A $q$-analog of the exponential formula

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Abstract

A $q$-analog of functional composition for Eulerian generating functions is introduced and applied to the enumeration of permutations by inversions and distribution of left-right maxima.

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1. Introduction

If $f(x) = \sum_{n=1}^{\infty} f_n x^n/n!$ is the exponential generating function for a class of ‘labeled objects’, then

$$g(x) = e^{f(x)}$$

will be (under appropriate conditions) the exponential generating function for sets of these objects. For example, if $f(x) = \sum_{n=1}^{\infty} (n-1)!x^n/n!$ is the exponential generating function for cyclic permutations, then $g(x) = \sum_{n=0}^{\infty} n!x^n/n!$ is the exponential generating function for all permutations; if $f(x)$ is the exponential generating function for connected labeled graphs, then

$$g(x) = \sum_{n=0}^{\infty} 2^\left(\frac{n}{2}\right) \frac{x^n}{n!}$$

is the exponential generating function for all labeled graphs. For various approaches to the exponential formula, see [3,4,10,11].

It is well known that many properties of exponential generating functions have analogs for Eulerian generating functions of the form

$$\sum_{n=0}^{\infty} f_n \frac{x^n}{n!_q}$$

where $n!_q = 1 \cdot (1 + q) \cdot \cdots (1 + q + \cdots + q^{n-1})$, and $f_n$ is a polynomial in $q$. Note that $n!_q$ reduces to $n!$ for $q = 1$. Eulerian generating functions arise in several combinatorial applications, such as finite vector spaces [6] and partitions [1], but here we shall be concerned primarily with their use in counting permutations by inversions. (See [5,9].)

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We introduce a q-analog of functional composition and show that q-exponentiation can be used to count permutations by inversions and ‘basic components’, which are related to left-right maxima. Combinatorial interpretations are obtained for Gould’s q-Stirling numbers of the first kind [7] and the ‘continuous q-Hermite polynomials’ study by Askey and Ismail [2] and others. Finally, we count involutions by inversions, using a new property of a correspondence of Foata [4].

2. Notation

We define \((a; q)_n\) to be \(\prod_{i=0}^{n-1} (1 - aq^i)\), with \((a; q)_0 = 1\). We often write \((a)_n\) for \((a; q)_n\). Thus

\[
(q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n) = (1 - q)^n n!_q \quad \text{and} \quad (a)_\infty = \prod_{i=0}^{\infty} (1 - aq^i).
\]

The q-binomial coefficient, which is a polynomial in q, is defined by:

\[
\left[\begin{array}{c} n \\ k \end{array}\right] = \frac{n!_q}{k!_q (n-k)!_q} = \frac{(q)_n}{(q)_k (q)_{n-k}}.
\]

We write \(n_q\) for \(\left[\begin{array}{c} n \\ 1 \end{array}\right] = 1 + q + \cdots + q^{n-1}\) and \(n\) for the set \(\{1, 2, \ldots, n\}\). All power series may be considered as formal, so that questions of convergence do not arise.

3. A q-analog of functional composition

The q-analog \(D\) of the derivative is defined by

\[
D f(x) = \frac{f(x) - f(qx)}{(1 - q)x}.
\]

Thus \(D 1 = 0\) and for \(n > 0\),

\[
D x^n = \frac{x^{n-1} n!_q}{(n-1)!_q}.
\]

(Note that for \(q = 1\), \(D\) reduces to the ordinary derivative.) We shall often write \(f'\) for \(D f\).

We now define a q-analog of the map \(f \mapsto f^k / k!\) for exponential generating functions.

**Definition 3.1.** Suppose that \(f(0) = 0\). Then for \(k \geq 0\), \(f^{[k]}\) is defined by \(f^{[0]} = 1\) and for \(k > 0\),

\[
D f^{[k]} = f' \cdot f^{[k-1]}, \quad \text{with} \quad f^{[k]}(0) = 0. \tag{3.1}
\]

Formula (3.1) is equivalent to the following recursion: let

\[
f^{[k]}(x) = \sum_{n=0}^{\infty} \frac{f_{n,k} x^n}{n!_q}.
\]

Then

\[
f_{n+1,k} = \sum_{j=0}^{\infty} \left[\begin{array}{c} n \\ j \end{array}\right] f_{n-j+1,k} f_{j,k-1}.
\]

It is clear that \(f_{n,k} = 0\) for \(n < k\).

As an example, take \(f(x) = x^n / m!_q\). Then

\[
\left(\frac{x^n}{m!_q}\right)^{[k]} = \left[\begin{array}{c} m k - 1 \\ m - 1 \end{array}\right] \left[\begin{array}{c} m(k - 1) - 1 \\ m - 1 \end{array}\right] \cdots \left[\begin{array}{c} m - 1 \\ m - 1 \end{array}\right] \frac{x^{mk}}{(mk)!_q} = \frac{(mk)!_q}{(m!_q)^k \cdot 1 \cdot (1 + q^m) \cdots (1 + q^m + q^{2m} + \cdots + q^{(k-1)m})} (mk)!_q x^{mk}.
\]
Note that for \( m = 1 \) we have \( x^{[k]} = x^k/k!q \), and for \( q = 1 \), \((x^m/m!q)^{[k]}\) reduces to
\[
\frac{(mk)!}{m!k!(mk)!} x^{mk}.
\]

**Definition 3.2.** Suppose that \( g(x) = \sum_{n=0}^{\infty} g_n(x^n/n!q) \) and \( f(0) = 0 \). Then the \( q \)-composition \( g[f] \) is defined to be
\[
\sum_{n=0}^{\infty} g_nf^{[n]}.
\]

Note that \( g[x] = g(x) \). The following is straightforward.

**Proposition 3.3 (The chain rule).** \( \mathcal{D} g[f] = g'[f]f' \).

Unfortunately \( q \)-composition is neither associative nor distributive over multiplication, i.e., in general \((fg)[h] \neq f[h] \cdot g[h]\).

Now let \( e(x) = \sum_{n=0}^{\infty} x^n/n!q \) be the \( q \)-analog of the exponential function. Since \( e'(x) = e(x) \), we have \( \mathcal{D} e[f] = e[f]f' \).

Equating coefficients gives a recurrence for the coefficients of \( e[f] \) in terms of the coefficients of \( f \):

**Proposition 3.4.** Let \( f(x) = \sum_{n=1}^{\infty} f_n(x^n/n!q) \) and let \( g(x) = \sum_{n=0}^{\infty} g_n(x^n/n!q) = e[f] \). Then \( g_0 = 1 \) and for \( n \geq 0 \),
\[
g_{n+1} = \sum_{k=0}^{n} \binom{n}{k} g_{n-k} f_k + 1.
\]

We can also express \( e[f] \) as an infinite product:

**Proposition 3.5.** Suppose \( f(0) = 0 \). Then
\[
e[f] = \prod_{k=0}^{\infty} [1 - (1 - q)q^k x f'(q^k x)]^{-1}.
\]

**Proof.** Let \( g = e[f] \). Then \( g'(x) = f'(x)g(x) \), so
\[
\frac{g(x) - g(qx)}{(1 - q)x} = f'(x)g(x)
\]
and thus
\[
g(x) = [1 - (1 - q)x f'(x)]^{-1} g(qx).
\]

Iterating (3.3) yields (3.2). \( \square \)

For \( f(x) = x \), Proposition 3.5 yields the well-known infinite product
\[
e(x) = e[x] = \prod_{k=0}^{\infty} [1 - (1 - q)q^k x]^{-1}.
\]

Since \( e[tf(x)] = \sum_{n=0}^{\infty} t^n f^{[n]} \), we have an alternative characterization of \( f^{[n]} \) as the coefficient of \( t^n \) in
\[
\prod_{k=0}^{\infty} [1 - (1 - q)q^k x f'(q^k x)t]^{-1}.
\]
4. Permutations

By a permutation of a set $A$ of positive integers we mean a linear arrangement $a_1a_2\cdots a_n$ of the elements of $A$. The length of $a_1a_2\cdots a_n$ is $n$. A permutation is basic if it begins with its greatest element. (By convention the ‘empty permutation’ of length zero is not basic.) We denote by $S_n$ and $B_n$ the sets of all permutations and of basic permutations of $n$. (Thus $|S_n|=n!$ for all $n$ and $|B_n|=(n-1)!$ for $n\geq 1$, with $|B_0|=0$.) A left-right maximum of a permutation $a_1a_2\cdots a_n$ is an $a_j$ such that $i<j$ implies $a_i<a_j$. For any nonempty permutation $\sigma$ we write $L(\sigma)$ for the first element of $\sigma$. The following is straightforward.

Lemma 4.1. Suppose the permutation $\pi=a_1a_2\cdots a_n$ has the factorization $\pi=\beta_1\beta_2\cdots\beta_k$, where the $\beta_i$ are nonempty permutations. Then the following are equivalent:

(i) Each $\beta_i$ is basic and $L(\beta_1)<L(\beta_2)<\cdots<L(\beta_k)$.
(ii) $a_i=L(\beta_s)$ for some $s$ if and only if $a_j$ is a left-right maximum.

It follows from the lemma that every permutation $\pi$ has a unique factorization $\beta_1\beta_2\cdots\beta_k$ satisfying (i) which we call the basic decomposition of $\pi$, and we call the $\beta_i$ the basic components of $\pi$. We note that any set $\{\beta_1,\ldots,\beta_k\}$ of basic permutations with no elements in common can be ordered in exactly one way to form the basic decomposition of some permutation. Thus we have a bijection between permutations and sets of disjoint basic permutations.

We call a permutation reduced if it is in $S_n$ for some $n\geq 0$. To any permutation $\pi=a_1a_2\cdots a_n$ we may associate a reduced permutation, $\text{red}(\pi)$, by replacing in $\pi$, for each $i=1,2,\ldots,n$, the $i$th smallest element of $\{a_1,a_2,\ldots,a_n\}$ by $i$. Thus $\text{red}(7926)=3412$. The content of the permutation $\pi=a_1a_2\cdots a_n$ is $\text{con}(\pi)=\{a_1,a_2,\ldots,a_n\}$. We note that a permutation is determined by its reduction and its content.

A function $\omega$ defined on permutations (with values in some commutative algebra over the rationals) is multiplicative if for all permutations $\pi$:

(i) $\omega(\pi)=\omega(\text{red}(\pi))$.
(ii) If $\beta_1\beta_2\cdots\beta_k$ is the basic decomposition of $\pi$, then

$$\omega(\pi)=\omega(\beta_1)\omega(\beta_2)\cdots\omega(\beta_k).$$

Thus a multiplicative function is determined by its values on reduced basic permutations, and these may be chosen arbitrarily. (We note that (ii) implies $\omega(\emptyset)=1$.)

5. Inversions of permutations

If $V$ is a subset of $n$ we denote by $I_n(V)$ the number of pairs $(v,w)$ with $v\in V$, $w\in n-V$, and $v>w$.

Lemma 5.1. Let

$$Q(n,k)=\sum_{V} q^{I_n(V)}$$

where the sum is over all $V\subseteq n$ with $|V|=n-k$. Then $Q(n,k)=\left[{ n \atop k}\right]$.

Proof. It is clear that $Q(n,n)=Q(n,0)=1$ for all $n\geq 0$. Then by considering the two cases $n\in V$ and $n\notin V$ we find the recurrence

$$Q(n,k)=q^k Q(n-1,k)+Q(n-1,k-1),$$

for $0<k<n$. Since $\left[{ n \atop k}\right]$ satisfies the same recurrence and boundary conditions, $Q(n,k)=\left[{ n \atop k}\right]$. \hfill $\Box$

An inversion of the permutation $\pi=a_1a_2\cdots a_n$ is a pair $(i,j)$ with $i<j$ and $a_i>a_j$. We write $I(\pi)$ for the number of inversions of $\pi$. Note that $I(\pi)=I(\text{red}(\pi))$. 

Theorem 5.2. Let \( \omega \) be a multiplicative function on permutations. Let \( g_n = \sum_{\pi \in S_n} \omega(\pi)q^{I(\pi)} \) and let \( f_n = \sum_{\beta \in B_n} \omega(\beta)q^{I(\beta)} \). Then

\[
\sum_{n=0}^{\infty} g_n \frac{x^n}{n!q^n} = e \left[ \sum_{n=0}^{\infty} f_n \frac{x^n}{n!q^n} \right].
\]

Proof. In view of Proposition 3.4, we need only prove

\[
g_{n+1} = \sum_{k=0}^{n} \binom{n}{k} g_{n-k} f_{k+1}.
\]

(5.1)

We shall prove (5.1) by showing that \( \binom{n}{k} g_{n-k} f_{k+1} \) counts those permutations counted by \( g_{n+1} \) whose last basic component has length \( k+1 \). Such a permutation may be factored as \( \pi = \sigma \beta \) where \( \sigma \) is of length \( n-k \), \( \beta \) is of length \( k+1 \), and the disjoint union of \( \text{con} (\sigma) \) and \( \text{con}(\beta) \) is \( n+1 \). The condition that \( \beta \) is the last basic component of \( \pi \) is equivalent to the condition that \( \beta \) is basic and \( \text{con}(\beta) \) contains \( n+1 \). Thus to determine \( \pi \) we choose \( V = \text{con}(\sigma) \) as an arbitrary \((n-k)\)-subset of \( n \) and choose \( \text{red}(\sigma) \in S_{n-k} \) and \( \text{red}(\beta) \in B_{k+1} \). It is easily seen that \( I(\pi) = I(\sigma) + I(\beta) + I_n(V) \). Thus the contribution to \( g_{n+1} \) of these \( \pi \) is

\[
\sum_{V} \sum_{\sigma \in S_{n-k}} \sum_{\beta \in B_{k+1}} \omega(\sigma) \omega(\beta)q^{I(\sigma) + I(\beta) + I_n(V)}
\]

\[
= \left[ \sum_{V} q^{I_n(V)} \right] \left[ \sum_{\sigma \in S_{n-k}} \omega(\sigma)q^{I(\sigma)} \right] \left[ \sum_{\beta \in B_{k+1}} \omega(\beta)q^{I(\beta)} \right]
\]

\[
= \left[ \binom{n}{k} \right] g_{n-k} f_{k+1}, \quad \text{by Lemma 5.1}. \]

Corollary 5.3. Let \( t_1, t_2, \ldots, \) be arbitrary, and set \( T(x) = \sum_{n=0}^{\infty} t_{n+1} x^n \). Define the multiplicative function \( \omega \) by

\[
\omega(\pi) = t_{b_1} t_{b_2} \cdots
\]

where \( \pi \) has \( b_i \) basic components of length \( i \). Let

\[
g_n = \sum_{\pi \in S_n} \omega(\pi)q^{I(\pi)}
\]

and let

\[
g(x) = \sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty} \frac{x^n}{n!q^n}.
\]

Then

\[
g(x) = \prod_{k=0}^{\infty} [1 - (1 - q)q^k x T(q^{k+1} x)]^{-1}.
\]

(5.2)

Proof. Let \( f_n = \sum_{\beta \in B_n} \omega(\beta)q^{I(\beta)} = t_n \sum_{\beta \in B_n} q^{I(\beta)} \). Every \( \beta \) in \( B_n \) is obtained by inserting \( n \) at the beginning of an element of \( S_{n-1} \); thus,

\[
\sum_{\beta \in B_n} q^{I(\beta)} = q^{n-1} \sum_{\pi \in S_{n-1}} q^{I(\pi)} = q^{n-1}(n-1)!_q,
\]

by a well-known result of Rodrigues [8], easily proved by induction. Thus,

\[
f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!q^n} = \sum_{n=0}^{\infty} t_n q^{n-1}(n-1)!_q \frac{x^n}{n!q^n},
\]
so
\[ f'(x) = \sum_{n=0}^{\infty} t_{n+1} q^n x^n = T(qx). \]

Then (5.2) follows from Theorem 5.2 and Proposition 3.5. \(\square\)

6. Examples

We first look at two trivial cases of Theorem 5.2. If we take \(t_i = 1\) for all \(i\), then \(T(x) = (1 - x)^{-1}\) and
\[
g(x) = \prod_{k=0}^{\infty} [1 - (1 - q)q^k x (1 - q^{k+1} x)^{-1}]^{-1}
   = \prod_{k=0}^{\infty} \frac{1 - q^{k+1} x}{1 - q^k x}
   = \frac{1}{1 - x} = \sum_{n=0}^{\infty} \frac{n! q^n}{n! q^n}.
\]

If we take \(t_1 = 1\) and \(t_i = 0\) for \(i > 1\), then \(T(x) = 1\) and
\[
g(x) = \prod_{k=0}^{\infty} [1 - (1 - q)q^k x]^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{n! q^n}.
\]

A more interesting example is that in which \(t_i = t\) for all \(i\). In the case \(q = 1\) we have
\[
g(x) = \exp \left( t \sum_{n=1}^{\infty} \frac{x^n}{n} \right) = (1 - x)^{-t}
   = \sum_{n,k=0}^{\infty} c(n,k)t^k \frac{x^n}{n!}
\]
where \(c(n,k) = |s(n,k)|\) is the unsigned Stirling number of the first kind.

For general \(q\), we have \(T(x) = t (1 - x)^{-1}\), and thus
\[
g(x) = \prod_{k=0}^{\infty} [1 - (1 - q)q^k x t (1 - q^{k+1} x)^{-1}]^{-1}
   = \prod_{k=0}^{\infty} \frac{1 - q^{k+1} x}{1 - (q + (1 - q)t) q^k x}
   = \frac{(qx)_\infty}{((q + (1 - q)t)x)_\infty}.
\]

We can expand this product with the \(q\)-binomial theorem [1, p. 17]:
\[
\frac{(ax)_\infty}{(x)_\infty} = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(q)_n},
\]
which with \(\beta x\) for \(x\) and \(\alpha \beta^{-1}\) for \(a\), gives
\[
\frac{(ax)_\infty}{(\beta x)_\infty} = \sum_{n=0}^{\infty} \left[ \prod_{i=0}^{n-1} (\beta - \alpha q^i) \right] \frac{x^n}{(q)_n},
\]
where as usual the empty product is one. Then (6.1) becomes

\[
\begin{align*}
g(x) &= \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} [(1 - q)t + q - q^{i+1}] \frac{x^n}{(q)_n} \\
&= \sum_{n=0}^{\infty} \prod_{i=0}^{n-1} [t + q(1 + q + \cdots + q^{i+1})] \frac{x^n}{n!_q} \\
&= 1 + \sum_{n=1}^{\infty} t(t + q \cdot 1_q)(t + q \cdot 2_q) \cdots (t + q(n - 1)_q) \frac{x^n}{n!_q}. \tag{6.2}
\end{align*}
\]

It should be noted that a direct combinatorial proof of (6.2) is not difficult. It follows from (6.2) that the coefficients of \( g(x) \) are essentially the same \( q \)-Stirling numbers as those studied by Gould [7].

With the help of the formula [1, p. 36]

\[
\prod_{i=0}^{n-1} (\alpha + \beta q^i) = \sum_{j=0}^{n} q^{\binom{j}{2}} \binom{n}{j} \alpha^{n-j} \beta^j,
\]

one can obtain the explicit formula

\[
c_q(n, k) = \left( \frac{q}{1 - q} \right)^{n-k} \sum_{j=0}^{n} (-1)^j \binom{n-j}{k} q^{\binom{j}{2}} \binom{n}{j}. \tag{6.3}
\]

(See Gould [7].)

It is remarkable that there seems to be no formula for the \( (q = 1) \) Stirling numbers of the first kind as simple as (6.3).

As a generalization, we may count permutations in which every basic component has length divisible by some positive integer \( r \), according to the number of basic components. (The last example is the case \( r = 1 \).) Here we have \( T(x) = tx^{r-1}/(1 - x^r) \) and a straightforward computation yields

\[
g(x) = \left( q^r x^r; q^r \right)_\infty \\
= \sum_{n=0}^{\infty} q^{(r-1)n} \frac{(nr)_q}{(nr)_q} \prod_{i=0}^{n-1} [t + q(r_i)_q] \frac{x^{nr}}{(nr)_q}. 
\]

Next, let us consider the case where all basic components have length one or two. Then we may set \( t_1 = t \), \( t_2 = 1 \), and \( t_i = 0 \) for \( i > 2 \). (Letting \( t_2 \) be an indeterminate would give us no additional information.) Then \( T(x) = t + x \) and we have

\[
g(x) = e^{\left[ tx + \frac{x^2}{2!q} \right]}. 
\]

Proposition 3.4 gives the recurrence

\[
g_{n+1} = t g_n + q n_q g_{n-1}
\]

from which the first few values of \( g_n \) are easily computed:

\[
g_0 = 1,
g_1 = t,
g_2 = t^2 + q,
g_3 = t^3 + (2q + q^2)t,
g_4 = t^4 + (3q + 2q^2 + q^3)t^2 + q^2 + q^3 + q^4.
\]
The infinite product for \( g(x) \) is

\[
g(x) = \prod_{k=0}^{\infty} [1 - (1 - q)q^k x(t + q^{k+1} x)]^{-1}. \tag{6.4}
\]

To find a formula for the coefficients of \( g(x) \) we introduce the ‘continuous \( q \)-Hermite polynomials’ \( H_n(u \mid q) \) defined by

\[
\prod_{k=0}^{\infty} (1 - 2uzq^k + z^2 q^{2k})^{-1} = \sum_{n=0}^{\infty} H_n(u \mid q) \frac{z^n}{(q)_n} \tag{6.5}
\]

These polynomials have been studied by Askey and Ismail \cite{2} and others. We find a formula for their coefficients by setting \( u = \cos \theta, x = e^{i\theta}, \) and \( \beta = e^{-i\theta}. \) Then \( 1 - 2uzq^k + z^2 q^{2k} = (1 - xzq^k)(1 - \beta zq^k) \) so

\[
\prod_{k=0}^{\infty} (1 - 2uzq^k + z^2 q^{2k})^{-1} = (xz)^{-1} (\beta z)^{-1}
\]

\[
= \left[ \sum_{n=0}^{\infty} \frac{z^n}{(q)_n} \right] \left[ \sum_{n=0}^{\infty} \beta^n \frac{z^n}{(q)_n} \right].
\]

Equating coefficients of \( z^n/(q)_n \) and using the well-known formula

\[
\cos \frac{r}{2} = \sum_{m=0}^{[r/2]} (-1)^m 2^{r-2m-1} \binom{r}{m} \cos^{r-2m} \theta
\]

for \( r > 0, \) we obtain

\[
H_n(u \mid q) = \sum_{j=0}^{[n/2]} (2u)^{n-j} \sum_{k=0}^{j} (-1)^j \binom{n-2k}{n-k-j} \frac{1}{n-k-j} n_{k} \left[ \begin{array}{c} n \\ k \end{array} \right] + E_n. \tag{6.6}
\]

where

\[
E_n = \left\{ \begin{array}{ll} \frac{n}{2} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd.} \end{array} \right.
\]

It will be convenient to consider the polynomials \( \tilde{H}_n(u \mid q) = i^n H_n(-iu \mid q) \), where \( i = \sqrt{-1}. \) Then (6.5) and (6.6) lead to

\[
\prod_{k=0}^{\infty} (1 - 2uzq^k - z^2 q^{2k})^{-1} = \sum_{n=0}^{\infty} \tilde{H}_n(u \mid q) \frac{z^n}{(q)_n} \tag{6.7}
\]

and

\[
\tilde{H}_n(u \mid q) = \sum_{j=0}^{[n-1/2]} (2u)^{n-2j} \sum_{k=0}^{j} (-1)^j \binom{n-2k}{n-k-j} \frac{1}{n-k-j} n_{k} \left[ \begin{array}{c} n \\ k \end{array} \right] + E_n. \tag{6.8}
\]

Now in (6.4), set \( z^2 = (1 - q)q x^2, \) so \( x = [(1 - q)q]^{-1/2} \). Then

\[
g(x) = \sum_{n=0}^{\infty} \tilde{H}_n \left( \frac{1}{2} \left( \frac{1 - q}{q} \right)^{1/2} t \mid q \right) \frac{z^n}{(q)_n}
\]

\[
= \sum_{n=0}^{\infty} [q/(1 - q)]^{n/2} \tilde{H}_n \left( \frac{1}{2} \left( \frac{1 - q}{q} \right)^{1/2} t \mid q \right) \frac{x^n}{n!}. \]
Theorem 7.1. Suppose 

Thus 

Then if 

Proof. We proceed by induction on the length of \( n \). The theorem is trivially true for lengths zero and one. Now let \( n \geq 2 \) and assume the truth of the theorem for all shorter lengths. Let \( n' \) be obtained from \( n \) by removing the last basic component, and let \( b'_2 \) be the number of basic components of length \( n' \) of length two. If the last basic component of \( n \) has length one, then \( \Psi(n) \) is \( \Psi(n') \) with \( n \) adjoined at the end, so \( I(\pi) = I(\pi') \), \( I(\Psi(\pi)) = I(\Psi(\pi')) \), and \( b_2 = b'_2 \).

Thus \( I(\Psi(\pi)) - 2I(\pi) + b_2 = I(\Psi(\pi')) - 2I(\pi') + b'_2 = 0 \).

To deal with the case in which the last basic component of \( \pi \) has length two, we first observe that Foata’s correspondence \( \Psi \) can be extended in the obvious way to permutations that are not reduced: \( \Psi(\sigma) \) is defined by \( \text{con}(\Psi(\sigma)) = \text{con}(\sigma) \) and \( \text{red}(\Psi(\sigma)) = \Psi(\text{red}(\sigma)) \).

If the last basic component of \( \pi \) has length two, then it must be \( nk \) for some \( k \). Then \( I(\pi) = I(\pi') + n - k \). If \( \Psi(\pi') = a_1a_2\cdots a_{n-2} \), then \( \Psi(\pi) \) is obtained from it by inserting \( n \) between \( a_{k-1} \) and \( a_k \) (or at the beginning, if \( k = 1 \)), and inserting \( k \) at the end. It is then easily seen that \( I(\Psi(\pi)) = I(\Psi(\pi')) + 2n - 2k - 1 \). Since \( b_2 = b'_2 + 1 \), we have

\[
I(\Psi(\pi)) - 2I(\pi) + b_2 = I(\Psi(\pi')) + 2n - 2k - 1 - 2[1(\pi') + n - k] + b'_2 + 1
\]

Thus \( I(\Psi(\pi')) + 2I(\pi') + b'_2 = 0 \). \( \square \)
It follows that if $g_n = g_n(t \mid q)$ is given by (6.9), then the number of involutions of $n$ with $r$ fixed points and $I$ inversions is the coefficient of $t^r q^I$ in $q^{-n/2} g_n(t q^{1/2} \mid q^2)$.

References


