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Local Whittle estimator for anisotropic random fields

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ABSTRACT

A local Whittle estimator is developed to simultaneously estimate the long memory parameters for stationary anisotropic scalar random fields. It is shown that these estimators are consistent and asymptotically normal, under some weak technical conditions. A brief simulation study illustrates a practical application of the estimator.

1. Introduction

Stationary scalar random fields with spatial long memory are useful in many diverse areas of applications (see, e.g., [2,8,21] and references therein). The Hurst parameter codes the extent of long memory, i.e., the power-law decay of autocorrelation as a function of separation distance in space. In many applications, it is unreasonable to employ an isotropic model, and hence there is a different Hurst index in each coordinate direction. In studies of ground water flow and contaminant transport, essential physical properties such as hydraulic conductivity are commonly modelled as scalar random fields with long memory. Estimates of the Hurst (long memory) index typically yield a larger value in the direction of flow, and a smaller value in the direction transverse to the flow.

In this paper, we develop a robust method to simultaneously estimate the Hurst index in each scaling direction. Our local Whittle estimator is based on spectral methods, essentially the idea that the power spectrum grows as a power law near zero if the autocorrelation decays as a power law near infinity. If the autocorrelations decay at a different power law rate in each spatial coordinate direction, then the spectral density grows as a different power law in each coordinate of the frequency. The local Whittle method assumes only the power-law asymptotics of the spectral density at the origin, making it extremely robust. The usual Whittle estimator estimates the Hurst index using the entire spectral density, and consequently the bias and standard deviation of the full Whittle estimator are comparable. One advantage of the local Whittle method is that the bias is always negligible with respect to the standard deviation, see Guyon [15].

Most commonly used random field models with long memory are isotropic [1,27]. The prototypical example is the fractional Brownian random field with moving average representation

$$B(x) = \int_{\mathbb{R}^d} \left(\|x-y\|^{H-d/2} - \|y\|^{H-d/2}\right) W(\mathrm{d}y) \tag{1.1}$$

where $0 < H < 1$ and $W(\mathrm{d}y)$ is an independently scattered Gaussian random measure on $\mathbb{R}^d$. This random field is self-similar $B(cx) \sim c^H B(x)$ (same finite dimensional distributions) with stationary increments $X(x) = \Delta B(x)$ where

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$\Delta = \Pi_{i=1}^{d}(I - L_i)$ and $L_i X(x) = X(x - b_i)$ with $b_1 \cdots b_d$ the standard basis for $\mathbb{R}^d$. A more flexible anisotropic random field [4] can be defined by

$$B(x) = \int_{\mathbb{R}^d} \left(\varphi(x-y)^{H-q/2} - \varphi(-y)^{H-q/2}\right) W(\mathrm{d}y) \tag{1.2}$$

where $\varphi(c^E x) = c\varphi(x)$ and $q = \mathrm{trace}(E)$, with $c^E = \exp(E \log c)$ and $\exp(A) = I + A + A^2/2! + \cdots$ the exponential operator. This random field is operator self-similar $B(c^E x) \sim c^H B(x)$, so that if $E = \mathrm{diag}(a_1, \dots, a_d)$ then $B_i(t) = B(tb_i)$ is self-similar $B_i(ct) \sim c^{H/a_i} B_i(t)$ with a Hurst index that varies with coordinate. For example, to get an isotropic fractional Brownian random field, take $E = I$ the identity matrix, and use the filter $\varphi(x) = \|x\|$. Operator self-similar random fields provide a more flexible model for physical quantities that exhibit significant anisotropy and long range dependence, with a different Hurst index in each coordinate.

In order to fit one of these anisotropic random field models to real data, the first step is usually to detrend and/or difference the data to obtain a stationary spatial process. Then we estimate the Hurst parameters for the stationary spatial process. For estimation of the Hurst index in one dimension, there are quite a few methods available, see Beran [3], Taqqu and Teverovsky [28], and the book of Robinson [25]. One could apply one of these estimators of the Hurst index for one dimensional sections $B_i(t)$ of the data, but this produces a different estimator for each slice. In this paper, we develop estimators based on all the data. In order to simplify the presentation, we state and prove our results in dimension $d = 2$. However, the extension to high dimensions is not difficult.

Let $X_{(s,t)}$ be the weakly stationary spatial moving average

$$X_{(s,t)} = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \tau_{l,k} \xi_{(s-l,t-k)}, \tag{1.3}$$

and assume that

$$\tau_{l,k} \sim l^{-3/2+H_1} k^{-3/2+H_2} \quad \text{as both } (l,k) \to \infty, \tag{1.4}$$

for some constants $1/2 < H_1, H_2 < 1$, and where $\{\xi_{(l,k)}, (l,k) \in R_n\}$ are i.i.d random variables with mean zero and variance one. Define

$$\tau(x,y) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \tau_{l,k} \mathrm{e}^{\mathrm{i}(lx+ky)}, \tag{1.5}$$

the Discrete Fourier Transform of $\tau_{l,k}$. Then the spectral density function $f(x,y) = |\tau(x,y)|^2/4\pi^2$ of the spatial series (1.3) will satisfy

$$f(x,y) \sim G x^{1-2H_1} y^{1-2H_2} \quad \text{as both } (x,y) \to (0,0). \tag{1.6}$$

For example, let L_j be the lag operator

$$L_1 X_{s,t} = X_{s-1,t}, \qquad L_2 X_{s,t} = X_{s,t-1}.$$

An ARMA field is generated from

$$P(L_1, L_2) X_{s,t} = Q(L_1, L_2) \varepsilon_{s,t}.$$

If $P(\mathrm{e}^{\mathrm{i}\lambda}) \neq 0$ for all $\lambda \in [-\pi, \pi]^2$, then there is an unique stationary solution $X_{s,t}$; e.g., see [26]. A straightforward extension of [10, Theorem 4.4.2] to spatial series shows that the spectral density function of $X_{s,t}$ is

$$f(x,y) = \frac{\sigma_\varepsilon^2}{4\pi^2} \frac{|Q(\mathrm{e}^{-\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}y})|^2}{|P(\mathrm{e}^{-\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}y})|^2}.$$

For another example, take $1/2 > d_1, d_2 > 0$, and consider

$$(1 - L_1)^{d_1} (1 - L_2)^{d_2} P(L_1, L_2) X_{s,t} = Q(L_1, L_2) \varepsilon_{s,t},$$

where $(1 - L)^d f(x) = \sum_{j=0}^{\infty} w_j f(x - j)$ with $w_j = \Gamma(j - d)/[\Gamma(d+1)\Gamma(-d)]$ the usual fractional difference operator. A straightforward extension of [10, Theorem 13.2.1] to spatial series shows that there is an unique stationary solution $X_{s,t}$ with spectral density function

$$f(x,y) = \frac{\sigma_\varepsilon^2}{4\pi^2} |1 - \mathrm{e}^{-\mathrm{i}x}|^{-2d_1} |1 - \mathrm{e}^{-\mathrm{i}y}|^{-2d_2} \frac{|Q(\mathrm{e}^{-\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}y})|^2}{|P(\mathrm{e}^{-\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}y})|^2}.$$

Then [10, page 522] shows that (1.6) holds as $(x,y) \to (0,0)$, where $d_i = H_i - 1/2$ for $i = 1, 2$. Refer to Lavancier [19] for more examples. Boissy [6] considers the special case where $P \equiv Q \equiv 1$, and then one can explicitly compute the auto-covariance

$$\gamma(k,l) = \frac{(-1)^{k+l} \Gamma(1 - 2d_2) \Gamma(1 - 2d_2) \sigma^2}{\Gamma(k + d_1 + 1) \Gamma(1 - k - d_1) \Gamma(l + d_2 + 1) \Gamma(1 - l - d_2)},$$

which is a product of the covariance of two ARFIMA $(0, d_1, 0)$ and ARFIMA $(0, d_2, 0)$ time series.

A stationary spatial series satisfying (1.3) and (1.5) is anisotropic when $H_1 \neq H_2$. If $H_i \in (1/2, 1)$ for $i = 1, 2$, then the restriction of this process to any slice along the ith coordinate axis is a long memory processes with Hurst parameter H_i. Several estimators of the long memory parameter H_i exist, and could be applied to any slice. However, in this paper, we develop an estimator of $H = (H_1, H_2)$ that uses all the data, combining the information in each slice.

It is known that in the one dimension case, the two long memory conditions

$$f(x) \sim Gx^{1-2H} \quad \text{as } x \to 0, \tag{1.7}$$

and

$$\gamma(k) \sim gk^{2H-2} \quad \text{as } k \to \infty, \tag{1.8}$$

are closely related [24]: when $g = 2G\,\Gamma(2-2H)\cos(\pi H)$, it is known that for $0 < H < 1/2$, (1.8) implies (1.7) (see Yong [30, page 90]), whereas for $1/2 < H < 1$, (1.7) and (1.8) are equivalent if the $\gamma(k)$ are quasi-monotonically convergent to zero, that is, $\gamma(k) \to 0$, as $k \to \infty$, and for some $C < \infty$, $\gamma(k+1) \le \gamma(k)(1+C/k)$ for all large enough k (see Yong [30, page 75]).

In the two dimensional case, if (1.4) holds, then it is not hard to show that the autocovariance function of the spatial series (1.3) satisfies

$$\gamma(k_1, k_2) \sim G^* k_1^{2H_1-2} k_2^{2H_2-2}, \quad \text{as both } (k_1, k_2) \to (\infty, \infty), \tag{1.9}$$

and then (1.6) follows. This paper is organised as follows: in Section 2, we introduce the local Whittle method. In Section 3, we prove the consistency of the local Whittle estimators of the long memory parameters. Section 4 includes some technical lemmas. Throughout the rest of the paper, we use Π^k to denote the cube $[-\pi, \pi]^k$ in $\mathbb{R}^k$.

2. Local Whittle method

The Whittle method estimates the Hurst index of self-similarity based on the asymptotic properties of the spectral density near the origin. The Whittle estimator in one dimension is asymptotically efficiency, in the sense that it achieves the same asymptotic variance as the exact MLE does when the process is Gaussian. Additional information and details can be found in Robinson [24], Beran [3] and Fox and Taqqu [12]. An essential ingredient in this approach is the discrete Fourier transform (DFT) of the process. Under the semi-parametric setup, where only (1.7) is assumed, both the DFT and the tapered DFT can be used to estimate the long memory parameters, see for example Dahlhaus [11], Lahiri [18], and Velasco [29].

Assume that we observe $X_{s,t}$ on a regular grid $\mathbb{R}_N = \{1, \dots, n\} \times \{1, \dots, n\}$ with sample size $N = n^2$, and $E[X_{s,t}] = 0$. Let $\bar{X}_n = N^{-1} \sum_{s,t=1}^{n} X_{s,t}$ the sample mean, and $\gamma(k, l) = E[X_{s,t} X_{s+k,t+l}]$ the auto-covariance. We can estimate $\gamma(k, l)$ by its sample version $\gamma_n(k, l) = N^{-1} \sum X_{s,t} X_{s+k,t+l}$ where the sum is over $1 \le s, t \le n$ such that $1 \le s + k, t + l \le n$ as well. Define the periodogram as (see [26, page 63])

$$I_n(x, y) = \frac{1}{(2\pi n)^2} \left| \sum_{s,t} \left(X(s, t) - \bar{X}_n \right) \mathrm{e}^{\mathrm{i}(sx+ty)} \right|^2. \tag{2.1}$$

The idea of the Whittle method originated from maximum likelihood estimation. If, for the time being, we assume $X = (X_t, t = 1, \dots, N) \sim \mathcal{N}(0, \Sigma)$, where $\Sigma = \Sigma(\theta)$, and θ represents the unknown parameter(s) in the covariance matrix, then the log-likelihood function of X is

$$L_N(\theta) = -\frac{N}{2} \log 2\pi - \frac{1}{2} \log |\Sigma| - \frac{1}{2} X' \Sigma^{-1} X.$$

Grenander and Szegö [14, Eq. (12) on page 65] show that

$$\lim_{N\to\infty} \log |\Sigma(\theta)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(x; \theta) \mathrm{d}x.$$

and Bleher [5] proves that

$$X' \Sigma^{-1} X \to \int_{-\pi}^{\pi} \frac{I(x)}{f(x; \theta)} \mathrm{d}x.$$

Therefore, L_N can be approximated by

$$-\frac{N}{2} \log 2\pi - \frac{N}{2} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(x; \theta) \mathrm{d}x - \int_{-\pi}^{\pi} \frac{I(x)}{f(x; \theta)} \mathrm{d}x. \tag{2.2}$$

By Kolmogorov's formula for the one-step mean square prediction error [10, page 184],

$$\sigma_\xi^2 = 2\pi \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(x; \theta_0) \mathrm{d}x \right\},$$

where $\sigma_\xi^2 = E[\xi^2]$, and θ_0 is the true parameter value. Hence, in the bivariate case of a spatial series, the Whittle estimator of θ minimizes

$$\tilde{Q}_n(\theta) = \int_{\Pi^2} \frac{I_n(x,y)}{f(x,y;\theta)} \mathrm{d}x\mathrm{d}y.$$

A further approximation replaces this quantity by its discrete version

$$Q_n(\theta) = 2\sum_{i,j=1}^{[n/2]} \frac{I_n(x_i,y_j)}{f(x_i,y_j;\theta)} \frac{4\pi^2}{n^2}, \tag{2.3}$$

where $x_i = 2\pi i/n$ and $y_j = 2\pi j/n$ for $i, j = 1, \ldots, n$ are the discrete Fourier frequencies. We write $I_{i,j} = I_n(x_i, y_j)$ for short, and likewise $f_{i,j} = f(x_i, y_j)$. In one dimensional case, $I(x_i)$ does not change if $X_{s,t} - \bar{X}$ is replaced by $X_{s,t}$. As mentioned in Robinson [24], the discrete version (2.3) is preferred because of computational efficiency, and invariance of $I_{i,j}$ with respect to the unknown mean of $X_{s,t}$. Results on the consistency and asymptotic normality of the Whittle estimator for an ARFIMA time series with long memory can be found in Fox and Taqqu [12], Giraitis and Surgailis [13]; and for one special case of a spatial ARFIMA series, see Boissy et al. [7], where properties of Toeplitz matrices are used, following the ideas used in Hannan [17].

In the current semi-parametric setup, we consider a spatial ARIMA process with spectral density $f(x, y)$ satisfying (1.6), and we only consider frequencies close to zero. Hence the local Whittle estimator $(G, \hat{H}_1, \hat{H}_2)$ is the minimiser of

$$Q_n(G, H_1, H_2) = \frac{1}{m^2} \sum_{i,j=1}^{m} \left\{ \log G x_i^{1-2H_1} y_j^{1-2H_2} + \frac{I_n(x_i, y_j)}{G x_i^{1-2H_1} y_j^{1-2H_2}} \right\},$$

where $H_i \in [\delta, 1-\delta]$ for some arbitrary small $\delta > 0$ (to avoid the boundary of the parameter set), and an integer m which satisfies $m/n \to 0$ (so that we only consider frequencies close to zero). For $1/2 < \Delta_1 < \Delta_2 < 1$, We can also write

$$\begin{aligned}
(\hat{H}_1, \hat{H}_2) &= \arg \min_{\Delta_1 \le H_i \le \Delta_2} R(H_1, H_2), \\
R(H_1, H_2) &= \log \hat{G}(H_1, H_2) + \frac{1-2H_1}{m} \sum_{i=1}^{m} \log x_i + \frac{1-2H_2}{m} \sum_{j=1}^{m} \log y_j, \\
\hat{G}(H_1, H_2) &= \frac{1}{m^2} \sum_{i,j=1}^{m} I_{i,j} x_i^{2H_1-1} y_j^{2H_2-1}.
\end{aligned} \tag{2.4}$$

Robinson [24] proved consistency and asymptotic normality of the local Whittle estimator for univariate moving averages with long memory. For a spatial moving average with long memory, we aim to give natural extensions of those results. We note that for classical ARMA spatial series with spatial dimension $d = 2$, the sample autocovariance $\gamma_n(k)$ is a biased estimator of $\gamma(k)$, with $E[\gamma_n(k) - \gamma(k)] = O(n)$, which is the same order as the variance of $\gamma_n(k)$. Therefore, to make the bias negligible, a modified sample covariance is used in calculating $I_n(x, y)$, see [26] for details. When using the Whittle method to achieve root n consistency of estimators, a similar technique is used by several authors, see [20,26]; otherwise, the bias is not negligible, see also [7]. When the local Whittle method is applied, we expect the rate of convergence (for consistency) to be root $\sqrt{m}$, where $m/n \to 0$. Hence the bias is asymptotically negligible, and therefore, we consider the usual periodogram without any modification.

3. Consistency

In this section, we present our results on the consistency of the local Whittle estimator of the long memory parameters $\boldsymbol{H} = (H_1, H_2)$ for anisotropic fields in two spatial dimensions. The arguments depend heavily on the properties of Fourier Transformations of spectral density functions. Extensions to three dimensions or higher are straightforward, but we state and prove our results in two dimensions for ease of notation. To prove consistency of $\hat{\boldsymbol{H}}$, we need some technical assumptions:

A1. The random field $X_{s,t}$ is covariance stationary with spectral density function $f(x, y) = g(x, y)h(x, y)$ for $x, y \in [-\pi, \pi]$, where:
- (i) g is continuous, non-negative function and $g(0, 0) = G_0 > 0$;
- (ii) $h(x, y) = h_1(x)h_2(y)$ for $x, y \in [-\pi, \pi]$ is integrable, with $h_1(x) \sim x^{-2d_1}$ and $h_2(y) \sim y^{-2d_2}$ as $x, y \to 0$ for some constants $d_1, d_2 \in (0, 1/2)$.

A2. In $[-\pi, \pi]$, $g(x, y)$ is differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}x} \log h_1(x) = O(x^{-1}), \quad x \to 0; \qquad \frac{\mathrm{d}}{\mathrm{d}y} \log h_2(y) = O(y^{-1}), \quad y \to 0.$$

A3. As $n \to 0$,

$$\frac{1}{m} + \frac{m}{n} \to 0.$$

Denote $\boldsymbol{H} = (H_1, H_2)$, $\hat{\boldsymbol{H}} = (\hat{H}_1, \hat{H}_2)$, and $\boldsymbol{H}_0 = (H_{10}, H_{20})$ the true parameter. Also write the parameter space $\Theta = [\Delta_1, \Delta_2]^2$.

Theorem 3.1. *Suppose (1.3) and that assumptions* **A1**–**A3** *hold. Then*

$$\hat{\boldsymbol{H}} \to_p \boldsymbol{H}_0 \quad \text{as } n \to \infty.$$

Proof. For $0 < \delta < 1/2$, let $N_\delta = \{\boldsymbol{H} : |H_1 - H_{10}| \le \delta, |H_2 - H_{20}| \le \delta\}$, N_δ^c is the complement of N_δ. Let $S(\boldsymbol{H}) = R(\boldsymbol{H}) - R(\boldsymbol{H}_0)$. Then

$$P(\hat{\boldsymbol{H}} \in N_\delta^c \cap \Theta) = P(\inf_{N_\delta^c \cap \Theta} R(\boldsymbol{H}) \le \inf_{N_\delta \cap \Theta} R(\boldsymbol{H})) \le P(\inf_{N_\delta^c \cap \Theta} S(\boldsymbol{H}) \le 0),$$

since $\boldsymbol{H}_0 \in N_\delta \cap \Theta$. Let

$$\begin{aligned} T(\boldsymbol{H}) \;=\; & -\log \frac{\hat{G}(\boldsymbol{H})}{G(\boldsymbol{H})} + \log \frac{\hat{G}(\boldsymbol{H}_0)}{G(\boldsymbol{H}_0)} \\ & -\log \left\{ \frac{[2(H_1 - H_{10}) + 1][2(H_2 - H_{20}) + 1]}{m^2} \sum_{i=1}^m \sum_{j=1}^m \left(\frac{i}{m}\right)^{2(H_1 - H_{10})} \left(\frac{j}{m}\right)^{2(H_2 - H_{20})} \right\} \\ & + 2(H_1 - H_{10}) \left(\frac{1}{m} \sum_{i=1}^m \log i - \log m + 1 \right) + 2(H_2 - H_{20}) \left(\frac{1}{m} \sum_{j=1}^m \log j - \log m + 1 \right), \end{aligned}$$

$$U(\boldsymbol{H}) = 2(H_1 - H_{10}) - \log[2(H_1 - H_{10}) + 1] + 2(H_2 - H_{20}) - \log[2(H_2 - H_{20}) + 1],$$

and

$$G(\boldsymbol{H}) = \frac{G_0}{m^2} \sum_{i=1}^m \sum_{j=1}^m x_i^{2(H_1 - H_{10})} y_j^{2(H_2 - H_{20})}.$$

One can see $S(\boldsymbol{H}) = -T(\boldsymbol{H}) + U(\boldsymbol{H})$. Then $P(\hat{\boldsymbol{H}} \in N_\delta^c \cap \Theta)$ equals

$$P(\inf_{N_\delta^c \cap \Theta} \{U(\boldsymbol{H}) - T(\boldsymbol{H})\} \le 0) \le P(\sup_\Theta |T(\boldsymbol{H})| \ge \inf_{N_\delta^c \cap \Theta} U(\boldsymbol{H})).$$

Note that $U(\boldsymbol{H})$ is the deterministic part of $S(\boldsymbol{H})$ obtained by replacing I_{ij} by $Gx_i^{1-2H_1} y_j^{1-2H_2}$. Since for $0 < x < 1$, $x - \log(1+x) \ge x^2/6$ and $-x - \log(1-x) \ge x^2/2$, we have

$$\inf_{\bar{N}_\delta \cap \Theta} U(\boldsymbol{H}) \ge \delta^{2/3}. \tag{3.1}$$

Therefore, to prove the consistency of $\hat{\boldsymbol{H}}$, it suffices to show

$$\sup_\Theta |T(\boldsymbol{H})| \to_p 0. \tag{3.2}$$

For a non-negative random variable Y, $|\log Y| \to_p 0$ is equivalent to $|Y - 1| \to_p 0$. Hence, (3.2) in turn is implied by the following:

$$\sup_\Theta \left| \frac{\hat{G}(\boldsymbol{H}) - G(\boldsymbol{H})}{G(\boldsymbol{H})} \right| = o_p(1); \tag{3.3}$$

$$\begin{aligned} & \sup_\Theta \left| \frac{[2(H_1 - H_{10}) + 1]}{m} \sum_{i=1}^m \left(\frac{i}{m}\right)^{2(H_1 - H_{10})} - 1 \right| = o(1). \\ & \sup_\Theta \left| \frac{[2(H_2 - H_{20}) + 1]}{m} \sum_{j=1}^m \left(\frac{j}{m}\right)^{2(H_2 - H_{20})} - 1 \right| = o(1) \end{aligned} \tag{3.4}$$

and

$$\left| \frac{1}{m} \sum_{i=1}^m \log i - (\log m - 1) \right| = o(1). \tag{3.5}$$

By Lemma 1 and Lemma2 in Robinson [24], (3.4) $= O(m^{-2(\Delta_1 - H_{i0})-1}) \to 0$ as $m \to \infty$ for $i = 1, 2$; and (3.5) $= O(\log m/m)$. To prove (3.3), write

$$\frac{\hat{G}(\boldsymbol{H}) - G(\boldsymbol{H})}{G(\boldsymbol{H})} = \frac{A(\boldsymbol{H})}{B(\boldsymbol{H})}$$

where

$$A(\boldsymbol{H}) = \kappa/m^2 \sum_{i=1}^{m}\sum_{j=1}^{m}\left(\frac{i}{m}\right)^{2(H_1-H_{10})}\left(\frac{j}{m}\right)^{2(H_2-H_{20})}\left(\frac{I_{ij}}{g_{ij}} - 1\right),$$

$$B(\boldsymbol{H}) = \frac{2(H_1 - H_{10}) + 1}{m}\sum_{i=1}^{m}\left(\frac{i}{m}\right)^{2(H_1-H_{10})}\frac{2(H_2 - H_{20}) + 1}{m}\sum_{j=1}^{m}\left(\frac{j}{m}\right)^{2(H_2-H_{20})}$$

with $\kappa = [2(H_1 - H_{10}) + 1][2(H_2 - H_{20}) + 1]$ and $g_{ij} = G_0 x_i^{1-2H_{10}} y_j^{1-2H_{20}}$. By Lemma 1 in Robinson [24], $B(\boldsymbol{H}) \geq 1/2$. Hence to prove the consistency of $\hat{\boldsymbol{H}}$, it is sufficient to show

$$A(\boldsymbol{H}) \to_p 0. \tag{3.6}$$

For this, we need the following formula of double summation by parts:

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}a_i b_j &= a_n b_n \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} + a_n \sum_{k=1}^{n-1}(b_k - b_{k+1})\left(\sum_{j=1}^{k}\sum_{i=1}^{n} C_{ij}\right) + b_n \sum_{k=1}^{n-1}(a_k - a_{k+1})\left(\sum_{i=1}^{k}\sum_{j=1}^{n} C_{ij}\right)\\ &\quad + \sum_{k=1}^{n-1}\sum_{h=1}^{n-1}(a_k - a_{k+1})(b_h - b_{h+1})\sum_{i=1}^{k}\sum_{j=1}^{h} C_{ij}.\end{aligned} \tag{3.7}$$

Let $C_{ij} = m^{-2}(I_{ij}/g_{ij} - 1)$, $a_i = (i/m)^{2(H_1-H_{10})}$, and $b_j = (j/m)^{2(H_2-H_{20})}$. Then

$$\begin{aligned}A(\boldsymbol{H}) = {}& \kappa m^{-2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(\frac{I_{ij}}{g_{ij}} - 1\right) + \kappa\sum_{k=1}^{m-1}\left(\left(\frac{k}{m}\right)^{2(H_2-H_{20})} - \left(\frac{k+1}{m}\right)^{2(H_2-H_{20})}\right)\sum_{j=1}^{k}\left(\sum_{i=1}^{m}\left(\frac{I_{ij}}{g_{ij}} - 1\right)\right)m^{-2}\\ &+ \kappa\sum_{k=1}^{m-1}\left(\left(\frac{k}{m}\right)^{2(H_1-H_{10})} - \left(\frac{k+1}{m}\right)^{2(H_1-H_{10})}\right)\sum_{i=1}^{k}\left(\sum_{j=1}^{m}\left(\frac{I_{ij}}{g_{ij}} - 1\right)\right)m^{-2}\\ &+ \kappa\sum_{k=1}^{m-1}\sum_{h=1}^{m-1}\left(\left(\frac{k}{m}\right)^{2(H_1-H_{10})} - \left(\frac{k+1}{m}\right)^{2(H_1-H_{10})}\right)\\ &\times\left(\left(\frac{k}{m}\right)^{2(H_2-H_{20})} - \left(\frac{k+1}{m}\right)^{2(H_2-H_{20})}\right)\left(\sum_{i=1}^{k}\sum_{j=1}^{h}\left(\frac{I_{ij}}{g_{ij}} - 1\right)\right)m^{-2}\\ =:{}& a(\boldsymbol{H}) + b(\boldsymbol{H}) + c(\boldsymbol{H}) + d(\boldsymbol{H}).\end{aligned}$$

In the following argument, we use Lemma A.1, and the subsequent derivations (A.23)–(A.29) in Appendix, which are generalisations of Theorem 2 in Robinson [24] to the two dimensional case.

Consider $a(\boldsymbol{H})$. Using (A.23) to break $I_{ij}/g_{ij} - 1$ into three parts. By (A.23) and (A.24), one sees that

$$m^{-2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left|1 - \frac{g_{ij}}{f_{ij}}\right| E\left|\frac{I_{ij}}{g_{ij}}\right| \leq C\eta$$

for any $\eta > 0$. By (A.27),

$$m^{-2}\sum_{i=1}^{m}\sum_{j=1}^{m}E\left|I_{ij}-\alpha_{ij}I_{ij\varepsilon}\right| \le m^{-2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sqrt{\frac{\log i}{i}+\frac{\log j}{j}}$$
$$\le \frac{C}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\{\sqrt{\frac{\log i}{i}}+\sqrt{\frac{\log j}{j}}\right\} \le \frac{C}{m}\int_1^m \frac{\log x}{x}\mathrm{d}x \le C\frac{\log^2 m}{m} \to 0.$$

By (A.29),

$$m^{-2}\sum_{i=1}^{m}\sum_{j=1}^{m}E|4\pi^2 I_{ij\varepsilon}-1| \to 0,$$

and hence $a(\boldsymbol{H}) = o_p(1)$.

Now, we consider $b(\boldsymbol{H})$. Because

$$\left|\left(1+\frac{1}{r}\right)^{2(H_i-H_{i0})}-1\right| \le \frac{2}{r} \tag{3.8}$$

on Θ when $r > 0$, we have

$$|b(\boldsymbol{H})| \le C\sum_{k=1}^{m-1}\left(\frac{k}{m}\right)^{2(H_2-H_{20})+1}\frac{1}{k^2}\left|\sum_{j=1}^{k}\frac{1}{m}\sum_{i=1}^{m}\left(\frac{I_{ij}}{g_{ij}}-1\right)\right|.$$

Using (A.23) again to decompose the above expression into three corresponding terms b_1, b_2 and b_3. One can see that

$$b_1 = \sum_{k=1}^{m-1}\left(\frac{k}{m}\right)^{2(H_2-H_{20})+1}\frac{1}{k^2}\sum_{j=1}^{k}\frac{1}{m}\sum_{i=1}^{m}\left|1-\frac{g_{ij}}{f_{ij}}\right|\left(E\left|\frac{I_{ij}}{g_{ij}}\right|\right)$$
$$\le C\eta\frac{1}{m}\sum_{k=1}^{m}\left(\frac{k}{m}\right)^{2(H_2-H_{20})} \le C\eta,$$

by (A.23) and (A.24); the second term

$$b_2 \le C\sum_{k=1}^{m-1}\left(\frac{k}{m}\right)^{2(H_2-H_{20})+1}\frac{1}{k^2}\frac{1}{m}\sum_{j=1}^{k}\sum_{i=1}^{m}\sqrt{\frac{\log j}{j}+\frac{\log i}{i}}$$
$$\le C\sum_{k=1}^{m-1}\left(\frac{k}{m}\right)^{2(H_2-H_{20})+1}\left[\frac{\sqrt{\log k}}{k^2}\sum_{j=1}^{k}\sqrt{\frac{1}{j}}+\frac{\sqrt{\log m}}{km}\sum_{i=1}^{m}\sqrt{\frac{1}{i}}\right]$$
$$\le C\sum_{k=1}^{m-1}\left(\frac{k}{m}\right)^{2(H_2-H_{20})+1}\left\{\frac{\sqrt{\log k}}{k^{3/2}}+\frac{\sqrt{\log m}}{km^{1/2}}\right\}$$
$$\le C\frac{\sqrt{\log m}}{m^{2(H_2-H_{20})+1}}\left\{\sum_{k=1}^{m-1}k^{2(H_2-H_{20})+1-3/2}+\frac{1}{m^{1/2}}\sum_{k=1}^{m-1}k^{2(H_2-H_{20})+1-1}\right\}$$
$$\le C\sqrt{\frac{\log m}{m}} \to 0.$$

by (A.27); the third term b_3 tends to zero in probability by (A.29). Therefore, $b(\boldsymbol{H}) \to_p 0$.

Similarly, $c(\boldsymbol{H}) \to_p 0$.

$$d(\boldsymbol{H}) \le C\frac{1}{m}\sum_{k=1}^{m-1}\left(\frac{k}{m}\right)^{2(H_1-H_{10})}\frac{1}{m}\sum_{h=1}^{m-1}\left(\frac{h}{m}\right)^{2(H_2-H_{20})}\frac{1}{kh}\sum_{i=1}^{k}\sum_{j=1}^{h}\left(\frac{I_{ij}}{g_{ij}}-1\right),$$

by using (A.23) and the arguments for $b(\boldsymbol{H})$, one can also show that $d(\boldsymbol{H}) \to_p 0$. Thus the consistency of $\hat{\boldsymbol{H}}$ is proved. □

4. Asymptotic distribution

To obtain the asymptotic distribution of the local Whittle estimators, we need additional assumptions.

AA1. The assumption **A1** is true. In addition, for some $\beta_1, \beta_2 \in (0, 2]$,

$$h_1(x) \sim x^{1-2H_{10}}(1 + O(x^{\beta_1})), \qquad h_2(y) \sim y^{1-2H_{20}}(1 + O(y^{\beta_2})), \quad x, y \to 0.$$

AA2. The assumption **A2** is true. In addition, in a neighbourhood of the origin, $\tau(x, y)$ is differentiable and as $(x, y) \to (0, 0)$,

$$\frac{\partial \tau(x, y)}{\partial x} = O\left(\frac{|\tau(x, y)|}{x}\right), \qquad \frac{\partial \tau(x, y)}{\partial y} = O\left(\frac{|\tau(x, y)|}{y}\right).$$

AA3. As $n \to 0$,

$$\frac{1}{m} + \frac{m^2}{n} + m \log m \left(\left(\frac{m}{n}\right)^{\beta_1} + \left(\frac{m}{n}\right)^{\beta_2} \right) \to 0.$$

AA4. The fourth moment of ε_1 is finite.

Theorem 4.1. *Suppose (1.3) and* **AA1–AA4** *hold. Then*

$$m(\hat{\boldsymbol{H}} - \boldsymbol{H}_0) + \Sigma_n^{-1}\mathbf{A}_n \Longrightarrow \frac{1}{2}N(\mathbf{0}, \boldsymbol{I}), \quad n \to \infty,$$

where

$$\Sigma_n = \frac{\partial^2}{\partial \boldsymbol{H}^2} R(\bar{\boldsymbol{H}}),$$

$\bar{\boldsymbol{H}}$ *belongs to the line segment between* $\hat{\boldsymbol{H}}$ *and* $\boldsymbol{H}_0$,

$$\boldsymbol{A}_n = \left(\frac{2}{m} \sum_{i,j=1}^{m} \left(\log i - \frac{1}{m} \sum_{i=1}^{m} \log i \right) \left(\frac{I_{ij}}{g_{ij}} - 4\pi^2 I_{ij\varepsilon} \right), \frac{2}{m} \sum_{i,j=1}^{m} \left(\log j - \frac{1}{m} \sum_{j=1}^{m} \log j \right) \left(\frac{I_{ij}}{g_{ij}} - 4\pi^2 I_{ij\varepsilon} \right) \right),$$

$\mathbf{0} = (0, 0)'$, *and* $\boldsymbol{I}$ *is the identity matrix.*

Remark 4.2. Using Lemma A.4 and **AA3**, we can show that $\boldsymbol{A}_n = O_p\left(m^{1/4}(\log m)^{3/2}\right)$, which is weaker than the bound $O_p((\log m)^3)$ for the bias in [7]. However, our results also apply to non-Gaussian processes, so they are more general than the case considered in [7].

Proof. Since Theorem 3.1 holds, we can apply the Mean Value Theorem to $R(\boldsymbol{H})$, and write

$$\frac{\partial}{\partial \boldsymbol{H}} R(\hat{\boldsymbol{H}}) - \frac{\partial}{\partial \boldsymbol{H}} R(\boldsymbol{H}_0) = \frac{\partial^2}{\partial \boldsymbol{H}^2} R(\bar{\boldsymbol{H}})(\hat{\boldsymbol{H}} - \boldsymbol{H}_0), \tag{4.1}$$

where $\bar{\boldsymbol{H}}$ belongs to the line segment between $\hat{\boldsymbol{H}}$ and $\boldsymbol{H}_0$. From (4.1), it suffices to show that

$$\frac{\partial^2}{\partial \boldsymbol{H}^2} R(\bar{\boldsymbol{H}}) \to_p 4\boldsymbol{I}, \tag{4.2}$$

$$\frac{m}{2} \frac{\partial}{\partial \boldsymbol{H}} R(\boldsymbol{H}_0) - \frac{1}{2}\mathbf{A}_n \Longrightarrow N(0, \boldsymbol{I}), \tag{4.3}$$

since $\partial R(\hat{\boldsymbol{H}})/\partial \boldsymbol{H} = 0$.

We have

$$\frac{\partial}{\partial \boldsymbol{H}} R(\boldsymbol{H}) = \left(\frac{\partial}{\partial H_1} R(\boldsymbol{H}), \frac{\partial}{\partial H_2} R(\boldsymbol{H}) \right)',$$

$$= \left(\frac{2}{\hat{G}(\boldsymbol{H})} \hat{G}_1^1(\boldsymbol{H}) - \frac{2}{m} \sum_{i=1}^{m} \log x_i, \frac{2}{\hat{G}(\boldsymbol{H})} \hat{G}_2^1(\boldsymbol{H}) - \frac{2}{m} \sum_{j=1}^{m} \log y_j \right)'$$

and

$$\frac{\partial^2}{\partial \boldsymbol{H}^2} R(\boldsymbol{H}) = \left(R_{k,l}\right), \quad k, l = 1, 2,$$

with

$$\hat{G}_1^k(\boldsymbol{H}) = \frac{1}{m^2} \sum_{i,j=1}^{m} I_{ij} x_i^{2H_1 - 1} y_j^{2H_2 - 1} (\log x_i)^k;$$

$$\hat{G}_2^k(\boldsymbol{H}) = \frac{1}{m^2}\sum_{i,j=1}^{m} I_{ij}x_i^{2H_1-1}y_j^{2H_2-1}(\log y_j)^k;$$

$$\hat{G}_3(\boldsymbol{H}) = \frac{1}{m^2}\sum_{i,j=1}^{m} I_{ij}x_i^{2H_1-1}y_j^{2H_2-1}(\log x_i)(\log y_j)$$

and

$$R_{i,i} = \frac{4}{(\hat{G}(\boldsymbol{H}))^2}\left(\hat{G}_i^2(\boldsymbol{H})\hat{G}(\boldsymbol{H}) - (\hat{G}_i^1(\boldsymbol{H}))^2\right),$$

$$R_{1,2} = R_{2,1} = \frac{4}{(\hat{G}(\boldsymbol{H}))^2}\left(\hat{G}_3(\boldsymbol{H})\hat{G}(\boldsymbol{H}) - (\hat{G}_1^1(\boldsymbol{H}))(\hat{G}_2^1(\boldsymbol{H}))\right).$$

Define the following terms, for $k = 0, 1, 2$,

$$F_1^k(\boldsymbol{H}) = \frac{1}{m^2}\sum_{ij}^{m} I_{ij}x_i^{2H_1-1}y_j^{2H_2-1}(\log i)^k;$$

$$F_2^k(\boldsymbol{H}) = \frac{1}{m^2}\sum_{ij}^{m} I_{ij}x_i^{2H_1-1}y_j^{2H_2-1}(\log j)^k;$$

$$F_3(\boldsymbol{H}) = \frac{1}{m^2}\sum_{ij}^{m} I_{ij}x_i^{2H_1-1}y_j^{2H_2-1}(\log i)(\log j);$$

$$E_1^k(\boldsymbol{H}) = \frac{1}{m^2}\sum_{ij}^{m} I_{ij}i^{2H_1-1}j^{2H_2-1}(\log i)^k;$$

$$E_2^k(\boldsymbol{H}) = \frac{1}{m^2}\sum_{ij}^{m} I_{ij}i^{2H_1-1}j^{2H_2-1}(\log j)^k;$$

$$E_3(\boldsymbol{H}) = \frac{1}{m^2}\sum_{ij}^{m} I_{ij}i^{2H_1-1}j^{2H_2-1}(\log i)(\log j).$$

Let $F^0(\boldsymbol{H}) = F_1^0(\boldsymbol{H}) = F_2^0(\boldsymbol{H})$, and $E^0(\boldsymbol{H}) = E_1^0(\boldsymbol{H}) = E_2^0(\boldsymbol{H})$. Sometimes, we omit $(\boldsymbol{H})$ in these notations for easy writing. Then one can observe that

$$\begin{aligned} R_{i,i} &= \frac{4}{(F^0)^2}\left(F_i^2F^0 - (F_i^1)^2\right) = \frac{4}{(E^0)^2}\left(E_i^2E^0 - (E_i^1)^2\right), \\ R_{1,2} &= \frac{4}{(F^0)^2}\left(F_3F^0 - F_1^1F_2^1\right) = \frac{4}{(E^0)^2}\left(E_3E^0 - E_1^1E_2^1\right). \end{aligned} \quad (4.4)$$

To show (4.2), it suffices to prove the following two claims:

$$\frac{\partial^2}{\partial \boldsymbol{H}^2}R(\bar{\boldsymbol{H}}) = \frac{\partial^2}{\partial \boldsymbol{H}^2}R(\boldsymbol{H}_0) + o_p(1); \quad (4.5)$$

$$\frac{\partial^2}{\partial \boldsymbol{H}^2}R(\boldsymbol{H}_0) \to_p 4\boldsymbol{I}. \quad (4.6)$$

By the proof of Lemma A.6, for $i = 1, 2$,

$$R_{i,i} = 4\left\{\frac{1}{m}\sum_{j=1}^{m}\log^2 j - \left(\frac{1}{m}\sum_{j=1}^{m}\log j\right)^2\right\} + o_p(1) = 4 + o_p(1);$$

$$R_{1,2} = R_{2,1} = o_p(1).$$

Therefore, (4.6) is true.

Let $q = 2H_{10} + 2H_{20} - 2$. By the proof of Lemmas A.6 and A.7, as $n \to \infty$

$$\begin{aligned} R_{jj}(\bar{\boldsymbol{H}}) &= 4\frac{[E_j^2(\boldsymbol{H}_0) + o_p(n^q)][E^0(\boldsymbol{H}_0) + o_p(n^q)] - [E_j^1(\boldsymbol{H}_0) + o_p(n^q)]^2}{\{E^0(\boldsymbol{H}_0) + o_p(n^q)\}^2} \\ &= \frac{4}{(F^0(\boldsymbol{H}_0))^2}\{F_j^2(\boldsymbol{H}_0)F^0(\boldsymbol{H}_0) - (F_j^1(\boldsymbol{H}_0))^2\} + o_p(1), \quad j = 1, 2; \end{aligned}$$

and

$$R_{12}(\bar{\boldsymbol{H}}) = 4\frac{[E_3(\boldsymbol{H}_0)+o_p(n^q)][E^0(\boldsymbol{H}_0)+o_p(n^q)]-[E_1^1(\boldsymbol{H}_0)E_2^1(\boldsymbol{H})+o_p(n^q)]}{\{E^0(\boldsymbol{H}_0)+o_p(n^q)\}^2}$$
$$= \frac{4}{(F^0(\boldsymbol{H}_0))^2}\{F_3(\boldsymbol{H}_0)F^0(\boldsymbol{H}_0)-F_1^1(\boldsymbol{H}_0)F_2^1(\boldsymbol{H}_0)\}+o_p(1).$$

Thus (4.5) follows by using (4.4) and the claim (4.2) is proved.

To prove (4.3), let $\nu_i = \log x_i - \frac{1}{m}\sum_{i=1}^m \log x_i$ and write

$$\frac{m}{2}\frac{\partial R(\boldsymbol{H}_0)}{\partial H_1} = m\left\{\frac{1}{\hat{G}(\boldsymbol{H}_0)}\left(\frac{1}{m^2}\sum_{i,j=1}^m x_i^{2H_{10}-1}y_j^{2H_{20}-1}I_{ij}(\log x_i)\right)-\frac{1}{m}\sum_{i=1}^m \log x_i\right\}$$
$$= \frac{1}{m}\sum_{i,j=1}^m \log x_i \frac{x_i^{2H_{10}-1}y_j^{2H_{20}-1}I_{ij}}{\hat{G}(\boldsymbol{H}_0)} - \sum_{i=1}^m \log x_i \frac{1}{m^2}\sum_{i,j=1}^m \frac{x_i^{2H_{10}-1}y_j^{2H_{20}-1}I_{ij}}{\hat{G}(\boldsymbol{H}_0)}$$
$$= \frac{1}{m}\sum_{i,j=1}^m \frac{x_i^{2H_{10}-1}y_j^{2H_{20}-1}I_{ij}}{\hat{G}(\boldsymbol{H}_0)}\nu_i = \frac{1}{m}\sum_{i,j=1}^m \frac{G_0}{\hat{G}(\boldsymbol{H}_0)}\frac{I_{ij}}{g_{ij}}\nu_i$$
$$= \frac{1}{m}\sum_{i,j=1}^m \nu_i\left(\frac{I_{ij}}{g_{ij}}-1\right)(1+o_p(1)) = \frac{1}{m}\sum_{i,j=1}^m \nu_i\left(\frac{I_{ij}}{g_{ij}}-1\right)+o_p(1),$$

using $\sum_{i=1}^m \nu_i = 0$, $\hat{G}(\boldsymbol{H}_0) = G_0 + O_p((\log m)^{1/2}/m^{3/4} + (m/n)^{\beta 1} + (m/n)^{\beta 2})$ from Lemma A.6 and summation by parts together with **AA3**, Lemmas A.4 and A.5 and $\sum_1^{m-1}|\nu_i - \nu_{i+1}| = O(\log m)$. Thus, we have

$$\frac{m}{2}\frac{\partial R(\boldsymbol{H}_0)}{\partial H_1} - \frac{1}{m}\sum_{i,j=1}^m \nu_i\left(\frac{I_{ij}}{g_{ij}}-4\pi^2 I_{ij\varepsilon}\right) = \frac{1}{m}\sum_{i,j=1}^m \nu_i\{(2\pi)^2 I_{ij\varepsilon}-1\}+o_p(1)$$
$$= 2\sum_{s,t=1}^n Z_{s,t} + o_p(1), \tag{4.7}$$

where

$$Z_{s,t} = \varepsilon_{s,t}\frac{1}{n^2 m}\sum_{u=1}^{s-1}\sum_{v=1}^n \varepsilon_{u,v}\left(\sum_{i=1}^m \nu_i\cos(s-u)x_i\sum_{j=1}^m \cos(t-v)y_j - \sum_{i=1}^m \nu_i \sin(s-u)x_i \sum_{j=1}^m \sin(t-v)y_j\right)$$

for $s \neq 1$ and $Z_{1,t} = 0$. Let $c_\nu(t) = (1/n\sqrt{m})\sum_{i=1}^m \nu_i \cos tx_i$, $c(t) = (1/n\sqrt{m})\sum_{i=1}^m \cos tx_i$, $s_\nu(t) = (1/n\sqrt{m})\sum_{i=1}^m \nu_i \sin tx_i$ and $s(t) = (1/n\sqrt{m})\sum_{i=1}^m \sin tx_i$.

Then, we can rewrite $Z_{s,t}$ as

$$Z_{s,t} = \varepsilon_{s,t}\sum_{u=1}^{s-1}\sum_{v=1}^n \varepsilon_{u,v}(c_\nu(s-u)c(t-v) - s_\nu(s-u)s(t-v)).$$

Next, we bound $c_\nu(s)$, $s_\nu(s)$, $c(s)$ and $s(s)$. From (4.20) and (4.21) in [24], $|c_\nu(s)|$ and $|s_\nu(s)|$ are bounded by $O(\frac{\sqrt{m}\log m}{n})$, for $1 \le s \le n/2$ and $O(\frac{\log m}{\sqrt{ms}})$, for $n/m \le s \le n/2$. Similarly, $|c(s)|$ and $|s(s)|$ are bounded by $O(\frac{\sqrt{m}}{n})$, for $1 \le s \le n/2$ and $O(\frac{1}{\sqrt{ms}})$, for $n/m \le s \le n/2$. Also, $|c_\nu(n-s)| = |c_\nu(s)|$, $|s_\nu(n-s)| = |s_\nu(s)|$, $|c(n-s)| = |c(s)|$ and $|s(n-s)| = |s(s)|$. Thus, we introduce $A(s)$ for $s = 1, \ldots, n$ such that

$$A(s) = \begin{cases} \frac{1}{\sqrt{ms}}, & \frac{n}{m} \le s \le \frac{n}{2} \\ \frac{\sqrt{m}}{n}, & 1 \le s \le \frac{n}{m} \\ A(n-s), & s > \frac{n}{2}. \end{cases} \tag{4.8}$$

Then, we have

$$\begin{aligned}
&|c_\nu(s)|, |s_\nu(s)| \le C \log m A(s), \\
&|c(s)|, |s(s)| \le CA(s), \\
&\sum_{s=1}^{n} |A(s)|^2 = O\left(\frac{1}{n}\right).
\end{aligned} \tag{4.9}$$

The last equality can be shown as in (4.22) of [24].

Similar to (4.7), we obtain

$$\begin{aligned}
\frac{m}{2}\frac{\partial R(\boldsymbol{H}_0)}{\partial H_2} - \frac{1}{m}\sum_{i,j=1}^{m} \nu_j \left(\frac{I_{ij}}{g_{ij}} - 4\pi^2 I_{ij\varepsilon}\right) &= 2\sum_{s,t=1} \varepsilon_{s,t} \left\{ \sum_{u=1}^{s-1} \sum_{\substack{v=1,\\ v\neq t}}^{n} \varepsilon_{u,v}(c(s-u)c_\nu(t-v) - s(s-u)s_\nu(t-v)) \right\} \\
&\quad + 2\sum_{s=1}^{n}\sum_{t=2}^{n} \varepsilon_{s,t} \left\{ \frac{\sqrt{m}}{n} \sum_{v=1}^{t-1} \varepsilon_{s,v} c_\nu(t-v) \right\} + o_p(1), \\
&= 2\sum_{s,t=1}^{n} Z^*_{s,t} + o_p(1)
\end{aligned} \tag{4.10}$$

where

$$Z^*_{s,t} = \varepsilon_{s,t} \sum_{u=1}^{s-1} \sum_{\substack{v=1,\\ v\neq t}}^{n} \varepsilon_{u,v}(c(s-u)c_\nu(t-v) - s(s-u)s_\nu(t-v)), \tag{4.11}$$

for $s \neq 1$ and $Z^*_{1,t} = 0$ since

$$2\sum_{s=1}^{n}\sum_{t=2}^{n} \varepsilon_{s,t} \left\{ \frac{\sqrt{m}}{n} \sum_{v=1}^{t-1} \varepsilon_{s,v} c_\nu(t-v) \right\} = O_p\left(\sqrt{\frac{m}{n}} \log m\right) = o_p(1),$$

which can be shown using (4.8) and (4.9). Now (4.3) follows using Lemma A.10 along with (4.7) and (4.10), which completes the proof. □

5. Simulation study

A brief simulation study was conducted to illuminate the finite-sample behaviour of our estimator, and to illustrate its practical application. We apply the estimator to simulated random fields $X_{s,t}$ with a different Hurst index in each coordinate. Specifically, we simulate a fractionally integrated noise

$$X_{s,t} = (1 - L_1)^{-d_1}(1 - L_2)^{-d_2}\varepsilon_{s,t},$$

where $(1 - L)^d f(x) = \sum_{j=0}^{\infty} w_j f(x - j)$ with $w_j = \Gamma(j - d)/[\Gamma(d + 1)\Gamma(-d)]$ is the usual fractional difference operator, and $\varepsilon_{s,t} \sim N(0, 1)$ is an IID white noise field. The fractional integration weights w_j were computed using the recursive formula [27, (7.13.1)]. The simulation used orders of fractional integration $d = 0.1, 0.25, 0.4$, so that the corresponding values of the Hurst index $H = d + 1/2$ are 0.6, 0.75, 0.9. Since we allow H to vary with coordinates, there are nine different values of $\boldsymbol{H} = (H_1, H_2)$. We collapse to six cases with $H_1 \le H_2$, without loss of generality. The size of the simulated random field was $n \times n$ where $n = 128$ is relatively small, and we truncated the fractional integration sum at $J = 200$ terms. We synthesise an ensemble of $r = 1000$ replications for each value $\boldsymbol{H}$. For each replication, we compute $\hat{\boldsymbol{H}}$ for each $m = 1, 2, 3, \ldots,$ defining the number of Fourier frequencies in each coordinate used to compute the estimator. We also compute the one dimensional local Whittle estimators, $\tilde{\boldsymbol{H}} = (\tilde{H}_1, \tilde{H}_1)$, using a single row and column of the simulated data, respectively. Then we compare the behaviour of this well known estimator with that of our new estimator, which uses all the data.

As an example, Fig. 1 shows the ensemble average Bias^2, variance, and mean squared error for our estimator $\hat{\boldsymbol{H}}$, and the one dimensional local Whittle estimator $\tilde{\boldsymbol{H}}$, when the true $\boldsymbol{H} = (0.6, 0.75)$. The magnitudes of these three diagnostic quantities for our estimator $\hat{\boldsymbol{H}}$ are, in general, smaller than those of the estimator $\tilde{\boldsymbol{H}}$, and naturally this conclusion holds for any different row or column of the data used to compute $\tilde{\boldsymbol{H}}$ (not shown). When m is between 40 and 60 [10%–20% of total frequency vectors], our estimator $\hat{\boldsymbol{H}}$ generally produces a reliable estimate of $\boldsymbol{H}$, with a lower MSE than the one dimensional estimator $\tilde{\boldsymbol{H}}$, for all values of $\boldsymbol{H}$ tested. The minimum MSE in each graph gives a rough indication of the optimal choice of m for this case.

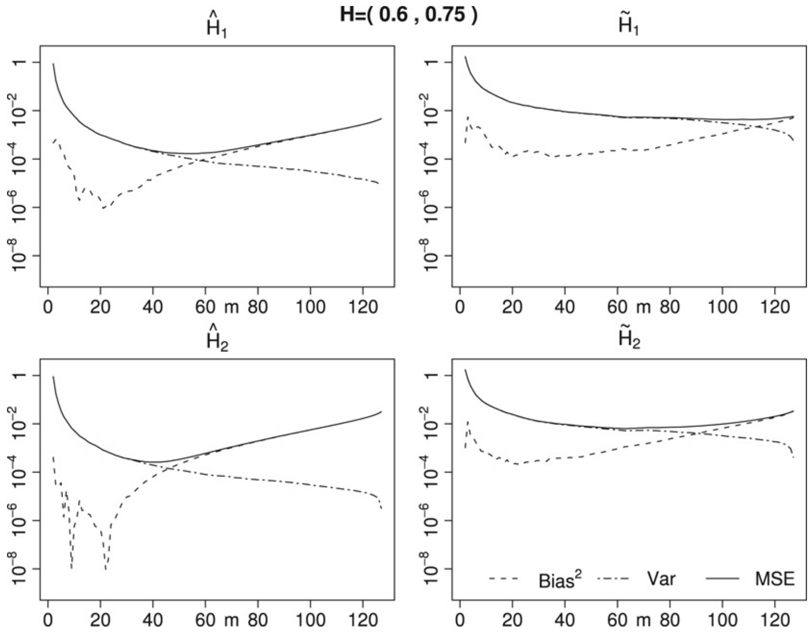

Fig. 1. Bias^2, variance and mean squared error versus the number m of Fourier frequencies used, for our estimator $\hat{\boldsymbol{H}}$ (left) and the usual one variable local Whittle estimator $\tilde{\boldsymbol{H}}$ (right). Note that the MSE of our estimator is significantly lower.

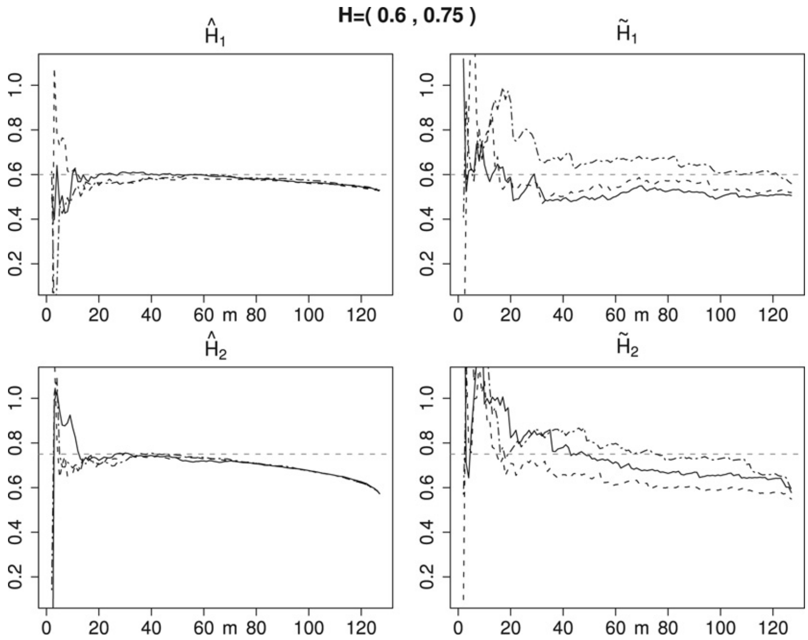

Fig. 2. Plot of our estimator $\hat{H}_i$ (left) and the usual one variable local Whittle estimator $\tilde{H}_i$ (right) versus the number m of Fourier frequencies used. Three representative realisations are shown, and the grey horizontal line indicates the true value of H_i. Note that our estimator stabilises near the true value over a broad range of m.

Since the local Whittle estimator depends on the number m of Fourier frequencies used, we also explored the behaviour of our estimator as it relates to the choice of m. Fig. 2 shows three representative plots of $\hat{\boldsymbol{H}}$ and $\tilde{\boldsymbol{H}}$ as a function of m in the case $\boldsymbol{H} = (0.6, 0.75)$. In practical applications, a reasonable approach is to plot the local Whittle estimator against m, and visually determine a region in which the estimator stabilises. Typically, in our simulations, the estimators $\hat{H}_i$ for both coordinates stabilise at a value close to the true H_i for m near 40, and the stable range is fairly broad. The one dimensional Whittle estimator $\tilde{\boldsymbol{H}}$ is more variable, and of course this conclusion persists using any row or column of the data to compute $\tilde{\boldsymbol{H}}$.

Table 1
Average bias and standard deviation vectors from $r = 1000$ simulations for our estimator $\hat{\boldsymbol{H}}$, and the one dimensional local Whittle estimator $\tilde{\boldsymbol{H}}$, using $m = 40$ Fourier frequencies. Our estimator has less bias and a smaller standard deviation in all cases.

(H_1, H_2)	Bias	SD
	$\hat{\boldsymbol{H}} = (\hat{H}_1, \hat{H}_2)$	
(0.60, 0.60)	(−0.0041, −0.0059)	(0.0146, 0.0140)
(0.60, 0.75)	(−0.0044, −0.0087)	(0.0140, 0.0136)
(0.60, 0.90)	(−0.0043, −0.0092)	(0.0145, 0.0142)
(0.75, 0.75)	(−0.0094, −0.0097)	(0.0139, 0.0143)
(0.75, 0.90)	(−0.0089, −0.0090)	(0.0148, 0.0141)
(0.90, 0.90)	(−0.0084, −0.0092)	(0.0143, 0.0143)
	$\tilde{\boldsymbol{H}} = (\tilde{H}_1, \tilde{H}_2)$	
(0.60, 0.60)	(−0.0183, −0.0123)	(0.0966, 0.0946)
(0.60, 0.75)	(−0.0115, −0.0198)	(0.0934, 0.0940)
(0.60, 0.90)	(−0.0148, −0.0250)	(0.0933, 0.0963)
(0.75, 0.75)	(−0.0229, −0.0211)	(0.0996, 0.0931)
(0.75, 0.90)	(−0.0209, −0.0172)	(0.0969, 0.0994)
(0.90, 0.90)	(−0.0192, −0.0176)	(0.0959, 0.0952)

Table 1 illustrates how the ensemble average bias and standard deviation of our estimator $\hat{\boldsymbol{H}}$, and of the one variable local Whittle estimator $\tilde{\boldsymbol{H}}$, vary with $\boldsymbol{H}$. For example, in the case $H_1 = H_2 = 0.6$, the average bias of $\hat{H}_1$ is -0.0041, and the average bias of $\hat{H}_2$ is -0.0059. The bias of both estimators inflates as H_i increases from 0.5 to 1.0, while the standard deviations are comparable for all H_i. For our estimator, the observed standard deviations are close to the asymptotic value of $1/(2m) = 0.125$ from Theorem 4.1. The one variable local Whittle estimator $\tilde{H}_i$ has a significantly higher bias, and a significantly larger standard deviation, as compared with our estimator $\hat{H}_i$, in all cases. Given the relatively small sample size of $n = 128$, this provides some confidence that our estimator can be useful for real data analysis, as an improvement over the one variable local Whittle estimator, in applications where the spatial data has significant anisotropy, so that the Hurst index varies with coordinate. We also note that, since the standard deviation of our estimator is much smaller, the opportunity to detect such anisotropy is greatly enhanced.

Acknowledgment

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Appendix

The following lemmas generalise Theorem 2 of Robinson [23] to the two dimensional case.
Let

$$\omega(x, y) = \frac{1}{2\pi n} \sum_{s,t=1}^{n} X_{s,t} \mathrm{e}^{\mathrm{i}(sx+ty)}.$$

Lemma A.1. *Suppose* **A1–A3** *hold. Then, uniformly for $j, k, l, h = 1, 2, \ldots, m$, and $j < l$, $k < h$,*

$$E[\omega(x_j, y_k)\bar{\omega}(x_j, y_k)] - f(x_j, y_k) = O\left(f(x_j, y_k)\left(\frac{\log j}{j} + \frac{\log k}{k}\right)\right). \tag{A.1}$$

$$E[\omega(x_j, y_k)\omega(x_j, y_k)] = O\left(f(x_j, y_k)\left(\frac{\log j}{j} + \frac{\log k}{k}\right)\right). \tag{A.2}$$

$$E[\omega(x_j, y_k)\bar{\omega}(x_l, y_h)] = O\left(f(x_j, y_k)\left(\frac{\log l}{j} + \frac{\log h}{k}\right)\right). \tag{A.3}$$

$$E[\omega(x_j, y_k)\omega(x_l, y_h)] = O\left(f(x_j, y_k)\left(\frac{\log l}{j} + \frac{\log h}{k}\right)\right). \tag{A.4}$$

Proof. We show (A.1) first. It is equivalent to

$$E[\omega(x_j, y_k)\bar{\omega}(x_j, y_k)] - f(x_j, y_k) = O(\epsilon_{jk}), \tag{A.5}$$

where

$$\epsilon_{jlkh} = \left(\frac{\log l}{j} + \frac{\log h}{k}\right) x_j^{-d_1} x_l^{-d_1} y_k^{-d_2} y_h^{-d_2}, \quad \text{and} \quad \epsilon_{jk} = \epsilon_{jjkk}.$$

Note that $\gamma_{s,t} = (2\pi)^{-2}\int_{\Pi^2} f(x,y)\mathrm{e}^{-\mathrm{i}(xs+yt)}\mathrm{d}x\mathrm{d}y$. The left hand side of (A.5) equals

$$\frac{1}{16\pi^4 n^2}\sum_{s,t}\sum_{l,h}\int_{\Pi^2} f(x,y)\mathrm{e}^{-\mathrm{i}(s-l)x-\mathrm{i}(t-h)y}\mathrm{d}x\mathrm{d}y\mathrm{e}^{\mathrm{i}(s-l)x_j+\mathrm{i}(t-h)y_k} - f(x_j,y_k)$$
$$= \frac{1}{4\pi^2}\int_{\Pi^2}\left(f(x,y)-f(x_j,y_k)\right)K\left((x-x_j),(y-y_k)\right)\mathrm{d}x\mathrm{d}y, \tag{A.6}$$

where $K(x,y)$ is the product of Fejér's kernels, that is,

$$K(x,y) = F(x)\cdot F(y),$$
$$F(x) = \frac{1}{2\pi n}|D(x)|^2 = \frac{1}{2\pi n}\sum_{s,l=1}^{n}\mathrm{e}^{\mathrm{i}(s-l)x},$$

where $D(x)$ is called the Dirichlet kernel. Choose $\epsilon > 0$ and n such that for $\lambda \in (-\epsilon,\epsilon)\setminus\{0\}$, for $i = 1, 2$,

$$|h_i(\lambda)| \le C|\lambda|^{-2d_i}, \qquad |h_i'(\lambda)| \le C|\lambda|^{-1-2d_i}.$$

The proof is carried out by the following partition:

$$\int_{-\pi}^{\pi} = \int_{-\pi}^{-\epsilon} + \int_{-\epsilon}^{-2a} + \int_{-2a}^{-a/2} + \int_{-a/2}^{a/2} + \int_{a/2}^{2a} + \int_{2a}^{\epsilon} + \int_{\epsilon}^{\pi}, \tag{A.7}$$

with $a = x_i$ and y_j respectively, and by **A1** and **A2**, especially the fact that $f(x,y)$ is a product of two separate function $h_1(x)$ and $h_2(y)$ near origin. We consider the following typical terms. The following inequalities of Diriclet Kernel and Fejer's kernel [10], are used repeatedly.

$$\left|\sum_{s=1}^{n}\mathrm{e}^{\mathrm{i}sx}\right| \le Cx^{-1}, \quad x\in\Pi,$$
$$(2\pi n)^{-1}\sum_{s,l=1}^{n}\mathrm{e}^{\mathrm{i}(s-l)x} \le C(nx^2)^{-1}, \quad x\in\Pi, \tag{A.8}$$

for sufficient large n; and that (Brillinger, or other time series text books),

$$\int_{\Pi} F(\lambda)\mathrm{d}\lambda \equiv \int_{\Pi}\frac{1}{n}\left[\frac{\sin(n\lambda/2)}{\sin(\lambda/2)}\right]^2\mathrm{d}\lambda = 2\pi. \tag{A.9}$$

First, applying (A.8) to K in (A.6),

$$\left|\int_{\epsilon}^{\pi}\int_{\epsilon}^{\pi}\right| \le \left(\frac{C}{n\epsilon^2}\right)^2\int_0^{\pi}\int_0^{\pi}[f(x,y)+g(x_j,y_k)x_j^{-2d_1}y_k^{-2d_2}]\mathrm{d}x\mathrm{d}y$$
$$\le Cn^{-2}x_j^{-2d_1}y_k^{-2d_2} = o(\epsilon_{jk});$$

Applying (A.8) again,

$$\left|\int_{\epsilon}^{\pi}\int_{2y_k}^{\epsilon}\right| \le C\int_{\epsilon}^{\pi}h_1(x)F(x-x_j)\mathrm{d}x\int_{2y_k}^{\epsilon}h_2(y)F(y-y_k)\mathrm{d}y + C\int_{\epsilon}^{\pi}h_1(x_j)F(x-x_j)\mathrm{d}x\int_{2y_k}^{\epsilon}h_2(y_k)F(y-y_k)\mathrm{d}y$$
$$\le \frac{C}{n\epsilon^2}\left\{\max_{2y_k\le y\le\epsilon}\left|\frac{h_2(y)}{y^{(1-2d_2)/2}}\right|\right\}\int_{2y_k}^{\epsilon}y^{(1-2d_2)/2}F(y-y_k)\mathrm{d}y + \frac{C}{n\epsilon^2}y_k^{-2d_2}\int_{2y_k}^{\epsilon}\frac{1}{n(y-y_k)^2}\mathrm{d}y$$
$$\le \frac{C}{n\epsilon^2}y_k^{-1/2-d_2}\int_{2y_k}^{\epsilon}y^{(1-2d_2)/2}\frac{1}{n(y-y_k)^2}\mathrm{d}y + \frac{C}{n\epsilon^2}y_k^{-2d_2}\int_{2y_k}^{\epsilon}\frac{1}{n(y-y_k)^2}\mathrm{d}y$$
$$\le \frac{C}{n\epsilon^2}y_k^{-1/2-d_2}\int_{2y_k}^{\epsilon}y^{(1-2d_2)/2}\frac{1}{ny^2}\mathrm{d}y + \frac{C}{n\epsilon^2}y_k^{-2d_2}\int_{2y_k}^{\epsilon}\frac{1}{ny^2}\mathrm{d}y$$
$$\le \frac{C}{n^2\epsilon^2}y_k^{(-1/2-d_2)+(-1/2-d_2)} + \frac{C}{n^2\epsilon^2}y_k^{-1-2d_2}$$
$$= \frac{C}{nk}y_k^{-2d_2} = o(\epsilon_{jk}).$$

For $x \in [\epsilon, \pi]$, $y \in [y_k/2, 2y_k]$,

$$\begin{aligned} f(x, y) - f(x_j, y_k) &= f(x, y) - f(x, y_k) + f(x, y_k) - f(x_j, y_k) \\ &= f_2'(x, \tilde{y})(y - y_k) + f_1'(\tilde{x}, y_k)(x - x_j), \end{aligned} \quad (A.10)$$

for $\tilde{x}$ between x and x_j, and $\tilde{y}$ between y and y_k. By **A2**,

$$\begin{aligned} f_1'(x, y) &= g_1'(x, y)h_1(x)h_2(y) + g(x, y)h_1'(x)h_2(y) = O(y^{-2d_2}); \\ f_2'(x, y) &= g_2'(x, y)h_1(x)h_2(y) + g(x, y)h_1(x)h_2'(y) = O(y^{-2d_2-1}). \end{aligned} \quad (A.11)$$

By (A.8) and (A.9), one has

$$\begin{aligned} \left| \int_\epsilon^\pi \int_{y_k/2}^{2y_k} f_1'(\tilde{x}, y_k)(x - x_j)F(x - x_j)F(y - y_k)\mathrm{d}x\mathrm{d}y \right| &\le C \int_\epsilon^\pi \int_{y_k/2}^{2y_k} y_k^{-2d_2}(x - x_j)F(x - x_j)F(y - y_k)\mathrm{d}x\mathrm{d}y \\ &\le C \int_\epsilon^\pi (x - x_j)F(x - x_j)\mathrm{d}x \int_{y_k/2}^{2y_k} y_k^{-2d_2} F(y - y_k)\mathrm{d}y \\ &\le C \left(\int_\epsilon^\pi \frac{1}{n|x - x_j|}\mathrm{d}x \right) \left(y_k^{-2d_2} \int_{y_k/2}^{2y_k} |F(y - y_k)|\mathrm{d}y \right) \\ &\le \frac{C}{n\epsilon} y_k^{-2d_2} = o(\epsilon_{jk}), \end{aligned} \quad (A.12)$$

by **A3.** In addition, by the fact that

$$\int_{-cy_k}^{cy_k} |D(y)|\mathrm{d}y = O(\log k) \quad (A.13)$$

in Lemma 5 of [22], one has

$$\begin{aligned} \left| \int_\epsilon^\pi \int_{y_k/2}^{2y_k} f_2'(x, \tilde{y})(y - y_k)F(x - x_j)F(y - y_k)\mathrm{d}x\mathrm{d}y \right| &\le C \int_\epsilon^\pi |F(x - x_j)|\mathrm{d}x \int_{y_k/2}^{2y_k} |y_k^{-2d_2-1}(y - y_k)F(y - y_k)|\mathrm{d}x\mathrm{d}y \\ &\le \frac{C}{n\epsilon^2} y_k^{-2d_2-1} \int_{y_k/2}^{2y_k} \left| \frac{1}{n} D(y - y_k) \right| \mathrm{d}x\mathrm{d}y \\ &\le \frac{C}{n\epsilon^2} y_k^{-2d_2-1} \frac{C \log k}{n} = \frac{C}{n} \frac{\log k}{k} y_k^{-2d_2} = o(\epsilon_{jk}). \end{aligned}$$

Hence, $| \int_\epsilon^\pi \int_{y_k/2}^{2y_k} | = o(\epsilon_{jk})$.

$$\begin{aligned} \left| \int_\epsilon^\pi \int_{-y_k/2}^{y_k/2} \right| &\le C \int_\epsilon^\pi \int_{-y_k/2}^{y_k/2} |h_1(x)h_2(y) + h_1(x_j)h_2(y_k)|F(x - x_j)F(y - y_k)\mathrm{d}x\mathrm{d}y \\ &\le C \int_\epsilon^\pi F(x - x_j)\mathrm{d}x \int_{-y_k/2}^{y_k/2} h_2(y)F(y - y_k)\mathrm{d}y + Cx_j^{-2d_1} y_k^{-2d_2} \int_\epsilon^\pi F(x - x_j)\mathrm{d}x \int_{-y_k/2}^{y_k/2} F(y - y_k)\mathrm{d}y \\ &\le \frac{C}{n\epsilon^2} \int_{-y_k/2}^{y_k/2} y^{-2d_2} F(y - y_k)\mathrm{d}y + \frac{C}{n\epsilon^2} x_j^{-2d_1} y_k^{-2d_2} \int_{-y_k/2}^{y_k/2} F(y - y_k)\mathrm{d}y \\ &\le \frac{C}{n\epsilon^2} \left(y_k^{-2d_2+1} \max_{|y| \le y_k/2} F(y - y_k) + x_j^{-2d_1} y_k^{-2d_2} \frac{C}{n} \int_0^{y_k/2} y_k^{-2}\mathrm{d}y \right) \\ &\le \frac{C}{n\epsilon^2} \left(y_k^{-2d_2+1} \frac{1}{ny_k^2} + x_j^{-2d_1} y_k^{-2d_2} \frac{1}{ny_k} \right) = o\left(\epsilon_{jk}\right) \end{aligned}$$

by applying (A.8). The proofs of terms $\int_\epsilon^\pi \int_{-2y_k}^{-y_k/2}$, $\int_\epsilon^\pi \int_{-\epsilon}^{-2y_k}$, and $\int_\epsilon^\pi \int_{-\pi}^{-\epsilon}$ are easier than the above three terms.

Now we prove those terms when $x \in [2x_j, \epsilon]$. The term $\int_{2x_j}^\epsilon \int_\epsilon^\pi$ can be handled in a similar way as in $\int_\epsilon^\pi \int_{2y_k}^\epsilon$.

$$\begin{aligned} \left| \int_{2x_j}^\epsilon \int_{2y_k}^\epsilon \right| &\le C \max_{2x_j \le x \le \epsilon} \left| \frac{h_1(x)}{x^{(1-2d_1)/2}} \right| \int_{2x_j}^\epsilon x^{(1-2d_1)/2} F(x - x_j)\mathrm{d}x \max_{2y_k \le y \le \epsilon} \left| \frac{h_2(y)}{y^{(1-2d_2)/2}} \right| \int_{2y_k}^\epsilon y^{(1-2d_2)/2} F(y - y_k)\mathrm{d}y \\ &\quad + Cx_j^{-2d_2} y_k^{-2d_2} \int_{2x_j}^\pi \int_{2y_k}^\pi F(x - x_j)F(y - y_k)\mathrm{d}x\mathrm{d}y \end{aligned}$$

$$\leq \frac{C}{n^2} x_j^{-1-2d_1} y_k^{-1-2d_2} = o(\epsilon_{jk}).$$

Applying (A.10),

$$\begin{aligned}\left|\int_{2x_j}^{\epsilon}\int_{y_k/2}^{2y_k}\right| &\leq C\int_{2x_j}^{\epsilon}\int_{y_k/2}^{2y_k}|f_1(\tilde{x}, y_k)(x-x_j)|F(x-x_j)F(y-y_k)\mathrm{d}x\mathrm{d}y \\ &\quad + C\int_{2x_j}^{\epsilon}\int_{y_k/2}^{2y_k}|f_2(x_j, \tilde{y})(y-y_k)|F(x-x_j)F(y-y_k)\mathrm{d}x\mathrm{d}y \\ &=: I + II.\end{aligned}$$

Then

$$\begin{aligned} I &\leq C\int_{2x_j}^{\epsilon} x^{-2d_1-1}|x-x_j|F(x-x_j)\mathrm{d}x\int_{y_k/2}^{2y_k} h_2(y_k)F(y-y_k)\mathrm{d}y \\ &\leq \frac{C}{n}x_j^{-2d_1-1}(\log j)y_k^{-2d_2}\int_{y_k/2}^{2y_k}F(y-y_k)\mathrm{d}y = O\left(\frac{\log j}{j}x_j^{-2d_1}y_k^{-2d_2}\right),\end{aligned}$$

by (A.9). Consider the second term II.

$$\begin{aligned} II &\leq \int_{2x_j}^{\epsilon}\int_{y_k/2}^{2y_k}|f_2'(x_j, \tilde{y})(y-y_k)F(x-x_j)F(y-y_k)|\mathrm{d}x\mathrm{d}y \\ &\leq Cx_j^{-2d_1}\int_{2x_j}^{\epsilon}\frac{1}{n|x-x_j|^2}\mathrm{d}x y_k^{-2d_2-1}\int_{y_k/2}^{2y_k}|y-y_k|F(y-y_k)\mathrm{d}y \\ &\leq C\frac{x_j^{-2d_1}}{j}y_k^{-2d_2-1}\int_{y_k/2}^{2y_k}|y-y_k|F(y-y_k)\mathrm{d}y \\ &= C\frac{x_j^{-2d_1}}{j}\frac{y_k^{-2d_2}\log k}{k} = O(\epsilon_{jk})\end{aligned}$$

by (A.8) and (A.13). A similar argument implies

$$\begin{aligned}\left|\int_{x_j/2}^{2x_j}\int_{y_k/2}^{2y_k}\right| &\leq \int_{x_j/2}^{2x_j}\int_{y_k/2}^{2y_k}|f_1'(\tilde{x}, y_k)(x-x_j)|F(x-x_j)F(y-y_k)\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{x_j/2}^{2x_j}\int_{y_k/2}^{2y_k}|f_2'(x_j, \tilde{y})(y-y_k)|F(x-x_j)F(y-y_k)\mathrm{d}x\mathrm{d}y \\ &\leq C\frac{x_j^{-2d_1}\log j}{j}y_k^{-2d_2} + Cx_j^{-2d_1}\frac{y_k^{-2d_2}\log k}{k}.\end{aligned}$$

$$\begin{aligned}\left|\int_{x_j/2}^{2x_j}\int_{-y_k/2}^{y_k/2}\right| &\leq \int_{x_j/2}^{2x_j}\int_{-y_k/2}^{y_k/2}\left|f_1'(\tilde{x}, y)(x-x_j) + f_2'(x_j, \tilde{y})(y-y_k)\right|F(x-x_j)F(y-y_k)\mathrm{d}x\mathrm{d}y \\ &\leq C\int_{x_j/2}^{2x_j}x_j^{-2d_1-1}(x-x_j)F(x-x_j)\mathrm{d}x\int_{-y_k/2}^{y_k/2}y^{-2d_2}F(y-y_k)\mathrm{d}y \\ &\quad + C\int_{x_j/2}^{2x_j}x_j^{-2d_1}F(x-x_j)\mathrm{d}x\int_{-y_k/2}^{y_k/2}y_k^{-2d_2-1}(y-y_k)F(y-y_k)\mathrm{d}y \\ &\leq \frac{Cx_j^{-2d_1}\log j}{j}\max_{|y|\leq y_k/2}F(y-y_k)\int_{-y_k/2}^{y_k/2}y^{-2d_2}\mathrm{d}y + Cx_j^{-2d_1}\frac{y_k^{-2d_2}\log k}{k}\int_{x_j/2}^{2x_j}F(x-x_j)\mathrm{d}x \\ &\leq \frac{Cx_j^{-2d_1}\log j}{j}\frac{y_k^{-2d_2}}{k} + Cx_j^{-2d_1}\frac{y_k^{-2d_2}\log k}{k} = O(\epsilon_{jk}).\end{aligned}$$

Hence (A.1) follows.

To prove (A.2), we note that the left hand side of (A.2) equals

$$\frac{1}{4\pi^2n^2}\sum_{s,t=1}^{n}\sum_{u,v=1}^{n}EX_{st}X_{uv}\mathrm{e}^{\mathrm{i}[(s+u)x_j+(t+v)y_k]} = \frac{1}{4\pi^2n^2}\sum_{s,t=1}^{n}\sum_{u,v=1}^{n}\int_{\Pi^2}f(x,y)\mathrm{e}^{-\mathrm{i}[(s-u)x+(t-v)y]}\mathrm{d}x\mathrm{d}y\ \mathrm{e}^{\mathrm{i}[(s+u)x_j+(t+v)y_k]}$$

$$= \frac{1}{4\pi^2 n^2}\int_{\Pi^2} f(x,y)\sum_{s,t=1}^{n}\sum_{u,v=1}^{n} e^{i[s(x_j-x)+u(x_j+x)+t(y_k-y)+v(y+y_k)]}dxdy$$

$$= \frac{1}{4\pi^2}\int_{\Pi^2}\{f(x,y)-f(x_j,y_k)\}\frac{1}{n^2}D(x_j-x)D(x_j+x)D(y_k-y)D(y+y_k)dxdy,$$

since

$$\int_{\Pi} D(x_j-x)D(-x_k+x) = \sum_{s=1}^{n}\sum_{u=1}^{n} e^{i(sx_j+ux_k)}\int_{\Pi} e^{ix(u-s)}dx = \sum_{s=1}^{n} e^{i(2sx_j)} \equiv 0, \tag{A.14}$$

by the fact that $\sum_{s=1}^{n} e^{i(x_j-x_k)s} \equiv 0$ for $j \neq k$ (page 322, [10]). Decompose and argue as in the proof of (A.1), we can obtain the result.

Now we prove (A.3).

$$E[\omega(x_j,y_k)\bar{\omega}(x_l,y_h)] = \frac{1}{4\pi^2 n^2}\sum_{s,t}\sum_{u,v}E(X_{s,t}X_{u,v})e^{i(sx_j+ty_k)-i(ux_l+vy_h)}$$
$$= \int_{\Pi^2} f(x,y)\frac{1}{4\pi^2 n^2}\sum_{s,t}\sum_{u,v} e^{i\{s(x_j-x)+t(y_k-y)+u(x-x_l)+v(y-y_h)\}}dxdy$$
$$= \int_{\Pi^2} f(x,y)E_{jl}(x)E_{kh}(y)dxdy,$$

where

$$E_{jl}(x) = \frac{1}{2\pi n}D(x_j-x)D(x-x_l).$$

We write $\boldsymbol{E} = E_{jl}(x)E_{kh}(y)$ for short, and also we assume $j < l$ and $k < h$ for simplicity. Using (A.14), we decompose the integral with respect to x into the following four parts as follow:

$$\int_{\Pi} dy \left\{ \int_{\frac{x_j+x_l}{2}}^{2x_l}[f(x,y)-f(x_l,y)]\boldsymbol{E}dx + \int_{x_j/2}^{\frac{x_j+x_l}{2}}[f(x,y)-f(x_j,y)]\boldsymbol{E}dx \right.$$
$$\left. - [f(x_l,y)-f(x_j,y)]\int_{x_j/2}^{\frac{x_j+x_l}{2}}\boldsymbol{E}dx + \left(\int_{2x_l}^{\pi}+\int_{-\pi}^{x_j/2}\right)[f(x,y)-f(x_l,y)]\boldsymbol{E}dx \right\}. \tag{A.15}$$

Again, by applying the same rule, we decompose the integral with respect to y in (A.15) into four parts, the resulting first term is

$$\int_{(y_k+y_h)/2}^{2y_h}\int_{\frac{x_j+x_l}{2}}^{2x_l}[f(x,y)-f(x_l,y)-f(x,y_h)+f(x_l,y_h)]\boldsymbol{E}dxdy$$
$$+ \int_{(y_k+y_h)/2}^{2y_h}\int_{x_j/2}^{(x_j+x_l)/2}[f(x,y)-f(x_j,y)-f(x,y_h)+f(x_j,y_h)]\boldsymbol{E}dxdy$$
$$+ \int_{(y_k+y_h)/2}^{2y_h}\int_{x_j/2}^{(x_j+x_l)/2}[f(x_l,y)-f(x_j,y)-f(x_l,y_h)+f(x_j,y_h)]\boldsymbol{E}dxdy$$
$$+ \int_{(y_k+y_h)/2}^{2y_h}\left(\int_{2x_l}^{\pi}+\int_{-\pi}^{x_j/2}\right)[f(x,y)-f(x_l,y)-f(x,y_h)+f(x_l,y_h)]\boldsymbol{E}dxdy; \tag{A.16}$$

the second one is

$$\int_{y_k/2}^{(y_k+y_h)/2}\int_{\frac{x_j+x_l}{2}}^{2x_l}[f(x,y)-f(x_l,y)-f(x,y_k)+f(x_l,y_k)]\boldsymbol{E}dxdy$$
$$+ \int_{y_k/2}^{(y_k+y_h)/2}\int_{x_j/2}^{(x_j+x_l)/2}[f(x,y)-f(x_j,y)-f(x,y_k)+f(x_j,y_k)]\boldsymbol{E}dxdy$$

$$+\int_{y_k/2}^{(y_k+y_h)/2}\int_{x_j/2}^{(x_j+x_l)/2}[f(x_l,y)-f(x_j,y)-f(x_l,y_k)+f(x_l,y_k)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y$$
$$+\int_{y_k/2}^{(y_k+y_h)/2}\left(\int_{2x_l}^{\pi}+\int_{-\pi}^{x_j/2}\right)[f(x,y)-f(x_l,y)-f(x,y_k)+f(x_l,y_k)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y; \quad (A.17)$$

the third one is the product of (-1) and

$$\int_{y_k/2}^{(y_k+y_h)/2}\int_{\frac{x_j+x_l}{2}}^{2x_l}[f(x,y_h)-f(x_l,y_h)-f(x,y_k)+f(x_l,y_k)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y$$
$$+\int_{y_k/2}^{(y_k+y_h)/2}\int_{x_j/2}^{(x_j+x_l)/2}[f(x,y_h)-f(x_j,y_h)-f(x,y_k)+f(x_j,y_k)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y$$
$$+\int_{y_k/2}^{(y_k+y_h)/2}\int_{x_j/2}^{(x_j+x_l)/2}[f(x_l,y_h)-f(x_j,y_h)-f(x_l,y_k)+f(x_j,y_k)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y$$
$$+\int_{y_k/2}^{(y_k+y_h)/2}\left(\int_{2x_l}^{\pi}+\int_{-\pi}^{x_j/2}\right)[f(x,y_h)-f(x_l,y_h)-f(x,y_k)+f(x_l,y_k)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y; \quad (A.18)$$

and the last term is

$$\left(\int_{2y_h}^{\pi}+\int_{-\pi}^{y_k/2}\right)\int_{\frac{x_j+x_l}{2}}^{2x_l}[f(x,y)-f(x_l,y)-f(x,y_h)+f(x_l,y_h)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y$$
$$+\left(\int_{2y_h}^{\pi}+\int_{-\pi}^{y_k/2}\right)\int_{x_j/2}^{(x_j+x_l)/2}[f(x,y)-f(x_j,y)-f(x,y_h)+f(x_j,y_h)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y$$
$$+\left(\int_{2y_h}^{\pi}+\int_{-\pi}^{y_k/2}\right)\int_{x_j/2}^{(x_j+x_l)/2}[f(x_l,y)-f(x_j,y)-f(x_l,y_h)+f(x_l,y_h)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y$$
$$+\left(\int_{2y_h}^{\pi}+\int_{-\pi}^{y_k/2}\right)\left(\int_{2x_l}^{\pi}+\int_{-\pi}^{x_j/2}\right)[f(x,y)-f(x_l,y)-f(x,y_h)+f(x_l,y_h)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y. \quad (A.19)$$

We choose ϵ and n as in the proof of (A.1). Also, because of **A1**, the spectral density function $f(x,y)$ is symmetric with respect to x and y near origin, we only calculate typical terms in (A.15). First, we consider each term in (A.16). By the Mean Value Theorem, **A1**, **A2**, and (A.8), (A.9) and (A.13),

$$\left|\int_{(y_k+y_h)/2}^{2y_h}\int_{\frac{x_j+x_l}{2}}^{2x_l}[f(x,y)-f(x_l,y)-f(x,y_h)+f(x_l,y_h)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y\right|$$
$$\leq \max_{\substack{(x_j+x_l)/2\leq x\leq 2x_l\\(y_k+y_h)/2\leq y\leq 2y_h}}|f_1'(x,y)|\int_{(y_k+y_l)/2}^{2y_h}E_{kh}(y)\mathrm{d}y\int_{(x_j+x_l)/2}^{2x_l}\frac{|D(x-x_j)|}{n}\mathrm{d}x$$
$$\leq Cx_l^{-2d_1-1}y_h^{-2d_2}\frac{\log l}{n}=O(\epsilon_{jlkh}). \quad (A.20)$$

And similarly, the second and the third terms in (A.16) are $O(\epsilon_{jlkh})$. To calculate the fourth term in (A.16), we choose an ϵ as in the proof of (A.1),

$$\int_{(y_k+y_h)/2}^{2y_h}\int_{2x_l}^{\pi}[f(x,y)-f(x_l,y)-f(x,y_h)+f(x_l,y_h)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y$$
$$=\int_{(y_k+y_h)/2}^{2y_h}\left\{\int_{\epsilon}^{\pi}+\int_{2x_l}^{\epsilon}\right\}[f(x,y)-f(x_l,y)-f(x,y_h)+f(x_l,y_h)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y.$$

The component $|\int_{(y_k+y_h)/2}^{2y_h}\int_{2x_l}^{\epsilon}[f(x,y)-f(x_l,y)-f(x,y_h)+f(x_l,y_h)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y|=O(\epsilon_{jlkh})$ can be obtained as before. Using the Mean Value theorem, **A1** and **A2**,

$$\left|\int_{(y_k+y_h)/2}^{2y_h}\int_{\epsilon}^{\pi}[f(x,y)-f(x_l,y)-f(x,y_h)+f(x_l,y_h)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y\right|$$
$$\leq\int_{\epsilon}^{\pi}E_{jl}(x)\mathrm{d}x\left\{\left|\int_{(y_k+y_h)/2}^{2y_h}(f(x,y)-f(x,y_h))E_{kh}(y)\mathrm{d}y\right|+\left|\int_{(y_k+y_h)/2}^{2y_h}(f(x_l,y)-f(x_l,y_h))E_{kh}(y)\mathrm{d}y\right|\right\}$$

$$\leq \int_{\epsilon}^{\pi} E_{jl}(x)\mathrm{d}x \left\{ \left| \int_{(y_k+y_h)/2}^{2y_h} f_2'(x,\tilde{y})(y-y_h)E_{kh}(y)\mathrm{d}y \right| + \left| \int_{(y_k+y_h)/2}^{2y_h} f_2'(x_l,\tilde{y}')(y-y_h)E_{kh}(y)\mathrm{d}y \right| \right\} \tag{A.21}$$

where $\tilde{y}, \tilde{y}'$ are between y and y_h. By **A2**, (A.8), (A.11) and (A.13), the above quantity is bounded by

$$\frac{C}{n^2\epsilon^2} \max_{(y_k+y_h)/2\leq y\leq 2y_h} y^{-2d_2-1} \left| \int_{(y_k+y_h)/2}^{2y_h} (y-y_h)D(y-y_h)D(y_k-y)\mathrm{d}y \right| \leq \frac{C}{n\epsilon^2 y_h^{2d_2}} \frac{\log h}{h} \leq \frac{C}{n}\frac{\log h}{k}\frac{1}{y_k^{d_2}y_h^{d_2}} = o(\epsilon_{jlkh}).$$

To calculate the last term in (A.16) due to the integral on $[x_j/2, -\pi]$,

$$\begin{aligned}
&\int_{(y_k+y_h)/2}^{2y_h}\int_{-\pi}^{x_j/2} [f(x,y)-f(x_l,y)-f(x,y_h)+f(x_l,y_h)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y \\
&= \int_{(y_k+y_h)/2}^{2y_h} \left\{ \int_{-x_j/2}^{x_j/2} + \int_{-\epsilon}^{-x_j/2} + \int_{-\pi}^{-\epsilon} \right\} [f(x,y)-f(x_l,y)-f(x,y_h)+f(x_l,y_h)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y \\
&= I + II + III.
\end{aligned}$$

The last two terms II and III are $O(\epsilon_{jlkh})$ by similar arguments to (A.20) and (A.21), respectively. Again by the Mean Value Theorem, **A2**, (A.8), (A.9) and (A.13),

$$\begin{aligned}
&\left| \int_{(y_k+y_h)/2}^{2y_h}\int_{-x_j/2}^{x_j/2} [f(x,y)-f(x_l,y)-f(x,y_h)+f(x_l,y_h)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y \right| \\
&\leq \left| \int_{(y_k+y_h)/2}^{2y_h} E_{kh}(y)\mathrm{d}y \left\{ \int_{-x_j/2}^{x_j/2} f_2'(x,\tilde{y})(y-y_h)E_{jl}(x)\mathrm{d}x + \int_{-x_j/2}^{x_j/2} f_2'(x,\tilde{y})(y-y_h)E_{jl}(x)\mathrm{d}x \right\} \right| \\
&\leq C\left(\max_{(y_k+y_h)/2\leq y\leq y_h} y^{-2d_2-1} \left| \int_{(y_k+y_h)/2}^{2y_h} E_{kh}(y)(y-y_h)\mathrm{d}y \right| \right) \max_{|x|\leq x_j/2} \frac{1}{n}|D(x_j-x)D(x-x_l)| \int_{-x_j/2}^{x_j/2} x^{-2d_1}\mathrm{d}x \\
&\leq C\left(y_h^{-2d_2-1}\frac{\log h}{n} \right)\left(x_j^{-2d_1+1}\frac{1}{nx_jx_l} \right) \\
&\leq \frac{C}{l}x_j^{-2d_1}\frac{1}{h}y_h^{-2d_2} \\
&\leq C\epsilon_{jlkh}\left(\frac{y_k}{y_h}\right)^{d_2}\frac{k}{h} = O(\epsilon_{jlkh}).
\end{aligned}$$

This proves that (A.16) $= O(\epsilon_{jlkh})$.

For (A.16)–(A.19), the proof is similar. We only show calculations of some typical terms not covered above.

$$\begin{aligned}
&\left| \int_{y_k/2}^{(y_k+y_h)/2}\int_{\epsilon}^{\pi} [f(x,y)-f(x_l,y)-f(x,y_k)+f(x_l,y_k)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y \right| \\
&\leq \int_{\epsilon}^{\pi} E_{jl}(x)\mathrm{d}x \left\{ \left| \int_{y_k/2}^{(y_k+y_h)/2} f_2'(x,\tilde{y})(y-y_k)E_{kh}(y)\mathrm{d}y \right| + \left| \int_{y_k/2}^{(y_k+y_h)/2} f_2'(x_l,\tilde{y}')(y-y_k)E_{kh}(y)\mathrm{d}y \right| \right\} \\
&\leq \frac{C}{n^2\epsilon^2} \max_{y_k/2\leq y\leq (y_k+y_h)/2} y^{-2d_2-1} \left| \int_{(y_k+y_h)/2}^{2y_h} (y-y_k)D(y-y_h)D(y_k-y)\mathrm{d}y \right| \\
&\leq \frac{C}{n\epsilon^2 y_k^{2d_2}}\frac{\log h}{k} = o(\epsilon_{jlkh}),
\end{aligned}$$

since

$$\frac{1}{n\epsilon^2 y_k^{2d_2}}\frac{\log h}{k}/\epsilon_{jlkh} = \frac{(\log h)y_k^{-d_2}y_h^{d_2}x_j^{d_1}x_l^{d_1}}{nk(\log l/j + \log h/k)} \to 0.$$

Also

$$\begin{aligned}
&\left| \int_{y_k/2}^{(y_k+y_h)/2}\int_{\epsilon}^{\pi} [f(x,y_h)-f(x_l,y_h)-f(x,y_k)+f(x_l,y_k)]\boldsymbol{E}\mathrm{d}x\mathrm{d}y \right| \\
&\leq \int_{y_k/2}^{(y_k+y_h)/2}\int_{\epsilon}^{\pi} |E_{jl}(x)|\{|f_2'(x,\tilde{y})| + |f_2'(x_l,\tilde{y}')|\}|(y_l-y_k)E_{kh}(y)|\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$\le \frac{C}{n\epsilon^2} \max_{y_k/2 \le y \le (y_k+y_h)/2} y^{-2d_2-1} \int_{y_k/2}^{(y_k+y_h)/2} |y_h - y_k| \frac{1}{n|y_h - y|} |D(y_k - y)| \mathrm{d}y$$
$$\le \frac{C}{n\epsilon^2} y_k^{-2d_2-1} \frac{\log h}{n} \le \frac{C}{n\epsilon^2} \frac{\log h}{k} y_k^{-2d_2} = o(\epsilon_{jlkh}).$$

It is easy to see that from (A.8) that

$$\left| \int_{\epsilon}^{\pi} \int_{\epsilon}^{\pi} [f(x, y_h) - f(x_l, y_h) - f(x, y_k) + f(x_l, y_k)] \boldsymbol{E} \mathrm{d}x\mathrm{d}y \right| = O(n^{-2}) = o(e_{jlkh}).$$

Summarise the above arguments, we obtain (A.3). We skip the proof of (A.4), which is similar to and easier than that of (A.3).

Corollary A.2. *Under* **AA1** *and* **AA2** *, uniformly for* $1 \le i, j \le m$ *with* $m/n \to 0$,

$$\int_{\Pi^2} \left| \frac{\tau(x, y)}{\alpha_{ij}} - 1 \right|^2 K(x - x_i, y - y_j) \mathrm{d}x\mathrm{d}y = O\left(\frac{1}{i} + \frac{1}{j}\right).$$

Proof. This corollary is a generalisation of Lemma 3 in [24]. It can be proved by similar arguments as in Lemma A.1. The new technique here is, when $x \notin [x_i/2, 2x_i]$ or $y \notin [y_j/2, 2y_j]$, we use the fact

$$|\tau(x, y) - \alpha_{ij}|^2 \le 2\{f(x, y) + f(x_i, y_j)\}$$

to proceed. □

Remark A.3. (A.3) and (A.4) also hold when $j < l$, $k = h$ or $j = l$, $k < h$. For example, we can write

$$E[\omega(x_j, y_k)\bar{\omega}(x_l, y_k)] = \frac{1}{4\pi^2} \int_{\Pi^2} f(x, y) E_{jl}(x) F(y - y_k) \mathrm{d}x\mathrm{d}y. \tag{A.22}$$

Using (A.14), decompose (A.22) with respect to x into four parts as in (A.15):

$$\int_{\Pi} \mathrm{d}y \left\{ \int_{\frac{x_j+x_l}{2}}^{2x_l} [f(x, y) - f(x_l, y)] E_{jl}(x) F(y - y_k) \mathrm{d}x + \int_{x_j/2}^{\frac{x_j+x_l}{2}} [f(x, y) - f(x_j, y)] E_{jl}(x) F(y - y_k) \mathrm{d}x \right.$$
$$\left. - [f(x_l, y) - f(x_j, y)] \int_{x_j/2}^{\frac{x_j+x_l}{2}} E_{jl}(x) F(y - y_k) \mathrm{d}x + \left(\int_{2x_l}^{\pi} + \int_{-\pi}^{x_j/2} \right) [f(x, y) - f(x_l, y)] E_{jl}(x) F(y - y_k) \mathrm{d}x \right\}.$$

Also consider the partition of the range of y as in (A.7). Similar techniques in the proof of Lemma A.1 then apply for each partition.

The following results are used in the proof of consistency of $\hat{\boldsymbol{H}}$. Note that

$$\frac{I_{ij}}{g_{ij}} - 1 = \left(1 - \frac{g_{ij}}{f_{ij}}\right) \frac{I_{ij}}{g_{ij}} + \frac{1}{f_{ij}} (I_{ij} - |\alpha_{ij}|^2 I_{ij\varepsilon}) + (4\pi^2 I_{ij\varepsilon} - 1),$$

where $\alpha_{ij} = \tau(x_i, y_j)$ and $\tau(x, y)$ is defined in (1.5). For any $\eta > 0$,

$$\left| 1 - \frac{g_{ij}}{f_{ij}} \right| \le \eta, \quad i, j = 1, \ldots, m. \tag{A.23}$$

and

$$E \left| \frac{I_{ij}}{g_{ij}} \right| \le C, \quad i, j = 1, \ldots, m. \tag{A.24}$$

by Lemma A.1 and (1.6). By the Cauchy–Schwarz inequality,

$$E|\omega_{ij} - \alpha_{ij}\omega_{ij\varepsilon}| \cdot |\omega_{ij} + \alpha_{ij}\omega_{ij\varepsilon}| \le \left(E[I_{ij}] - \alpha_{ij} E[\omega_{ij\varepsilon}\bar{\omega}_{ij}] - \bar{\alpha}_{ij} E[\bar{\omega}_{ij\varepsilon}\omega_{ij}] + |\alpha_{ij}|^2 E[I_{ij\varepsilon}]\right)^{1/2}$$
$$\times \left(E[I_{ij}] + \alpha_{ij} E[\omega_{ij\varepsilon}\bar{\omega}_{ij}] + \bar{\alpha}_{ij} E[\bar{\omega}_{ij\varepsilon}\omega_{ij}] + |\alpha_{ij}|^2 E[I_{ij\varepsilon}]\right)^{1/2}. \tag{A.25}$$

From Lemma A.1,

$$E[I_{ij}] = f_{ij} \left(1 + O\left(\frac{\log i}{i} + \frac{\log j}{j}\right)\right).$$

Also note that $E[I_{ij\varepsilon}] = (2\pi)^{-2}$ and

$$E[\omega_{ij}\bar{\omega}_{ij\varepsilon}] = \frac{\alpha_{ij}}{4\pi^2}\left(1 + O\left(\frac{\log i}{i} + \frac{\log j}{j}\right)\right), \tag{A.26}$$

which can be shown in a similar way to prove Lemma A.1.

Thus, for $i, j = 1, \ldots, m$,

$$\frac{1}{f_{ij}}(I_{ij} - |\alpha_{ij}|^2 I_{ij\varepsilon}) = O_P\left(\sqrt{\frac{\log i}{i} + \frac{\log j}{j}}\right). \tag{A.27}$$

For the third term, $E[4\pi^2 I_{ij\varepsilon} - 1] = 0$, and

$$4\pi^2 I_{ij\varepsilon} - 1 = \frac{1}{n^2}\sum_{s,t=1}^{n}(\varepsilon_{s,t}^2 - 1) + \frac{1}{n^2}\sum\sum_{(s,t)\neq(h,l)} \varepsilon_{s,t}\varepsilon_{h,l} e^{i(s-h)x_i + (t-l)y_j}. \tag{A.28}$$

Because of i.i.d. of $\varepsilon_{s,t}$,

$$\frac{1}{n^2}\sum_{s,t=1}^{n}(\varepsilon_{s,t}^2 - 1) \to_p 0.$$

The expected value of the square of the second term in (A.28) is bounded by

$$\frac{C}{n^4}\left|\sum\sum_{(s,t)\neq(h,l)} e^{2i[(s-h)x_i + (t-l)y_j]}\right| \le C\left(\frac{1}{n^2}\left|\sum_{s,h} e^{2i(s-h)x_i}\right|\right)\left(\frac{1}{n^2}\left|\sum_{t,l} e^{2i(t-l)y_j}\right|\right) \le \frac{C}{i^2 j^2},$$

by the inequality of Fejer's kernel; e.g., see Zygmund [31, page 90]. Therefore,

$$4\pi^2 I_{ij\varepsilon} - 1 \to_p 0. \tag{A.29}$$

The following lemmas are needed for the proof of Theorem 4.1 regarding the asymptotic distribution of the local Whittle estimator.

Lemma A.4. *Under the assumption of Theorem 4.1, as $n \to \infty$*

$$\sum_{i=1}^{q}\sum_{j=1}^{r}\left(\frac{I_{ij}}{g_{ij}} - 4\pi^2 I_{ij\varepsilon}\right) = O_p\left(\rho^{5/4}\log^{1/2}\rho + \rho^2\left(\left(\frac{\rho}{n}\right)^{\beta_1} + \left(\frac{\rho}{n}\right)^{\beta_2}\right)\right),$$

for $1 \le q, r \le m$, and $\rho = \max\{q, r\}$.

Proof. Assume $q \ge r$. Choose an $l < r$. By (A.24), and the fact $E(4\pi^2 I_{ij\varepsilon}) = 1$, we have

$$E\left|\sum_{i,j=1}^{l}\left(\frac{I_{ij}}{g_{ij}} - 4\pi^2 I_{ij\varepsilon}\right)\right| \le Cl^2 \quad n \to \infty. \tag{A.30}$$

By **AA1.** and (A.24),

$$E\left|\sum_{i=l+1}^{q}\sum_{j=l+1}^{r}\left(\frac{I_{ij}}{g_{ij}} - \frac{I_{ij}}{f_{ij}}\right)\right| \le C\sum_{i=l+1}^{q}\sum_{j=l+1}^{r}\left|1 - \frac{g_{ij}}{f_{ij}}\right| \le Cq^2\left(\left(\frac{q}{n}\right)^{\beta_1} + \left(\frac{q}{n}\right)^{\beta_2}\right) \tag{A.31}$$

and

$$E\left|\left\{\sum_{i=1}^{l}\sum_{j=l+1}^{r} + \sum_{i=l+1}^{q}\sum_{j=1}^{l}\right\}\left(\frac{I_{ij}}{g_{ij}} - \frac{I_{ij}}{f_{ij}}\right)\right| \le C\left\{\sum_{i=1}^{l}\sum_{j=l+1}^{r} + \sum_{i=l+1}^{q}\sum_{j=1}^{l}\right\}\left|1 - \frac{g_{ij}}{f_{ij}}\right| \le Cq^2\left(\left(\frac{q}{n}\right)^{\beta_1} + \left(\frac{q}{n}\right)^{\beta_2}\right).$$

Let $u_{ij} = 2\pi\omega_{ij}/|\alpha_{ij}|$, $v_{ij} = 2\pi\omega_{ij\varepsilon}$. Now we consider

$$E\left\{\sum_{i=l+1}^{q}\sum_{j=l+1}^{r}\left(\frac{I_{ij}}{f_{ij}} - 4\pi^2 I_{ij\varepsilon}\right)\right\}^2 = \left\{\sum_{i=l+1}^{q}\sum_{j=l+1}^{r}[E|u_{ij}|^4 - 2E|u_{ij}v_{ij}|^2 + E|v_{ij}|^4] + \sum_{(i,j)\neq(k,h)}[E|u_{ij}u_{kh}|^2 - E|u_{ij}v_{kh}|^2 - E|u_{kh}v_{ij}|^2 + E|v_{ij}v_{kh}|^2]\right\} =: \boldsymbol{a} + \boldsymbol{b},$$

where $\boldsymbol{a} = \boldsymbol{a_1} + \boldsymbol{a_2}$, $\boldsymbol{b} = \boldsymbol{b_1} + \boldsymbol{b_2}$,

$$\boldsymbol{a}_1 = \sum_{i=l+1}^{q}\sum_{j=l+1}^{r}\{2(E|u_{ij}|^2)^2 + |Eu_{ij}^2|^2 - 2|Eu_{ij}v_{ij}|^2 - 2|Eu_{ij}\bar{v}_{ij}|^2 - 2E|u_{ij}|^2E|v_{ij}|^2 + 2(E|v_{ij}|^2)^2 + |Ev_{ij}^2|^2\},$$

$$\boldsymbol{a}_2 = \sum_{i=l+1}^{q}\sum_{j=l+1}^{r}\{\mathrm{cum}(u_{ij}, u_{ij}, \bar{u}_{ij}, \bar{u}_{ij}) + \mathrm{cum}(v_{ij}, v_{ij}, \bar{v}_{ij}, \bar{v}_{ij}) - 2\,\mathrm{cum}(u_{ij}, v_{ij}, \bar{u}_{ij}, \bar{v}_{ij})\},$$

and

$$\boldsymbol{b}_1 = 2\left\{\sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j<h}}^{r} + \sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j>h}}^{r} + \sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j=h}}^{r} + \sum_{\substack{i=l+1,\\ i=k}}^{q}\sum_{\substack{j=l+1\\ j<h}}^{r}\right\}\left\{(E|u_{ij}|^2 - 1)(E|u_{kh}|^2 - 1) + |Eu_{ij}u_{kh}|^2 + |Eu_{ij}\bar{u}_{kh}|^2 - |Eu_{ij}v_{kh}|^2 - |Eu_{ij}\bar{v}_{kh}|^2 - |Ev_{ij}u_{kh}|^2 - |Ev_{ij}\bar{u}_{kh}|^2 + |Ev_{ij}v_{kh}|^2 + |Ev_{ij}\bar{v}_{kh}|^2\right\}$$

$$\boldsymbol{b}_2 = 2\left\{\sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j<h}}^{r} + \sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j>h}}^{r} + \sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j=h}}^{r} + \sum_{\substack{i=l+1,\\ i=k}}^{q}\sum_{\substack{j=l+1,\\ j<h}}^{r}\right\}\left\{\mathrm{cum}(u_{ij}, u_{kh}, \bar{u}_{ij}, \bar{u}_{kh}) - \mathrm{cum}(u_{ij}, v_{kh}, \bar{u}_{ij}, \bar{v}_{kh}) - \mathrm{cum}(v_{ij}, u_{kh}, \bar{v}_{ij}, \bar{u}_{kh}) + \mathrm{cum}(v_{ij}, v_{kh}, \bar{v}_{ij}, \bar{v}_{kh})\right\}.$$

Definition of cumulants can be found in [9] (Page 18). We have

$$\begin{aligned}
&E|v_{ij}|^2 = E[I_{ij\varepsilon}]/4\pi^2 = 1\\
&Ev_{ij}^2 = O\left(\frac{1}{ij}\right)\\
&Ev_{ij}v_{kh} = \begin{cases} O\left(\frac{1}{i+k}\frac{1}{j+h}\right), & i\neq k, j\neq h\\ O\left(\frac{1}{i+k}\frac{1}{j}\right), & i\neq k, j=h\\ O\left(\frac{1}{i}\frac{1}{j+h}\right), & i=k, j\neq h\end{cases}\\
&Ev_{ij}\bar{v}_{kh} = \begin{cases} O\left(\frac{1}{|k-i|}\frac{1}{|h-j|}\right), & i\neq k, j\neq h\\ O\left(\frac{1}{|k-i|}\right), & i\neq k, j=h\\ O\left(\frac{1}{|h-j|}\right), & i=k, j\neq h\end{cases}
\end{aligned} \tag{A.32}$$

for $i, j, k, h = 1, \ldots m$ by straightforward calculation using (A.8) as in the proof of Lemma A.1.

As in (A.26), by an argument similar to Lemma A.1 but for the cross-spectral density, we can show that

$$\begin{aligned}|Eu_{ij}\bar{v}_{ij}| &= 4\pi^2\left|E\frac{\omega(x_i, y_j)}{|\alpha_{ij}|}\bar{\omega}_\varepsilon(x_i, y_j)\right|\\ &= 1 + O\left(\frac{\log i}{i} + \frac{\log j}{j}\right),\end{aligned} \tag{A.33}$$

$$
\begin{aligned}
|Eu_{ij}v_{ij}| &= O\left(\frac{\log i}{i} + \frac{\log j}{j}\right), \\
|Eu_{ij}v_{kh}| &= O\left(\frac{\log k}{i} + \frac{\log h}{j}\right), \\
|Eu_{ij}\bar{v}_{kh}| &= O\left(\frac{\log k}{i} + \frac{\log h}{j}\right),
\end{aligned} \tag{A.34}
$$

when $i < k \leq m, j < h \leq m$ or one of inequalities is replaced with equality since the cross spectral density between X and ε is $\frac{\tau(x,y)}{4\pi^2}$ and

$$
\begin{aligned}
\left|\frac{\tau(x, y)}{4\pi^2}\right| &= \frac{\sqrt{f(x, y)}}{2\pi} = \frac{\sqrt{g(x, y)h_1(x)h_2(y)}}{2\pi} \\
&\sim x^{-d_1}y^{-d_2} \quad \text{for } x, y \to 0.
\end{aligned}
$$

Thus, together with Lemma A.1 applied to u_{ij}, we have

$$
\begin{aligned}
\boldsymbol{a}_1 &= \sum_{i=l+1}^{q}\sum_{j=l+1}^{r}\{2(E|u_{ij}|^2 - 1)^2 + 2(E|u_{ij}|^2 - 1) + |Eu_{ij}^2|^2 - 2|Eu_{ij}v_{ij}|^2 \\
&\quad - 2|Eu_{ij}\bar{v}_{ij} - 1|^2 - 2(E\bar{v}_{ij}u_{ij} - 1) - 2(E\bar{u}_{ij}v_{ij} - 1) + |Ev_{ij}^2|^2\} \\
&\leq C\sum_{i=l+1}^{q}\sum_{j=l+1}^{r}\left\{\frac{\log i}{i} + \frac{\log j}{j}\right\} \leq Cq\log^2 q,
\end{aligned}
$$

where $\log^k q = (\log q)^k$.

The first component of $\boldsymbol{b}_1$ is

$$
\begin{aligned}
&\sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j<h}}^{r}\left\{(E|u_{ij}|^2 - 1)(E|u_{kh}|^2 - 1) + |Eu_{ij}u_{kh}|^2 + |Eu_{ij}\bar{u}_{kh}|^2\right. \\
&\quad \left. - |Eu_{ij}v_{kh}|^2 - |Eu_{ij}\bar{v}_{kh}|^2 - |Ev_{ij}u_{kh}|^2 - |E\bar{v}_{ij}u_{kh}|^2 + |Ev_{ij}v_{kh}|^2 + |Ev_{ij}\bar{v}_{kh}|^2\right\} \\
&= O\left(\sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j<h}}^{r}\left\{\left(\frac{\log i}{i} + \frac{\log j}{j}\right)\left(\frac{\log k}{k} + \frac{\log h}{h}\right) + \left(\frac{\log h}{j} + \frac{\log k}{i}\right)^2 + \left(\frac{1}{i+k}\frac{1}{j+h}\right)^2 + \left(\frac{1}{k-i}\frac{1}{h-j}\right)^2\right\}\right) \\
&= O\left(\frac{q^3\log^2 q}{l} + q^2\log^4 q\right),
\end{aligned}
$$

since, for example,

$$
\begin{aligned}
&\sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j<h}}^{r}\frac{\log i}{i}\frac{\log k}{k} \leq Cr^2\log^2 q\sum_{k=l+2}^{q}\sum_{i=l+1}^{k-1}\frac{1}{ik} \leq C\frac{q^3\log^2 q}{l}, \\
&\sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j<h}}^{r}\frac{\log i}{i}\frac{\log h}{h} \leq Cq^2\log^2 q\sum_{i=l+2}^{q}\sum_{h=l+1}^{r}\frac{1}{i}\frac{1}{h} \leq Cq^2\log^4 q.
\end{aligned}
$$

Similarly, the second component of $\boldsymbol{b}_1$ is

$$
\begin{aligned}
&\sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j>h}}^{r}\left\{(E|u_{ij}|^2 - 1)(E|u_{kh}|^2 - 1) + |Eu_{ij}u_{kh}|^2 + |Eu_{ij}\bar{u}_{kh}|^2\right. \\
&\quad \left. - |Eu_{ij}v_{kh}|^2 - |Eu_{ij}\bar{v}_{kh}|^2 - |Ev_{ij}u_{kh}|^2 - |E\bar{v}_{ij}u_{kh}|^2 + |Ev_{ij}v_{kh}|^2 + |Ev_{ij}\bar{v}_{kh}|^2\right\} \\
&= O\left(\frac{q^3\log^2 q}{l} + q^2\log^4 q\right).
\end{aligned}
$$

Similarly, we can also show that the third and fourth component of $\boldsymbol{b}_1$ is $O\left(q^2\log^2 q\right)$, since, for example,

$$
\sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{j=l+1}^{r}\{(E|u_{ij}|^2 - 1)(E|u_{kj}|^2 - 1) + |Eu_{ij}u_{kj}|^2 + |Eu_{ij}\bar{u}_{kj}|^2
$$

$$- |Eu_{ij}v_{kj}|^2 - |Eu_{ij}\bar{v}_{kj}|^2 - |Ev_{ij}u_{kj}|^2 - |E\bar{v}_{ij}u_{kj}|^2 + |Ev_{ij}v_{kj}|^2 + |Ev_{ij}\bar{v}_{kj}|^2\}$$
$$= O\left(\sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{j=l+1}^{r}\left\{\left(\frac{\log i}{i}+\frac{\log j}{j}\right)\left(\frac{\log k}{k}+\frac{\log j}{j}\right)+\left(\frac{\log j}{j}+\frac{\log k}{i}\right)^2+\left(\frac{1}{i+k}\frac{1}{j}\right)^2+\left(\frac{1}{k-i}\right)^2\right\}\right)$$
$$= O\left(q^2\log^2 q\right).$$

Thus, we have

$$\boldsymbol{b}_1 = O\left(\frac{q^3\log^2 q}{l}+q^2\log^4 q\right).$$

Now, consider the term $\boldsymbol{b}_2$. By generalising (2.6.4) and (2.10.3) of Brillinger [9] to 2-dimensional linear random fields defined as (1.3) based on Section 4 [26] and Section 4 [10], the summand of $\boldsymbol{b}_2$ is

$$\frac{C}{n^4}\int_{\Pi^6}\left\{\frac{\tau(x_1+y_1+z_1,x_2+y_2+z_2)\tau(-y_1,-y_2)}{|\alpha_{ij}|^2}-1\right\}$$
$$\times\left\{\frac{\tau(-x_1,-x_2)\tau(-z_1,-z_2)}{|\alpha_{kh}|^2}-1\right\}D_{ijkh}(x_1,x_2,y_1,y_2,z_1,z_2)\mathrm{d}(x,y,z), \tag{A.35}$$

where $w_l = x_l + y_l + z_l$, for $l = 1, 2$, $\mathrm{d}(x, y, z) = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}z_1\mathrm{d}z_2$,

$$D_{ijkh}(x_1,x_2,y_1,y_2,z_1,z_2) = D(x_i-w_1)D(y_j-w_2)D(x_k+x_1)D(y_h+x_2)$$
$$\times D(-x_i+y_1)D(-y_j+y_2)D(-x_k+z_1)D(-y_h+z_2).$$

We use $\boldsymbol{D}$ to denote this term for short when there is no confusion. After applying the identity (page 1649 of Robinson [24]):

$$(a_1a_2-1)(a_3a_4-1) = \prod_1^4(a_j-1)+\sum_{i=1}^4\prod_{j=1,j\neq i}^4(a_j-1)+\sum\sum_{i,j=1}^2(a_i-1)(a_{j+2}-1),$$

we observe that (A.35) has components of three types. The first component is

$$\frac{C}{n^4}\int_{\Pi^6}\left(\frac{\tau(w_1,w_2)}{\alpha_{ij}}-1\right)\left(\frac{\tau(-y_1,-y_2)}{\bar{\alpha}_{ij}}-1\right)\left(\frac{\tau(-x_1,-x_2)}{\alpha_{kh}}-1\right)\left(\frac{\tau(-z_1,-z_2)}{\bar{\alpha}_{kh}}-1\right)\boldsymbol{D}\mathrm{d}(x,y,z)$$

which is bounded in absolute value by

$$CP_{ij}P_{kh},$$

by Schwarz inequality, where

$$P_{ij} = \int_{\Pi^2}\left|\frac{\tau(x_1,x_2)}{\alpha_{ij}}-1\right|^2\frac{|D(x_i-x_1)D(y_j-x_2)|^2}{n^2}\mathrm{d}x_1\mathrm{d}x_2.$$

The second type of component is

$$\frac{C}{n^4}\int_{\Pi^6}\left(\frac{\tau(x_1,x_2)}{\alpha_{kh}}-1\right)\left(\frac{\tau(z_1,z_2)}{\bar{\alpha}_{kh}}-1\right)\left(\frac{\alpha(y_1,y_2)}{\bar{\alpha}_{ij}}-1\right)\boldsymbol{D}\mathrm{d}(x,y,z)\le CP_{kh}P_{ij}^{1/2}.$$

The third type of component is

$$\frac{C}{n^4}\int_{\Pi^6}\left(\frac{\tau(y_1,y_2)}{\alpha_{ij}}-1\right)\left(\frac{\tau(z_1,z_2)}{\alpha_{kh}}-1\right)\boldsymbol{D}\mathrm{d}(x,y,z)\le\frac{C}{n}P_{ij}^{1/2}P_{kh}^{1/2}.$$

Therefore, by Corollary A.2 and $\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$, $a, b\ge 0$, we have

$$\boldsymbol{b}_2 = O\left(\left\{\sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j<h}}^{r}+\sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j>h}}^{r}+\sum_{\substack{i=l+1,\\ i<k}}^{q}\sum_{\substack{j=l+1,\\ j=h}}^{r}+\sum_{\substack{i=l+1,\\ i=k}}^{q}\sum_{\substack{j=l+1,\\ j<h}}^{r}\right\}\left\{\left(\frac{1}{i}+\frac{1}{j}\right)\left(\frac{1}{k}+\frac{1}{h}\right)\right.\right.$$
$$\left.\left.+\left(\frac{1}{i}+\frac{1}{j}\right)\sqrt{\frac{1}{k}+\frac{1}{h}}+\left(\frac{1}{k}+\frac{1}{h}\right)\sqrt{\frac{1}{i}+\frac{1}{j}}+\frac{1}{n}\sqrt{\left(\frac{1}{i}+\frac{1}{j}\right)\left(\frac{1}{k}+\frac{1}{h}\right)}\right\}\right)$$
$$= O\left(q^2\log^2 q+q^{5/2}\log q+\frac{q^3}{n}\right)$$
$$= O\left(q^{5/2}\log q\right).$$

$\boldsymbol{a}_2 = O(q^{1/2}\log q)$ which can be calculated similarly with $k = i$, $h = j$. We thus have

$$\boldsymbol{a} + \boldsymbol{b} = O\left(q\log^2 q + \frac{q^3\log^2 q}{l} + q^2\log^4 q + q^{5/2}\log q + q^{1/2}\log q\right) = O\left(\frac{q^3\log^2 q}{l} + q^{5/2}\log q\right).$$

Next, we consider

$$E\left\{\sum_{i=1}^{l}\sum_{j=l+1}^{r}\left(\frac{I_{ij}}{f_{ij}} - 4\pi^2 I_{ij\varepsilon}\right)\right\}^2 = \left\{\sum_{i=1}^{l}\sum_{j=l+1}^{r}[E|u_{ij}|^4 - 2E|u_{ij}v_{ij}|^2 + E|v_{ij}|^4] + \sum_{(i,j)\neq(k,h)}[E|u_{ij}u_{kh}|^2 - E|u_{ij}v_{kh}|^2 - E|u_{kh}v_{ij}|^2 + E|v_{ij}v_{kh}|^2]\right\} =: \boldsymbol{c} + \boldsymbol{d},$$

where $\boldsymbol{c} = \boldsymbol{c_1} + \boldsymbol{c_2}$, $\boldsymbol{d} = \boldsymbol{d_1} + \boldsymbol{d_2}$,

$$\boldsymbol{c}_1 = \sum_{i=1}^{l}\sum_{j=l+1}^{r}\{2(E|u_{ij}|^2)^2 + |Eu_{ij}^2|^2 - 2|Eu_{ij}v_{ij}|^2 - 2|Eu_{ij}\bar{v}_{ij}|^2 - 2E|u_{ij}|^2E|v_{ij}|^2 + 2(E|v_{ij}|^2)^2 + |Ev_{ij}^2|^2\},$$

$$\boldsymbol{c}_2 = \sum_{i=1}^{l}\sum_{j=l+1}^{r}\{\mathrm{cum}(u_{ij}, u_{ij}, \bar{u}_{ij}, \bar{u}_{ij}) + \mathrm{cum}(v_{ij}, v_{ij}, \bar{v}_{ij}, \bar{v}_{ij}) - 2\,\mathrm{cum}(u_{ij}, v_{ij}, \bar{u}_{ij}, \bar{v}_{ij})\},$$

and

$$\boldsymbol{d}_1 = 2\left\{\sum_{\substack{i=1,\\ i<k}}^{l}\sum_{\substack{j=l+1,\\ j<h}}^{r} + \sum_{\substack{i=1,\\ i<k}}^{l}\sum_{\substack{j=l+1,\\ j>h}}^{r} + \sum_{\substack{i=1,\\ i<k}}^{l}\sum_{\substack{j=l+1,\\ j=h}}^{r} + \sum_{\substack{i=1,\\ i=k}}^{l}\sum_{\substack{j=l+1\\ j<h}}^{r}\right\}\Big\{(E|u_{ij}|^2 - 1)(E|u_{kh}|^2 - 1) + |Eu_{ij}u_{kh}|^2 + |Eu_{ij}\bar{u}_{kh}|^2 - |Eu_{ij}v_{kh}|^2 - |Eu_{ij}\bar{v}_{kh}|^2 - |Ev_{ij}u_{kh}|^2 - |Ev_{ij}\bar{u}_{kh}|^2 + |Ev_{ij}v_{kh}|^2 + |Ev_{ij}\bar{v}_{kh}|^2\Big\}$$

$$\boldsymbol{d}_2 = 2\left\{\sum_{\substack{i=1,\\ i<k}}^{l}\sum_{\substack{j=l+1,\\ j<h}}^{r} + \sum_{\substack{i=1,\\ i<k}}^{l}\sum_{\substack{j=l+1,\\ j>h}}^{r} + \sum_{\substack{i=1,\\ i<k}}^{l}\sum_{\substack{j=l+1,\\ j=h}}^{r} + \sum_{\substack{i=1,\\ i=k}}^{l}\sum_{\substack{j=l+1,\\ j<h}}^{r}\right\}\Big\{\mathrm{cum}(u_{ij}, u_{kh}, \bar{u}_{ij}, \bar{u}_{kh}) - \mathrm{cum}(u_{ij}, v_{kh}, \bar{u}_{ij}, \bar{v}_{kh}) - \mathrm{cum}(v_{ij}, u_{kh}, \bar{v}_{ij}, \bar{u}_{kh}) + \mathrm{cum}(v_{ij}, v_{kh}, \bar{v}_{ij}, \bar{v}_{kh})\Big\}.$$

In a similar way to get a bound for $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, we have

$$\boldsymbol{c}_1 = O\left(q\log^2 q\right)$$
$$\boldsymbol{c}_2 = O\left(q^{5/2}\log q\right)$$
$$\boldsymbol{d}_1 = O\left(q^2\log^4 q\right)$$
$$\boldsymbol{d}_2 = O\left(q^{5/2}\log q\right)$$
$$\boldsymbol{c} + \boldsymbol{d} = O\left(q^{5/2}\log q\right).$$

The bound of $E\left\{\sum_{i=l+1}^{q}\sum_{j=1}^{l}\left(\frac{I_{ij}}{f_{ij}} - 4\pi^2 I_{ij\varepsilon}\right)\right\}^2$ can be computed similarly.

By setting $l^2 = \sqrt{q^3\log^2 q/l}$ in (A.30), we choose $l = q^{3/5}\log^{2/5} q$, then

$$E\left\{\sum_{i=l+1}^{q}\sum_{j=l+1}^{r}\left(\frac{I_{ij}}{f_{ij}} - 4\pi^2 I_{ij\varepsilon}\right)\right\}^2 + E\left\{\sum_{i=1}^{l}\sum_{j=l+1}^{r}\left(\frac{I_{ij}}{f_{ij}} - 4\pi^2 I_{ij\varepsilon}\right)\right\}^2 + E\left\{\sum_{i=l+1}^{q}\sum_{j=1}^{l}\left(\frac{I_{ij}}{f_{ij}} - 4\pi^2 I_{ij\varepsilon}\right)\right\}^2 = O\left(q^{12/5}\log^{8/5} q + q^{5/2}\log q\right) = O\left(q^{5/2}\log q\right). \tag{A.36}$$

The lemma follows by summarising (A.30), (A.31) and (A.36). □

Lemma A.5. *Under the assumption of Theorem 4.1, as $n \to \infty$*

$$\sum_{i=1}^{q}\sum_{j=1}^{r}\left(4\pi^2 I_{ij\varepsilon}-1\right)=O_p(\rho),$$

for $1 \le q, r \le m$ and $\rho = \max\{q, r\}$.

Proof. Without loss of generality, we assume $r \le q$. Note that

$$\sum_{i=1}^{q}\sum_{j=1}^{r}\left(4\pi^2 I_{ij\varepsilon}-1\right)=\sum_{i=1}^{q}\sum_{j=1}^{r}\frac{1}{n^2}\sum_{s,t=1}^{n}(\varepsilon_{s,t}^2-1)+\frac{1}{n^2}\sum_{(s,t)\neq(u,v)}\varepsilon_{s,t}\varepsilon_{u,v}\sum_{i=1}^{q}\sum_{j=1}^{r}\exp\{\mathrm{i}[(s-u)x_i+(t-v)y_j]\}.$$

The first term has mean zero and variance $O(q^4/n^2)$. The second term also has mean zero, its variance is bounded above by

$$\begin{aligned}
&\frac{C}{n^4}\left\{\sum_{s=1}^{n-1}(n-s)|d_q(s)|^2\sum_{s=1}^{n-1}(n-s)|d_r(s)|^2+nr^2\sum_{s=1}^{n-1}(n-s)|d_q(s)|^2+nq^2\sum_{s=1}^{n-1}(n-s)|d_r(s)|^2\right\}\\
&\le\frac{C}{n^4}\left\{n^2\sum_{s=1}^{n-1}|d_q(s)|^2\sum_{s=1}^{n-1}|d_r(s)|^2+n^2r^2\sum_{s=1}^{n-1}|d_q(s)|^2+n^2q^2\sum_{s=1}^{n-1}|d_r(s)|^2\right\}\\
&\le\frac{C}{n^4}\left\{n^2\left\{\sum_{s=1}^{[n/q]}+\sum_{s>[n/q]}\right\}|d_q(s)|^2\left\{\sum_{s=1}^{[n/r]}+\sum_{s>[n/r]}\right\}|d_r(s)|^2\right.\\
&\quad\left.+n^2r^2\left\{\sum_{s=1}^{[n/q]}+\sum_{s>[n/q]}\right\}|d_q(s)|^2+n^2q^2\left\{\sum_{s=1}^{[n/r]}+\sum_{s>[n/r]}\right\}|d_r(s)|^2\right\}\\
&=O(q^2),
\end{aligned}$$

where $d_q(s)=\sum_{j=1}^{q}\mathrm{e}^{\mathrm{i}sx_j}$ and we have $|d_q(s)| \le Cq$ for $s \le n/q$ and $|d_q(s)| \le Cn/s$ for $s > n/q$. $\square$

Lemma A.6. *Under the assumption of Theorem 4.1,*

$$|F_l^k(\boldsymbol{H}_0)-G_0\frac{1}{m^2}\sum_{i,j=1}^{m}(\log j)^k|=o_p(1);\quad l=1,2,k=0,1,2.$$

$$|F_3(\boldsymbol{H}_0)-\frac{G_0}{m^2}\sum_{i,j}^{m}\log i\log j|=o_p(1).$$

Proof. Using summation by parts,

$$\begin{aligned}
\left|F_1^k(\boldsymbol{H}_0)-\frac{G_0}{m}\sum_{j=1}^{m}(\log j)^k\right| &= \frac{G_0}{m}\left|\sum_{i=1}^{m}(\log i)^k\left(\frac{1}{m}\sum_{j=1}^{m}\frac{I_{ij}}{g_{ij}}-1\right)\right|\\
&\le G_0(\log m)^k\frac{1}{m^2}\left|\sum_{i,j=1}^{m}\left(\frac{I_{ij}}{g_{ij}}-1\right)\right|\\
&\quad+\frac{G_0}{m}\sum_{l=1}^{m-1}\{(\log(l+1))^k-(\log l)^k\}\frac{1}{m}\left|\sum_{i=1}^{l}\sum_{j=1}^{m}\left(\frac{I_{ij}}{g_{ij}}-1\right)\right|\\
&=O_p\left(\frac{(\log m)^k}{m^2}\left\{m^{5/4}(\log m)^{1/2}+m^2\left(\left(\frac{m}{n}\right)^{\beta_1}+\left(\frac{m}{n}\right)^{\beta_2}\right)\right\}\right)\\
&=O_p\left(\frac{(\log m)^{k+1/2}}{m^{3/4}}+(\log m)^k\left(\left(\frac{m}{n}\right)^{\beta_1}+\left(\frac{m}{n}\right)^{\beta_2}\right)\right)
\end{aligned}$$

by Lemmas A.4 and A.5. Similarly, it holds for $l = 2$. Using a double summation by parts (3.7),

$$\begin{aligned}
F_3(\boldsymbol{H}_0)-\frac{G_0}{m^2}\sum_{i,j}^{m}\log i\log j &= \frac{G_0}{m^2}(\log m)^2\sum_{i,j=1}^{m}\left(\frac{I_{ij}}{g_{ij}}-1\right)\\
&\quad+\frac{G_0}{m^2}\log m\sum_{l=1}^{m-1}(\log l-\log(l+1))\left(\sum_{i=1}^{l}\sum_{j=1}^{m}\left(\frac{I_{ij}}{g_{ij}}-1\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{G_0}{m^2} \log m \sum_{l=1}^{m-1} (\log l - \log(l+1)) \left(\sum_{j=1}^{l} \sum_{i=1}^{m} \left(\frac{I_{ij}}{g_{ij}} - 1 \right) \right) \\
&+ \frac{G_0}{m^2} \sum_{l=1}^{m-1} \sum_{h=1}^{m-1} (\log l - \log(l+1))(\log h - \log(h+1)) \left(\sum_{j=1}^{l} \sum_{i=h}^{m} \left(\frac{I_{ij}}{g_{ij}} - 1 \right) \right) \\
&= O_p \left(\frac{(\log m)^{2+1/2}}{m^{3/4}} + (\log m)^2 \left(\left(\frac{m}{n} \right)^{\beta_1} + \left(\frac{m}{n} \right)^{\beta_2} \right) \right)
\end{aligned}
$$

by Lemmas A.4 and A.5 again. □

Lemma A.7. *Under the assumption of Theorem 4.1, for $l = 1, 2$ and $k = 0, 1, 2$, as $n \to \infty$,*

$$E_l^k(\bar{\boldsymbol{H}}) - E_l^k(\boldsymbol{H}_0) = o_p(n^{(2H_{10}-1)+(2H_{20}-1)}(\log m)^{-2}). \tag{A.37}$$

Also

$$E_3(\bar{\boldsymbol{H}}) - E_3(\boldsymbol{H}_0) = o_p(n^{(2H_{10}-1)+(2H_{20}-1)}(\log m)^{-2}). \tag{A.38}$$

Proof. We first notice that $\left(\frac{2\pi}{n}\right)^{(2H_{10}-1)+(2H_{20}-1)} E^0(\boldsymbol{H}_0) = F^0(\boldsymbol{H}_0) = \hat{G}(\boldsymbol{H}_0) \to_p G_0 > 0$ by (3.3) and the fact that $G(\boldsymbol{H}_0) = G_0$. For $k = 0, 1, 2$,

$$
\begin{aligned}
|E_1^k(\boldsymbol{H}) - E_1^k(\boldsymbol{H}_0)| &\le \frac{1}{m^2} \left| \sum_{i,j=1}^{m} (\log i)^k (i^{2H_1-1} j^{2H_2-1} - i^{2H_{10}-1} j^{2H_{20}-1}) I_{ij} \right| \\
&\le \frac{(\log m)^k}{m^2} \sum_{i,j=1}^{m} \left| i^{2H_1-2H_{10}} j^{2H_2-2H_{20}} - 1 \right| I_{ij} i^{2H_{10}-1} j^{2H_{20}-1}.
\end{aligned}
$$

Since $|i^{2(H_1-H_{10})} - 1|/|H_1 - H_{10}| \le 2(\log i) m^{2|H_1-H_{10}|}$,

$$
\begin{aligned}
|i^{2H_1-2H_{10}} j^{2H_2-2H_{20}} - 1| &\le 4m^{2|H_1-H_{10}|+2|H_2-H_{20}|} (\log i)(\log j) |H_1 - H_{10}||H_2 - H_{20}| \\
&\quad + 2m^{2|H_1-H_{10}|} (\log i) |H_1 - H_{10}| + 2m^{2|H_1-H_{10}|} (\log j) |H_2 - H_{20}| \\
&\le C m^{2|H_1-H_{10}|+2|H_2-H_{20}|} \left\{ (\log m)^2 |H_1 - H_{10}||H_2 - H_{20}| \right. \\
&\quad \left. + (\log m)|H_1 - H_{10}| + (\log m)|H_2 - H_{20}| \right\}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
|E_1^k(\boldsymbol{H}) - E_1^k(\boldsymbol{H}_0)| &\le C m^{2|H_1-H_{10}|+2|H_2-H_{20}|} \left\{ (\log m)^2 |H_1 - H_{10}||H_2 - H_{20}| \right. \\
&\quad \left. + (\log m)|H_1 - H_{10}| + (\log m)|H_2 - H_{20}| \right\} (\log m)^k E_1^0(\boldsymbol{H}_0).
\end{aligned}
$$

For a fixed $\epsilon > 0$, define

$$M = \{(H_1, H_2) : (\log m)^5 |H_1 - H_{10}| < \epsilon, (\log m)^5 |H_2 - H_{20}| < \epsilon\}.$$

For all m sufficiently large, note that $2\epsilon < (\log m)^4$. Hence, on M,

$$
\begin{aligned}
|E_1^k(\boldsymbol{H}) - E_1^k(\boldsymbol{H}_0)| &\le C \left\{ \epsilon^2 (\log m)^{k-8} + 2\epsilon (\log m)^{k-4} \right\} E_1^0(\boldsymbol{H}_0) \\
&\le C\epsilon (\log m)^{k-4} E_1^0(\boldsymbol{H}_0).
\end{aligned} \tag{A.39}
$$

To prove the desired result in this lemma, it suffices to show that, for $\eta > 0$,

$$P \left(|E_1^k(\bar{\boldsymbol{H}}) - E_1^k(\boldsymbol{H}_0)| > \eta \left(\frac{2\pi}{n} \right)^{1-2H_{10}} \left(\frac{2\pi}{n} \right)^{1-2H_{20}} (\log m)^{-2} \right) \to 0. \tag{A.40}$$

The left hand side of (A.40) is bounded by

$$
\begin{aligned}
&P \left(C\epsilon (\log m)^{k-4} E_1^0(\boldsymbol{H}_0) > \eta \left(\frac{2\pi}{n} \right)^{1-2H_{10}} \left(\frac{2\pi}{n} \right)^{1-2H_{20}} (\log m)^{-2} \right) + P \left(\bar{\boldsymbol{H}} \in M^c \cap \Theta \right) \\
&\le P \left(\hat{G}(\boldsymbol{H}_0) > \frac{\eta}{C\epsilon} (\log m)^{2-k} \right) + P \left(\bar{\boldsymbol{H}} \in M^c \cap \Theta \right),
\end{aligned}
$$

where $M^c = \mathbb{R}^2 \setminus M$. The above first probability tends to zero for sufficiently small ϵ, since $\hat{G}(\boldsymbol{H}_0) \to G_0 > 0$. Using the notation from the proof of Theorem 3.1, the above second probability is bounded by

$$P\left(\inf_{\Theta\cap N_\delta\cap M^c} S(\boldsymbol{H}) \le 0\right) + P\left(\inf_{\Theta\cap N_\delta^c} S(\boldsymbol{H}) \le 0\right).$$

We have shown that $P\left(\inf_{\Theta\cap N_\delta^c} S(\boldsymbol{H}) \le 0\right) \to 0$ in Theorem 3.1. Therefore, to prove (A.40), it suffices to show that

$$P\left(\sup_{\Theta\cap N_\delta} |T(\boldsymbol{H})| \ge \inf_{\Theta\cap N_\delta\cap M^c} U(\boldsymbol{H})\right) \to 0. \tag{A.41}$$

Since we have

$$\inf_{\Theta\cap N_\delta\cap M^c} U(\boldsymbol{H}) \ge \inf_{\Theta\cap M^c} U(\boldsymbol{H}) \ge C\epsilon^2(\log m)^{-10}$$

by applying (3.1), it is enough to show that

$$\sup_{\Theta\cap N_\delta} T(\boldsymbol{H}) = o_p((\log m)^{-10}). \tag{A.42}$$

Since (3.4) and (3.5) with $o(1)$ replaced by $o((\log m)^{-10})$ follow from the proof of Theorem 3.1, (A.42) follows from

$$\sup_{\Theta\cap N_\delta} \left|\frac{\hat{G}(\boldsymbol{H}) - G(\boldsymbol{H})}{G(\boldsymbol{H})}\right| = o_p((\log m)^{-10}). \tag{A.43}$$

Again, using the notation $A(\boldsymbol{H})$ and $B(\boldsymbol{H})$ in the proof of Theorem 3.1 with the fact that $\inf_{\Theta\cap N_\delta} B(\boldsymbol{H}) \ge \inf_\Theta B(\boldsymbol{H}) \ge 1/2$ for large enough m, to prove (A.43) it suffices to show that

$$A(\boldsymbol{H}) = a(\boldsymbol{H}) + b(\boldsymbol{H}) + c(\boldsymbol{H}) + d(\boldsymbol{H}) = o_p((\log m)^{-10}).$$

By Lemmas A.4 and A.5, and (3.8), we can establish by an argument similar to the proof of Theorem 3.1 that

$$A(\boldsymbol{H}) = o_p((\log m)^{-10}). \tag{A.44}$$

By (A.44) and (A.43)holds and similarly, we can show that (A.37) for $l = 2$ and (A.38) hold. □

We state the trigonometric identities as Lemma for convenience.

Lemma A.8. *For $x_i = \frac{2\pi i}{n}$ and $x_j = \frac{2\pi j}{n}$ such that $i \ne j$,*

$$\sum_{k=2}^{n}\sum_{l=1}^{k-1} \cos^2 lx_i = \frac{\sin(2n-1)x_i}{8\sin x_i} + \frac{1}{8}(2(n-1)^2 - 1),$$

$$\sum_{k=2}^{n}\sum_{l=1}^{k-1} \sin^2 lx_i = -\frac{\sin(2n-1)x_i}{8\sin x_i} + \frac{1}{8}(2n^2 - 1),$$

$$\sum_{k=2}^{n}\sum_{l=1}^{k-1} \cos lx_i \sin lx_i = \frac{n}{4}\frac{\cos x_i}{\sin x_i} - \frac{\cos x_i \sin(2n-1)x_i}{8\sin^2 x_i} - \frac{\cos(2n-1)x_i}{8\sin x_i},$$

$$\sum_{k=2}^{n}\sum_{l=1}^{k-1} \cos lx_i \cos lx_j = \frac{\sin(2n-1)(x_i - x_j)/2}{8\sin(x_i - x_j)/2} + \frac{\sin(2n-1)(x_i + x_j)/2}{8\sin(x_i + x_j)/2} - \frac{2n-1}{4},$$

$$\sum_{k=2}^{n}\sum_{l=1}^{k-1} \sin lx_i \sin lx_j = \frac{\sin(2n-1)(x_i - x_j)/2}{8\sin(x_i - x_j)/2} - \frac{\sin(2n-1)(x_i + x_j)/2}{8\sin(x_i + x_j)/2},$$

$$\begin{aligned}\sum_{k=2}^{n}\sum_{l=1}^{k-1} \cos lx_i \sin lx_j = &\left(\frac{n}{4}\frac{\cos(x_i + x_j)/2}{\sin(x_i + x_j)/2} - \frac{\cos(x_i + x_j)/2\sin(2n-1)(x_i + x_j)/2}{8\sin^2(x_i + x_j)/2} - \frac{\cos(2n-1)(x_i + x_j)/2}{8\sin(x_i + x_j)/2}\right)\\ &-\left(\frac{n}{4}\frac{\cos(x_i - x_j)/2}{\sin(x_i - x_j)/2} - \frac{\cos(x_i - x_j)/2\sin(2n-1)(x_i - x_j)/2}{8\sin^2(x_i - x_j)/2} - \frac{\cos(2n-1)(x_i - x_j)/2}{8\sin(x_i - x_j)/2}\right).\end{aligned}$$

Proof. The proof follows by an elementary argument using

$$\sum_{k=1}^{n} \cos k\theta = \frac{\sin(n+1/2)\theta}{2\sin\theta/2} - \frac{1}{2},$$
$$\sum_{k=1}^{n} \sin k\theta = \frac{\cos\theta/2 - \cos(n+1/2)\theta}{2\sin\theta/2}$$

from [31].

To obtain the asymptotic normality, we need a two-parameter (or spatial) martingale central limit theorem. Consider the ordering of two dimensional indices such that $\bar{i} = (i_1, i_2) > \bar{j} = (j_1, j_2)$ if $i_1 > j_1$ or $i_1 = j_1, i_2 > j_2$. Denote $\bar{1} = (1, 1)$ and $\bar{n} = (n, n)$. Let $V_{\bar{n},\bar{k}} = \{\bar{j} \in \mathbb{Z}^2 \mid \bar{j} \le \bar{k}, \bar{1} \le \bar{j} \le \bar{n}\}$, $\bar{1} \le \bar{k} \le \bar{k}_{\bar{n}}$ be the lattice subsets on the plane, $S_{\bar{n},\bar{k}} = \sum_{\bar{j} \in V_{\bar{n},\bar{k}}} \xi_{\bar{n},\bar{j}}$ and $\mathcal{F}_{\bar{n},\bar{k}}$ be the σ-field of events generated by $\xi_{\bar{n},\bar{j}}$ for $\bar{j} \in V_{\bar{n},\bar{k}}$. Also denote $V_{\bar{n}} := V_{\bar{n},\bar{n}}$. Assume that $S_{\bar{n},\bar{k}}$ are square integrable and $\xi_{\bar{n},\bar{k}}$ satisfy

$$E[\xi_{\bar{n},\bar{k}}] = 0, \quad \bar{k} \in V_{\bar{n}},$$
$$E[\xi_{\bar{n},\bar{k}} | \mathcal{F}_{\bar{n},\bar{k}-1}] = 0, \quad \text{a.s.}$$

so that

$$E[S_{\bar{n},\bar{k}} | \mathcal{F}_{\bar{n},\bar{k}-1}] = \sum_{\bar{j} \in V_{\bar{n},\bar{k}-1}} \xi_{\bar{n},\bar{j}} = S_{\bar{n},\bar{k}-1}$$

and $\{S_{\bar{n},\bar{k}}, \mathcal{F}_{\bar{n},\bar{k}}, \ \bar{1} \le \bar{k} \le \bar{k}_{\bar{n}}\}$ be a zero-mean square integrable martingale array for each $n \ge 1$. Note that $\xi_{\bar{n},\bar{k}} = S_{\bar{n},\bar{k}} - S_{\bar{n},\bar{k}-1}$ are martingale differences. We state the Martingale Central Limit Theorem [16] for convenience.

Theorem A.9. *Let $\{S_{\bar{n},\bar{k}}, \mathcal{F}_{\bar{n},\bar{k}}, \ \bar{1} \le \bar{k} \le \bar{k}_{\bar{n}}\}$ be a square integrable martingale array defined as above. In addition,*

$$\sum_{\bar{k}=\bar{1}}^{\bar{k}_{\bar{n}}} E[\xi^2_{\bar{n},\bar{k}}] \to \sigma^2, \tag{A.45}$$

$$\sum_{\bar{k}=\bar{1}}^{\bar{k}_{\bar{n}}} \left(E[\xi^2_{\bar{n},\bar{k}} | \mathcal{F}_{\bar{n},\bar{k}-1}] - E[\xi^2_{\bar{n},\bar{k}}] \right) \to_p 0 \quad \textit{and} \tag{A.46}$$

$$\sum_{\bar{k}=\bar{1}}^{\bar{k}_{\bar{n}}} E[\xi^2_{\bar{n},\bar{k}} I(|\xi_{\bar{n},\bar{k}}| > \delta)] \to 0, \tag{A.47}$$

for all $\delta > 0$. Then

$$S_{\bar{n},\bar{k}_{\bar{n}}} \Longrightarrow N(0, \sigma^2).$$

Lemma A.10. *Under the conditions of Theorem 4.1 we have, as $n \to \infty$, that*

$$\left(\sum_{s,t=1}^{n} Z_{s,t}, \sum_{s,t=1}^{n} Z^*_{s,t} \right)' \Longrightarrow \frac{1}{2} N(\mathbf{0}, \boldsymbol{I}). \tag{A.48}$$

Proof. To show (A.48), we consider Cramer–Wold device. That is, we show the asymptotic normality of

$$a \sum_{s,t=1}^{n} Z_{s,t} + b \sum_{s,t=1}^{n} Z^*_{s,t} = \sum_{s,t=1}^{n} (aZ_{s,t} + bZ^*_{s,t})$$
$$=: \sum_{s,t=1}^{n} W_{s,t} \tag{A.49}$$

for arbitrary a and b, not all zero. Using the notations in Theorem A.9, by letting $\xi_{\bar{n},\bar{k}} = W_{s,t}$, for $\bar{k} = (s, t)$, it can be observed that $\{\sum_{\bar{j} \in V_{\bar{n},\bar{k}}} \xi_{\bar{n},\bar{j}}, \mathcal{F}_{\bar{n},\bar{k}}, \ \bar{1} \le \bar{k} \le \bar{k}_{\bar{n}}\}$ is a zero-mean martingale array so that it is enough to verify conditions in Theorem A.9.

First we show the asymptotic covariance matrix of $(\sum_{s,t=1}^n Z_{s,t}, \sum_{s,t=1}^n Z_{s,t}^*)'$ is $\frac{1}{4}I$, which also proves $\sum_{s,t=1}^n W_{s,t}$ is square integrable and (A.45).

$$\begin{aligned}\mathrm{Var}\left(\sum_{s,t=1}^n Z_{s,t}\right) &= E\left(\sum_{s,t=1}^n Z_{s,t}\right)^2 = \sum_{s,t=1}^n EZ_{s,t}^2 \\ &= \sum_{s,t=1}^n \sum_{u=1}^{s-1}\sum_{v=1}^n (c_\nu(s-u)c(t-v) - s_\nu(s-u)s(t-v))^2 \\ &= \sum_{s=2}^n\sum_{u=1}^{s-1} c_\nu(u)^2\left(\frac{m}{n} + 2\sum_{t=2}^n\sum_{v=1}^{t-1} c(v)^2\right) + 2\sum_{s=2}^n\sum_{u=1}^{s-1} s_\nu(u)^2 \sum_{t=2}^n\sum_{v=1}^{t-1} s(v)^2\end{aligned}$$

since $EZ_{s,t} = 0$, $EZ_{s,t}Z_{s',t'} = 0$ for $(s,t) \neq (s',t')$ and $\sum_{t=1}^n\sum_{v=1}^n c(t-v)s(t-v) = 0$.

By Lemma A.8, note that

$$\begin{aligned}&\frac{1}{n^2}\sum_{s=2}^n\sum_{u=1}^{s-1}\cos^2(ux_i) = \frac{1}{4} + o(1) \\ &\frac{1}{n^2}\sum_{s=2}^n\sum_{u=1}^{s-1}\sin^2(ux_i) = \frac{1}{4} + o(1) \\ &\sum_{s=2}^n\sum_{u=1}^{s-1}\cos(ux_i)\cos(ux_j) = O(n) \\ &\sum_{s=2}^n\sum_{u=1}^{s-1}\sin(ux_i)\sin(ux_j) = O(n)\end{aligned}$$

for $i \neq j$. Because

$$\frac{1}{m}\sum_{i=1}^m \nu_i^2 = 1 + O\left(\frac{(\log m)^2}{m}\right) \quad \text{and} \quad -\sum_{i\neq j}\nu_i\nu_j = \sum_{i=1}^m \nu_i^2,$$

$$\begin{aligned}\sum_{s=2}^n\sum_{u=1}^{s-1} c_\nu(u)^2 &= \frac{1}{n^2 m}\sum_{i,j}^m \nu_i\nu_j\sum_{s=2}^n\sum_{u=1}^{s-1}\cos(ux_i)\cos(ux_j) \\ &= \frac{1}{n^2 m}\sum_i^m \nu_i^2\sum_{s=2}^n\sum_{u=1}^{s-1}\cos^2(ux_i) + \frac{1}{n^2 m}\sum_{i\neq j}\nu_i\nu_j\sum_{s=2}^n\sum_{u=1}^{s-1}\cos(ux_i)\cos(ux_j) \\ &= \frac{1}{4} + o(1), \\ \sum_{t=2}^n\sum_{v=1}^{t-1} c(v)^2 &= \frac{1}{n^2 m}\sum_{i,j}^m\sum_{t=2}^n\sum_{v=1}^{t-1}\cos(vx_i)\cos(vx_j) \\ &= \frac{1}{n^2 m}\sum_i^m\sum_{t=2}^n\sum_{v=1}^{t-1}\cos^2(vx_i) + \frac{1}{n^2 m}\sum_{i\neq j}\sum_{t=2}^n\sum_{v=1}^{t-1}\cos(vx_i)\cos(vx_j) \\ &= \frac{1}{4} + o(1),\end{aligned}$$

similarly,

$$\begin{aligned}&\sum_{s=2}^n\sum_{u=1}^{s-1} s_\nu(u)^2 = \frac{1}{4} + o(1), \\ &\sum_{t=2}^n\sum_{v=1}^{t-1} s(v)^2 = \frac{1}{4} + o(1).\end{aligned}$$

Thus,

$$\mathrm{Var}\left(\sum_{s,t=1}^n Z_{s,t}\right) = \sum_{s,t=1}^n EZ_{s,t}^2 = \frac{1}{4} + o(1).$$

In a similar way, we can show

$$\mathrm{Var}\left(\sum_{s,t=1}^{n} Z_{s,t}^{*}\right) = \sum_{s,t=1}^{n} E{Z_{s,t}^{*}}^{2} = \frac{1}{4} + o(1).$$

Next, we consider

$$\begin{aligned} E\left(\sum_{s,t=1}^{n} Z_{s,t} \sum_{s,t=1}^{n} Z_{s,t}^{*}\right) &= \sum_{s,t=1}^{n} EZ_{s,t}Z_{s,t}^{*} \\ &= \sum_{s,t=1}^{n}\sum_{u=1}^{s-1}\sum_{v=1,v\neq t}^{n} (c_\nu(s-u)c(t-v) - s_\nu(s-u)s(t-v)) \\ &\quad \times (c(s-u)c_\nu(t-v) - s(s-u)s_\nu(t-v)) \\ &= 2\left(\sum_{s=2}^{n}\sum_{u=1}^{s-1} c_\nu(u)c(u)\right)^2 + 2\left(\sum_{s=2}^{n}\sum_{u=1}^{s-1} s_\nu(u)s(u)\right)^2 \end{aligned}$$

since $\sum_{t\neq v}^{n} c(t-v)s_\nu(t-v) = 0$ and $\sum_{t\neq v}^{n} s(t-v)c_\nu(t-v) = 0$. By Lemma A.8 and $\sum_{i=1}^{m} \nu_i = 0$, we have

$$\begin{aligned} \sum_{s=2}^{n}\sum_{u=1}^{s-1} c_\nu(u)c(u) &= \frac{1}{n^2 m}\sum_{i=1}^{m}\nu_i\sum_{s=2}^{n}\sum_{u=1}^{s-1}\cos^2(ux_i) + \frac{1}{n^2 m}\sum_{i\neq j}^{m}\nu_i\sum_{s=2}^{n}\sum_{u=1}^{s-1}\cos(ux_i)\cos(ux_j) \\ &= \frac{1}{n^2 m}\sum_{i=1}^{m}\nu_i\left(\frac{\sin(2n-1)x_i}{8\sin x_i} + \frac{1}{8}(2(n-1)^2 - 1)\right) + \frac{1}{n^2 m}\sum_{i\neq j}^{m}\nu_i O(n) \\ &= \frac{1}{n^2 m}\sum_{i=1}^{m}\nu_i O(n) + \frac{1}{n^2 m}\sum_{i\neq j}^{m}\nu_i O(n) \\ &= O\left(\frac{\log m}{n} + \frac{m\log m}{n}\right) = o(1) \end{aligned}$$

and similarly

$$\sum_{s=2}^{n}\sum_{u=1}^{s-1} s_\nu(u)s(u) = O\left(\frac{\log m}{n} + \frac{m\log m}{n}\right) = o(1).$$

Thus,

$$E\left(\sum_{s,t=1}^{n} Z_{s,t} \sum_{s,t=1}^{n} Z_{s,t}^{*}\right) = o(1)$$

and thus the asymptotic covariance matrix of $(\sum_{s,t=1}^{n} Z_{s,t}, \sum_{s,t=1}^{n} Z_{s,t}^{*})'$ is $\frac{1}{4}I$ and (A.45) also holds.

To verify (A.46), it is enough to show that

$$\sum_{s,t=1}^{n}\left(E(Z_{s,t}^2|\mathcal{F}_{\bar{n},\bar{k}-1}) - E(Z_{s,t}^2)\right) \to_p 0 \tag{A.50}$$

$$\sum_{s,t=1}^{n}\left(E({Z_{s,t}^{*}}^2|\mathcal{F}_{\bar{n},\bar{k}-1}) - E({Z_{s,t}^{*}}^2)\right) \to_p 0 \tag{A.51}$$

$$\sum_{s,t=1}^{n}\left(E(Z_{s,t}Z_{s,t}^{*}|\mathcal{F}_{\bar{n},\bar{k}-1}) - E(Z_{s,t}Z_{s,t}^{*})\right) \to_p 0, \tag{A.52}$$

where $\bar{k} = (s, t)$. We have

$$\sum_{s,t=1}^{n} \left(E(Z_{s,t}^2|\mathcal{F}_{\bar{n},\bar{k}-1}) - E(Z_{s,t}^2)\right) = \sum_{s,t=1}^{n} \left(\sum_{u=1}^{s-1}\sum_{v=1}^{n}(\epsilon_{u,v}^2 - 1)(c_\nu(s-u)c(t-v) - s_\nu(s-u)s(t-v))^2\right)$$
$$+ \sum_{s,t=1}^{n} \left(\sum_{u\neq u'}^{s-1}\sum_{v\neq v'}^{n} \epsilon_{u,v}\epsilon_{u',v'}(c_\nu(s-u)c(t-v) - s_\nu(s-u)s(t-v)) \times (c_\nu(s-u')c(t-v') - s_\nu(s-u')s(t-v'))\right)$$
$$+ \sum_{s,t=1}^{n} \left(\left\{\sum_{u=u'}^{s-1}\sum_{v\neq v'}^{n} + \sum_{u\neq u'}^{s-1}\sum_{v=v'}^{n}\right\} \epsilon_{u,v}\epsilon_{u',v'}(c_\nu(s-u)c(t-v) - s_\nu(s-u)s(t-v)) \times (c_\nu(s-u')c(t-v') - s_\nu(s-u')s(t-v'))\right) =: I + II + III.$$

$$\begin{aligned} I &= \sum_{u=1}^{n-1}\sum_{s=u+1}^{n}\sum_{v=1}^{n}\sum_{t=1}^{n}(\epsilon_{u,v}^2 - 1)(c_\nu(s-u)c(t-v) - s_\nu(s-u)s(t-v))^2 \\ &= \sum_{u=1}^{n-1}\sum_{k=1}^{n-u}\left\{\sum_{v<t} + \sum_{v>t} + \sum_{v=t}\right\}(\epsilon_{u,v}^2 - 1)(c_\nu(k)c(t-v) - s_\nu(k)s(t-v))^2 \\ &= \frac{m}{n^2}\sum_{u=1}^{n-1}\sum_{v=1}^{n}(\epsilon_{u,v}^2 - 1)\sum_{k=1}^{n-u}c_\nu(k)^2 + \sum_{u=1}^{n-1}\sum_{v=1}^{n-1}(\epsilon_{u,v}^2 - 1)\sum_{k=1}^{n-u}\sum_{l=1}^{n-v}(c_\nu(k)c(l) - s_\nu(k)s(l))^2 \\ &\quad + \sum_{u=1}^{n-1}\sum_{v=2}^{n}(\epsilon_{u,v}^2 - 1)\sum_{k=1}^{n-u}\sum_{l=1}^{v-1}(c_\nu(k)c(l) + s_\nu(k)s(l))^2 =: I_1 + I_2 + I_3. \end{aligned}$$

By (4.9), we have

$$\begin{aligned} EI_1^2 &= \left(\frac{m}{n^2}\right)^2 E\left(\sum_{u=1}^{n-1}\sum_{v=1}^{n}(\epsilon_{u,v}^2 - 1)\sum_{k=1}^{n-u}c_\nu(k)^2\right)^2 \\ &\le C\left(\frac{m}{n^2}\right)^2\sum_{u=1}^{n-1}\sum_{v=1}^{n}\left(\sum_{k=1}^{n-u}c_\nu(k)^2\right)^2 \le C\left(\frac{m}{n^2}\right)^2\sum_{u=1}^{n-1}\sum_{v=1}^{n}\left(\frac{(\log m)^2}{n}\right)^2 = O\left(\frac{m(\log m)^2}{n^2}\right)^2, \\ EI_2^2 &= C\sum_{u=1}^{n-1}\sum_{v=1}^{n}\left(\sum_{k=1}^{n-u}c_\nu(k)^2\sum_{l=1}^{n-v}c(l)^2 + \sum_{k=1}^{n-u}s_\nu(k)^2\sum_{l=1}^{n-v}s(l)^2 - 2\sum_{k=1}^{n-u}c_\nu(k)s_\nu(k)\sum_{l=1}^{n-v}c(l)s(l)\right)^2 \\ &\le C\sum_{u=1}^{n-1}\sum_{v=1}^{n}\left(\left(\sum_{k=1}^{n-u}c_\nu(k)^2\right)^2\left(\sum_{l=1}^{n-v}c(l)^2\right)^2 + \left(\sum_{k=1}^{n-u}s_\nu(k)^2\right)^2\left(\sum_{l=1}^{n-v}s(l)^2\right)^2\right. \\ &\quad \left. + \left(\sum_{k=1}^{n-u}c_\nu(k)s_\nu(k)\right)^2\left(\sum_{l=1}^{n-v}c(l)s(l)\right)^2\right) \\ &= O\left(n^2\left(\frac{(\log m)^2}{n}\right)^2\left(\frac{1}{n}\right)^2\right) = O\left(\frac{(\log m)^2}{n}\right)^2 \end{aligned}$$

and similarly

$$EI_3^2 = O\left(\frac{(\log m)^2}{n}\right)^2.$$

Thus, $I = I_1 + I_2 + I_3 = o_p(1)$.

$$EII^2 = E\left(\sum_{s,t=1}^{n}\sum_{u\neq u'}^{s-1}\sum_{v\neq v'}^{n}\epsilon_{u,v}\epsilon_{u',v'}(c_\nu(s-u)c(t-v) - s_\nu(s-u)s(t-v)) \times (c_\nu(s-u')c(t-v') - s_\nu(s-u')s(t-v'))\right)^2$$

$$
\begin{aligned}
&\le C\sum_{s,t=1}^{n}\sum_{s',t'=1}^{n}\Bigg(\sum_{u\neq u'}^{\min\{s-1,s'-1\}}\sum_{v\neq v'}^{n}(c_\nu(s-u)c(t-v)-s_\nu(s-u)s(t-v))(c_\nu(s-u')c(t-v')-s_\nu(s-u')\\
&\quad\times s(t-v'))(c_\nu(s'-u)c(t'-v)-s_\nu(s'-u)s(t'-v))(c_\nu(s'-u')c(t'-v')-s_\nu(s'-u')s(t'-v'))\Bigg)\\
&= C\sum_{s,t=1}^{n}\left(\sum_{u\neq u'}^{s-1}\sum_{v\neq v'}^{n}(c_\nu(s-u)c(t-v)-s_\nu(s-u)s(t-v))^2(c_\nu(s-u')c(t-v')-s_\nu(s-u')s(t-v'))^2\right)\\
&\quad+C\left\{\sum_{s<s'}^{n}\sum_{t<t'}^{n}+\sum_{s<s'}^{n}\sum_{t>t'}^{n}\right\}\Bigg(\sum_{u\neq u'}^{\min\{s-1,s'-1\}}\sum_{v\neq v'}^{n}(c_\nu(s-u)c(t-v)\\
&\quad-s_\nu(s-u)s(t-v))(c_\nu(s-u')c(t-v')-s_\nu(s-u')s(t-v'))\\
&\quad\times(c_\nu(s'-u)c(t'-v)-s_\nu(s'-u)s(t'-v))(c_\nu(s'-u')c(t'-v')-s_\nu(s'-u')s(t'-v'))\Bigg)\\
&\quad+C\left\{\sum_{s=s'}^{n}\sum_{t<t'}^{n}+\sum_{s<s'}^{n}\sum_{t=t'}^{n}\right\}\Bigg(\sum_{u\neq u'}^{\min\{s-1,s'-1\}}\sum_{v\neq v'}^{n}(c_\nu(s-u)c(t-v)\\
&\quad-s_\nu(s-u)s(t-v))(c_\nu(s-u')c(t-v')-s_\nu(s-u')s(t-v'))\\
&\quad\times(c_\nu(s'-u)c(t'-v)-s_\nu(s'-u)s(t'-v))(c_\nu(s'-u')c(t'-v')-s_\nu(s'-u')s(t'-v'))\Bigg)\\
&=: II_1+II_2+II_3.
\end{aligned}
$$

Using $A(s)$ defined in (4.8) with (4.9),

$$
\begin{aligned}
II_1 &\le C\sum_{s=2}^{n}\sum_{u\neq u'}^{s-1}\sum_{t=1}^{n}\sum_{v\neq v'}^{n}\left(c_\nu(s-u)^2c(t-v)^2+s_\nu(s-u)^2s(t-v)^2\right)\\
&\quad\times\left(c_\nu(s-u')^2c(t-v')^2+s_\nu(s-u')^2s(t-v')^2\right)\\
&\le C(\log m)^4\sum_{s=2}^{n}\sum_{u\neq u'}^{s-1}A(s-u)^2A(s-u')^2\sum_{t=1}^{n}\sum_{v\neq v'}^{n}A(t-v)^2A(t-v')^2\\
&\le C(\log m)^4\sum_{s=3}^{n}\sum_{u'=2}^{s-1}A(s-u')^2\sum_{u=1}^{u'-1}A(s-u)^2\sum_{t=1}^{n}\left(\sum_{v=1}^{n}A(t-v)^2\right)^2\\
&= O\left(\frac{(\log m)^4}{n^2}\right).
\end{aligned}
$$

The first component of II_2 is

$$
\begin{aligned}
&\sum_{s'=3}^{n}\sum_{s=2}^{s'-1}\sum_{t'=2}^{n}\sum_{t=1}^{t'-1}\sum_{u\neq u'}^{s-1}\sum_{v\neq v'}^{n}(c_\nu(s-u)c(t-v)-s_\nu(s-u)s(t-v))\\
&\quad\times(c_\nu(s-u')c(t-v')-s_\nu(s-u')s(t-v'))(c_\nu(s'-u)c(t'-v)-s_\nu(s'-u)s(t'-v))\\
&\quad\times(c_\nu(s'-u')c(t'-v')-s_\nu(s'-u')s(t'-v'))
\end{aligned}
$$

and this is bounded by

$$
\begin{aligned}
&C(\log m)^4\sum_{s'=3}^{n}\sum_{s=2}^{s'-1}\sum_{u\neq u'}^{s-1}A(s-u)A(s-u')A(s'-u)A(s'-u')\sum_{t'=2}^{n}\sum_{t=1}^{t'-1}\sum_{v\neq v'}^{n}A(t-v)A(t-v')A(t'-v)A(t'-v')\\
&\le C(\log m)^4\sum_{s'=3}^{n}\sum_{s=2}^{s'-1}\sum_{u=1}^{s-1}A(s-u)^2\sum_{u=1}^{s-1}A(s'-u)^2\sum_{t'=2}^{n}\sum_{t=1}^{t'-1}\sum_{v=1}^{n}A(t-v)^2\sum_{v=1}^{n}A(t'-v)^2\\
&= O\left(\frac{(\log m)^4}{m^{1/3}}\right).
\end{aligned}
$$

The last equality can be shown similarly as (4.23) in [24]. In a similar way, we can show the second component of II_2 is also bounded by $O\left(\frac{(\log m)^4}{m^{1/3}}\right)$, each component of II_3 is bounded by $O\left(\frac{(\log m)^4}{n}\right)$. Thus,

$$EII^2 = O\left(\frac{(\log m)^4}{m^{1/3}}\right), \tag{A.53}$$

which implies $II = o_p(1)$. Again, we can show $III = o_p(1)$ in a similar way, and consequently (A.50) holds. Similar to (A.50) and (A.51) also holds.

To show (A.52) holds, we have

$$\begin{aligned}
&\sum_{s,t=1}^{n} \left(E(Z_{s,t}Z_{s,t}^{*}|\mathcal{F}_{\bar{n},\bar{k}-1}) - E(Z_{s,t}Z_{s,t}^{*})\right) \\
&= \sum_{s,t=1}^{n} \left(\sum_{u=1}^{s-1} \sum_{\substack{v=1,\\ v\neq t}}^{n} (\epsilon_{u,v}^2 - 1)(c_\nu(s-u)c(t-v) - s_\nu(s-u)s(t-v))(c(s-u)c_\nu(t-v) - s(s-u)s_\nu(t-v)) \right) \\
&\quad + \sum_{s,t=1}^{n} \left(\frac{\sqrt{m}}{n} \sum_{u=1}^{s-1} \epsilon_{u,t} c_\nu(s-u) \sum_{u'=1}^{s-1} \sum_{v'\neq t} \epsilon_{u',v'}(c(s-u')c_\nu(t-v') - s(s-u')s_\nu(t-v')) \right) \\
&\quad + \sum_{s,t=1}^{n} \left(\sum_{\substack{u,u',\\ u\neq u'}}^{s-1} \sum_{\substack{v,v',\\ v\neq v',v,v'\neq t}}^{n} \epsilon_{u,v}\epsilon_{u',v'}(c_\nu(s-u)c(t-v) - s_\nu(s-u)s(t-v)) \right. \\
&\quad \left. \times\ (c(s-u')c_\nu(t-v') - s(s-u')s_\nu(t-v')) \right) \\
&\quad + \sum_{s,t=1}^{n} \left\{ \sum_{u=u'}^{s-1} \sum_{\substack{v,v',\\ v\neq v',v,v'\neq t}}^{n} + \sum_{\substack{u,u',\\ u\neq u'}}^{s-1} \sum_{\substack{v=v',\\ v,v'\neq t}}^{n} \right\} \left(\epsilon_{u,v}\epsilon_{u',v'}(c_\nu(s-u)c(t-v) - s_\nu(s-u)s(t-v)) \right. \\
&\quad \left. \times (c(s-u')c_\nu(t-v') - s(s-u')s_\nu(t-v'))\right) \\
&=: I + II + III + IV.
\end{aligned}$$

I can be decomposed into two parts:

$$\begin{aligned}
I = &\sum_{s=2}^{n} \sum_{u=1}^{s-1} \left\{ \sum_{v<t} + \sum_{v>t} \right\} (\epsilon_{u,v}^2 - 1)(c_\nu(s-u)c(t-v) - s_\nu(s-u)s(t-v)) \\
&\times (c(s-u)c_\nu(t-v) - s(s-u)s_\nu(t-v)) \\
= &\sum_{u=1}^{n-1} \sum_{v=1}^{n} (\epsilon_{u,v}^2 - 1) \sum_{k=1}^{n-u} \sum_{l=1}^{n-v} (c_\nu(k)c(l) - s_\nu(k)s(l))(c(k)c_\nu(l) - s(k)s_\nu(l)) \\
&+ \sum_{u=1}^{n-1} \sum_{v=1}^{n} (\epsilon_{u,v}^2 - 1) \sum_{k=1}^{n-u} \sum_{l=1}^{v-1} (c_\nu(k)c(l) - s_\nu(k)s(l))(c(k)c_\nu(l) - s(k)s_\nu(l)) \\
=: &\ I_1 + I_2.
\end{aligned}$$

By the Cauchy–Schwarz inequality and (4.9), we have

$$\begin{aligned}
EI_1^2 &= C \sum_{u=1}^{n-1} \sum_{v=1}^{n} \left(\sum_{k=1}^{n-u} \sum_{l=1}^{n-v} (c_\nu(k)c(l) - s_\nu(k)s(l))(c(k)c_\nu(l) - s(k)s_\nu(l)) \right)^2 \\
&\leq C(\log m)^4 \sum_{u=1}^{n-1} \sum_{v=1}^{n} \left(\sum_{k=1}^{n-u} A(k)^2 \right)^2 \left(\sum_{l=1}^{n-v} A(l)^2 \right)^2 \\
&= O\left(\frac{(\log m)^4}{n^2}\right).
\end{aligned}$$

In a similar way, we can show $EI_2^2 = O\left((\log m)^4/n^2\right)$. Thus $I = o_p(1)$.

For II,

$$EII^2 = \frac{m}{n^2}E\left(\sum_{s,t=1}^{n}\sum_{u=1}^{s-1}\sum_{u'=1}^{s-1}\sum_{\substack{v'=1\\ v'\neq t}}^{n}\epsilon_{u,t}\epsilon_{u',v'}c_\nu(s-u)(c(s-u')c_\nu(t-v')-s(s-u')s_\nu(t-v'))\right)^2$$
$$\leq C\frac{(\log m)^4 m}{n^2}\sum_{s,s'=2}^{n}\sum_{u,u'=1}^{\min\{s-1,s'-1\}}A(s-u)A(s'-u)A(s-u')A(s'-u')\sum_{\substack{t,v,\\ t\neq v}}A(t-v)^2$$
$$\leq C\frac{(\log m)^4 m}{n^2}\sum_{s\leq s'}\sum_{u=1}^{s-1}A(s-u)^2\sum_{u=1}^{s-1}A(s'-u)^2\sum_{t,v}A(t-v)^2$$
$$= O\left(\frac{(\log m)^4 m}{n^2}\right).$$

Similar to the proof of (A.53), we can show $III = o_p(1)$ and $IV = o_p(1)$. Thus, (A.52) holds and so we have proved that (A.46).

Next, to prove that (A.47) holds, in view of (A.49) we consider a sufficient condition

$$\sum_{s,t=1}^{n} EW_{s,t}^4 \to 0$$

and this can be shown by

$$\sum_{s,t=1}^{n} E(Z_{s,t}^4) \to 0, \quad \text{and} \tag{A.54}$$

$$\sum_{s,t=1}^{n} E({Z_{s,t}^*}^4) \to 0 \tag{A.55}$$

since $W_{s,t}^4 \leq C(Z_{s,t}^4 + {Z_{s,t}^*}^4)$.

By the Cauchy–Schwarz inequality and (4.9), we have

$$\sum_{s,t=1}^{n} EZ_{s,t}^4 = \sum_{s,t=1}^{n} E\left(\epsilon_{s,t}^4\left(\sum_{u=1}^{s-1}\sum_{v=1}^{n}\epsilon_{u,v}(c_\nu(s-u)c(t-v)-s_\nu(s-u)s(t-v))\right)^4\right)$$
$$\leq C\sum_{s,t=1}^{n}\left(\sum_{u=1}^{s-1}\sum_{v=1}^{n}(c_\nu(s-u)c(t-v)-s_\nu(s-u)s(t-v))^4\right.$$
$$\left.+\sum_{(u,v)\neq(u',v')}(c_\nu(s-u)c(t-v)-s_\nu(s-u)s(t-v))^2\ (c_\nu(s-u')c(t-v')-s_\nu(s-u')s(t-v'))^2\right)$$
$$\leq C\sum_{s,t=1}^{n}\left(\sum_{u=1}^{s-1}\sum_{v=1}^{n}(c_\nu(s-u)^4c(t-v)^4+s_\nu(s-u)^4s(t-v)^4)\right.$$
$$\left.+\left(\sum_{u=1}^{s-1}\sum_{v=1}^{n}(c_\nu(s-u)^2c(t-v)^2+s_\nu(s-u)^2s(t-v)^2)\right)^2\right)$$
$$\leq C(\log m)^4\sum_{s,t=1}^{n}\left(\sum_{u=1}^{s-1}A(s-u)^4\sum_{v=1}^{n}A(t-v)^4+\left(\sum_{u=1}^{s-1}A(s-u)^2\sum_{v=1}^{n}A(t-v)^2\right)^2\right)$$
$$= O\left((\log m)^4\left(\frac{m^4}{n^4}+\frac{1}{n^2}\right)\right) = o(1).$$

Also (A.55) can be shown in a similar way. Thus (A.47) holds, and the proof is complete. □

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