_{d-i+1})$$

for $0 \leqslant i \leqslant d$. In the literature on TD pairs there is a scalar β that is used to describe the eigenvalues of A and A^* [2, Sections 10 and 11]. We show that each of $\vartheta_1, \vartheta_2, \ldots, \vartheta_d$ is nonzero if and only if neither of the following holds: (i) $\beta = -2$, d is odd, and $\mathrm{Char}(\mathbb{K}) \neq 2$; (ii) $\beta = 0$, $d = 3$, and $\mathrm{Char}(\mathbb{K}) = 2$. We conclude the paper with a few comments on further research.

2. Preliminaries

When working with a tridiagonal pair, it is useful to consider a closely related object called a tridiagonal system. In order to define this, we first recall some facts from elementary linear algebra.

Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $\mathrm{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all linear transformations from V to V. Let A denote a diagonalizable element in $\mathrm{End}(V)$. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A. For $0 \leqslant i \leqslant d$ let θ_i be the eigenvalue of A corresponding to V_i. Define $E_i \in \mathrm{End}(V)$ by $(E_i - I)V_i = 0$, $E_iV_j = 0$ if $j \neq i$ $(0 \leqslant j \leqslant d)$. In other words, E_i is the projection map from V onto V_i. We refer to E_i as the *primitive idempotent* of A associated with θ_i. By elementary linear algebra, we have that (i) $AE_i = E_iA = \theta_iE_i$ $(0 \leqslant i \leqslant d)$; (ii) $E_iE_j = \delta_{ij}E_i$ $(0 \leqslant i, j \leqslant d)$; (iii) $V_i = E_iV$ $(0 \leqslant i \leqslant d)$; (iv) $I = \sum_{i=0}^d E_i$. One readily checks that

$$E_i = \prod_{\substack{0 \leqslant j \leqslant d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \quad (0 \leqslant i \leqslant d).$$

Let M denote the $\mathbb{K}$-subalgebra of $\mathrm{End}(V)$ generated by A. We note that each of $\{A^i\}_{i=0}^d$, $\{E_i\}_{i=0}^d$ is a basis for the $\mathbb{K}$-vector space M.

Given a TD pair A, A^* on V, an ordering of the eigenspaces of A (resp. A^*) is said to be *standard* whenever (1) (resp. (2)) holds. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A. By [2, Lemma 2.4], the ordering $\{V_{d-i}\}_{i=0}^d$ is standard and no further ordering is standard. A similar result holds for the eigenspaces of A^*. An ordering of the primitive idempotents of A (resp. A^*) is said to be *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard.

Definition 2.1 [14, Definition 2.1]**.** Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *tridiagonal system* (or *TD system*) on V, we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(iii) below.

(i) A, A^* is a tridiagonal pair on V.
(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A.
(iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*.

We call d the *diameter* of Φ, and say Φ is *over* $\mathbb{K}$. For notational convenience, set $E_{-1} = 0$, $E_{d+1} = 0$, $E^*_{-1} = 0$, $E^*_{d+1} = 0$.

In Definition 2.1 we do not assume that the primitive idempotents $\{E_i\}_{i=0}^d$, $\{E^*_i\}_{i=0}^d$ all have rank 1. A TD system for which each of these primitive idempotents does have rank 1 is called a Leonard system [18]. The Leonard systems are classified up to isomorphism [18, Theorem 1.9].

For the rest of the present paper, we fix a TD system Φ as in Definition 2.1.

Definition 2.2. For $0 \leqslant i \leqslant d$ let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E^*_i). We refer to $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) as the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ.

A given TD system can be modified in a number of ways to get a new TD system. For example, given the TD system Φ in Definition 2.1, the sequence

$$\Phi^{\Downarrow} = (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$$

is a TD system on V. Following [2, Section 3], we call $\Phi^{\Downarrow}$ the *second inversion* of Φ. When discussing $\Phi^{\Downarrow}$, we use the following notational convention. For any object f associated with Φ we let $f^{\Downarrow}$ denote the corresponding object for $\Phi^{\Downarrow}$.

For later use, we associate with Φ two families of polynomials as follows. Let x be an indeterminate. Let $\mathbb{K}[x]$ denote the $\mathbb{K}$-algebra consisting of the polynomials in x that have all coefficients in $\mathbb{K}$. For $0 \leqslant i \leqslant j \leqslant d+1$, we define the polynomials $\tau_{ij} = \tau_{ij}(\Phi)$, $\eta_{ij} = \eta_{ij}(\Phi)$ in $\mathbb{K}[x]$ by

$$\tau_{ij} = (x - \theta_i)(x - \theta_{i+1}) \cdots (x - \theta_{j-1}), \tag{3}$$

$$\eta_{ij} = (x - \theta_{d-i})(x - \theta_{d-i-1}) \cdots (x - \theta_{d-j+1}). \tag{4}$$

We interpret $\tau_{i,i-1} = 0$ and $\eta_{i,i-1} = 0$. Note that each of τ_{ij}, η_{ij} is monic with degree $j - i$. In particular, $\tau_{ii} = 1$ and $\eta_{ii} = 1$. We remark that $\tau^{\Downarrow}_{ij} = \eta_{ij}$ and $\eta^{\Downarrow}_{ij} = \tau_{ij}$.

Observe that for $0 \leqslant i \leqslant j \leqslant k \leqslant d+1$,

$$\tau_{ij}\tau_{jk} = \tau_{ik}, \quad \eta_{ij}\eta_{jk} = \eta_{ik}. \tag{5}$$

As we proceed through the paper, we will focus on τ_{ij}. We will develop a number of results concerning τ_{ij}. Similar results hold for η_{ij}, although we will not state them explicitly.

Lemma 2.3. *For $0 \leqslant i \leqslant j \leqslant d+1$, the kernel of $\tau_{ij}(A)$ is*

$$E_iV + E_{i+1}V + \cdots + E_{j-1}V.$$

Proof. For $0 \leqslant h \leqslant d$, E_hV is the eigenspace of A corresponding to θ_h. The result follows from this and (3). □

For $0 \leqslant j \leqslant d+1$, we abbreviate $\tau_j = \tau_{0j}$ and $\eta_j = \eta_{0j}$. Thus

$$\tau_j = (x - \theta_0)(x - \theta_1) \cdots (x - \theta_{j-1}), \tag{6}$$

$$\eta_j = (x - \theta_d)(x - \theta_{d-1}) \cdots (x - \theta_{d-j+1}). \tag{7}$$

In our discussion of Ψ, the following scalars will be useful.

Definition 2.4 [18, Section 10]. For $0 \leqslant i \leqslant d+1$, define

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d}.$$

We observe that

$$\vartheta_{i+1} - \vartheta_i = \frac{\theta_i - \theta_{d-i}}{\theta_0 - \theta_d} \quad (0 \leqslant i \leqslant d). \tag{8}$$

These scalars will be discussed further in Section 13.

3. The first split decomposition of *V*

We continue to discuss the TD system Φ from Definition 2.1.

We use the following concept. By a *decomposition* of V, we mean a sequence of subspaces whose direct sum is V. For example, $\{E_iV\}_{i=0}^d$ and $\{E_i^*V\}_{i=0}^d$ are decompositions of V. There are two more decompositions of V of interest called the first and second split decomposition. In this section, we discuss the first split decomposition of V. In Section 4, we will discuss the second split decomposition of V.

Definition 3.1. For $0 \leqslant i \leqslant d$ define the subspace $U_i \subseteq V$ by

$$U_i = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_iV + E_{i+1}V + \cdots + E_dV).$$

For notational convenience, define $U_{-1} = 0$ and $U_{d+1} = 0$.

We recall a few facts about the sequence $\{U_i\}_{i=0}^d$. By [2, Theorem 4.6], the sequence $\{U_i\}_{i=0}^d$ is a decomposition of V. We refer to the sequence $\{U_i\}_{i=0}^d$ as the *first split decomposition* of V. By [2, Theorem 4.6],

$$(A - \theta_i I)U_i \subseteq U_{i+1} \quad (0 \leqslant i \leqslant d-1), \qquad (A - \theta_d I)U_d = 0, \tag{9}$$

$$(A^* - \theta_i^* I)U_i \subseteq U_{i-1} \quad (1 \leqslant i \leqslant d), \qquad (A^* - \theta_0^* I)U_0 = 0. \tag{10}$$

Consequently, for $0 \leqslant i \leqslant d$ both

$$A^k U_i \subseteq U_i + U_{i+1} + \cdots + U_{i+k} \quad (0 \leqslant k \leqslant d-i), \tag{11}$$

$$(A^*)^k U_i \subseteq U_i + U_{i-1} + \cdots + U_{i-k} \quad (0 \leqslant k \leqslant i). \tag{12}$$

By [2, Theorem 4.6], for $0 \leqslant i \leqslant d$ both

$$U_i + U_{i+1} + \cdots + U_d = E_iV + E_{i+1}V + \cdots + E_dV, \tag{13}$$

$$U_0 + U_1 + \cdots + U_i = E_0^*V + E_1^*V + \cdots + E_i^*V. \tag{14}$$

By [2, Corollary 5.7], for $0 \leqslant i \leqslant d$ the dimensions of E_iV, E_i^*V, U_i coincide. Denoting this common dimension by ρ_i, we have $\rho_i = \rho_{d-i}$. We refer to the sequence $\{\rho_i\}_{i=0}^d$ as the *shape* of Φ. Note that Φ and $\Phi^{\Downarrow}$ have the same shape.

For $0 \leqslant i \leqslant d$ define $F_i \in \mathrm{End}(V)$ by

$$(F_i - I)U_i = 0, \tag{15}$$

$$F_iU_j = 0 \quad \text{if} \quad j \neq i, \quad (0 \leqslant j \leqslant d). \tag{16}$$

In other words, F_i is the projection map from V onto U_i. For notational convenience, define $F_{-1} = 0$ and $F_{d+1} = 0$. By construction, both

$$F_iF_j = \delta_{ij}F_i \quad (0 \leqslant i, j \leqslant d), \tag{17}$$

$$I = \sum_{i=0}^{d} F_i. \tag{18}$$

Following [2, Definition 6.1], we define

$$R = A - \sum_{h=0}^{d} \theta_h F_h, \quad L = A^* - \sum_{h=0}^{d} \theta_h^* F_h. \tag{19}$$

We refer to R (resp. L) as the *raising map* (resp. *lowering map*) for Φ. Observe that for $0 \leqslant i \leqslant d$ the following hold on U_i.

$$R = A - \theta_i I, \quad L = A^* - \theta_i^* I. \tag{20}$$

Combining (20) with (9), (10) we obtain the following result.

Lemma 3.2. *Both*

$$RU_i \subseteq U_{i+1} \quad (0 \leqslant i \leqslant d-1), \qquad RU_d = 0, \tag{21}$$

$$LU_i \subseteq U_{i-1} \quad (1 \leqslant i \leqslant d), \qquad LU_0 = 0. \tag{22}$$

Corollary 3.3. *The expression $R^{j-i} - \tau_{ij}(A)$ vanishes on U_i for $0 \leqslant i \leqslant j \leqslant d+1$.*

Proof. Use (3), (20), and (21). □

Lemma 3.4. *For $0 \leqslant i \leqslant j \leqslant d+1$, we have $\tau_{ij}(A)U_i \subseteq U_j$.*

Proof. Use (3) and (9). □

The following result is a reformulation of [2, Lemma 6.5].

Lemma 3.5 [2, Lemma 6.5]**.** *For $0 \leqslant i \leqslant j \leqslant d$ the linear transformation $U_i \to U_j$, $v \mapsto \tau_{ij}(A)v$ is an injection if $i+j \leqslant d$, a bijection if $i+j = d$, and a surjection if $i+j \geqslant d$.*

Proof. By [2, Lemma 6.5] the linear transformation $U_i \to U_j$, $v \mapsto R^{j-i}v$ is an injection if $i+j \leqslant d$, a bijection if $i+j = d$, and a surjection if $i+j \geqslant d$. The result follows from this and Corollary 3.3. □

Corollary 3.6. *The restriction of $A - \theta_i I$ to U_i is injective for $0 \leqslant i < d/2$.*

4. The second split decomposition of V

We continue to discuss the TD system Φ from Definition 2.1. Since $\Phi^{\Downarrow}$ is a TD system on V, all the results from Section 3 apply to it. For later use, we now emphasize a few of these results. By definition,

$$U_i^{\Downarrow} = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_0V + E_1V + \cdots + E_{d-i}V) \tag{23}$$

for $0 \leqslant i \leqslant d$. Applying [2, Theorem 4.6] to $\Phi^{\Downarrow}$ we obtain the following facts. The subspaces $\{U_i^{\Downarrow}\}_{i=0}^d$ form a decomposition of V which we call the *second split decomposition* of V. We also have that

$$(A - \theta_{d-i}I)U_i^{\Downarrow} \subseteq U_{i+1}^{\Downarrow} \quad (0 \leqslant i \leqslant d-1), \qquad (A - \theta_0 I)U_d^{\Downarrow} = 0,$$
$$(A^* - \theta_i^* I)U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \quad (1 \leqslant i \leqslant d), \qquad (A^* - \theta_0^* I)U_0^{\Downarrow} = 0.$$

In addition, for $0 \leqslant i \leqslant d$ both

$$U_i^{\Downarrow} + U_{i+1}^{\Downarrow} + \cdots + U_d^{\Downarrow} = E_0V + E_1V + \cdots + E_{d-i}V, \tag{24}$$
$$U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_i^{\Downarrow} = E_0^*V + E_1^*V + \cdots + E_i^*V. \tag{25}$$

Lemma 4.1. *For* $0 \leqslant i \leqslant d$,

$$U_0 + U_1 + \cdots + U_i = U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_i^{\Downarrow}.$$

Proof. Both sides equal $E_0^*V + E_1^*V + \cdots + E_i^*V$ by (14) and (25). □

We now make some comments concerning $\{F_i^{\Downarrow}\}_{i=0}^d$ and $R^{\Downarrow}$. For $0 \leqslant i \leqslant d$, $F_i^{\Downarrow}$ is the projection of V onto $U_i^{\Downarrow}$. Observe that

$$R^{\Downarrow} = A - \sum_{h=0}^{d} \theta_{d-h} F_h^{\Downarrow}. \tag{26}$$

For $0 \leqslant i \leqslant j \leqslant d+1$, the action of $\left(R^{\Downarrow}\right)^{j-i}$ on $U_i^{\Downarrow}$ agrees with the action of $\eta_{ij}(A)$ on $U_i^{\Downarrow}$. In addition,

$$R^{\Downarrow}U_i^{\Downarrow} \subseteq U_{i+1}^{\Downarrow} \quad (0 \leqslant i \leqslant d-1), \quad R^{\Downarrow}U_d^{\Downarrow} = 0.$$

5. The projections F_i, $F_i^{\Downarrow}$

We continue to discuss the TD system Φ from Definition 2.1. In this section, we consider how the maps $\{F_i\}_{i=0}^d$ and $\{F_i^{\Downarrow}\}_{i=0}^d$ interact. In [2, Section 5], there are a number of results concerning how the maps $\{E_i\}_{i=0}^d$ and $\{F_i\}_{i=0}^d$ interact. The results given in this section are reformulations of these results.

Lemma 5.1. *For* $0 \leqslant i < j \leqslant d$ *both*

$$F_jF_i^{\Downarrow} = 0, \quad F_j^{\Downarrow}F_i = 0. \tag{27}$$

Proof. We first verify the equation on the left in (27). Consider the action of $F_jF_i^{\Downarrow}$ on V. Using the fact that $F_i^{\Downarrow}V = U_i^{\Downarrow}$ along with Lemma 4.1, we find that

$$F_jF_i^{\Downarrow}V \subseteq F_j\left(U_0 + U_1 + \cdots + U_i\right). \tag{28}$$

Since $i < j$, it follows from (16) that the space on the right in (28) equals 0. So $F_jF_i^{\Downarrow}$ vanishes on V.

The proof for the equation on the right in (27) is similar. □

Lemma 5.2. *For* $0 \leqslant i \leqslant d$ *both*

$$F_i F_i^{\Downarrow} F_i = F_i, \quad F_i^{\Downarrow} F_i F_i^{\Downarrow} = F_i^{\Downarrow}. \tag{29}$$

Proof. We first show the equation on the left in (29). By (17), (18), and Lemma 5.1,

$$F_i = F_i F_i = F_i \left(\sum_{h=0}^{d} F_h^{\Downarrow} \right) F_i = F_i F_i^{\Downarrow} F_i.$$

The proof of the equation on the right in (29) is similar. □

Lemma 5.3. *For* $0 \leqslant i \leqslant d$ *the restrictions*

$$F_i^{\Downarrow}|_{U_i} : U_i \to U_i^{\Downarrow}, \quad F_i|_{U_i^{\Downarrow}} : U_i^{\Downarrow} \to U_i$$

are bijections. Moreover, these bijections are inverses.

Proof. We first show that the map $F_i F_i^{\Downarrow}$ acts as the identity on U_i. Let $v \in U_i$. By (15) and the equation on the left in (29),

$$F_i F_i^{\Downarrow} v = F_i F_i^{\Downarrow} F_i v = F_i v = v.$$

We have shown $F_i F_i^{\Downarrow}$ acts as the identity on U_i. One can show similarly that $F_i^{\Downarrow} F_i$ acts as the identity on $U_i^{\Downarrow}$. The result follows. □

Lemma 5.4 [2, Lemma 6.4]**.** *We have*

(i) $RF_i = F_{i+1}R \quad (0 \leqslant i \leqslant d-1), \quad RF_d = 0, \quad F_0 R = 0.$
(ii) $LF_i = F_{i-1}L \quad (1 \leqslant i \leqslant d), \quad LF_0 = 0, \quad F_d L = 0.$

Lemma 5.5. *For* $0 \leqslant i \leqslant d-1$, *we have* $R^{\Downarrow} F_i^{\Downarrow} F_i = F_{i+1}^{\Downarrow} F_{i+1} R$.

Proof. We show $R^{\Downarrow} F_i^{\Downarrow} F_i - F_{i+1}^{\Downarrow} F_{i+1} R = 0$. By Lemma 5.4(i) (applied to both Φ and $\Phi^{\Downarrow}$),

$$\begin{aligned} R^{\Downarrow} F_i^{\Downarrow} F_i - F_{i+1}^{\Downarrow} F_{i+1} R &= F_{i+1}^{\Downarrow} R^{\Downarrow} F_i - F_{i+1}^{\Downarrow} R F_i \\ &= F_{i+1}^{\Downarrow} \left(R^{\Downarrow} - R \right) F_i. \end{aligned} \tag{30}$$

By (19) and (26),

$$R^{\Downarrow} - R = \sum_{h=0}^{d} \theta_h F_h - \sum_{h=0}^{d} \theta_{d-h} F_h^{\Downarrow}. \tag{31}$$

Eliminate $R^{\Downarrow} - R$ in (30) using (31). Simplify the resulting expression using (17) (applied to both Φ and $\Phi^{\Downarrow}$) and Lemma 5.1 to get 0. □

6. The subspaces K_i

We continue to discuss the TD system Φ from Definition 2.1. Shortly we will define the linear transformation Ψ. In our discussion of Ψ, it will be useful to consider a certain refinement of the first and second split decomposition of V. This refinement was introduced in [11]. In order to describe this refinement, we introduce a sequence of subspaces $\{K_i\}_{i=0}^r$, where $r = \lfloor d/2 \rfloor$.

Definition 6.1. For $0 \leqslant i \leqslant d/2$, define the subspace $K_i \subseteq V$ by

$$K_i = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_iV + E_{i+1}V + \cdots + E_{d-i}V).$$

Observe that $K_0 = E_0^*V = U_0$.

Lemma 6.2. *We have $K_i = U_i \cap U_i^{\Downarrow}$ for $0 \leqslant i \leqslant d/2$.*

Proof. Use (23), Definition 3.1, and Definition 6.1. □

Lemma 6.3 [11, Lemma 4.1(iii)]**.** *For $0 \leqslant i \leqslant d/2$, the restriction of $\tau_{i,d-i+1}(A)$ to U_i has kernel K_i.*

Proof. Use Lemma 2.3 and Definition 3.1. □

We now consider the spaces $\tau_{ij}(A)K_i$ where $0 \leqslant i \leqslant d/2$ and $i \leqslant j \leqslant d-i$. We start with an observation.

Lemma 6.4 [11, Lemma 4.1(vi)]**.** *For $0 \leqslant i \leqslant d/2$ and $i \leqslant j \leqslant d-i$, the linear transformation $K_i \to \tau_{ij}(A)K_i$, $v \mapsto \tau_{ij}(A)v$ is a bijection.*

Proof. By construction the map is surjective. By Lemma 3.5 the restriction of $\tau_{ij}(A)$ to K_i is injective. The result follows. □

From Lemma 6.4 we draw two corollaries.

Corollary 6.5. *For $0 \leqslant i \leqslant d/2$ and $i \leqslant j \leqslant k \leqslant d-i$, the linear transformation $\tau_{ij}(A)K_i \to \tau_{ik}(A)K_i$, $v \mapsto \tau_{jk}(A)v$ is a bijection.*

Proof. Use Lemma 6.4 and the equation on the left in (5). □

Corollary 6.6. *For $0 \leqslant i \leqslant d/2$ and $i \leqslant j \leqslant d-i$, the dimension of $\tau_{ij}(A)K_i$ coincides with the dimension of K_i.*

7. Concerning the decomposition $\{U_i\}_{i=0}^d$

We continue to discuss the TD system Φ from Definition 2.1. Recall the first split decomposition $\{U_i\}_{i=0}^d$ of V from Definition 3.1. We know that $K_0 = U_0$ and $K_i \subseteq U_i$ for $1 \leqslant i \leqslant d$. We will use these facts along with information about the raising map R to give a decomposition of each U_i.

The following result is essentially due to Nomura [11, Theorem 4.2]. We give an alternate proof.

Lemma 7.1 [11, Theorem 4.2]**.** *For $1 \leqslant i \leqslant d/2$, each of the following sums is direct.*

$$U_i = K_i + RU_{i-1}, \quad U_i = K_i + (A - \theta_{i-1}I)U_{i-1}. \tag{32}$$

Proof. We first show that $U_i = K_i + RU_{i-1}$. By Lemma 3.2 and Lemma 6.2, $U_i \supseteq K_i + RU_{i-1}$. We now show $U_i \subseteq K_i + RU_{i-1}$. Let $v \in U_i$. By Lemma 3.2 we get $R^{d-2i+1}v \in U_{d-i+1}$. By Corollary 3.3 and Lemma 3.5 there exists $w \in U_{i-1}$ such that $R^{d-2i+2}w = R^{d-2i+1}v$. Rearranging terms we obtain $R^{d-2i+1}(Rw - v) = 0$. So $Rw - v$ is in the kernel of R^{d-2i+1}. By Lemma 3.2, $Rw - v \in U_i$. By Corollary 3.3

and Lemma 6.3, K_i is the intersection of U_i and the kernel of R^{d-2i+1}. By these comments $Rw - v \in K_i$. Therefore

$$v = -(Rw - v) + Rw \in K_i + RU_{i-1}.$$

Hence $U_i \subseteq K_i + RU_{i-1}$. We have shown $U_i = K_i + RU_{i-1}$. We now show that this sum is direct. Let $v \in K_i \cap RU_{i-1}$. Since $v \in RU_{i-1}$, there exists $w \in U_{i-1}$ such that $v = Rw$. Recall $v \in K_i$ so $R^{d-2i+1}v = 0$. Therefore $R^{d-2i+2}w = 0$. By Lemma 3.5, the restriction of R^{d-2i+2} to U_{i-1} is injective. So $w = 0$ and thus $v = 0$. We have shown that the sum $U_i = K_i + RU_{i-1}$ is direct.

Now consider the space on the right in (32). Observe that $RU_{i-1} = (A - \theta_{i-1}I)U_{i-1}$. The result follows from the above comments. □

Combining Lemma 3.5 and Lemma 7.1, we see that $\rho_i \leqslant \rho_{i+1}$ for $0 \leqslant i < d/2$ and $\rho_i \geqslant \rho_{i+1}$ for $d/2 \leqslant i \leqslant d-1$. This fact was previously obtained in [2, Corollary 6.6].

From Lemma 7.1 we also obtain the following corollary.

Corollary 7.2 [11, Lemma 4.3]**.** *For $1 \leqslant i \leqslant d/2$, the dimension of K_i equals $\rho_i - \rho_{i-1}$ (this dimension could be zero). Moreover, the dimension of K_0 equals ρ_0.*

Lemma 7.3 [11, Theorem 4.7]**.**

(i) *For $0 \leqslant i \leqslant d/2$, the following sum is direct.*

$$U_i = K_i + \tau_{i-1,i}(A)K_{i-1} + \tau_{i-2,i}(A)K_{i-2} + \cdots + \tau_{0i}(A)K_0. \tag{33}$$

(ii) *For $d/2 \leqslant i \leqslant d$, the following sum is direct.*

$$U_i = \tau_{d-i,i}(A)K_{d-i} + \tau_{d-i-1,i}(A)K_{d-i-1} + \cdots + \tau_{0i}(A)K_0.$$

Proof. (i) Recall $U_0 = K_0$. By Lemma 7.1, the sum $U_j = K_j + (A - \theta_{j-1}I)U_{j-1}$ is direct for $1 \leqslant j \leqslant i$. Combining these equations and simplifying the result using (3), we get (33). The directness of the sum (33) follows in view of Corollary 3.6.
(ii) Observe that $0 \leqslant d-i \leqslant d/2$. So (33) gives a decomposition of U_{d-i}. By Lemma 3.5, the restriction of $\tau_{d-i,i}(A)$ to U_{d-i} gives a bijection $U_{d-i} \to U_i$. Apply this bijection to each term in the above mentioned decomposition for U_{d-i} and simplify the result using the equation on the left in (5). □

Combining parts (i) and (ii) of Lemma 7.3 we have

$$U_j = \sum_{i=0}^{\min\{j,d-j\}} \tau_{ij}(A)K_i \quad \text{(direct sum)} \tag{34}$$

for $0 \leqslant j \leqslant d$.

Corollary 7.4 [11, Theorem 4.8]**.** *The following sum is direct.*

$$V = \sum_{i=0}^{r} \sum_{j=i}^{d-i} \tau_{ij}(A)K_i, \tag{35}$$

where $r = \lfloor d/2 \rfloor$.

Proof. Recall the direct sum $V = \sum_{h=0}^{d} U_h$ from Section 3. Evaluate each summand using (34). In the resulting double summation, invert the order of summation. □

8. The subalgebra *M*

We continue to discuss the TD system Φ from Definition 2.1. Recall from Section 2 the subalgebra M of $\mathrm{End}(V)$ generated by A. In our discussion of M, we mentioned that each of $\{E_i\}_{i=0}^d$, $\{A^i\}_{i=0}^d$ is a basis for M. In this section, we give a third basis for M and use it to realize V as a direct sum of M-modules.

Lemma 8.1. *For $0 \leqslant i \leqslant d/2$, the vector space M has basis*

$$\{E_0, E_1, \ldots, E_{i-1}\} \cup \{E_{d-i+1}, E_{d-i+2}, \ldots, E_d\} \cup \{\tau_{ij}(A) | i \leqslant j \leqslant d-i\}. \tag{36}$$

Proof. By [14, Lemma 5.1],

$$\{E_0, E_1, \ldots, E_{i-1}\} \cup \{E_{d-i+1}, E_{d-i+2}, \ldots, E_d\} \cup \{A^{j-i} | i \leqslant j \leqslant d-i\}$$

is a basis for M. By the comments following (4),

$$\mathrm{Span}\{A^{j-i} | i \leqslant j \leqslant d-i\} = \mathrm{Span}\{\tau_{ij}(A) | i \leqslant j \leqslant d-i\}.$$

The result follows. $\square$

For the rest of this section, we view V as an M-module. For $0 \leqslant i \leqslant d/2$ let MK_i denote the M-submodule of V generated by K_i. Our goal in this section is to show that the sum $V = \sum_{i=0}^r MK_i$ is direct, where $r = \lfloor d/2 \rfloor$. We start by giving a detailed description of the MK_i.

Lemma 8.2. *For $0 \leqslant i \leqslant d/2$ such that $K_i \neq 0$, the sum*

$$MK_i = K_i + \tau_{i,i+1}(A)K_i + \tau_{i,i+2}(A)K_i + \cdots + \tau_{i,d-i}(A)K_i \tag{37}$$

is direct. Moreover $\tau_{i,d-i+1}$ is the minimal polynomial for the action of A on MK_i.

Proof. For the basis of M given in (36), apply each element to K_i. By Definition 6.1, each primitive idempotent in (36) vanishes on K_i. This gives Eq. (37). We now show the sum on the right in (37) is direct. By Lemma 3.4, we have $\tau_{ij}(A)K_i \subseteq U_j$ for $i \leqslant j \leqslant d-i$. The sum (37) is direct by this and the fact that $\{U_j\}_{j=0}^d$ is a decomposition of V.

It remains to show that $\tau_{i,d-i+1}$ is the minimal polynomial for the action of A on MK_i. Let P denote the minimal polynomial for the action of A on MK_i and let k denote the degree of P. By Lemma 2.3 and Definition 6.1, $\tau_{i,d-i+1}(A)K_i = 0$. Since $A \in M$ and M is commutative, it follows that $\tau_{i,d-i+1}(A)MK_i = 0$. So P divides $\tau_{i,d-i+1}$ and hence $k \leqslant d-2i+1$.

Suppose now that $k < d-2i+1$ to get a contradiction. Since the degree of P is k,

$$MK_i = K_i + AK_i + \cdots + A^{k-1}K_i. \tag{38}$$

By (11), the right-hand side of (38) is contained in $U_i + U_{i+1} + \cdots + U_{i+k-1}$. By Lemma 6.4, the restriction of $\tau_{i,d-i}(A)$ to K_i is an injection. It follows from this and $K_i \neq 0$ that $\tau_{i,d-i}(A)K_i \neq 0$. Recall that $\tau_{i,d-i}(A)K_i \subseteq U_{d-i}$. By (37) and the above comments we find that $\tau_{i,d-i}(A)K_i$ is contained in the intersection of $U_i + U_{i+1} + \cdots + U_{i+k-1}$ and U_{d-i}. This intersection is zero since $k < d-2i+1$ and $\{U_j\}_{j=0}^d$ is a decomposition of V. Therefore $\tau_{i,d-i}(A)K_i = 0$ for a contradiction. Thus $k = d-2i+1$ and therefore $P = \tau_{i,d-i+1}$ since $\tau_{i,d-i+1}$ is monic. $\square$

Corollary 8.3. *For $0 \leqslant i \leqslant d/2$ and $0 \neq v \in K_i$, the vector space Mv has basis*

$$v, \quad \tau_{i,i+1}(A)v, \quad \tau_{i,i+2}(A)v, \quad \ldots, \quad \tau_{i,d-i}(A)v.$$

Lemma 8.4. *The following is a direct sum of M-modules.*

$$V = \sum_{i=0}^{r} MK_i, \tag{39}$$

where $r = \lfloor d/2 \rfloor$.

Proof. Eq. (39) follows from Corollary 7.4 and Lemma 8.2. The directness of the sum follows from the directness of the sum in Corollary 7.4. $\square$

9. The linear transformation Δ

We continue to discuss the TD system Φ from Definition 2.1. In this section we will construct a linear transformation $\Delta \in \mathrm{End}(V)$ that has certain properties which we find attractive. It will turn out that Δ is the unique element of $\mathrm{End}(V)$ such that both

$$(\Delta - I)E_i^*V \subseteq E_0^*V + E_1^*V + \cdots + E_{i-1}^*V,$$
$$\Delta(E_iV + E_{i+1}V + \cdots + E_dV) = E_0V + E_1V + \cdots + E_{d-i}V$$

for $0 \leqslant i \leqslant d$.

Definition 9.1. Define $\Delta \in \mathrm{End}(V)$ by $\Delta = \sum_{h=0}^{d} F_h^{\Downarrow} F_h$, where $F_h, F_h^{\Downarrow}$ are from Section 3.

Lemma 9.2. *With reference to Definition 9.1, we have* $F_i^{\Downarrow}\Delta = \Delta F_i$ *for* $0 \leqslant i \leqslant d$.

Proof. Use (17) and Definition 9.1. $\square$

Lemma 9.3. *With reference to Definition 9.1,* Δ^{-1} *exists and* $\Delta^{-1} = \Delta^{\Downarrow}$.

Proof. Observe that $\Delta^{\Downarrow} = \sum_{h=0}^{d} F_h F_h^{\Downarrow}$. Consider the product $\Delta\Delta^{\Downarrow}$. Simplify this product using (17), (18), and Lemma 5.2 to obtain $\Delta\Delta^{\Downarrow} = I$. $\square$

Lemma 9.4. *With reference to Definition 9.1,*

$$\Delta U_i = U_i^{\Downarrow} \quad (0 \leqslant i \leqslant d), \tag{40}$$

$$(\Delta - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1} \quad (0 \leqslant i \leqslant d). \tag{41}$$

Proof. Line (40) follows from (15), (16), Lemma 5.3, and Definition 9.1.

We now verify (41). By (15) and (16), it suffices to show that $F_j\,(\Delta - I)\,U_i = 0$ for $i \leqslant j \leqslant d$. For $i = j$, this follows from (15), (16), Definition 9.1, and the equation on the left in (29). For $i+1 \leqslant j \leqslant d$, this follows from (15), (16), (27), and Definition 9.1. $\square$

We now show that (40), (41) characterize Δ.

Lemma 9.5. *Given* $\Delta' \in \mathrm{End}(V)$ *such that*

$$\Delta' U_i \subseteq U_i^{\Downarrow} \quad (0 \leqslant i \leqslant d), \tag{42}$$
$$(\Delta' - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1} \quad (0 \leqslant i \leqslant d). \tag{43}$$

Then $\Delta' = \Delta$.

Proof. Since $\{U_i\}_{i=0}^d$ is a decomposition of V, it suffices to show that Δ, Δ' agree on U_i for $0 \leqslant i \leqslant d$. Let i be given. By (40) and (42),

$$\left(\Delta - \Delta'\right) U_i \subseteq U_i^{\Downarrow}. \tag{44}$$

By (41), (43), and Lemma 4.1,

$$\left(\Delta - \Delta'\right) U_i \subseteq U_0 + U_1 + \cdots + U_{i-1} = U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_{i-1}^{\Downarrow}. \tag{45}$$

Combining (44) and (45) we find that $(\Delta - \Delta')\, U_i$ is contained in the intersection of $U_i^{\Downarrow}$ and $U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_{i-1}^{\Downarrow}$. This intersection is zero since $\{U_j^{\Downarrow}\}_{j=0}^d$ is a decomposition of V. Therefore $(\Delta - \Delta')U_i = 0$. So Δ, Δ' agree on U_i. $\square$

Lemma 9.6. *With reference to Definition 9.1,*

$$(\Delta^{-1} - I)U_i \subseteq U_0 + U_1 + \cdots + U_{i-1} \quad (0 \leqslant i \leqslant d).$$

Proof. Apply Δ^{-1} to both sides in (41). In the resulting containment, simplify the right-hand side using Lemma 4.1 and (40). $\square$

Lemma 9.7. *With reference to Definition 9.1,*

$$(\Delta - I)U_i^{\Downarrow} \subseteq U_0^{\Downarrow} + U_1^{\Downarrow} + \cdots + U_{i-1}^{\Downarrow} \quad (0 \leqslant i \leqslant d).$$

Proof. Apply Lemma 9.6 to $\Phi^{\Downarrow}$. Use Lemma 9.3 to simplify the result. $\square$

We now obtain the characterization of Δ given in the Introduction.

Lemma 9.8. *With reference to Definition 9.1,*

$$(\Delta - I)E_i^*V \subseteq E_0^*V + E_1^*V + \cdots + E_{i-1}^*V \quad (0 \leqslant i \leqslant d), \tag{46}$$

$$\Delta(E_iV + E_{i+1}V + \cdots + E_dV) = E_0V + E_1V + \cdots + E_{d-i}V \quad (0 \leqslant i \leqslant d). \tag{47}$$

Proof. We first show (46). By (14), $E_i^*V \subseteq U_0 + U_1 + \cdots + U_i$. Using this fact along with (41), we see that $(\Delta - I)E_i^*V \subseteq U_0 + U_1 + \cdots + U_{i-1}$. The result follows from this and (14).

We now show (47). From (13) and (40), we see that

$$\Delta(E_iV + E_{i+1}V + \cdots + E_dV) = U_i^{\Downarrow} + U_{i+1}^{\Downarrow} + \cdots + U_d^{\Downarrow}.$$

The result follows from this comment along with (24). $\square$

We now show that (46), (47) characterize Δ.

Lemma 9.9. *Given* $\Delta' \in \mathrm{End}(V)$ *such that*

$$(\Delta' - I)E_i^*V \subseteq E_0^*V + E_1^*V + \cdots + E_{i-1}^*V \quad (0 \leqslant i \leqslant d), \tag{48}$$

$$\Delta'(E_iV + E_{i+1}V + \cdots + E_dV) \subseteq E_0V + E_1V + \cdots + E_{d-i}V \quad (0 \leqslant i \leqslant d). \tag{49}$$

Then $\Delta' = \Delta$.

Proof. By Lemma 9.5, it suffices to show that Δ' satisfies (42) and (43). These lines are routinely verified using (13), (14), and Lemma 4.1. $\square$

We now derive some relations involving Δ that will be of use later.

Lemma 9.10. *With reference to Definition 9.1, $R^{\Downarrow}\Delta = \Delta R$.*

Proof. In the expression $R^{\Downarrow}\Delta - \Delta R$, eliminate Δ using Definition 9.1. Simplify the result using Lemma 5.4(i) and Lemma 5.5 to obtain $R^{\Downarrow}\Delta - \Delta R = 0$. $\square$

Lemma 9.11. *With reference to Definition 9.1,*

$$\Delta A - A\Delta = \sum_{h=0}^{d} (\theta_h - \theta_{d-h}) F_h^{\Downarrow} F_h. \tag{50}$$

Proof. By Lemma 9.10, we have $\Delta R - R^{\Downarrow}\Delta = 0$. In this equation, eliminate R and $R^{\Downarrow}$ using (19) and (26) to get

$$\Delta A - A\Delta = \sum_{h=0}^{d} \theta_h \Delta F_h - \sum_{h=0}^{d} \theta_{d-h} F_h^{\Downarrow} \Delta. \tag{51}$$

Simplify the right-hand side of (51) using Definition 9.1 and (17) to get the result. $\square$

We now express Lemma 9.11 from a slightly different perspective.

Corollary 9.12. *With reference to Definition 9.1,*

$$A - \Delta^{-1} A \Delta = \sum_{h=0}^{d} (\theta_h - \theta_{d-h}) F_h.$$

Proof. Apply Δ^{-1} to both sides of (50). Simplify the resulting right-hand side using Lemma 5.2, Definition 9.1, and (17). $\square$

Lemma 9.13. *With reference to Definition 9.1,*

$$L^{\Downarrow}\Delta - \Delta L = A^*\Delta - \Delta A^*. \tag{52}$$

Proof. In the left-hand side of (52), eliminate L and $L^{\Downarrow}$ using (19). Evaluate the result using Lemma 9.2. $\square$

Lemma 9.14. *With reference to Definition 9.1,*

$$(\Delta^{-1} A^* \Delta - A^*) U_i \subseteq U_{i-1} \quad (1 \leqslant i \leqslant d), \quad (\Delta^{-1} A^* \Delta - A^*) U_0 = 0.$$

Proof. By Lemma 9.13,

$$\Delta^{-1} A^* \Delta - A^* = \Delta^{-1} L^{\Downarrow} \Delta - L. \tag{53}$$

Let $1 \leqslant i \leqslant d$. By (22) (applied to $\Phi^{\Downarrow}$) and (40), $\Delta^{-1} L^{\Downarrow} \Delta U_i \subseteq U_{i-1}$. By (22), $LU_i \subseteq U_{i-1}$. Thus $(\Delta^{-1} L^{\Downarrow} \Delta - L) U_i \subseteq U_{i-1}$. By these comments, $(\Delta^{-1} A^* \Delta - A^*) U_i \subseteq U_{i-1}$.

To obtain $(\Delta^{-1} A^* \Delta - A^*) U_0 = 0$, use (22) (applied to both Φ and $\Phi^{\Downarrow}$), (40), and (53). $\square$

10. More on Δ

We continue to discuss the TD system Φ from Definition 2.1. Recall the decomposition of V given in Corollary 7.4. In this section, we consider the action of Δ on each of the summands of this decomposition.

Lemma 10.1. *Let $0 \leqslant i \leqslant d/2$. For $v \in K_i$ and $i \leqslant j \leqslant d-i$, both*

$$F_j^{\Downarrow}\tau_{ij}(A)v = \eta_{ij}(A)v, \quad F_j\eta_{ij}(A)v = \tau_{ij}(A)v. \tag{54}$$

Proof. We first show the equation on the left in (54). First suppose $i = j$. Use (15), Lemma 6.2, and the fact that both τ_{ii} and η_{ii} equal 1. Now suppose $i < j$. By the comments following (4), $\tau_{ij} - \eta_{ij}$ has degree at most $j-i-1$ and is therefore in $\mathrm{Span}\{\eta_{ih}\}_{h=i}^{j-1}$. From this and Lemma 3.4 (applied to $\Phi^{\Downarrow}$) we find that

$$\left(\tau_{ij}(A) - \eta_{ij}(A)\right) v \in U_i^{\Downarrow} + U_{i+1}^{\Downarrow} + \cdots + U_{j-1}^{\Downarrow}. \tag{55}$$

Apply $F_j^{\Downarrow}$ to each side of (55). By (16), $F_j^{\Downarrow}$ applied to the right-hand side of (55) is zero. So $F_j^{\Downarrow}(\tau_{ij}(A) - \eta_{ij}(A))v = 0$. By (15) and Lemma 3.4 (applied to $\Phi^{\Downarrow}$), $F_j^{\Downarrow}\eta_{ij}(A)v = \eta_{ij}(A)v$. The result follows from the above comments.

The equation on the right in (54) is similarly obtained. $\square$

Lemma 10.2. *For $0 \leqslant i \leqslant d/2$ and $i \leqslant j \leqslant d-i$, let Δ_{ij} denote the restriction of Δ to the subspace $\tau_{ij}(A)K_i$. Then the following diagram commutes.*

$$\begin{tikzcd} & K_i \arrow[dl, "\tau_{ij}(A)"'] \arrow[dr, "\eta_{ij}(A)"] & \\ \tau_{ij}(A)K_i \arrow[rr, "\Delta_{ij}"] & & \eta_{ij}(A)K_i \end{tikzcd}$$

Proof. Let $v \in K_i$. We push v around the diagram. Observe that $\Delta_{ij}\tau_{ij}(A)v = \Delta\tau_{ij}(A)v$. Consider $\Delta\tau_{ij}(A)v$. In this expression, eliminate Δ using Definition 9.1. Then simplify the result using (15), (16), Lemma 3.4 (applied to both Φ and $\Phi^{\Downarrow}$), and Lemma 10.1. By these comments we find $\Delta\tau_{ij}(A)v = \eta_{ij}(A)v$. $\square$

We emphasize a point for later use. By Lemma 10.2, we see that for $0 \leqslant i \leqslant d/2$ and $i \leqslant j \leqslant d-i$,

$$\Delta\tau_{ij}(A)v = \eta_{ij}(A)v \quad (v \in K_i). \tag{56}$$

Setting $j = i$ in the above argument, we see that $(\Delta - I)K_i = 0$.

11. The linear transformation Ψ

We continue to discuss the TD system Φ from Definition 2.1. We now introduce a certain linear transformation $\Psi \in \mathrm{End}(V)$ which has properties that we find attractive. To define Ψ we give its action on each summand in the decomposition of V from Corollary 7.4. It will turn out that Ψ is the unique linear transformation such that both

$$\Psi E_iV \subseteq E_{i-1}V + E_iV + E_{i+1}V,$$
$$\left(\Psi - \frac{\Delta - I}{\theta_0 - \theta_d}\right) E_i^*V \subseteq E_0^*V + E_1^*V + \cdots + E_{i-2}^*V$$

for $0 \leqslant i \leqslant d$. This characterization of Ψ will be discussed in Section 16.

Lemma 11.1. *There exists a unique linear transformation $\Psi \in \mathrm{End}(V)$ such that*

$$\Psi \tau_{ij}(A) - (\vartheta_j - \vartheta_i)\, \tau_{i,j-1}(A) \tag{57}$$

vanishes on K_i for $0 \leqslant i \leqslant d/2$ and $i \leqslant j \leqslant d-i$. Recall that $\tau_{i,i-1} = 0$.

Proof. Recall the decomposition of V given in Corollary 7.4. In the statement of the lemma, we specified the action of Ψ on each summand and therefore Ψ exists. The uniqueness assertion is clear. $\square$

We clarify the meaning of Ψ. Fix an integer i $(0 \leqslant i \leqslant d/2)$. Lemma 11.1 implies that $\Psi K_i = 0$. More generally, for $i \leqslant j \leqslant d-i$ and $v \in K_i$,

$$\Psi \tau_{ij}(A)v = (\vartheta_j - \vartheta_i)\tau_{i,j-1}(A)v. \tag{58}$$

We look at Ψ from several perspectives.

Lemma 11.2. *With reference to Lemma 11.1,*

$$\Psi U_j \subseteq U_{j-1} \quad (1 \leqslant j \leqslant d), \quad \Psi U_0 = 0.$$

Proof. We first show $\Psi U_j \subseteq U_{j-1}$ for $1 \leqslant j \leqslant d$. Let j be given. Recall from (34) the direct sum $U_j = \sum_{i=0}^{\min\{j,d-j\}} \tau_{ij}(A)K_i$. Referring to this sum, we will show Ψ sends each summand into U_{j-1}. Consider the i^{th} summand $\tau_{ij}(A)K_i$. First suppose $i = j$. Then Ψ sends this summand to zero because $\Psi K_i = 0$. Next suppose $i < j$. Using Lemma 3.4 and (58), we obtain

$$\Psi \tau_{ij}(A)K_i \subseteq \tau_{i,j-1}(A)K_i \subseteq U_{j-1}.$$

We now show $\Psi U_0 = 0$. Recall that $\Psi K_0 = 0$. The result follows since $K_0 = U_0$. $\square$

Lemma 11.3. *With reference to Lemma 11.1,*

$$F_i\Psi = \Psi F_{i+1} \quad (0 \leqslant i \leqslant d-1), \quad \Psi F_0 = 0, \quad F_d\Psi = 0. \tag{59}$$

Proof. We first show that $F_i\Psi = \Psi F_{i+1}$ for $0 \leqslant i \leqslant d-1$. Let i be given. Recall the decomposition $\{U_j\}_{j=0}^d$ of V from Section 3. We will show that $F_i\Psi - \Psi F_{i+1}$ vanishes on each U_j. Observe that

$$F_i\Psi - \Psi F_{i+1} = (F_i - I)\Psi - \Psi(F_{i+1} - I). \tag{60}$$

The right-hand side of (60) vanishes on U_j by (15), (16), and Lemma 11.2. Thus $F_i\Psi - \Psi F_{i+1}$ vanishes U_j and hence on V. The equation on the left in (59) follows from the above comments.

The assertions $\Psi F_0 = 0$, $F_d\Psi = 0$ follow from Lemma 11.2. $\square$

Lemma 11.4. *With reference to Definition 9.1 and Lemma 11.1, for $0 \leqslant j \leqslant d$ apply either of*

$$\Delta - I - (\theta_0 - \theta_d)\Psi, \quad \Delta^{-1} - I + (\theta_0 - \theta_d)\Psi \tag{61}$$

to U_j and consider the image. This image is contained in $U_0 + U_1 + \cdots + U_{j-2}$ if $j \geqslant 2$ and equals 0 if $j < 2$.

Proof. We first consider the expression on the left in (61). Referring to the decomposition of V given in Corollary 7.4, consider any summand $\tau_{ij}(A)K_i$. We show that the image of $\tau_{ij}(A)K_i$ under the expression on the left in (61) is contained in $U_0 + U_1 + \cdots + U_{j-2}$ if $j \geqslant 2$ and equals 0 if $j < 2$. By (56) and Lemma 11.1, the actions of the expression on the left in (61) times $\tau_{ij}(A)$ and

$$\eta_{ij}(A) - \tau_{ij}(A) - (\theta_0 - \theta_d)(\vartheta_j - \vartheta_i)\tau_{i,j-1}(A) \tag{62}$$

agree on K_i. By (3), (4), and Definition 2.4, (62) is a polynomial in A of degree at most $j-i-2$ if $j \geqslant i+2$ and equals 0 if $j < i+2$. The result follows from the above comments and (11).

We now consider the expression on the right in (61). We will use the fact that the result holds for the expression on the left in (61). Observe that

$$\Delta^{-1} - I + (\theta_0 - \theta_d)\Psi = \Delta^{-1}(\Delta - I)^2 - \Delta + I + (\theta_0 - \theta_d)\Psi.$$

The result follows from the above comments, (41) and Lemma 9.6. $\square$

Lemma 11.5. *With reference to Lemma 11.1, Ψ satisfies*

$$\Psi R - R\Psi = \sum_{h=0}^{d} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} F_h. \tag{63}$$

Proof. Referring to the decomposition of V given in Corollary 7.4, consider any summand $\tau_{ij}(A)K_i$. We apply each side of (63) to this summand. We claim that on this summand, each side of (63) acts as $(\theta_j - \theta_{d-j})(\theta_0 - \theta_d)^{-1}I$.

The claim holds for the right-hand side of (63) by (15), (16), and the fact that $\tau_{ij}(A)K_i \subseteq U_j$. Concerning the left-hand side of (63), we routinely carry out this application using (3), (8), (20), and Lemma 11.1. $\square$

Corollary 11.6. *With reference to Definition 9.1 and Lemma 11.1,*

$$\frac{A - \Delta^{-1}A\Delta}{\theta_0 - \theta_d} = \Psi R - R\Psi.$$

Proof. Use Corollary 9.12 and Lemma 11.5. $\square$

We now give a characterization of Ψ.

Lemma 11.7. *Given $\Psi' \in \mathrm{End}(V)$ such that*

$$\Psi' R - R\Psi' = \sum_{h=0}^{d} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} F_h \tag{64}$$

and $\Psi' K_i = 0$ for $0 \leqslant i \leqslant d/2$. Then $\Psi' = \Psi$.

Proof. Recall from Corollary 7.4 the decomposition $V = \sum_{i=0}^{r} \sum_{j=i}^{d-i} \tau_{ij}(A)K_i$, where $r = \lfloor d/2 \rfloor$. We show that $\Psi - \Psi'$ vanishes on each summand by fixing i and inducting on j. Let i be given. Recall that $\Psi K_i = 0$. Thus $\Psi - \Psi'$ vanishes on $\tau_{ii}(A)K_i = K_i$. Now suppose $\Psi - \Psi'$ vanishes on $\tau_{ij}(A)K_i$. We show that $\Psi - \Psi'$ vanishes on $\tau_{i,j+1}(A)K_i$. By (63) and (64), we see that $(\Psi - \Psi')R = R(\Psi - \Psi')$. By the above comments, $\Psi - \Psi'$ vanishes on $R\tau_{ij}(A)K_i$. By (3) and (20), $R\tau_{ij}(A)K_i = \tau_{i,j+1}(A)K_i$. Thus $\Psi - \Psi'$ vanishes on $\tau_{i,j+1}(A)K_i$. So $\Psi - \Psi'$ vanishes on V. $\square$

Lemma 11.8. *With reference to Definition 9.1 and Lemma 11.1, $\Delta^{-1}A^*\Delta - A^*$ acts on U_i as $(\theta^*_{i-1} - \theta^*_i)(\theta_0 - \theta_d)\Psi$ for $1 \leqslant i \leqslant d$ and as 0 for $i = 0$.*

Proof. First assume $1 \leqslant i \leqslant d$. For notational convenience, we abbreviate $\Omega = (\theta_0 - \theta_d)\Psi$. We will show that

$$\Delta^{-1}A^*\Delta - A^* - (\theta^*_{i-1} - \theta^*_i)\Omega \tag{65}$$

vanishes on U_i. To accomplish this, we show that the image of U_i under (65) is contained in both U_{i-1} and $\sum_{h=0}^{i-2} U_h$.

We first show that the image of U_i under (65) is contained in U_{i-1}. This follows from Lemma 9.14 and Lemma 11.2.

We now show that the image of U_i under (65) is contained in $\sum_{h=0}^{i-2} U_h$. Observe that (65) is equal to

$$\begin{aligned} &\theta^*_{i-1}(\Delta^{-1} - I)\Omega \;+\; \Delta^{-1}(A^* - \theta^*_{i-1}I)\Omega \;+\; (\Delta^{-1} - I)(A^* - \theta^*_i I) \\ &\quad + \Delta^{-1}A^*(\Delta - I - \Omega) \;+\; \theta^*_i(\Delta^{-1} - I + \Omega). \end{aligned} \tag{66}$$

We will argue that each of the five terms in this sum sends U_i into $\sum_{h=0}^{i-2} U_h$. We begin by recalling some facts. For $0 \leqslant j \leqslant d$ each of

$$A^* - \theta^*_j I, \quad \Delta - I, \quad \Delta^{-1} - I, \quad \Omega$$

sends U_j into $\sum_{h=0}^{j-1} U_h$. This is a consequence of (10), (41), Lemma 9.6, and Lemma 11.2 respectively. It follows from these comments that for $0 \leqslant j \leqslant d$, each of A^*, Δ, Δ^{-1}, Ω sends U_j into $\sum_{h=0}^{j} U_h$. Using the above facts we find that each of

$$(\Delta^{-1} - I)\Omega, \quad \Delta^{-1}(A^* - \theta^*_{i-1}I)\Omega, \quad (\Delta^{-1} - I)(A^* - \theta^*_i I)$$

sends U_i into $\sum_{h=0}^{i-2} U_h$. Thus each of the first three terms in the sum (66) sends U_i into $\sum_{h=0}^{i-2} U_h$. By Lemma 11.4, each of

$$\Delta - I - \Omega, \quad \Delta^{-1} - I + \Omega$$

sends U_i into $\sum_{h=0}^{i-2} U_h$. By the above facts, each of the last two terms in the sum (66) sends U_i into $\sum_{h=0}^{i-2} U_h$. We have now shown that each of the five terms in the sum (66) sends U_i into $\sum_{h=0}^{i-2} U_h$. Therefore, the image of U_i under (66) is contained in $\sum_{h=0}^{i-2} U_h$.

We have now shown that image of U_i under (65) is contained in the intersection of U_{i-1} and $\sum_{h=0}^{i-2} U_h$. Since $\{U_j\}_{j=0}^{d}$ is a decomposition of V, this intersection is zero and hence the expression (65) vanishes on U_i. The proof is complete for $1 \leqslant i \leqslant d$.

The case when $i = 0$ follows from Lemma 9.14. □

Combining Lemma 11.8 with Lemma 9.13, we obtain the following corollary.

Corollary 11.9. *With reference to Definition 9.1 and Lemma 11.1, $\Delta^{-1}L^{\Downarrow}\Delta - L$ acts on U_i as $(\theta^*_{i-1} - \theta^*_i)(\theta_0 - \theta_d)\Psi$ for $1 \leqslant i \leqslant d$ and as 0 for $i = 0$.*

We mention how Δ, Ψ act on the spaces Mv from Section 8.

Lemma 11.10. *For $0 \leqslant i \leqslant d/2$ and $v \in K_i$, the space Mv is invariant under both Δ and Ψ.*

Proof. To show Mv is invariant under Δ, use Corollary 8.3, (56), and the fact that $\eta_{ij}(A) \in M$ for $i \leqslant j \leqslant d - i$. To show Mv is invariant under Ψ, use Corollary 8.3 and (58). □

12. The eigenvalue and dual eigenvalue sequences

We continue to discuss the TD system Φ from Definition 2.1. In Sections 15, 16, and 17, we will obtain some detailed results about Δ and Ψ. In order to do so, we must first recall some facts concerning the eigenvalues and dual eigenvalues of Φ.

Theorem 12.1 [2, Theorem 11.1]**.** *The expressions*

$$\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}, \quad \frac{\theta^*_{i-2}-\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i} \tag{67}$$

are equal and independent of i for $2 \leqslant i \leqslant d-1$.

Definition 12.2. We associate a scalar β with Φ as follows. If $d \geqslant 3$ let $\beta+1$ denote the common value of (67). If $d \leqslant 2$ let β denote any nonzero scalar in $\mathbb{K}$. We call β the *base* of Φ.

On occasion, it will be necessary to split into cases based on the scalar β. This motivates the following definition.

Definition 12.3 [3, Definition 4.4]**.** We assign to Φ a *type* as follows:

Type	Description
I	$\beta \neq 2, \beta \neq -2$
II	$\beta = 2$, $\mathrm{Char}(\mathbb{K}) \neq 2$
III^+	$\beta = -2$, $\mathrm{Char}(\mathbb{K}) \neq 2$, d even
III^-	$\beta = -2$, $\mathrm{Char}(\mathbb{K}) \neq 2$, d odd
IV	$\beta = 0$, $\mathrm{Char}(\mathbb{K}) = 2$

We say Φ has *type* III whenever Φ has type III^+ or III^-.

We refer the reader to [2, Theorem 11.2] for explicit formulas for the eigenvalues and dual eigenvalues for each type. These formulas are not necessary for the present paper. However we will need the following facts.

Lemma 12.4 [2, Theorem 11.2]**.** *With reference to Definition 12.2, the following* (i)–(iv) *hold.*

(i) *Suppose* Φ *has type* I. *Pick* $q \in \overline{\mathbb{K}}$ *such that* $q+q^{-1}=\beta$. *Then* $q^i \neq 1$ *for* $1 \leqslant i \leqslant d$.
(ii) *Suppose* Φ *has type* II. *Then* $\mathrm{Char}(\mathbb{K})=0$ *or* $\mathrm{Char}(\mathbb{K}) > d$.
(iii) *Suppose* Φ *has type* III. *Then* $\mathrm{Char}(\mathbb{K})=0$ *or* $\mathrm{Char}(\mathbb{K}) > d/2$.
(iv) *Suppose* Φ *has type* IV. *Then* $d=3$.

Lemma 12.5 [18, Lemma 9.4]**.** *With reference to Definition 12.2, pick integers* i, j, r, s $(0 \leqslant i, j, r, s \leqslant d)$ *and assume* $i+j=r+s$, $i \neq j$. *Then the following* (i)–(iv) *hold.*

(i) *Suppose* Φ *has type* I. *Then*

$$\frac{\theta_r-\theta_s}{\theta_i-\theta_j}=\frac{q^r-q^s}{q^i-q^j},$$

where $q+q^{-1}=\beta$.
(ii) *Suppose* Φ *has type* II. *Then*

$$\frac{\theta_r-\theta_s}{\theta_i-\theta_j}=\frac{r-s}{i-j}.$$

(iii) *Suppose* Φ *has type* III. *Then*

$$\frac{\theta_r - \theta_s}{\theta_i - \theta_j} = \begin{cases} (-1)^{r+i}\frac{r-s}{i-j} & \text{if } i+j \text{ is even}, \\ (-1)^{r+i} & \text{if } i+j \text{ is odd}. \end{cases}$$

(iv) *Suppose* Φ *has type* IV. *Then*

$$\frac{\theta_r - \theta_s}{\theta_i - \theta_j} = \begin{cases} 0 & \text{if } r = s, \\ 1 & \text{if } r \neq s. \end{cases}$$

13. Some scalars

We continue to discuss the TD system Φ from Definition 2.1. In Section 2, we used Φ to define the scalars $\{\vartheta_i\}_{i=0}^{d+1}$. In this section we discuss some properties of these scalars which will be of use later.

Recall from Definition 2.4 that

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (0 \leqslant i \leqslant d+1).$$

We remark that

$$\vartheta_0 = 0, \quad \vartheta_1 = 1, \quad \vartheta_d = 1, \quad \vartheta_{d+1} = 0.$$

Moreover,

$$\vartheta_i = \vartheta_{d-i+1} \quad (0 \leqslant i \leqslant d+1). \tag{68}$$

When we were working with the eigenvalues of Φ, a key feature was that they are mutually distinct. So it is natural to ask if there are any duplications in the sequence $\{\vartheta_i\}_{i=0}^{d+1}$. In (68) we already saw that $\vartheta_i = \vartheta_{d-i+1}$ for $0 \leqslant i \leqslant d+1$. So we would like to know if the $\{\vartheta_i\}_{i=0}^{r}$ are mutually distinct, where $r = \lfloor \frac{d+1}{2} \rfloor$. It turns out that this is false in general, but something can be said in certain cases. We now explain the details.

Lemma 13.1. *With reference to Definition 12.2, the following holds for* $0 \leqslant i, j \leqslant d+1$.

(i) *Suppose* Φ *has type* I. *Then*

$$\vartheta_i - \vartheta_j = \frac{\left(q^j - q^i\right)\left(1 - q^{d-i-j+1}\right)}{(1-q)\left(1-q^d\right)}.$$

(ii) *Suppose* Φ *has type* II. *Then*

$$\vartheta_i - \vartheta_j = \frac{(i-j)(d-i-j+1)}{d}.$$

(iii) *Suppose* Φ *has type* III$^-$. *Then*

$$\vartheta_i - \vartheta_j = \begin{cases} 0 & \text{if } i+j \text{ is even}, \\ (-1)^j & \text{if } i+j \text{ is odd}. \end{cases}$$

(iv) *Suppose* Φ *has type* III$^+$. *Then*

$$\vartheta_i - \vartheta_j = \begin{cases} (-1)^j \frac{i-j}{d} & \text{if } i+j \text{ is even,} \\ (-1)^j \frac{d-i-j+1}{d} & \text{if } i+j \text{ is odd.} \end{cases}$$

(v) *Suppose* Φ *has type* IV. *Then*

$$\vartheta_i - \vartheta_j = \begin{cases} 0 & \text{if } i+j \text{ is even,} \\ 1 & \text{if } i+j \text{ is odd.} \end{cases}$$

Proof. Use Lemma 12.4 and [18, Lemma 10.2]. □

Observe that we can obtain explicit formulas for the ϑ_i by setting $j = 0$ in the above lemma.

Lemma 13.2. *Suppose* Φ *has type* I, II, *or* III$^+$. *Then the following are equivalent for* $0 \leqslant i, j \leqslant d+1$.

(i) $\vartheta_i = \vartheta_j$.
(ii) $i = j$ *or* $i + j = d + 1$.

Proof. Use Lemma 12.4 and Lemma 13.1. □

Corollary 13.3. *Suppose* Φ *has type* I, II, *or* III$^+$. *Then* $\vartheta_i \neq 0$ *for* $1 \leqslant i \leqslant d$.

We finish this section with a comment.

Lemma 13.4. *For* $0 \leqslant i, j, r, s \leqslant d$ *we have*

$$(\theta_r - \theta_s)(\vartheta_i - \vartheta_j) = (\theta_i - \theta_j)(\vartheta_r - \vartheta_s),$$

provided that $i + j = r + s$.

Proof. Use Lemma 12.5 and Lemma 13.1. □

14. The scalars $[r, s, t]$

We continue to discuss the TD system Φ from Definition 2.1. To motivate our results in this section, for the moment fix an integer i ($0 \leqslant i \leqslant d/2$). As we proceed, it will be convenient to express each of $\{\tau_{ij}\}_{j=i}^{d-i}$ as a linear combination of $\{\eta_{ij}\}_{j=i}^{d-i}$. In order to describe the coefficients, we will use the following notation.

For all $a, q \in \overline{\mathbb{K}}$ define

$$(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n = 0, 1, 2, \ldots \tag{69}$$

and interpret $(a; q)_0 = 1$.

In [16] Terwilliger defined some scalars $[r, s, t]_q \in \mathbb{K}$ for nonnegative integers r, s, t such that $r + s + t \leqslant d$. By [16, Lemma 13.2] these scalars are rational functions of the base β. In this paper we are going to drop the subscript q and just write $[r, s, t]$. For further discussion of these scalars see [3,16].

Definition 14.1 [16, Lemma 13.2]**.** With reference to Definition 12.2, let r, s, t denote nonnegative integers such that $r + s + t \leqslant d$. We define $[r, s, t]$ as follows.

(i) Suppose Φ has type I. Then

$$[r,s,t]=\frac{(q;q)_{r+s}(q;q)_{r+t}(q;q)_{s+t}}{(q;q)_r(q;q)_s(q;q)_t(q;q)_{r+s+t}},$$

where $q+q^{-1}=\beta$.

(ii) Suppose Φ has type II. Then

$$[r,s,t]=\frac{(r+s)!\,(r+t)!\,(s+t)!}{r!\,s!\,t!\,(r+s+t)!}.$$

(iii) Suppose Φ has type III. If each of r,s,t is odd, then $[r,s,t]=0$. If at least one of r,s,t is even, then

$$[r,s,t]=\frac{\lfloor\frac{r+s}{2}\rfloor!\lfloor\frac{r+t}{2}\rfloor!\lfloor\frac{s+t}{2}\rfloor!}{\lfloor\frac{r}{2}\rfloor!\lfloor\frac{s}{2}\rfloor!\lfloor\frac{t}{2}\rfloor!\lfloor\frac{r+s+t}{2}\rfloor!}.$$

The expression $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.

(iv) Suppose Φ has type IV. If each of r,s,t equals 1, then $[r,s,t]=0$. If at least one of r,s,t equals 0, then $[r,s,t]=1$.

We make a few observations. The expression $[r,s,t]$ is symmetric in r,s,t. Also, $[r,s,t]=1$ if at least one of r,s,t equals zero.

Lemma 14.2 [3, Lemma 5.3]. *Let r,s,t,u denote nonnegative integers such that $r+s+t+u\leqslant d$. Then*

$$[r,s,t+u][t,u,r+s]=[s,u,r+t][r,t,s+u].$$

The following result is a modification of [13, Lemma 12.4].

Lemma 14.3. *Let $0\leqslant i\leqslant d/2$ and $i\leqslant j\leqslant d-i$. Both*

$$\tau_{ij}=\sum_{h=0}^{j-i}[h,j-i-h,d-i-j]\tau_{i,i+h}(\theta_{d-i})\eta_{i,j-h}, \tag{70}$$

$$\eta_{ij}=\sum_{h=0}^{j-i}[h,j-i-h,d-i-j]\eta_{i,i+h}(\theta_{i})\tau_{i,j-h}. \tag{71}$$

Proof. Apply [13, Lemma 12.4] to the sequence $\{\theta_k\}_{k=i}^{d-i}$. $\square$

Later in the paper, we will be doing some computations involving the coefficients in (70) and (71). The following results will aid in these computations.

Corollary 14.4. *For $0\leqslant i\leqslant d/2$ and $i+1\leqslant j\leqslant d-i$,*

$$(\theta_0-\theta_d)\left(\vartheta_j-\vartheta_i\right)=(\theta_i-\theta_{d-i})\,[1,j-i-1,d-i-j].$$

Proof. Let C denote the coefficient of x^{j-i-1} on either side of (70). From the left-hand side of (70), we see

$$C=-\sum_{h=i}^{j-1}\theta_h. \tag{72}$$

From the right-hand side of (70), we see

$$C = (\theta_{d-i} - \theta_i)\,[j-i-1,\,1,\,d-i-j] - \sum_{h=i}^{j-1} \theta_{d-h}. \tag{73}$$

Subtract (72) from (73) and invoke the symmetry of $[r, s, t]$ as well as Definition 2.4 to get the result. □

Lemma 14.5. *For $0 \leqslant i \leqslant d/2$ and $i+1 \leqslant j \leqslant d-i$ and $0 \leqslant h \leqslant j-i-1$,*

$$(\vartheta_j - \vartheta_i)\,[h, j-i-h-1, d-i-j+1] = (\vartheta_{j-h} - \vartheta_i)\,[h, j-i-h, d-i-j] \tag{74}$$

and

$$(\vartheta_j - \vartheta_i)\,[h, j-i-h-1, d-i-j+1] = (\vartheta_{i+h+1} - \vartheta_i)\,[h+1, j-i-h-1, d-i-j]. \tag{75}$$

Proof. For (74), use Lemma 14.2 with $r = 1$, $s = j-i-h-1$, $t = d-i-j$, $u = h$. Simplify the result using Corollary 14.4 and the fact that $[r, s, t]$ is symmetric in r, s, t.

Line (75) is similarly obtained. □

Corollary 14.6. *Suppose Φ has type* I, II, *or* III$^+$. *For $0 \leqslant i \leqslant d/2$ and $i \leqslant j \leqslant d-i$ and $0 \leqslant h \leqslant j-i$,*

$$[h, j-i-h, d-i-j] = \prod_{k=0}^{h-1} \frac{\vartheta_{j-k} - \vartheta_i}{\vartheta_{d-i-k} - \vartheta_i}. \tag{76}$$

In (76) the denominators are nonzero by Lemma 13.2.

Proof. Assume $h \geqslant 1$; otherwise both sides of (76) equal 1. From (75) we obtain

$$[h, j-i-h, d-i-j] = \frac{\vartheta_j - \vartheta_i}{\vartheta_{i+h} - \vartheta_i}[h-1, j-i-h, d-i-j+1].$$

Iterating this we get

$$[h, j-i-h, d-i-j] = \prod_{k=0}^{h-1} \frac{\vartheta_{j-k} - \vartheta_i}{\vartheta_{i+k+1} - \vartheta_i}.$$

Evaluating the denominator using Lemma 13.2 we obtain the result. □

15. The maps Δ, Ψ commute

We continue to discuss the TD system Φ from Definition 2.1. In Section 9, we introduced the linear transformation Δ and discussed some of its properties. In Section 11, we introduced the linear transformation Ψ and discussed some of its properties. We now discuss how Δ, Ψ relate to each other. Along this line we have two main results. They are Theorem 15.1 and Theorem 17.1. We prove Theorem 15.1 in this section. Before proving Theorem 17.1, it will be convenient to give the characterization of Ψ discussed in Section 1. This will be done in Section 16.

Theorem 15.1. *With reference to Definition 9.1 and Lemma 11.1, the operators Δ, Ψ commute.*

Proof. Recall the decomposition of V given in Corollary 7.4. We will show that $\Psi\Delta$, $\Delta\Psi$ agree on each summand $\tau_{ij}(A)K_i$.

First assume that $i = j$. Recall that τ_{ii} and η_{ii} both equal 1. Using the fact that both $\Delta - I$ and Ψ vanish on K_i, we routinely find that each of $\Psi\Delta$, $\Delta\Psi$ vanishes on $\tau_{ii}(A)K_i$.

Next assume that $i < j$. In order to show that $\Psi\Delta$, $\Delta\Psi$ agree on $\tau_{ij}(A)K_i$, it suffices to show that $\Psi\Delta\tau_{ij}(A)$ and $\Delta\Psi\tau_{ij}(A)$ agree on K_i. By (71), Lemma 10.2, and Lemma 11.1, the operators $\Psi\Delta\tau_{ij}(A)$ and

$$\sum_{h=0}^{j-i-1} (\vartheta_{j-h} - \vartheta_i)\,[h, j-i-h, d-i-j]\eta_{i,i+h}(\theta_i)\tau_{i,j-h-1}(A) \tag{77}$$

agree on K_i. By (71), Lemma 10.2, and Lemma 11.1, the operators $\Delta\Psi\tau_{ij}(A)$ and

$$(\vartheta_j - \vartheta_i)\sum_{h=0}^{j-i-1} [h, j-i-h-1, d-i-j+1]\eta_{i,i+h}(\theta_i)\tau_{i,j-h-1}(A) \tag{78}$$

agree on K_i. In order to show that (77), (78) agree on K_i, we will need the fact that

$$(\vartheta_{j-h} - \vartheta_i)\,[h, j-i-h, d-i-j]$$

and

$$(\vartheta_j - \vartheta_i)\,[h, j-i-h-1, d-i-j+1]$$

are equal for $0 \leqslant h \leqslant j-i-1$. This equality is (74). Therefore (77), (78) agree on K_i. Thus $\Psi\Delta\tau_{ij}(A)$ and $\Delta\Psi\tau_{ij}(A)$ agree on K_i. Hence $\Psi\Delta$, $\Delta\Psi$ agree on $\tau_{ij}(A)K_i$. By Corollary 7.4, $\Psi\Delta$, $\Delta\Psi$ agree on V. $\square$

From Theorem 15.1, we derive a number of corollaries.

Corollary 15.2. *With reference to Lemma 11.1, $\Psi^{\Downarrow} = \Psi$.*

Proof. We first show that $\Psi^{\Downarrow}\Delta = \Delta\Psi$. Recall the decomposition of V given in Corollary 7.4. We will show that $\Psi^{\Downarrow}\Delta$, $\Delta\Psi$ agree on each summand $\tau_{ij}(A)K_i$. By (56) and (58) (applied to both Φ and $\Phi^{\Downarrow}$), $\Psi^{\Downarrow}\Delta\tau_{ij}(A)$ and $\Delta\Psi\tau_{ij}(A)$ agree on K_i. Hence $\Psi^{\Downarrow}\Delta$, $\Delta\Psi$ agree on $\tau_{ij}(A)K_i$. By Corollary 7.4, $\Psi^{\Downarrow}\Delta$, $\Delta\Psi$ agree on V. Thus $\Psi^{\Downarrow}\Delta = \Delta\Psi$. Combine this fact with Theorem 15.1 and the fact that Δ is invertible to get the result. $\square$

Corollary 15.3. *With reference to Lemma 11.1, we have*

$$\Psi U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \quad (1 \leqslant i \leqslant d), \quad \Psi U_0^{\Downarrow} = 0.$$

Proof. Combine Corollary 15.2 with Lemma 11.2. $\square$

Corollary 15.4. *With reference to Lemma 11.1, we have*

$$\Psi E_i V \subseteq E_{i-1}V + E_iV + E_{i+1}V \quad (0 \leqslant i \leqslant d).$$

Proof. Let i be given. On the one hand, by (13) and Lemma 11.2, we have

$$\begin{aligned} \Psi E_iV &\subseteq \Psi(E_iV + E_{i+1}V + \cdots + E_dV) \\ &= \Psi(U_i + U_{i+1} + \cdots + U_d) \\ &\subseteq U_{i-1} + U_i + \cdots + U_d \\ &= E_{i-1}V + E_{i+1}V + \cdots + E_dV. \end{aligned} \tag{79}$$

On the other hand, by (24) and Corollary 15.3, we have

$$\begin{aligned} \Psi E_iV &\subseteq \Psi(E_0V + E_1V + \cdots + E_iV) \\ &= \Psi(U_{d-i}^{\Downarrow} + U_{d-i+1}^{\Downarrow} + \cdots + U_d^{\Downarrow}) \\ &\subseteq U_{d-i-1}^{\Downarrow} + U_{d-i}^{\Downarrow} + \cdots + U_d^{\Downarrow} \\ &= E_0V + E_1V + \cdots + E_{i+1}V. \end{aligned} \tag{80}$$

Observe that $\Psi E_i V$ is contained in the intersection of (79) and (80). This intersection equals $E_{i-1}V + E_iV + E_{i+1}V$, and the result follows. $\square$

16. A characterization of Ψ

We continue to discuss the TD system Φ from Definition 2.1. Our goal in this section is to obtain the characterization of Ψ given in the Introduction.

Lemma 16.1. *With reference to Lemma 11.1, we have*

$$\Psi E_i^* V \subseteq E_0^* V + E_1^* V + \cdots + E_{i-1}^* V \quad (0 \leqslant i \leqslant d).$$

Proof. By (14), $E_i^* V \subseteq U_0 + U_1 + \cdots + U_i$. Use this fact along with Lemma 11.2 to obtain $\Psi E_i^* V \subseteq U_0 + U_1 + \cdots + U_{i-1}$. The result follows from this fact along with (14). $\square$

Lemma 16.2. *With reference to Definition 9.1 and Lemma 11.1, for $0 \leqslant j \leqslant d$ apply either of*

$$\Delta - I - (\theta_0 - \theta_d)\Psi, \quad \Delta^{-1} - I + (\theta_0 - \theta_d)\Psi$$

to $E_j^* V$ and consider the image. This image is contained in $E_0^* V + E_1^* V + \cdots + E_{j-2}^* V$ if $j \geqslant 2$ and equals 0 if $j < 2$.

Proof. Use (14) and Lemma 11.4. $\square$

By Corollary 15.4 and Lemma 16.2, both

$$\Psi E_i V \subseteq E_{i-1}V + E_iV + E_{i+1}V,$$
$$\left(\Psi - \frac{\Delta - I}{\theta_0 - \theta_d}\right) E_i^* V \subseteq E_0^* V + E_1^* V + \cdots + E_{i-2}^* V$$

for $0 \leqslant i \leqslant d$. We show that these two properties characterize Ψ.

Lemma 16.3. *Given $\Psi' \in \mathrm{End}(V)$ such that both*

$$\Psi' E_i V \subseteq E_{i-1}V + E_iV + E_{i+1}V,$$
$$\left(\Psi' - \frac{\Delta - I}{\theta_0 - \theta_d}\right) E_i^* V \subseteq E_0^* V + E_1^* V + \cdots + E_{i-2}^* V$$

for $0 \leqslant i \leqslant d$. Then $\Psi' = \Psi$.

Proof. Recall that $\{U_i\}_{i=0}^d$ is a decomposition of V. So it suffices to show that Ψ, Ψ' agree on U_i for $0 \leqslant i \leqslant d$. Let i be given. Observe that

$$\Psi - \Psi' = \Psi - \frac{\Delta - I}{\theta_0 - \theta_d} - \Psi' + \frac{\Delta - I}{\theta_0 - \theta_d}. \tag{81}$$

Using (81) along with (14) and Lemma 16.2, we obtain

$$\begin{aligned}(\Psi - \Psi')U_i &\subseteq (\Psi - \Psi')(U_0 + U_1 + \cdots + U_i) \\ &= (\Psi - \Psi')(E_0^* V + E_1^* V + \cdots + E_i^* V) \\ &\subseteq E_0^* V + E_1^* V + \cdots + E_{i-2}^* V \\ &= U_0 + U_1 + \cdots + U_{i-2}.\end{aligned}$$

By (13) and Corollary 15.4,

$$\begin{aligned}(\Psi - \Psi')U_i &\subseteq (\Psi - \Psi')(U_i + U_{i+1} + \cdots + U_d)\\ &= (\Psi - \Psi')(E_iV + E_{i+1}V + \cdots + E_dV)\\ &\subseteq E_{i-1}V + E_iV + \cdots + E_dV\\ &= U_{i-1} + U_i + \cdots + U_d.\end{aligned}$$

Thus $(\Psi - \Psi')U_i$ is contained in the intersection of $U_0 + U_1 + \cdots + U_{i-2}$ and $U_{i-1} + U_i + \cdots + U_d$. This intersection is zero since $\{U_i\}_{i=0}^d$ is a decomposition of V. So $\Psi - \Psi'$ vanishes on U_i. Therefore Ψ, Ψ' agree on U_i. $\square$

17. In general, $\Delta^{\pm 1}$ are polynomials in Ψ

We continue to discuss the TD system Φ from Definition 2.1. Recall the map Δ from Definition 9.1 and the map Ψ from Lemma 11.1. In Section 15, we saw that Δ, Ψ commute. In this section, we show that $\Delta^{\pm 1}$ are polynomials in Ψ provided that each of $\vartheta_1, \vartheta_2, \ldots, \vartheta_d$ is nonzero.

Theorem 17.1. *Let $\Delta \in \mathrm{End}(V)$ be as in Definition 9.1 and let $\Psi \in \mathrm{End}(V)$ be as in Lemma 11.1. Assume that Φ has type* I, II, *or* III$^+$ *so that the scalars $\{\vartheta_i\}_{i=1}^d$ from Definition 2.4 are nonzero. Then both*

$$\Delta = I + \frac{\eta_1(\theta_0)}{\vartheta_1}\Psi + \frac{\eta_2(\theta_0)}{\vartheta_1\vartheta_2}\Psi^2 + \cdots + \frac{\eta_d(\theta_0)}{\vartheta_1\vartheta_2\cdots\vartheta_d}\Psi^d, \tag{82}$$

$$\Delta^{-1} = I + \frac{\tau_1(\theta_d)}{\vartheta_1}\Psi + \frac{\tau_2(\theta_d)}{\vartheta_1\vartheta_2}\Psi^2 + \cdots + \frac{\tau_d(\theta_d)}{\vartheta_1\vartheta_2\cdots\vartheta_d}\Psi^d. \tag{83}$$

Proof. We first show (82). Recall the decomposition of V from Corollary 7.4. We show that each side of (82) agrees on each summand $\tau_{ij}(A)K_i$. Let $v \in K_i$. We apply each side of (82) to the vector $\tau_{ij}(A)v$ and show that the results agree.

We first apply the left-hand side of (82) to $\tau_{ij}(A)v$. By Lemma 10.2 and (71), $\Delta\tau_{ij}(A)v$ is a linear combination of $\{\tau_{i,j-h}(A)v\}_{h=0}^{j-i}$ such that the coefficient of $\tau_{i,j-h}(A)v$ is

$$[h, j-i-h, d-i-j]\eta_{i,i+h}(\theta_i) \tag{84}$$

for $0 \leqslant h \leqslant j-i$. We now apply the right-hand side of (82) to $\tau_{ij}(A)v$. For the sum on the right-hand side of (82), the action of each term on $\tau_{ij}(A)v$ is computed using (58). From this computation, one finds that the right-hand side of (82) applied to $\tau_{ij}(A)v$ is a linear combination of $\{\tau_{i,j-h}(A)v\}_{h=0}^{j-i}$ such that the coefficient of $\tau_{i,j-h}(A)v$ is

$$\frac{\eta_h(\theta_0)}{\vartheta_1\vartheta_2\cdots\vartheta_h}\prod_{k=0}^{h-1}(\vartheta_{j-k} - \vartheta_i) \tag{85}$$

for $0 \leqslant h \leqslant j-i$. It remains to show that (84) is equal to (85) for $0 \leqslant h \leqslant j-i$. Let h be given. By (4) and Corollary 14.6, the scalar (84) is equal to

$$\prod_{k=0}^{h-1}\frac{(\theta_i - \theta_{d-i-k})(\vartheta_{j-k} - \vartheta_i)}{\vartheta_{d-i-k} - \vartheta_i}. \tag{86}$$

By (7) and since $\vartheta_\ell = \vartheta_{d-\ell+1}$ for $1 \leqslant \ell \leqslant h$, the scalar (85) is equal to

$$\prod_{k=0}^{h-1}\frac{(\theta_0 - \theta_{d-k})(\vartheta_{j-k} - \vartheta_i)}{\vartheta_{d-k}}. \tag{87}$$

By Lemma 13.4 and since $\vartheta_0 = 0$,

$$\frac{\theta_i - \theta_{d-i-k}}{\vartheta_{d-i-k} - \vartheta_i} = \frac{\theta_0 - \theta_{d-k}}{\vartheta_{d-k}} \quad (0 \leqslant k \leqslant h-1).$$

Using this we find that (86) is equal to (87). Therefore (84) is equal to (85) for $0 \leqslant h \leqslant j - i$ as desired. We have shown (82).

To get (83), apply (82) to $\Phi^{\Downarrow}$ and use Corollary 15.2 along with the fact that $\vartheta_k^{\Downarrow} = \vartheta_k$ for $1 \leqslant k \leqslant d$. $\square$

18. Comments

We now make a few comments regarding future work related to this paper.

The reader may have already noticed that the relation in Lemma 11.5 looks like one of the defining relations for the quantum sl_2. In fact, there exists a quantum sl_2-module structure here. We will treat this topic comprehensively in a future paper.

The reader may have also noticed some similarities between Δ and the switching element S from [13]. In spite of the superficial similarities, we see no connection between Δ and S.

We now give some suggestions for further research relating to this paper.

Problem 18.1. With reference to Lemma 11.1, what is $L\Psi - \Psi L$?

Problem 18.2. With reference to Lemma 11.1, are L and Ψ related in an interesting way? How about $L^{\Downarrow}$ and Ψ?

Problem 18.3. With reference to Definition 9.1 and Lemma 11.1, write Ψ as a polynomial in $\Delta - I$.

Acknowledgments

This paper was written while the author was a graduate student at the University of Wisconsin-Madison. The author would like to thank her advisor, Paul Terwilliger, for offering many valuable ideas and suggestions.

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