Boundedness of Solutions for Semilinear
Duffing Equations*

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In this paper, we are concerned with the boundedness of all the solutions and the
existence of quasi-periodic solutions for some semilinear Duffing equation

\[ 0 < K \leq \frac{g(x)}{x} \leq K < +\infty. \]

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1. INTRODUCTION

In this paper, we deal with the boundedness of all the solutions and the
existence of quasi-periodic solutions for a class of semilinear Duffing equations

\[ x'' + g(x) = e(t), \quad \left( t = \frac{d}{dt} \right) \quad (1.1) \]

where \( e(t) \) is a smooth 2\pi-periodic function and \( g: \mathbb{R} \rightarrow \mathbb{R} \) is a continuous
function, and the semilinearity means that

\[ 0 < K \leq \frac{g(x)}{x} \leq K < +\infty. \]

As one of the simplest but nontrivial conservative systems, Eq. (1.1) has
been widely investigated for a long time. For example, many authors
studied the existence and multiplicity of periodic solutions by various

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methods, such as critical point theory, phase-plane analysis combined with
the shooting methods or fixed point theorems of planar homeomorphisms,
and continuation methods based on degree theory. Since the 1970s, using
Moser’s twist theorem [12], several authors have studied more complicated
phenomena of solutions of (1.1), such as the existence of quasi-periodic
solutions and Mather set, and Lagrangian stability (that is, all the solutions
are bounded in the phase plane) which was proposed by Littlewood [5],
Markus [8], and Moser [11].

The first result of boundedness of solutions in superlinear case (i.e.,
g(x)/x → +∞ as |x| → +∞) was due to Morris [10]. He proved that
every solution of the equation
\[ x'' + 2x^3 = e(t) \]
is bounded, that is, if x(t) is a solution, then it is defined for all \( t \in \mathbb{R} \) and
\[ \sup_{t \in \mathbb{R}} (|x(t)| + |x'(t)|) < +\infty. \]

Dieckerhoff and Zehnder [1] obtained the same conclusion for a more
general equation
\[ x'' + x^{2n+1} + \sum_{i=0}^{2n} x'p_i(t) = 0, \]
where the 2π-periodic functions \( p_i(t) \) (i = 0, ..., 2n) are smooth functions.
For further developments of Lagrangian stability in superlinear case and
sublinear case (i.e., \( g(x)/x \to +0 \) as \( |x| \to +\infty \)), we refer to [3, 4, 6, 16]
and [2, 7], respectively.

Recently, Ortega [13] investigates a special form of (1.1)
\[ x'' + ax^+ - bx^- = 1 + \varepsilon p(t), \] (1.2)
where \( \varepsilon \) is a small parameter, \( a \) and \( b \) are positive constants \((a \neq b)\),
\( x^+ = \max\{x, 0\} \) and \( x^- = \max\{-x, 0\} \). He proves that all the solutions
are bounded if the periodic function \( p(t) \in C^4(\mathbb{R}) \) and \( |\varepsilon| \) is sufficiently
small. As far as we know, this is only one nontrivial result in the study of
Lagrangian stability for semilinear Duffing equations.

The idea for proving the boundedness of solutions of Eq. (1.1) is as
follows. By means of transformation theory, (1.1) is, outside of a large disc
\( \mathcal{D} = \{(x, x') \in \mathbb{R}^2 : x^2 + x'^2 \leq r^2\} \) in \( (x, x') \)-plane, transformed into a pertur-
bation of an integrable Hamiltonian system. The Poincaré map of the
transformed system is closed to a so-called twist map in \( \mathbb{R} \setminus \mathcal{D} \). Then
Moser’s twist theorem [12] guarantees the existence of arbitrarily large
invariant curves diffeomorphic to circles and surrounding the origin in the
(x, x‘)-plane. Every such curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space (x, x‘, t) ∈ R² × R, which confines the solutions in the interior and which leads to a bound of these solutions.

In this article, we consider the semilinear Duffing equations

\[ x'' + \lambda x' + \phi(x) = e(t), \tag{1.3} \]

where \( \phi(x) = o(x) \) as \( |x| \rightarrow +\infty \) and \( e(t) \) is a 2π-periodic function. For such equations we will prove the boundedness of all the solutions and the existence of quasi-periodic solutions under some suitable assumptions on \( \phi(x) \) and \( e(t) \) but without requiring any smallness condition.

In the following, we denote by \( \Phi(x) \) with \( \Phi(0) = 0 \) the integral of \( \phi(x) \), i.e.,

\[ \Phi(x) = \int_0^x \phi(s) \, ds. \]

We also denote by \( c < 1 \) and \( C > 1 \), respectively, two universal positive constants not concerning their quantities.

The main result is

**Theorem 1.** Suppose that \( \phi(x) \in C^5(R \setminus \{0\}) \cap C^0(R) \), \( \lambda > 0 \) is a constant and \( e(t) \in C^5(R) \) is 2π-periodic in \( t \). Moreover, we assume that the function \( \phi(x) \) satisfies the following conditions: for all \( x \neq 0 \),

\[ \gamma x \phi(x) \geq x^2 \phi'(x) > 0, \quad x \phi(x) \geq x \Phi(x), \tag{1.4} \]

with some constants \( 0 < \gamma < 1, 1 < x < 2 \) and

\[ \left| x^k \frac{d^k}{dx^k} \phi(x) \right| \leq C \cdot \Phi(x), \quad \text{for} \quad 3 \leq k \leq 6. \tag{1.5} \]

Then the following conclusions hold:

1. Every solution of (1.3) is bounded; that is, if \( x(t) \) is a solution of (1.3), then \( x(t) \) is defined for all \( t \in R \) and

   \[ \sup_{t \in R} (|x(t)| + |x'(t)|) < +\infty; \]

2. There exists \( \varepsilon_0 > 0 \) and a closed set \( \mathcal{A} \subset (\lambda/2\pi, (\lambda/2\pi) + \varepsilon_0) \) having a positive measure such that for any \( \omega \in \mathcal{A} \), there is a quasi-periodic solution of (1.3) with the basic frequency (ω, 1);
For any \( \omega \in (\lambda/2\pi, (\lambda/2\pi) + \epsilon_0) \), Eq. (1.3) has a solution \((x_\omega(t), x'_\omega(t))\) of Mather type with rotation number \(\omega\). More precisely,

- if \(\omega = p/q\) is rational, the solutions \((x_\omega(t+2i\pi), x'_\omega(t+2i\pi))\), \(0 \leq i \leq q-1\) are mutually unlinked periodic solutions of period \(2q\pi\);
- if \(\omega\) is irrational, the solution \((x_\omega(t), x'_\omega(t))\) is either a usual quasi-periodic solution or a generalized one.

Remarks. (1) A generalized quasiperiodic solution means that the closed set

\[ \{(x_\omega(2i\pi), x'_\omega(2i\pi)), i \in \mathbb{Z}\} \]

is a Denjoy's minimal set.

(2) Compare with Ortega's result, we do not require the assumption of the smallness of \(|\epsilon(t)|\) which plays an essential role in his proof.

(3) As is well-known, some bounded perturbation of \(\lambda^2 x\) causes the existence of unbounded solutions of (1.1). For instance, the equation

\[ x'' + (2m + 1)^2 x + \frac{1}{(2m + 1)^2} \arctan x = \sin(2m + 1) t \]

has no periodic solutions, which implies that all the solutions are unbounded solutions by the Massera's theorem. Note that all the conditions of Theorem 1 are satisfied for (1.6) except (1.4) which guarantees that the perturbation term \(\phi(x)\) is unbounded.

(4) From the first inequality in (1.4), it is easy to see that

\[ x\phi(x) \leq \beta \Phi(x), \]

where \(\beta := \gamma + 1 < 2\). This inequality will be used frequently in the next two sections.

The proof of Theorem 1 will be given in Section 3. Section 2 deals with some technical lemmas that are employed in the proof of the main result. In the last section, we state some results for the global existence and uniqueness of Eq. (1.3) which guarantee the Poincaré mapping of (1.3), outside of a large disc in the phase plane, is well-defined.

2. SOME TECHNICAL LEMMAS

In this section, we will state and prove some technical lemmas which will be used in the next section. Throughout this section, we assume the hypotheses of Theorem 1 hold.
Equation (1.3) is equivalent to the following planar Hamiltonian system
\[ x' = -\lambda y, \quad y' = \lambda x + \lambda^{-1} \phi(x) - \lambda^{-1} e(t). \] (2.1)

Under the standard symplectic polar transformation \((r, \theta) \mapsto (x, y), \ r > 0\) and \(\theta \mod 2\pi\), given by
\[ x = \sqrt{2} r^{3/2} \cos \theta, \quad y = \sqrt{2} r^{3/2} \sin \theta, \] (2.2)
the system (2.1) is transformed into another Hamiltonian system
\[ r' = \frac{\partial h}{\partial \theta}(r, \theta, t), \quad \theta' = \frac{\partial h}{\partial r}(r, \theta, t), \] (2.3)
where
\[ h(r, \theta, t) = \lambda x + I_1(r, \theta) + I_2(r, \theta, t), \] (2.4)
with
\[ I_1 = \frac{1}{\lambda} \phi(\sqrt{2} r^{3/2} \cos \theta), \quad I_2 = -\lambda^{-1} \sqrt{2} r^{3/2} \cos \theta \cdot e(t). \]

Note that \(I_1(r, \theta) \in C^1(\mathbb{R}^+ \times S^1)\) and \(I_2(r, \theta, \cdot) \in C^\infty(\mathbb{R}^+ \times S^1)\) where \(S^1 = \mathbb{R}/2\pi\mathbb{Z}\). Moreover, it is easy to show that for any fixed \(\theta\), the function \(r \mapsto I_1(r, \theta)\) is \(C^6\) in \(r\). Denote by \(J(r)\) the average value of \(I_1(r, \theta)\) over \(S^1\), that is
\[ J(r) = \frac{1}{2\pi} \int_0^{2\pi} I_1(r, \theta) d\theta. \]

Then we have

**Lemma 2.1.** The following inequalities hold:
\[ 0 \leq \frac{\alpha}{2} I_1(r, \theta) \leq r \frac{\partial I_1}{\partial r}(r, \theta) \leq \frac{\beta}{2} I_1(r, \theta), \]
\[ 0 \leq \frac{\alpha}{2} J(r) \leq r J'(r) \leq \frac{\beta}{2} J(r), \quad r^2 |J''(r)| \geq \gamma_1 \cdot J(r), \] (2.5)
\[ r^k \frac{\partial^k I_1}{\partial r^k}(r, \theta) \leq C \cdot I_1(r, \theta), \quad r^k \frac{\partial^k J}{\partial r^k}(r) \leq C \cdot J(r), \]
for \(0 \leq k \leq 6\) and \(\gamma_1 = (1 - \gamma) \pi/4 > 0\).
Proof. We give a proof of the inequality

$$r^2 |J'(r)| \geq \gamma_1 \cdot J(r).$$

The other inequalities are easily proved from (1.4), (1.5), and (1.7).

By the definition of $J(r)$, we have

$$r J'(r) = \frac{1}{4\pi^2} \int_0^{2\pi} x \phi(x) \, d\theta,$$

and

$$r^2 J''(r) + r J'(r) = \frac{1}{8\pi} \lambda^{-1} \int_0^{2\pi} [x \phi(x) + x^2 \phi'(x)] \, d\theta,$$

where $x = x(r, \theta)$ is given by (2.2). Hence

$$r^2 J''(r) = \frac{1}{8\pi} \lambda^{-1} \int_0^{2\pi} [x^2 \phi'(x) - x \phi(x)] \, d\theta.$$

From (1.4), it follows that

$$x^2 \phi'(x) - x \phi(x) \leq (\gamma - 1) x \phi(x) \leq 0,$$

which yields that

$$r^2 |J'(r)| \geq \frac{1}{8\pi} \lambda^{-1} \int_0^{2\pi} (1 - \gamma) x \phi(x) \, d\theta$$

$$\geq \frac{1}{8\pi} (1 - \gamma) x \lambda^{-1} \int_0^{2\pi} \phi(x) \, d\theta$$

$$= \gamma_1 \cdot J(r).$$

The proof is completed. 

From this lemma, for any $r \geq 1$, it is easy to see that

$$0 \leq I_1(r, \theta) \leq C \cdot r^{\beta-2}, \quad c \cdot r^{\alpha-2} \leq J(r) \leq C \cdot r^{\beta-2},$$

$$0 \leq c \cdot r^{(\alpha-2)} \leq J'(r) \leq C \cdot r^{(\beta-2)} \rightarrow 0 \quad \text{as} \quad r \to +\infty.$$  \hfill (2.6)  \hfill (2.7)

Lemma 2.2. For any fixed $\theta \in \mathbb{R}$ and $r > 0$, we have

$$I_1 \left( \frac{1}{2} r, \theta \right) \leq I_1(r, \theta) \leq I_1 \left( \frac{3}{2} r, \theta \right) \leq C \cdot I_1 \left( \frac{1}{2} r, \theta \right),$$

$$J \left( \frac{1}{2} r \right) \leq J(r) \leq J \left( \frac{3}{2} r \right) \leq C \cdot J \left( \frac{1}{2} r \right).$$  \hfill (2.8)  \hfill (2.9)
Proof. For any fixed $\theta$, it follows from Lemma 2.1 that the function $r \mapsto I_1(r, \theta)$ is non-decreasing in $r$, so

$$I_1\left(\frac{1}{2}r, \theta\right) \leq I_1(r, \theta) \leq I_1\left(\frac{3}{2}r, \theta\right).$$

Now we prove

$$I_1\left(\frac{1}{2}r, \theta\right) \leq C \cdot I_1\left(\frac{3}{2}r, \theta\right).$$

If $\cos \theta = 0$, then $I_1\left(\frac{1}{2}r, \theta\right) = I_1\left(\frac{3}{2}r, \theta\right) = \lambda^{-1} \Phi(0) = 0$. In the following, we assume that $\cos \theta \neq 0$. In this case, $x \neq 0$ and $\phi'(x)$ exists.

By the definition of $I_1$ and the condition (1.4), we have

$$r^2 \frac{\partial^2}{\partial r^2} I_1(r, \theta) = \frac{1}{42} \left( x^2 \phi'(x) - x \phi(x) \right) < 0,$$

which yields that the positive function $r \mapsto \partial I_1/\partial r$ is decreasing in $r$. Hence, by (2.5),

$$I_1\left(\frac{3}{2}r, \theta\right) - I_1\left(\frac{1}{2}r, \theta\right) \leq r \frac{\partial I_1}{\partial r} \left(\frac{1}{2}r, \theta\right) \leq \beta \cdot I_1\left(\frac{1}{2}r, \theta\right).$$

The inequalities in (2.9) is a direct consequence of (2.8).

The next lemma shows that the quantity of $I(r, \theta)$ can be controlled by the average value $J(r)$.

Lemma 2.3. $I_1(r, \theta) \leq C \cdot J(r)$.

Proof. From the conditions of Theorem 1, we know that, for any $x \in [-r, r]$,

$$\Phi(x) \leq \max\{\Phi(r), \Phi(-r)\}.$$ 

Similar to the proof of the previous lemma, one can show that

$$\left(1 - \frac{\beta r}{2}\right) \Phi(r) \leq \Phi\left(\frac{r}{2}\right) \leq \Phi(r)$$

and

$$\left(1 - \frac{\beta r}{2}\right) \Phi(-r) \leq \Phi\left(-\frac{r}{2}\right) \leq \Phi(-r).$$

By the definition of $I_1$, it suffices to prove that

$$\max\{\Phi(\sqrt{2} r^{1/2}), \Phi(-\sqrt{2} r^{1/2})\} \leq C \int_0^{2\pi} \Phi(\sqrt{2} r^{1/2} \cos \theta) \, d\theta.$$
Since $\Phi(\sqrt{2} r^{1/2} \cos \theta) \geq 0$, we have
\[
\int_{0}^{2\pi} \Phi(\sqrt{2} r^{1/2} \cos \theta) \, d\theta \\
\geq \int_{0}^{\pi/3} \Phi(\sqrt{2} r^{1/2} \cos \theta) \, d\theta \geq \frac{2}{3} \Phi(\sqrt{2} r^{1/2}/2) \geq \left(1 - \frac{\beta}{2}\right) \frac{2}{3} \Phi(\sqrt{2} r^{1/2}).
\]
Similarly, one can prove
\[
\int_{0}^{2\pi} \Phi(\sqrt{2} r^{1/2} \cos \theta) \, d\theta \geq \left(1 - \frac{\beta}{3}\right) \frac{2}{3} \Phi(\sqrt{2} r^{1/2}).
\]
This completes the proof. 

From Lemmas 2.1 and 2.3, we have
\[
\left| r^k \frac{\partial^h I_1}{\partial r^k} (r, \theta) \right| \leq C \cdot J(r)
\]
for $0 \leq k \leq 6$.

**Lemma 2.4.** Suppose that a smooth function $R(h, t, \theta)$ satisfies
\[
R(h, t, \theta) = \lambda^{-1} I_1(\lambda^{-1} h - R, \theta) - \lambda^{-2} \sqrt{2} (\lambda^{-1} h - R)^{1/2} \cos \theta \cdot e(t),
\]
(2.10)

**together** with $|R(h, t, \theta)| \leq \lambda^{-1} h/2$. Then for $h \gg 1$,
\[
\left| \frac{\partial^{k+\ell}}{\partial h^{k} \partial \theta^{\ell}} R(h, t, \theta) \right| \leq C \cdot h^{-k} J(\lambda^{-1} h),
\]
(2.11)

for $k + \ell \leq 6$ and $\ell \leq 5$.

**Proof.** (i) When $k + \ell = 0$, the conclusion follows from Lemmas 2.1–2.3 and

\[
J(\lambda^{-1} h) \geq c \cdot h^{\gamma/2} \geq c \cdot h^{1/2}.
\]

(ii) When $k + \ell = 1$, since $|R| \leq \lambda^{-1} h/2$ and $|(\partial I_1/\partial r)(r, \theta)| \leq C \cdot r^{\beta/2 - 1}$ with $\beta < 2$, we know that
\[
\left| \lambda^{-1} \frac{\partial I_1}{\partial r} (\lambda^{-1} h - R, \theta) \right| + \left| \frac{1}{\sqrt{2}} \lambda^{-2}(\lambda^{-1} h - R)^{-1/2} \cos \theta \cdot e(t) \right| \leq \frac{1}{2}
\]
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for \( h \gg 1 \). Define

\[
g_1(h, t, \theta) = -\lambda^{-2} \sqrt{2} (\lambda^{-1}h - R)^{1/2} \cos \theta \cdot e'(t),
g_2(h, t, \theta) = \lambda^{-1} \frac{\partial I_1}{\partial \theta} (\lambda^{-1}h - R, \theta) - \lambda^{-3} \cos \theta \sqrt{2} (\lambda^{-1}h - R)^{-1/2} e(t),
\]

(2.12)

\[
A(h, t, \theta) = 1 + \lambda^{-1} \frac{\partial I_1}{\partial \theta} (\lambda^{-1}h - R, \theta) - \frac{\cos \theta}{\sqrt{2}} \lambda^{-1} (\lambda^{-1}h - R)^{-1/2} e(t).
\]

Then \( A(h, t, \theta) \geq 1/2 \) for \( h \gg 1 \) and

\[
A \cdot \frac{\partial R}{\partial t}(h, t, \theta) = g_1(h, t, \theta), \quad A \cdot \frac{\partial R}{\partial h}(h, t, \theta) = g_2(h, t, \theta).
\]

(2.13)

By the previous lemmas, we obtain

\[
\left| \frac{1}{2} \frac{\partial}{\partial t} R(h, t, \theta) \right| \leq |A| \left| \frac{\partial}{\partial h} R(h, t, \theta) \right| = |\lambda^{-2} (\lambda^{-1}h - R)^{1/2} \cos \theta \cdot e'(t)| \\
\leq C \cdot (\lambda^{-1}h)^{1/2} \leq C \cdot J(\lambda^{-1}h),
\]

and

\[
\left| \frac{1}{2} \frac{\partial}{\partial h} R(h, t, \theta) \right| \leq |A| \left| \frac{\partial}{\partial h} R(h, t, \theta) \right| = |\lambda^{-2} \frac{\partial I_1}{\partial \theta} (\lambda^{-1}h - R, \theta) - \lambda^{-3} \cos \theta \sqrt{2} (\lambda^{-1}h - R)^{-1/2} \cdot e(t)| \\
\leq C \left( |\lambda^{-1}h - R, \theta)| + |(\lambda^{-1}h - R)^{-1/2}| \right) \\
\leq C \cdot h^{-1} \left( I_1 \left( \frac{3}{2} \lambda^{-1}h, \theta \right) + \left( \frac{3}{2} \lambda^{-1}h \right)^{1/2} \right) \\
\leq C \cdot h^{-1} J \left( \frac{3}{2} \lambda^{-1}h \right) \leq C \cdot h^{-1} J(\lambda^{-1}h).
\]

(iii) When \( k + \ell = 2 \), by (i) and (ii), we have

\[
\left| \frac{\partial}{\partial h} g_1(h, t, \theta) \right| \leq C \cdot h^{-1} J(\lambda^{-1}h), \quad \left| \frac{\partial}{\partial t} g_1(h, t, \theta) \right| \leq C \cdot J(\lambda^{-1}h),
\]

\[
\left| \frac{\partial}{\partial h} g_2(h, t, \theta) \right| \leq C \cdot h^{-2} J(\lambda^{-1}h), \quad \left| \frac{\partial}{\partial t} g_2(h, t, \theta) \right| \leq C \cdot h^{-1} J(\lambda^{-1}h),
\]
and
\[
\left| \frac{\partial}{\partial h} A(h, t, \theta) \right| \leq C \cdot h^{-2} J(\lambda^{-1} h), \quad \left| \frac{\partial}{\partial t} A(h, t, \theta) \right| \leq C \cdot h^{-1} J(\lambda^{-1} h).
\]

From (2.13) it follows that
\[
\frac{1}{2} \left| \frac{\partial^2 R(h, t, \theta)}{\partial t^2} \right| \leq |A| \left| \frac{\partial^2 R(h, t, \theta)}{\partial t^2} \right| \leq \left| \frac{\partial}{\partial t} g_1 \right| + \left| \left( \frac{\partial}{\partial t} A \right) \cdot \frac{\partial}{\partial t} R(h, t, \theta) \right|
\]
\[
\leq C \cdot (\lambda h^{-1} h + h^{-1} [J(\lambda^{-1} h)]^2) \leq C \cdot J(\lambda^{-1} h),
\]
and
\[
\frac{1}{2} \left| \frac{\partial^2 R(h, t, \theta)}{\partial h \partial t} \right| \leq |A| \left| \frac{\partial^2 R(h, t, \theta)}{\partial h \partial t} \right|
\]
\[
\leq \left| \frac{\partial}{\partial h} g_1 \right| + \left| \left( \frac{\partial}{\partial h} A \right) \cdot \frac{\partial}{\partial h} R(h, t, \theta) \right|
\]
\[
\leq C \cdot (h^{-1} J(\lambda^{-1} h) + h^{-2} [J(\lambda^{-1} h)]^2)
\]
\[
\leq C \cdot h^{-1} J(\lambda^{-1} h),
\]
where we have used the inequality \( J(\lambda^{-1} h) \leq C \cdot h \) which is a consequence of the second inequality in (2.6).

Differentiating on both sides of the second equality in (2.13) with respect to \( h \), one has
\[
A \frac{\partial^2 R}{\partial h^2} = \frac{\partial}{\partial h} g_2 - \left( \frac{\partial}{\partial h} A \right) \frac{\partial R}{\partial h},
\]
which implies that
\[
\frac{1}{2} \left| \frac{\partial^2 R(h, t, \theta)}{\partial h^2} \right| \leq |A| \left| \frac{\partial^2 R(h, t, \theta)}{\partial h^2} \right|
\]
\[
\leq \left| \frac{\partial}{\partial h} g_2 \right| + \left| \left( \frac{\partial}{\partial h} A \right) \frac{\partial R}{\partial h} \right|
\]
\[
\leq C \cdot (h^{-2} J(\lambda^{-1} h) + h^{-2} J(\lambda^{-1} h) \cdot h^{-1} J(\lambda^{-1} h))
\]
\[
\leq C \cdot h^{-2} J(\lambda^{-1} h).
\]

In general, if
\[
\left| \frac{\partial^{k+\ell}}{\partial h^k \partial t^\ell} R(h, t, \theta) \right| \leq C \cdot h^{-k} J(\lambda^{-1} h), \quad \text{for } k + \ell \leq p,
\]


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then
\[
\begin{align*}
\left| \frac{\partial^k h^{k/\ell}}{\partial h^k \partial t^{\ell}} A(h, t, \theta) \right| & \leq C \cdot h^{-k-1} J(\lambda^{-1} h), \\
\left| \frac{\partial^k h^{k/\ell}}{\partial h^k \partial t^{\ell}} g_1(h, t, \theta) \right| & \leq C \cdot h^{-k} J(\lambda^{-1} h), \\
\left| \frac{\partial^k h^{k/\ell}}{\partial h^k \partial t^{\ell}} g_2(h, t, \theta) \right| & \leq C \cdot h^{-k-1} J(\lambda^{-1} h),
\end{align*}
\]
for \(1 \leq k + \ell \leq p\). The proof of these estimates is similar to the proof of the claim in Lemma 2.5 below.

From (2.13) and the above estimates of \(A, g_1\) and \(g_2\) with a direct computation, one can get
\[
\left| \frac{\partial^k h^{k/\ell}}{\partial h^k \partial t^{\ell}} R(h, t, \theta) \right| \leq C \cdot h^{-k} J(\lambda^{-1} h), \quad \text{for } k + \ell \leq p + 1.
\]

In (2.10), let \(R(h, t, \theta) = \lambda^{-1} I_1(\lambda^{-1} h, \theta) + R_1(h, t, \theta)\). Then
\[
R_1 = -\int_0^1 \frac{\partial I_1}{\partial \theta}(\lambda^{-1} h - s R, \theta) R \, ds - \lambda^{-2} \sqrt{2} (\lambda^{-1} h - R)^{1/2} \cos \theta \cdot e(t).
\]

(2.14)

**Lemma 2.5.** For \(h \gg 1\), \(R_1(h, t, \theta)\) possesses the estimates:
\[
\left| \frac{\partial^k h^{k/\ell}}{\partial h^k \partial t^{\ell}} R_1(h, t, \theta) \right| \leq C \cdot h^{-k-1} \cdot \max \{ [J(\lambda^{-1} h)]^2, (\lambda^{-1} h)^{3/2} \},
\]
for any non-negative integers \(k\) and \(\ell\) with \(k + \ell \leq 5\).

**Proof.** The estimate (2.15) follows easily from a direct computation with (2.11) and the following claim.

**Claim.** For any \(k, \ell\) satisfying \(k + \ell \leq 5\) and \(h \gg 1\), we have
\[
\left| \frac{\partial^k h^{k/\ell}}{\partial h^k \partial t^{\ell}} \frac{\partial I_1}{\partial \theta}(\lambda^{-1} h - s R, \theta) \right| \leq C \cdot h^{-k-1} J(\lambda^{-1} h),
\]
and
\[
\left| \frac{\partial^k h^{k/\ell}}{\partial h^k \partial t^{\ell}} (\lambda^{-1} h - s R)^{1/2} \cos \theta \cdot e(t) \right| \leq C \cdot h^{-k-1} (\lambda^{-1} h)^{3/2}.
\]
Now we prove this claim.
Proof of (2.16). (i) When $k + \ell = 0$, we have
\[
\left| \frac{\partial^k I_l}{\partial^{kl} r} (\lambda^{-1}h - sR, \theta) \right| \leq C \cdot (\lambda^{-1}h - sR)^{-1} I_1 (\lambda^{-1}h - sR, \theta)
\]
\[
\leq C \cdot \left( \frac{1}{2} h \right)^{-1} J \left( \frac{3}{2} \lambda^{-1}h \right)
\]
\[
\leq C \cdot h^{-1} J(\lambda^{-1}h),
\]
where we have used Lemmas 2.12.3 and the inequalities
\[
|R(h, t, \theta)| \leq \frac{1}{2} \lambda^{-1}h, \quad 0 \leq s \leq 1, \quad I_1 \leq C \cdot J, \quad J(\lambda^{-1}h) \leq C \cdot J(\lambda^{-1}h).
\]

(ii) When $k > 0$, $\ell = 0$, the following equality holds
\[
\frac{\partial^k \partial_1 I}{\partial^{kl} r} (\lambda^{-1}h - sR, \theta) = \sum \frac{\partial^{q+1} I_1}{\partial^{q+1} r} (\lambda^{-1}h - sR, \theta) \cdot \frac{\partial^{q+1} u}{\partial^{q+1} t} \ldots \frac{\partial^{q+1} u}{\partial^{q+1} t}
\]
where $0 < q \leq k$, $j_1, \ldots, j_q > 0$, $j_1 + \cdots + j_q = k$ and $u := \lambda^{-1}h - sR$. Assume that there are $b(q)$ numbers in $\{ j_1, \ldots, j_q \}$ which equal to 1. Then
\[
\left| \frac{\partial^k \partial_1 I}{\partial^{kl} r} \right| \leq C \cdot (\lambda^{-1}h - sR)^{-q-1} I_1 (\lambda^{-1}h - sR, \theta)
\]
\[
\times h^{-j_1} \cdots h^{-j_q} \cdot [J(\lambda^{-1}h)]^{q-b}
\]
\[
\leq C \cdot h^{-k-1} \cdot J(\lambda^{-1}h) \cdot [h^{-1}J(\lambda^{-1}h)]^{q-b}
\]
\[
\leq C \cdot h^{-k-1} J(\lambda^{-1}h),
\]
since $J(\lambda^{-1}h) \leq C \cdot h^{1/2} \leq C \cdot h$.

(iii) When $k = 0$, $\ell > 0$. From the equality
\[
\frac{\partial^\ell}{\partial^\ell \theta} (\lambda^{-1}h - sR, \theta) = \sum \frac{\partial^{q+1} I_1}{\partial^{q+1} r} (\lambda^{-1}h - sR, \theta) \cdot \frac{\partial^{q+1} u}{\partial^{q+1} t} \ldots \frac{\partial^{q+1} u}{\partial^{q+1} t},
\]
where $p \leq \ell$, $\ell_1, \ldots, \ell_p > 0$ and $\ell_1 + \cdots + \ell_p = \ell$, it follows that
\[
\left| \frac{\partial^\ell}{\partial^\ell \theta} (\lambda^{-1}h - sR, \theta) \right| \leq C \cdot (\lambda^{-1}h - sR)^{-p-1} I_1 (\lambda^{-1}h - sR, \theta) [J(\lambda^{-1}h)]^p
\]
\[
= C \cdot h^{-1} \cdot J(\lambda^{-1}h) \cdot [h^{-1}J(\lambda^{-1}h)]^p
\]
\[
\leq C \cdot h^{-1} J(\lambda^{-1}h).
(iv) When $k, \ell > 0$. From a direct computation, we have

$$\frac{\partial^{k+\ell}}{\partial h^{k} \partial t^{\ell}} (\lambda^{-1} h - s R, \theta) = \sum \frac{\partial^{p+q+1}}{\partial h^{p} \partial h^{q} \partial t^{\ell}} (\lambda^{-1} h - s R, \theta) \frac{\partial^{k_{1}} u^{1}}{\partial h^{h_{1}}} \frac{\partial^{k_{2}} u^{2}}{\partial h^{h_{2}}} \cdots \frac{\partial^{k_{p}} u^{p}}{\partial h^{h_{p}}} \frac{\partial^{l_{1}} \ell_{1}}{\partial t^{l_{1}}} \cdots \frac{\partial^{l_{p}} \ell_{p}}{\partial t^{l_{p}}},$$

where $u = \lambda^{-1} h - s R$ and

$$0 \leq p \leq \ell, \quad 0 \leq q \leq k, \quad j_{1}, \ldots, j_{q}, \ell_{1}, \ldots, \ell_{p} > 0, \quad k_{1}, \ldots, k_{p} \geq 0, \quad j_{1} + \cdots + j_{q} + k_{1} + \cdots + k_{p} = k, \quad \ell_{1} + \cdots + \ell_{p} = \ell.$$

Assume that there are $b' (\leq q)$ numbers in $\{j_{1}, \ldots, j_{q}\}$ which equal to 1. Then

$$\left| \frac{\partial^{k+\ell}}{\partial h^{k} \partial t^{\ell}} \frac{\partial I_{1}}{\partial r} \right| \leq C \cdot (\lambda^{-1} h - s R)^{-(p+q+1)} I_{1}(\lambda^{-1} h - s R, \theta) h^{-((i_{1} + \cdots + i_{k}) - 1)} \times [J(\lambda^{-1} h)]^{q-h} \cdot h^{-i_{1}k_{1} + \cdots + i_{p}k_{p}} [J(\lambda^{-1} h)]^p

\leq C \cdot h^{-k - 1} \cdot J(\lambda^{-1} h) \cdot h^{-(p+q-h)} [J(\lambda^{-1} h)]^{p+q-h}

\leq C \cdot h^{-k - 1} [J(\lambda^{-1} h)]^{p+q-h}

$$
as $J(\lambda^{-1} h) \leq C \cdot h^{p+q} \leq C \cdot h$.

The proof of (2.7) is very analogous to the above one, we omit it here.

Remark. From the lemmas stated in Section 4, there is a large disc $D = \{(x, y) \in \mathbb{R}^{2}; x^{2} + y^{2} \leq d_{0}\}$ such that the Poincaré mapping of (2) is well-defined in $\mathbb{R}^{2} \setminus D$. In the next section, we will prove that this mapping has arbitrarily large invariant curves diffeomorphic to circles and surrounding the origin of the $(x, y)$-plane. The boundedness of solutions of (1.3) follows from the existence of such invariant curves.

3. PROOF OF THEOREM 1

Now we are concerned with the Hamiltonian system (2.3) with Hamiltonian function $h = h(r, 0, t) \in C^{1}(\mathbb{R}^{+} \times T^{2})$ given by (2.4). Observe that

$$r \, d\theta - h \, dt = -(h \, dt - r \, d\theta).$$
This means that if one can solve $r = r(h, t, \theta)$ from (2.4) as a function of $h, t$ and $\theta$, then

$$\frac{dh}{dt} = -\frac{\partial r}{\partial t}(h, t, \theta), \quad \frac{dt}{dh} = \frac{\partial r}{\partial h}(h, t, \theta),$$

(3.1)
i.e., (3.1) is a Hamiltonian system with Hamiltonian function $r = r(h, t, \theta)$ and now the action, angle, and time variables are $h, t, \theta$, respectively. This trick has been used in [3] and [4].

Remark. This observation is a key point in our proof. Notice that the Hamiltonian function $h(r, \theta, t)$ of (2.3), given by (2.4), is only $C^1$ in the angle variable $\theta$. This means that even if there is a symplectic transformation $(r, \theta) \rightarrow (\rho, \psi)^1$ such that the Poincaré mapping of the transformed system of (2.3) is a small perturbation of twist mapping, one could not use Moser’s twist theorem directly to prove the Lagrangian stability because such a perturbation is small in $C^0$-norm only, but not in $C^3 + \epsilon$, which is required in the twist theorem.

From (2.4) and Lemmas 2.1–2.3, it follows that

$$\lim_{r \to +\infty} \frac{h}{r} = \lambda > 0$$

and

$$\frac{\partial h}{\partial r} = \lambda + \frac{\partial}{\partial r} I_1(r, \theta) + \frac{\partial}{\partial r} I_2(r, \theta, t) > 0,$$

if $r \gg 1$. By the implicit function theorem, we know that there is a function $R = R(h, t, \theta)$ such that

$$r(h, t, \theta) = \lambda^{-1}h - R(h, t, \theta).$$

(3.2)

Moreover, for $h \gg 1$,

$$|R(h, t, \theta)| \leq \lambda^{-1}h/2$$

and $R(h, t, \theta)$ is $C^6$ in $h$ and $C^5$ in $t$.

From (2.4) and (3.2), it follows that $R$ satisfies (2.10), i.e.,

$$R(h, t, \theta) = \lambda^{-1}I_1(\lambda^{-1}h - R, \theta) - \lambda^{-2} \sqrt{2} (\lambda^{-1}h - R)^{1/2} \cos \theta \cdot e(t).$$

Such a transformation does exist and was used in [6].
By (2.14) and Lemma 2.5, one can write the function \( r \) into the form
\[
r(h, t, \theta) = \lambda^{-1} h - \lambda^{-1} I_1(\lambda^{-1} h, \theta) - R_1(h, t, \theta),
\]
and \( R_1 \) possesses the estimate
\[
\left| \frac{\partial^{k+\ell}}{\partial h^k \partial \theta^\ell} R_1(h, t, \theta) \right| \leq C \cdot h^{-k-1} \cdot \max \{ |J(\lambda^{-1} h)|^2, (\lambda^{-1} h)^{3/2} \},
\]
for \( k + \ell \leq 5 \). Now the system (3.1) can be written into the form
\[
\begin{align*}
\frac{dh}{dt} &= \lambda^{-1} - \lambda^{-1} \frac{\partial}{\partial h} I_1(\lambda^{-1} h, \theta) - \frac{\partial}{\partial h} R_1(h, t, \theta), \\
\frac{dh}{dt} &= R_1(h, t, \theta).
\end{align*}
\]
\[ (3.3) \]
\[ (3.4) \]

**Lemma 3.1.** There is a canonical transformation \( \Psi \) of the form
\[
\Psi: \quad h = \rho, \quad t = \tau + T(\rho, \theta)
\]
with \( T(\rho, \theta + 2\pi) = T(\rho, 2\pi) \) such that the transformed system of (3.5) is of the form
\[
\begin{align*}
\frac{d\rho}{d\tau} &= \lambda^{-1} \rho - \lambda^{-1} J(\lambda^{-1} \rho) - \bar{R}_1(\rho, \tau, \theta), \\
\frac{d\tau}{d\theta} &= \lambda^{-1} \mu - \lambda^{-1} J(\lambda^{-1} \mu) - \bar{R}_1(\rho, \tau, \theta).
\end{align*}
\]
For the new perturbation \( \bar{R}_1 \), we have
\[
\left| \frac{\partial^{k+\ell}}{\partial \rho^k \partial \tau^\ell} \bar{R}_1(\rho, \tau, \theta) \right| \leq C \cdot \rho^{-k-1} \cdot \max \{ |J(\lambda^{-1} \rho)|^2, (\lambda^{-1} \rho)^{3/2} \},
\]
for \( k + \ell \leq 5 \).
\[ (3.6) \]
\[ (3.7) \]

**Proof.** The transformation \( \Psi \) is defined implicitly in the following
\[
\rho = h + \frac{\partial S}{\partial \tau}(h, \tau, \theta), \quad t = \tau + \frac{\partial S}{\partial h}(h, \tau, \theta),
\]
where the generating function \( S = S(h, \tau, \theta) \) will be determined later. Under \( \Psi \), (3.5) is transformed into the system
\[
\begin{align*}
\frac{d\rho}{d\theta} &= \lambda^{-1} \rho - \lambda^{-1} J(\lambda^{-1} \rho) - \bar{R}_1(\rho, \tau, \theta), \\
\frac{d\tau}{d\theta} &= \lambda^{-1} \mu - \lambda^{-1} J(\lambda^{-1} \mu) - \bar{R}_1(\rho, \tau, \theta).
\end{align*}
\]
where the Hamiltonian function $\tilde{r}$ is of the form

$$\tilde{r} = \lambda^{-1}h - \lambda^{-1}I_1(\lambda^{-1}h, \theta) - R_1(h, t, \theta) + \frac{\partial S}{\partial \theta}. $$

Now we choose

$$S = \lambda^{-1} \int_0^\theta [I_1(\lambda^{-1}h, \theta) - J(\lambda^{-1}h)] \, d\theta. \quad (3.8)$$

Obviously, $S$ does not depend on $r$, and it is $2\pi$-periodic in $\theta$. Hence $\rho = h$. Let

$$T(h, \theta) = \partial S/\partial h.$$ 

Then the canonical transformation $\Psi$ is of the form

$$h = \rho, \quad t = \tau + T(\rho, \theta).$$

Let

$$\tilde{R}_1(\rho, \tau, \theta) = \tilde{R}_1(\rho, \tau + T(\rho, \theta), \theta). \quad (3.9)$$

Then the transformed Hamiltonian function $\tilde{r}$ is of the form

$$\tilde{r}(\rho, \tau, \theta) = \lambda^{-1}\rho - \lambda^{-1}J(\lambda^{-1}\rho) - \tilde{R}_1(\rho, \tau, \theta).$$

Now we prove (3.7). From (3.8), Lemmas 2.1–2.3, and the definition of $T$, it follows that

$$\left| \frac{\partial^k}{\partial \rho^k} T(\rho, \theta) \right| \leq C \cdot \rho^{-k-1} J(\lambda^{-1}\rho) \quad (3.10)$$

for $0 \leq k \leq 5$. In particular, by (2.6)

$$\left| \frac{\partial^k}{\partial \rho^k} T(\rho, \theta) \right| \leq C \cdot \rho^{-k-1 + \beta/2} \rightarrow 0 \quad \text{as} \quad \rho \rightarrow +\infty.$$ 

By a direct computation, we have

$$\frac{\partial^{p+q} \tilde{R}_1(\rho, \tau, \theta)}{\partial \rho^p \partial \tau^q} = \sum \frac{\partial^{p+q+\ell} \tilde{R}_1(\rho, \tau + T(\rho, \theta), \theta)}{\partial \rho^{\ell} \partial \tau^{p+q+\ell}} \cdot \frac{\partial^\ell T}{\partial \rho^\ell} \cdot \frac{\partial^\ell T}{\partial \rho^\ell},$$

where

$$p + q \leq k, \quad j_1, \ldots, j_q \geq 0, \quad j_1 + \cdots + j_q + p = k.$$
Hence, by (2.6), (3.4), and (3.10), it follows that, for \( k + \ell \leq 5 \),
\[
\left| \frac{\partial^{k+\ell}}{\partial \rho^k \partial \tau^\ell} \tilde{R}_1 (\rho, \tau, \theta) \right| \leq C \cdot \rho^{-\rho-1} \cdot \max \{ \left| J(\lambda^{-1} \rho) \right|^2, (\lambda^{-1} \rho)^{3/2} \} \\
\times \rho^{-\lambda-1} J(\lambda^{-1} \rho) \cdots \rho^{-\lambda-1} J(\lambda^{-1} \rho) \\
\leq C \cdot \rho^{-k-1} \cdot \max \{ \left| J(\lambda^{-1} \rho) \right|^2, (\lambda^{-1} \rho)^{3/2} \} \cdot \rho^{\beta(\rho-1)} \\
\leq C \cdot \rho^{-k-1} \cdot \max \{ \left| J(\lambda^{-1} \rho) \right|^2, (\lambda^{-1} \rho)^{3/2} \}.
\]
The proof of this lemma is completed.

In order to apply the Moser’s small twist theorem, we introduce a new variable \( v \) varying in the closed interval \([1/2, 5/2]\) and a small positive parameter \( \varepsilon \) by the formula
\[
J'(\lambda^{-1} \rho) = \lambda \varepsilon v.
\]
From (2.7) it follows that
\[
\rho^{-1} \leq C \cdot e^{2(1-\alpha)}, \quad \rho \gg 1 \iff \varepsilon \ll 1.
\]
Now the Hamiltonian system (3.6) is equivalent to the following system
\[
\frac{d\tau}{dt} = \lambda^{-1} - \lambda^{-1} \varepsilon v + f_1 (v, \tau, \theta, \varepsilon), \quad \frac{dv}{dt} = f_2 (v, \tau, \theta, \varepsilon),
\]
where
\[
f_1 (v, \tau, \theta, \varepsilon) = -\frac{\partial \tilde{R}_1}{\partial \rho} (\rho(\varepsilon, \tau, \theta), \theta),
\]
\[
f_2 (v, \tau, \theta, \varepsilon) = \lambda^{-2} \varepsilon^{-1} \cdot J'(\lambda^{-1} \rho) \cdot \frac{\partial \tilde{R}_1}{\partial \tau} (\rho(\varepsilon, \tau, \theta), \theta),
\]
and \( \rho = \rho(\varepsilon) \) is defined implicitly by (3.11).

**Lemma 3.2.** The functions \( f_1 (v, \tau, \theta, \varepsilon) \) and \( f_2 (v, \tau, \theta, \varepsilon) \) possess the estimates
\[
\left| \frac{\partial^{k+\ell}}{\partial \rho^k \partial \tau^\ell} f_1 \right|, \quad \left| \frac{\partial^{k+\ell}}{\partial \rho^k \partial \tau^\ell} f_2 \right| \leq C \cdot \varepsilon^{1+\sigma} \quad (3.13)
\]
for \( k + \ell \leq 4 \), where
\[
\sigma := \min \left\{ 1, \frac{\alpha - 1}{2 - \alpha} \right\} > 0.
\]
Proof. From (2.5), (3.11), and $1 \leq v \leq 2$, it follows that

$$\left( \frac{d}{dv} \right)^k \rho(xv) \leq C \cdot \rho, \quad \text{for} \quad 0 \leq k \leq 4. \quad (3.14)$$

The proof can be found in [4].

By the definition of $f_1$ we have, for $k \geq 1$,

$$\frac{\partial^{k+\ell}}{\partial v^k \partial \tau^\ell} f_1 = -\sum \frac{\partial^{k+\ell-1}}{\partial v^k \partial \tau^{\ell-1}} \tilde{R}_1 (\rho, \tau, \theta) \cdot \left( \frac{d}{dv} \right)^h \rho \cdots \left( \frac{d}{dv} \right)^{\ell} \rho,$$

where

$$p \leq k, \quad j_1, \ldots, j_{\ell} \geq 1, \quad j_1 + \cdots + j_{\ell} = k.$$

Hence, from (2.5), (2.7), (3.7), and (3.14) it follows that

$$\left| \frac{\partial^{k+\ell}}{\partial v^k \partial \tau^\ell} f_1 \right| \leq C \cdot \sum \rho^{-p-2} \max \left\{ [J(\lambda^{-1} \rho)]^2, (\lambda^{-1} \rho)^{3/2} \right\} \rho^p$$

$$\leq C \cdot \rho^{-2} \max \left\{ [J(\lambda^{-1} \rho)]^2, (\lambda^{-1} \rho)^{3/2} \right\}$$

$$\leq C \cdot \max \left\{ [\rho^{-1} \max \left\{ [J(\lambda^{-1} \rho)]^2, (\lambda^{-1} \rho)^{3/2} \right\}$$

$$\leq C \cdot \max \left\{ \varepsilon^2, [J(\lambda^{-1} \rho)]^{1/2 - \varepsilon} \right\}$$

$$\leq C \cdot \varepsilon^{1+\sigma}.$$

If $k = 0$, we have

$$\left| \frac{\partial^\ell}{\partial \tau^{\ell}} f_1 \right| = \left| \frac{\partial^{\ell-1}}{\partial \tau^{\ell-1}} \tilde{R}_1 (\rho, \tau, \theta) \right|$$

$$\leq C \cdot \rho^{-2} \max \left\{ [J(\lambda^{-1} \rho)]^2, (\lambda^{-1} \rho)^{3/2} \right\}$$

$$\leq C \cdot \max \left\{ [\rho^{-1} \max \left\{ [J(\lambda^{-1} \rho)]^2, (\lambda^{-1} \rho)^{3/2} \right\}$$

$$\leq C \cdot \max \left\{ \varepsilon^2, [J(\lambda^{-1} \rho)]^{1/2 - \varepsilon} \right\}$$

$$\leq C \cdot \varepsilon^{1+\sigma}.$$

For the proof of the estimates of $f_2$ in (3.13), we need the inequalities

$$\left| \frac{\partial^{k+\ell}}{\partial v^k \partial \tau^\ell} \tilde{R}_1 \right| \leq C \cdot \rho^{-1} \max \left\{ [J(\lambda^{-1} \rho)]^2, (\lambda^{-1} \rho)^{3/2} \right\}, \quad \text{for} \quad k + \ell \leq 5.$$

The proof of these inequalities is similar to the previous one, we omit it here.
From the definition of \( f_2 \), it follows that
\[
\frac{\partial^{k+l}}{\partial v^k \partial t^l} f_2 = \lambda^{-2} e^{-1} \sum_{k_1 + k_2 = k} \left( \frac{d}{dv} \right)^{k_1} J''(\lambda^{-1} \rho) \frac{\partial^{k_1} \mu_{k_2+1}}{\partial v^{k_2} \partial t^{k_2+1}} R_i.
\]
Now we give an estimate of \((d/dv)^{k_1} J''(\lambda^{-1} \rho)\). It is easy to see that
\[
\left| \left( \frac{d}{dv} \right)^{k_1} J''(\lambda^{-1} \rho) \right| = \sum_{j_1 + \ldots + j_k = k} \lambda^{-p} f^{(p+2)}(\lambda^{-1} \rho) \cdot \left( \frac{d}{dv} \right)^{j_1} \rho \ldots \left( \frac{d}{dv} \right)^{j_k} \rho \leq C \cdot \rho^{-p-2} J(\lambda^{-1} \rho) \cdot \rho^p = C \cdot \rho^{-2} J(\lambda^{-1} \rho),
\]
Hence, one has
\[
\left( \frac{\partial^{k+l}}{\partial v^k \partial t^l} f_2 \right) \leq C \cdot e^{-1} \cdot \rho^{-2} J(\lambda^{-1} \rho) \cdot \rho^{-1} \max\left\{ [J(\lambda^{-1} \rho)]^2, (\lambda^{-1} \rho)^{3/2} \right\}
\leq C \cdot \frac{J(\lambda^{-1} \rho)}{\epsilon \rho} \cdot \rho^{-2} \cdot \max\left\{ [J(\lambda^{-1} \rho)]^2, (\lambda^{-1} \rho)^{3/2} \right\}
\leq C \cdot \frac{\rho J(\lambda^{-1} \rho)}{\epsilon \rho} \cdot \rho^{-2} \cdot \max\left\{ [J(\lambda^{-1} \rho)]^2, (\lambda^{-1} \rho)^{3/2} \right\}
\leq C \cdot e^{1+\sigma},
\]
for \( k + \ell \leq 4 \).

Now we are in a position to prove Theorem 1 stated in Introduction.

**Proof of Theorem 1.** Since the functions \( f_1 \) and \( f_2 \) satisfy the estimates in (3.13), one verifies easily that the solutions of (3.12) do exist for \( 0 \leq \theta \leq 2\pi \) if the parameter \( \epsilon \) is sufficiently small. Integrating Eq. (3.12) from \( \theta = 0 \) to \( \theta = 2\pi \), we obtain the Poincaré mapping \( P \) of the form
\[
\tau_1 = \tau_0 + 2\pi \lambda^{-1} - 2\pi e^{-1} v_0 + \mathcal{E}_1(\tau_0, v_0, \epsilon), \quad v_1 = v_0 + \mathcal{E}_2(\tau_0, v_0, \epsilon),
\]
where \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) satisfy the estimates (3.13) as well as \( f_1 \) and \( f_2 \), that is,
\[
\left| \frac{\partial^{k+l}}{\partial v^k \partial t^l} \mathcal{E}_1 \right|, \quad \left| \frac{\partial^{k+l}}{\partial v^k \partial t^l} \mathcal{E}_2 \right| \leq C \cdot e^{1+\sigma}, \quad \text{for} \quad k + \ell \leq 4.
\]
Moreover, the mapping \( P \) possesses the intersection property in the annulus \( [1, 2] \times S^1 \), that is, if \( \Gamma \) is an embedded circle in \( [1, 2] \times S^1 \) homotopic to a circle \( v = \text{const.} \), then \( P(\Gamma) \cap \Gamma \neq \emptyset \). The proof can be found in [1].
Until now, we have verified that the mapping $P$ satisfies all the conditions of Moser’s small twist theorem [12, 15]. Hence for any $\hat{\omega} \in (2\pi \lambda^{-1} - 4\pi \lambda^{-1} \epsilon, 2\pi \lambda^{-1} - 2\pi \lambda^{-1} \epsilon)$ satisfying

$$\left| \hat{\omega} - \frac{P}{q} \right| \geq \epsilon_0 \cdot |q|^{-1/2},$$

(3.16)

there is an invariant curve $\Gamma$ of $P$ surrounding $v_0 = 1$ if $\epsilon \ll 1$. This means that there exist invariant curves of the Poincaré mapping of the system (2.1), which surround the origin $(x, y) = (0, 0)$ and are arbitrarily far from it. Statement (1) of Theorem 1 has been proved.

Because the solutions starting from an invariant curve $\Gamma$ are quasi-periodic with basic frequencies $(\hat{\omega}, 1)$, Statement (2) follows from the following well-known fact:

**The measure of the closed set

$$\Omega = \{ \hat{\omega} \in (2\pi \lambda^{-1} - 4\pi \lambda^{-1} \epsilon, 2\pi \lambda^{-1} - 2\pi \lambda^{-1} \epsilon); \hat{\omega} \text{ satisfying } (3.16) \}$$

is positive.**

Conclusion (3) is a direct consequence of Aubry–Mather theory, and the details of the proof can be found in [14]. The proof of Theorem 1 is completed.

**Remarks.**

1. In Theorem 1, for brevity of proof, we assume that all the inequalities in (1.4) and (1.5) hold for any $x \neq 0$. In fact, it is not necessary and can be replaced by a slightly weaker one:

   All the inequalities are valid for any $|x| \geq d$ with $d > 0$ a constant.

2. The conclusions of Theorem 1 are also valid for the equation

$$x'' + \lambda^2 x - \phi(x) = e(t),$$

(3.17)

where $\phi(x)$ and $e(t)$ satisfy (1.4) and (1.5). More precisely, we have

**Theorem 2.** Suppose that $\phi(x) \in C^4(\mathbb{R} \setminus \{0\}) \cap C^2(\mathbb{R})$, $\lambda > 0$ is a constant and $e(t) \in C^0(\mathbb{R})$ is $2\pi$-periodic in $t$. Moreover, we assume that the function $\phi(x)$ satisfies the following conditions: There is a constant $d > 0$ such that for all $|x| \geq d$,

$$\gamma x \phi(x) \geq x^2 \phi'(x) > 0, \quad x \phi(x) \geq x \Phi(x),$$
with some constants $\gamma \in (0, 1)$, $\alpha \in (1, 2)$ and

$$\left| x^k \frac{d^k \Phi(x)}{dx^k} \right| \leq C \cdot \Phi(x), \quad \text{for} \quad 3 \leq k \leq 6.$$ 

Then the following conclusions hold:

1. Every solution of (3.17) is bounded; that is, if $x(t)$ is a solution of (3.17), then $x(t)$ is defined for all $t \in \mathbb{R}$ and

$$\sup_{t \in \mathbb{R}} (|x(t)| + |x'(t)|) < +\infty.$$ 

2. There exist $\varepsilon_0 > 0$ and a closed set $\mathcal{A} \subset ((\lambda/2\pi) - \varepsilon_0, \lambda/2\pi)$ having a positive measure such that for any $\omega \in \mathcal{A}$, there is a quasi-periodic solution of (3.17) with the basic frequency $(\omega, 1)$.

3. For any $\omega \in ((\lambda/2\pi) - \varepsilon_0, \lambda/2\pi)$, Eq. (3.17) has a solution $(x_\omega(t), x'_\omega(t))$ of Mather type with rotation number $\omega$. More precisely,

- if $\omega = p/q$ is rational, the solutions $(x_\omega(t + 2\pi q), x'_\omega(t + 2\pi q))$, $0 \leq i \leq q - 1$ are mutually unlinked periodic solutions of period $2\pi q$;
- if $\omega$ is irrational, the solution $(x_\omega(t), x'_\omega(t))$ is either a usual quasi-periodic solution or a generalized one.

Example 1. All the solutions of

$$x'' + \lambda^2 x \pm |x|^{\alpha-1} x = e(t)$$

are bounded, where $\lambda > 0$, $0 < a < 1$ and $e(t) \in C^\alpha$ is $2\pi$-periodic in $t$.

Example 2. Every solution of

$$x'' + \lambda^2 x \pm x(1 + x^2)^{-1/3} = e(t)$$

is bounded if $\lambda > 0$ and $e(t) \in C^\alpha$ is $2\pi$-periodic in $t$.

4. APPENDIX: GLOBAL EXISTENCE AND UNIQUENESS

In this appendix, we will prove some results about global existence and uniqueness of solutions for Eq. (2.1), which is equivalent to (1.3).

Lemma 4.1. Suppose that all assumptions of Theorem 1 hold. Let $z(t) = (x(t), y(t))$ be a noncontinuable solution of (2.1). Then $z(t)$ is defined
for all $t \in \mathbb{R}$. Moreover, for any constant $C_1 > 0$, there exists a constant $C_2 > C_1$ depending on $C_1$ such that
\[ x^2(t_0) + y^2(t_0) \geq C_2 \Rightarrow x^2(t) + y^2(t) > C_1 \]
for all $t$ in the interval $|t - t_0| \leq 4\pi$.

**Proof.** Denote by $M$ the maximum of $|e(t)|$, i.e.,
\[ M = \max_{t \in [0, 2\pi]} |e(t)|. \]

In order to prove this lemma, we introduce the Liapunov-like function
\[ V(a, b) = \frac{1}{2}a^2 + \frac{1}{2}b^2 + \phi(a) \]
and
\[ V^*(t) = V(x(t), y(t)). \]

From the assumptions of Theorem 1, it follows that $V^*(t) > 0$ and
\[ \frac{d}{dt} V^*(t) = |e(t)| y(t) \leq M |y(t)| \leq \lambda^{-1} M V^*(t). \]

Hence, using elementary differential inequalities, we have that
\[ V^*(t_0) e^{-M |t - t_0|/\lambda} \leq V^*(t) \leq V^*(t_0) e^{M |t - t_0|/\lambda}. \]

(4.1)

So the global existence is easy proved by the second inequality in (4.1).

From the definition of $V(a, b)$, it follows that
\[ \lim_{a^2 + b^2 \to \infty} V(a, b) = +\infty. \]

Hence, for any $C_1 > 0$, there is $C_2 > C_1$ such that
\[ \inf_{a^2 + b^2 \geq C_2} V(a, b) > e^{4M\epsilon/\lambda}, \quad \sup_{a^2 + b^2 \leq C_1} V(a, b). \]

Therefore, if $x^2(t_0) + y^2(t_0) \geq C_2$, by (4.1), we have that, for $|t - t_0| \leq 4\pi$,
\[ V(x(t), y(t)) \geq V^*(t_0) e^{-M |t - t_0|/\lambda} \]
\[ \geq \inf_{a^2 + b^2 \geq C_2} V(a, b) \cdot e^{-4M\epsilon/\lambda} \]
\[ > \sup_{a^2 + b^2 \leq C_1} V(a, b). \]
This means that
\[ x^2(t) + y^2(t) > C_1 \]
for every \( t \in \{ t : |t - t_0| \leq 4\pi \} \).

Next, we prove the uniqueness of solutions.

**Lemma 4.2.** Under the hypotheses of Theorem 1, there is a constant \( d_0 > 0 \) such that for every point \((t_0, x_0, y_0) \in \mathbb{R}^3\) with \( x_0^2 + y_0^2 \geq d_0 \), the system (2.1) has a unique solution \( z(t) = (x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0)) \) for \( t \in (t_0 - 4\pi, t_0 + 4\pi) \) passing through \((t_0, x_0, y_0)\).

**Proof.** The existence of such a solution is a direct consequence of Peano’s theorem and Lemma 4.1. Now we prove the uniqueness.

Since the system (2.1) is transformed into (2.3) under the symplectic diffeomorphism (2.2), it suffices to prove that there is \( d_0 > 0 \) such that for \( t \in (t_0 - 4\pi, t_0 + 4\pi) \), the system (2.3) has a unique solution \((r(t; t_0, r_0, \theta_0), \theta(t; t_0, r_0, \theta_0))\) satisfying the initial condition

\[ (r(t_0; t_0, r_0, \theta_0), \theta(t_0; t_0, r_0, \theta_0)) = (r_0, \theta_0) \]

provided that \( r_0 \geq d_0/2 \).

The Choice of \( d_0 \). By Lemma 2.1, there is a constant \( d_1 > 0 \) such that, if \( r \geq d_1 \),
\[ \frac{\partial h}{\partial r} (r, \theta, t) = \dot{r} + \frac{\partial I_1}{\partial r} + \frac{\partial I_2}{\partial r} > 0, \quad h(r, \theta, t) > 0, \quad (4.2) \]
where the function \( h(r, \theta, t) \) is given by (2.4).

By Lemma 4.1, there is a constant \( d^* > d_1 > 0 \) such that if \( r_0 \geq d^* \), then
\[ r(t; t_0, r_0, \theta_0) > d_1 \]
for all \( t \in (t_0 - 4\pi, t_0 + 4\pi) \). Set
\[ d_0 = 2d^*. \]

Then (4.2) holds for \( |t - t_0| \leq 4\pi \) provided that \( r_0 \geq d_0/2 \).

Suppose that there are two different solutions \( z_1(t) = (r_1(t; t_0, r_0, \theta_0), \theta_1(t; t_0, r_0, \theta_0)) \) and \( z_2(t) = (r_2(t; t_0, r_0, \theta_0), \theta_2(t; t_0, r_0, \theta_0)) \) satisfying the same initial condition:
\[ z_1(t_0) = z_2(t_0) = (r_0, \theta_0) \]
for some \((t_0, r_0, \theta_0)\) with \(r_0 \geq d_0/2\). Furthermore, we assume that there is a \(t_i \in (t_0 - 4\pi, t_0 + 4\pi)\) such that
\[ z_1(t_1) \neq z_2(t_1). \]

Without loss of generality, we suppose \(t_0 < t_1 < t_0 + 4\pi\). Let
\[ \hat{t} = \max \{ t \in [t_0, t_1] | z_1(t) = z_2(t) \}. \]

Then \(t_0 \leq \hat{t} < t_1\). Moreover, we have \(z_1(\hat{t}) = z_2(\hat{t})\) and
\[ z_1(t) \neq z_2(t), \quad \text{for} \quad \hat{t} < t \leq t_1. \tag{4.3} \]

Since
\[
\theta_i(\hat{t}; t_0, r_0, \theta_0) = \theta_2(\hat{t}; t_0, r_0, \theta_0) := \bar{\theta}_i
\]
and
\[
r_i(\hat{t}; t_0, r_0, \theta_0) = r_2(\hat{t}; t_0, r_0, \theta_0) := \bar{r}_i \geq d_1
\]
and
\[
\frac{d\theta_i}{dt}(\hat{t}; t_0, r_0, \theta_0) = \frac{\partial h_i}{\partial r}(\hat{t}, \bar{\theta}_i, \bar{r}_i) > 0, \quad i = 1, 2,
\]

it follows from the inverse function theorem that there are two functions \(\tau_i(\theta; \bar{\theta}, \bar{r}, \bar{t})\) with \(|\theta - \bar{\theta}| < c_1\) such that
\[
\theta_i(\tau_i(\theta; \bar{\theta}, \bar{r}, \bar{t}); t_0, r_0, \theta_0) = \theta \tag{4.4}
\]
and
\[
\tau_i(\bar{\theta}; \bar{\theta}, \bar{r}, \bar{t}) = \hat{t},
\]
for \(i = 1, 2\). Moreover, we can choose \(c_1\) sufficiently small such that \(\tau_i(\theta; \bar{\theta}, \bar{r}, \bar{t}) < t_1\) for all \(|\theta - \bar{\theta}| \leq c_1\).

Let
\[
\eta_i(\theta) = r_i(\tau_i(\theta; \bar{\theta}, \bar{r}, \bar{t}); t_0, r_0, \theta_0) \tag{4.5}
\]
and
\[
h_i(\theta) = \dot{\eta}_1(\theta) + I_1(\eta_1(\theta), \bar{\theta}) + I_2(\eta_1(\theta), \theta, \tau_1(\theta)). \tag{4.6}
\]

Then \(\eta_i(\theta) \geq d_1\) and \(h_i(\theta) > 0\) for \(|\theta - \bar{\theta}| \leq c_1\).

By a direct computation, one can see that the vector-functions \((h_1(\theta), \tau_1(\theta))\) and \((h_2(\theta), \tau_2(\theta))\) are the solutions of (3.1) with the initial data:
\[
(h_1(\bar{\theta}), \tau_1(\bar{\theta})) = (h_2(\bar{\theta}), \tau_2(\bar{\theta})) = (h(\bar{r}, \bar{\theta}, \bar{t}), \bar{t}).
\]
Because the Hamiltonian function \( r(h, t, \theta) \) of (3.1) is, given by (3.2), \( C^6 \) in \( h \) and \( C^3 \) in \( t \), applying the uniqueness theorem to (3.1), we have that the vector-functions \( (h_1(\theta), \tau_1(\theta)) \) coincide with \( (h_2(\theta), \tau_2(\theta)) \) in some small interval \(|\theta - \bar{\theta}| \leq c_2\) with \( c_2 > 0 \). This means that
\[
h_1(\theta) = h_2(\theta), \quad \tau_1(\theta) = \tau_2(\theta) := \tau(\theta),
\]
for \(|\theta - \bar{\theta}| \leq c^*\) with \( c^* = \min\{c_1, c_2\} > 0 \).

From (4.6), (4.2) and the implicit function theorem, there is a positive constant \( c^{**} < c^* \) such that
\[
\eta_1(\theta) = \eta_2(\theta) := \eta(\theta)
\]
for \(|\theta - \bar{\theta}| \leq c^{**}\). Let \( t' = \tau(\bar{\theta} + c^{**}) \). Then \( t_i > t' > \tau(\bar{\theta}) = \bar{t}\) as \( dt_i/d\theta = 1/(dt/d\theta) = 1/(\partial h/\partial r) > 0 \). Moreover, by (4.5) and (4.6), we have
\[
r_i(t'; t_0, r_0, \theta_0) = r_2(t'; t_0, r_0, \theta_0) = \eta(\bar{\theta} + c^{**}),
\]
\[
\theta_i(t'; t_0, r_0, \theta_0) = \theta_2(t'; t_0, r_0, \theta_0) = \bar{\theta} + c^{**},
\]
which yields a contradiction to (4.3).

Note added in proof. After I submitted this manuscript, F. Zanolin called my attention to the related work of R. Ortega [Or]. In that paper, he studied the same problem for Eq. (1.3) with a bounded perturbation \( \phi(x) \).


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