Unsteady Flows of Viscoelastic Fluids with Inflow and Outflow Boundary Conditions

R. TALHOUK
Mathématiques, Université Paris XII-Val de Marne
94010 Créteil Cedex, France
and
Laboratoire d'Analyse Numérique, Bâtiment 425
Université Paris-Sud et CNRS, 91405 Orsay Cedex, France
(Received November 1995; accepted January 1996)

Abstract—In this paper, we prove an existence and uniqueness result for unsteady flows of viscoelastic fluids through a strip with inflow and outflow boundary conditions, which can be regarded as perturbations of rigid motions. The fluids obey a constitutive law of Oldroyd type.

Keywords—Viscoelastic fluids, Inflow and outflow boundary conditions, Local solutions.

1. INTRODUCTION

We consider the Jeffreys model for incompressible viscoelastic fluids (0 < ε < 1). The system of partial differential equations in nondimensional form is given as follows (see, for instance, [1,2]):

\[
\begin{align*}
\text{Re } & \left( \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) - (1 - \epsilon) \Delta u + \nabla p = f + \nabla \tau, \\
\text{div } u &= 0, \\
\text{We } & \left( \frac{\partial \tau}{\partial t} + (u \cdot \nabla) \tau + g(\nabla u, \tau) \right) + \tau = 2\epsilon D[u].
\end{align*}
\]

System (1) is satisfied in \( Q \times [0, T] \), where \( T \) is a given positive number. We suppose the flow is periodic in the y-direction with period \( L \). We set \( Q = [0, 1] \times [0, L] \) and \( Q_T = Q \times [0, T] \). The unknowns are the velocity field \( u \), the hydrodynamic pressure \( p \), and the (symmetric) extra-stress tensor (the total stress is given by \( \sigma = -pI + \tau \)). \( \text{Re} \) and \( \text{We} \) are the Reynolds and Weissenberg (positive) numbers, \( \epsilon \) is a positive retardation parameter satisfying \( 0 < \epsilon < 1 \). We have also used the notation \( g(\nabla u, \tau) = \tau W - W \tau - a(D\tau + TD) \), where \( W = (1/2)(\nabla u - \nabla u^T) \), \( D = (1/2)(\nabla u + \nabla u^T) \), and \( a \in [-1, 1] \). For \( f = 0 \), a solution of system (1) is given by the uniform flow \( u = (U, 0) \), \( p = p_0 \) and \( \tau = 0 \), with \( U \in \mathbb{R} \) and \( p_0 \in \mathbb{R}_+ \). We are looking for flows which are unsteady perturbations of such a flow.

Here we prescribe a nonzero body force \( f \), some boundary conditions for the velocity and the extra-stress tensor as well. The velocities on the entry boundary \( \Gamma_E = \{ X = (x, y) \in Q; x = 0 \} \), and on the exit boundary \( \Gamma_S = \{ X = (x, y) \in \bar{Q}; x = 1 \} \), are given:

\[
\begin{align*}
u(0, y, t) &= u_E = (U, 0) + v(y) = (U + v_1(y), v_2(y)), \\
u(1, y, t) &= u_S = (U, 0) + w(y) = (U + w_1(y), w_2(y)).
\end{align*}
\]
for all \( y \in ]0, L[ \) and \( t \in [0, T] \), and satisfy

\[
0 < u_S \cdot n|_{\Gamma_S}, \quad u_E \cdot n|_{\Gamma_E} < -\alpha, \quad \text{for some } \alpha > 0,
\]

where \( n \) is the outward normal unit vector to the boundary. Moreover, the extra stress tensor on the entry boundary is given

\[
\tau(0, y, t) = \tau_E(y), \quad y \in ]0, L[, \quad t \in [0, T].
\]

We also give \( u \) and \( \tau \) at the initial time \( t = 0 \),

\[
u(0, \cdot) = u_0, \quad \tau(0, \cdot) = \tau_0 \quad \text{in } Q.
\]

In [3], Renardy considers steady flows of viscoelastic fluids in a bounded channel. He proves that the Jeffreys model is well posed in the channel if all the components of \( \tau \) on \( \Gamma_E \) are given. In the Maxwell case and in two dimensions he shows in particular that the steady problem is well posed if the diagonal part of \( \tau \) is given. Similar results hold in three-dimensional flows (see [3,4]).

In this paper, we prove that, under suitable compatibility conditions for \( \tau_0, u_0, \tau_E \), problem (1)-(5) admits a unique local solution.

\[2. \quad \text{NOTATION AND RESULTS} \]

We use the following notation:

\[
C_{0, \text{per}}^\infty = \{ u \in (C^\infty([0,1] \times \mathbb{R}))^2 \text{ with } x \mapsto u(x, y) \in (C^\infty([0,1]))^2 \text{ and } y \mapsto u(x, y) \text{ periodic with period } L \},
\]

\[
D = \{ u \in C_{0, \text{per}}^\infty; \text{div } u = 0 \},
\]

\[
H_{\text{per}}^m(Q) = \{ u \in H_{\text{loc}}^m((0, 1) \times \mathbb{R}); y \mapsto u(x, y) \text{ periodic with period } L \}
\]

(with the convention \( H_{\text{loc}}^0 = L_{\text{loc}}^1 \)),

\[
H_{0, \text{per}}^1(Q) = \text{closure in } H^1(Q) \text{ of } C_{0, \text{per}}^\infty,
\]

\[
V = \{ u \in H_{0, \text{per}}^1(Q); \text{div } u = 0 \text{ in } (0, 1) \times \mathbb{R} \},
\]

\[
H = \{ u \in H_{\text{per}}^{1/2}(Q); \text{div } u = 0 \text{ in } (0, 1) \times \mathbb{R} \}.
\]

We denote by \( A = -P \Delta \) the Stokes operator with domain \( D(A) = H_{\text{per}}^2(Q) \cap V \), where \( P \) is the orthogonal projection operator of \( L_{\text{loc}}^2(Q) \) on \( H \). The initial data and the boundary conditions are supposed to satisfy the following compatibility conditions:

\[
u_0|_{\Gamma_S} = u_E, \quad u_0|_{\Gamma_E} = u_S, \quad \int_{\Gamma_E} v(y) dy = \int_{\Gamma_S} w(y) dy,
\]

\[
(Pf(0) + P \text{div } \tau_0 - (1 - \epsilon) A u_0 - P [(u_0 \cdot \nabla) u_0]) \in V,
\]

\[
\tau_0|_{\Gamma_E} = \tau_E,
\]

\[
[\text{We } ((u_0 \cdot \nabla) \tau_0 + g(\tau_0, \nabla u_0)) + \tau_0]_{\Gamma_E} = 2\epsilon \text{D } [u_0]_{\Gamma_E}.
\]

\[\text{THEOREM 1. (Existence of local regular solutions). Let } f \in C([0, T], H_{\text{per}}^1(Q)), f' \in L^2(0, T; H_{\text{per}}^1(Q)), u_0 \in H_{\text{per}}^2(Q), u_S, u_E \in H_{\text{per}}^{3/2}((0, L)) \text{ satisfying (3), } \tau_0 \in H_{\text{per}}^2(Q), \tau_E \in H_{\text{per}}^2((0, L)) \text{ satisfying the compatibility conditions (6). Then there exists } T^* > 0 \text{ such that problem (1)-(4) admits a unique solution } (u, p, \tau) \text{ satisfying the following:}
\]

\[
\begin{align*}
 u & \in L^2(0, T^*; H_{\text{per}}^3(Q)) \cap C([0, T^*]; H_{\text{per}}^2(Q)), \\
p & \in L^2(0, T^*; H_{\text{per}}^2(Q)), \\
u' & \in L^2(0, T^*; H_{\text{per}}^2(Q)) \cap C([0, T^*]; V), \\
\tau & \in C([0, T^*]; H_{\text{per}}^1(Q)) \cap L^\infty(0, T^*; H_{\text{per}}^2(Q)), \\
\tau' & \in L^\infty(0, T^*; H_{\text{per}}^1(Q)).
\end{align*}
\]
3. AUXILIARY PROBLEMS

We first study two linearized problems, one for the velocity \( u \), and the other one for the symmetric tensor \( \tau \). We recall some classical results for the time dependent Stokes problem, written as

\[
\begin{align*}
  u(.): & \in V \quad \text{a.e. in } (0,T), \\
  u' + (1 - \epsilon)Au &= PF(a.e. in (0,T), 2) \\
  u(0) &= u_0.
\end{align*}
\]

**Lemma 1.** We suppose \( F' \in L^2(0, T; \mathbb{H}^0_{\text{per}}(Q)) \), \( F \in C([0, T]; \mathbb{H}^1_{\text{per}}(Q)) \), \( u_0 \in \mathbb{H}^3_{\text{per}}(Q) \cap V \) and \((-1 -\epsilon)Au_0 + PF(0) \in V\). Then problem (7) admits a unique solution satisfying

\[
\begin{align*}
  u(t) &\in L^2(0, T; \mathbb{H}^3_{\text{per}}(Q)) \cap C([0, T]; \mathbb{H}^2_{\text{per}} \cap V), \\
  u' &\in L^2(0, T; \mathbb{H}^2_{\text{per}}(Q) \cap V) \cap C([0, T]; V), \quad p \in L^2(0, T; \mathbb{H}^2_{\text{per}}(Q)), \\
  u(0) &= u_0.
\end{align*}
\]

and there exists a constant \( c_2 = c(\epsilon, Q) \) with

\[
\begin{align*}
  \|u\|^2_{L^2(0, T; \mathbb{H}^3_{\text{per}}(Q))} &+ \|u'\|^2_{L^2(0, T; \mathbb{H}^2_{\text{per}}(Q) \cap V)} + \|p\|^2_{L^2(0, T; \mathbb{H}^2_{\text{per}}(Q))} \\
  &\leq c_2 \left\{ (\|u_0\|^2 + \|PF\|^2_{L^2(0, T; \mathbb{H}^1_{\text{per}}(Q))} + \|PF'\|^2_{L^2(0, T; \mathbb{H}^2_{\text{per}})}) + \|PF(0) - (1 - \epsilon)Au_0\|^2 \right. \\
  &\left. + T \right\}.
\end{align*}
\]

These results are proven in a similar way as for the classical cases of periodic and Dirichlet boundary conditions (see [5]).

We now turn to the study of the transport equation (1) verified by \( \tau \). We first show the existence and the uniqueness of a regular solution \( \tau \) to the following problem: Find a tensor \( \tau = \tau(x, y, t) \), periodic in the \( y \)-direction, solution of the transport equation

\[
\begin{align*}
  \frac{\partial \tau}{\partial t} + (\bar{u}.\nabla)\tau + g(\nabla \bar{u}, \tau) + \frac{\tau}{We} &= \frac{2\epsilon}{We} \mathbf{D} \bar{u} \quad \text{a.e. in } Q_T, \\
  \tau(., 0) &= \tau_0 \quad \text{a.e. in } Q, \\
  \tau(0, y, t) &= \tau_E(y) \quad (y, t) \text{ a.e. in } [0, L] \times [0, T].
\end{align*}
\]

The function \( \bar{u} \) is given and satisfies the regularity of the solution of the Stokes problem given by Lemma 1,

\[
\bar{u} \in L^2(0, T; \mathbb{H}^3_{\text{per}}(Q)) \cap C([0, T]; \mathbb{H}^2_{\text{per}}(Q)), \quad \bar{u}' \in L^2(0, T; \mathbb{H}^2_{\text{per}}(Q)), \tag{10}
\]

and \( \bar{u} \) satisfies the boundary conditions (2) and (3).

**Lemma 2.** Let \( \bar{u} \) satisfy hypotheses (2), (3) and (10). Let \( \tau_E \in \mathbb{H}^2_{\text{per}}(0, L) \) and \( \tau_0 \in \mathbb{H}^2_{\text{per}}(Q) \) satisfying the compatibility conditions (6)\textsubscript{3}, (6)\textsubscript{4}. Then there exists a unique solution of problem (9) such that

\[
\tau \in L^\infty([0, T]; \mathbb{H}^1_{\text{per}}(Q)) \cap C([0, T]; \mathbb{H}^0_{\text{per}}(Q)), \quad \tau' \in L^\infty(0, T; \mathbb{H}^1_{\text{per}}(Q))
\]

with the estimates

\[
\begin{align*}
  \|\tau\|_{L^\infty(0, T; \mathbb{H}^0_{\text{per}})} &\leq \left\{ \left( \frac{\phi_1 + \phi_2}{\|\bar{u}\|_{L^\infty(0, T; \mathbb{H}^3_{\text{per}})}} \right) \left( \frac{\phi_3}{\|\bar{u}\|_{L^\infty(0, T; \mathbb{H}^2_{\text{per}})}} + \frac{\tau_0}{\phi_3} + \left| \tau_0 \right|_2 \right) \times \exp \left( c_0 \|\bar{u}\|_{L^1(0, T; \mathbb{H}^1_{\text{per}})} + \frac{T}{We} \right) \right. \\
  &\left. + \phi_4 \|u'\|_{L^2(0, T; \mathbb{H}^2_{\text{per}})} \right\}, \\
  \|\tau'\|_{L^\infty(0, T; \mathbb{H}^0_{\text{per}})} &\leq \left( \frac{c_0}{\|\bar{u}\|_{L^\infty(0, T; \mathbb{H}^3_{\text{per}})}} + \frac{1}{c_0 We} \right) \left( \|\tau\|_{L^\infty(0, T; \mathbb{H}^0_{\text{per}})} + \frac{T}{W e} \right),
\end{align*}
\]

where \( \phi_1, \phi_2, \phi_3 \) and \( \phi_4 \) are some constants depending on \( \epsilon, We, Q, u_E \), and \( \tau_E \).
The existence of a unique solution is proven via the method of characteristics. We define the trajectory of some fluid element located at $X$ at time $t$ by the system of ordinary differential equations,

$$
\frac{dY}{ds}(X, t; s) = \tilde{u}(Y(X, t; s), s), \quad 0 \leq s \leq t,
$$

$$
Y(X, t; t) = X \in Q.
$$

System (13) defines a vector function $Y(X, t; s)$ which satisfies

(a) either $(Y(X, t; s) \in Q$ for all $s, 0 \leq s \leq t$,

(b) or $Y(X, t; t_*(X, t)) \in \Gamma_E$ for some $t_* \in [0, T]$.

We solve equation (9) along a trajectory defined by (13), and obtain two possible representation formulas for $\tau$, one for each case (a) or (b), that is

$$
\tau(X, t) = \tau_0(Y(X, t; 0)) + \frac{1}{\text{We}} \int_0^t \{2\alpha D[\tilde{u}] - \text{We g } \nabla \tilde{u}, \tau - \tau \} \{Y(X, t; s), s\} ds,
$$

or

$$
\tau(X, t) = \tau_E(Y(X, t; t_*(X, t))) + \frac{1}{\text{We}} \int_{t_*(X, t)}^t \{2\alpha D[\tilde{u}] - \text{We g } \nabla \tilde{u}, \tau - \tau \} \{Y(X, t; s), s\} ds.
$$

To prove the regularity of $\tau$ we can proceed in the same manner as Judović [6]. In fact, it is not too hard to see that $\tau \in C([0, T]; \mathbb{H}^2_{\text{per}}(Q))$ in $Q_T$ except on the surface $\{t_*(X, t) = 0\}$. By differentiating the representation formulas (15) and (16) and using the compatibility conditions (6)–(6)4, we prove that $\tau \in L^\infty(0, T; \mathbb{H}^2_{\text{per}}(Q))$ and $\tau' \in L^\infty(0, T; \mathbb{H}^1_{\text{per}})$.

Now the crucial point is to prove the energy estimates (11) and (12). To this end, we write a differential inequality satisfied by $\|\tau\|^2_2$ (see [7]),

$$
\frac{1}{2} \frac{d}{dt} \left(\text{We } \|\tau\|^2_2\right) + \|\tau\|^2_2 + \frac{\text{We}}{2} \int_{\Gamma_E} u_{E,n} \left\{ |\tau_E|^2 + \left| \frac{d\tau_E}{dy} \right|^2 + \left| \frac{\partial \tau}{\partial x} \right|^2 + \left( \frac{\partial^2 \tau}{\partial y^2} + \frac{\partial^2 \tau}{\partial x \partial y} \right) \right\} d\Gamma + ((\nabla \tilde{u} \cdot \nabla) \tau, \nabla \tau) + ((D^2 \tilde{u} \cdot \nabla) \tau, D^2 \tau)
$$

$$
+ 2 ((\partial_1 \tilde{u} \cdot \nabla) \partial_1 \tau, \partial_1 \tau) + ((g(\tau, v \tilde{u}), \tau))_2 \leq 2 \varepsilon ((D[\tilde{u}], \tau))_2.
$$

All the terms are classical to estimate except for the boundary integral term. Using equation (9) on $\Gamma_E$, we obtain

$$
\left\| \frac{\partial \tau}{\partial x} \right\|^2_{1, \Gamma_E} \leq \left\| \frac{1}{v_1 + U} \right\|^2_{1, \Gamma_E} \left( \frac{2}{\text{We}} \left\| D[\tilde{u}] \right\|^2_{1, \Gamma_E} + \left\| g(\nabla \tilde{u}, \tau_E) \right\|^2_{1, \Gamma_E}
$$

$$
+ \left\| v_2 \right\|^2_{1, \Gamma_E} \left\| \frac{d\tau_E}{dy} \right\|^2_{1, \Gamma_E} + \frac{1}{\text{We}} \left\| \tau_E \right\|^2_{1, \Gamma_E} \right),
$$

where thanks to hypothesis (3)2 the quantities $\chi_0 = \|1/(v_1 + U)\|^2_{1, \Gamma_E}$ and $\chi_1 = \|1/(v_1 + U)\|^2_{1, \Gamma_E}$ are finite. Next we differentiate (9) with respect to $x$ and deduce

$$
\left\| \frac{\partial^2 \tau}{\partial x^2} \right\|^2_{0, \Gamma_E} \leq \chi_0 \left\{ \left( \frac{2 \varepsilon}{\text{We}} \right)^2 + \frac{2}{\text{We}} \left\| \tau_E \right\|^2_{1, \Gamma_E} \right\} \left\| \tilde{u} \right\|^2_3 + \chi_1 \left( \frac{2}{\text{We}} \left\| \nabla \tilde{u} \right\|^2_{0, \Gamma_E} + \left\| v_2 \right\|^2_{1, \Gamma_E} \right) \left\| \frac{d\tau_E}{dy} \right\|^2_{1, \Gamma_E} + \left( \frac{1}{\text{We}} \right)^2 \left\| \tau_E \right\|^2_{1, \Gamma_E}
$$

$$
\times \left[ \left( \frac{2 \varepsilon}{\text{We}} \right)^2 + \frac{1}{\text{We}} \left\| \tau_E \right\|^2_{1, \Gamma_E} \right] \left\| \tilde{u} \right\|^2_3 + \left\| v_2 \right\|^2_{1, \Gamma_E} \left\| \frac{d\tau_E}{dy} \right\|^2_{1, \Gamma_E} + \left( \frac{1}{\text{We}} \right)^2 \left\| \tau_E \right\|^2_{1, \Gamma_E} \right]
$$

+ \chi_0 \left( \frac{2 \varepsilon}{\text{We}} \right)^2 \left\| \tau_E \right\|^2_{1, \Gamma_E} \left\| u'' \right\|^2_2.
$$

(19)
We rewrite (19) in the following way:

\[ \frac{\partial^2 T}{\partial x^2}^2 \leq \phi_1 \| \bar{u} \|^2 + \phi_2 \| \bar{u} \|^2 + \phi_3 \| u \|^2 + \phi_4, \]  

(20)

where \( \phi_1, \phi_2, \phi_3 \) and \( \phi_4 \) are some constants depending on \( \epsilon, W, u, \) and \( \tau E \). It is now easy to deduce estimate (11) from (17), (18) and (20). Finally, estimate (12) for \( \tau' \) follows from (11) and (9).

Remark 1. Lemma 2 can also be proven by approximating \( \bar{u} \) by a sequence of regular \( \bar{u}_n \), so as to obtain an estimate of \( r \) in the \( L^\infty(0, T; H^2_{\text{per}}) \) norm, which is independent of \( n \). A compactness argument shows the existence of a solution \( \tau \in L^\infty(0, T; H^2_{\text{per}}) \).

4. PROOF OF THEOREM 1

The proof of Theorem 1 is based on the Schauder fixed point theorem (see, e.g., [2,8] for a similar proof). We define an application in the following manner:

\[ \phi: \begin{cases} R_T \to X \\ \bar{u}(\tau) \to (u, \tau), \end{cases} \]  

(21)

where \( X = C([0, T]; V) \times C([0, T]; H^2_{\text{per}}) \), and

\[ R_T = \{ (\bar{u}(\tau), \bar{u}) \in C([0, T]; H^2_{\text{per}}(Q)) \cap V) \cap L^2(0, T; H^2_{\text{per}}(Q)), \]  

\[ \bar{u}' \in C([0, T]; V) \cap L^2(0, T; H^2_{\text{per}}(Q)), \]  

\[ \bar{\tau} \in L^\infty(0, T; H^2_{\text{per}}(Q)), \]  

\[ \bar{\tau}' \in L^\infty(0, T; H^1_{\text{per}}(Q)), \]  

\[ \bar{u}(0) = u_0 - r(ubd), \bar{\tau}(0) = \tau_0 = \bar{\tau}, |\Gamma_E = \tau E, \]  

\[ \| \bar{u} \|^2 \leq \beta_1, \]  

\[ \| \bar{\tau} \|^2 \leq \beta_2, \| \bar{\tau}' \|^2 \leq \beta_3. \]  

(22)

\( \beta_1, \beta_2, \) and \( \beta_3 \) are positive constants, \( r(ubd) \in H^3_{\text{per}}(Q) \) satisfies \( \text{div} r(ubd) = 0 \), and the boundary conditions for the velocity field \( u \) (2). The set \( R_T \) is not empty if \( \beta_1 \) and \( \beta_2 \) are sufficiently large.

In the definition (21) of \( \phi \), the functions \( u \) and \( \tau \) are the solution of the following problems:

\[ \text{Re} u' + (1 - \epsilon)Au = P \{ f + \text{div} \bar{\tau} - \text{Re} [(\bar{u} + r(ubd)) \cdot \nabla (\bar{u} + r(ubd))] \} \]  

\[ + (1 - \epsilon) \Delta r(ubd), \]  

a.e. in \( QT \),

\[ u(0) = u_0 - r(ubd) \]  

in \( Q \),

\[ \text{We} \{ r' + [(\bar{u} + r(ubd)) \cdot \nabla]r + g(\nabla(\bar{u} + r(ubd), r)) + r = 2\epsilon D [\bar{u} + r(ubd)] \} \]  

a.e. in \( QT \),

\[ r(0) = \tau_0 \]  

in \( Q \),

\[ r|_{\Gamma_E} = \tau_E \]  

on \( \Gamma_E \).

(23)

(24)

Lemma 3. There exists a time \( T_c > 0 \) such that \( \forall T \leq T_c, \phi(R_T) \subset R_T \).

Problems (23) and (24) admit a unique solution \( (u, \tau) \) as seen from Lemmas 1 and 2. To prove that \( (u, \tau) \) belongs to \( R_T \), we use estimates (8), (11) and (12). After some calculations, we deduce that the constants \( \beta_i, i = 1, 2, 3 \) and \( T \) have to satisfy

\[ \beta_1 \geq \max \left\{ 2c_2 \| A(u_0 - r(ubd)) \|_1, 2 \left[ \| A(u_0 - r(ubd)) \|^2_{0} + \| f \|^2_{L^2(0, T; H^2_{\text{per}})} \right] + 4 \left( \| \tau_0 \|^2_{2} + \| f(0) \|^2_{1} + \text{Re} \| u_0 \|^4_{2} + (1 - \epsilon) \| A u_0 \|^2_{1} \right) + 8 \| f' \|^2_{L^2(0, T; H^2_{\text{per}})} \right\}, \]  

(25)
\[
\beta_2 \geq 2 \left[ \phi_1 + \phi_2 \left( \beta_1^{1/2} + \|u_{bd}\|_2 \right)^2 + \phi_3 \beta_1^{1/2} + \|u_{bd}\|_2 e^1 \right], \\
\beta_3 \geq c_0 \left( \beta_1^{1/2} + \|u_{bd}\|_2 + \frac{1}{c_0} \frac{2e}{\text{We}} \right) \left( \beta_2 + \frac{2e}{c_0 \text{We}} \right), \\
T \leq \min \left\{ \left( -\frac{\beta_1^{1/2} + \|u_{bd}\|_2}{2} \right)^{1/2}, \frac{\beta_2}{2c_0}, \frac{\beta_3}{\psi(\beta_1, \beta_2, \beta_3, \|u_{bd}\|_2)} \right\},
\]

where \( \psi \equiv \psi_1 + \psi_2 + c_0 \left( \|u_{bd}\|_2^2 + \beta_3^2 + (1/2)(11 + \text{Re}^4)\beta_3^2 \right) + (c_2 + 11/2)\|u_{bd}\|_2^2 \). Some easy but tedious calculations show the continuity of the application \( \phi \) for the topology of \( X_{T_\varepsilon} \), and the compactness of \( R_T \) in \( X_{T_\varepsilon} \) implies the compactness of \( \phi \).

Finally, the uniqueness of a solution is proven by using the energy method.

REFERENCES