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On the recognition of fuzzy circular interval graphs

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ABSTRACT

Fuzzy circular interval graphs are a generalization of proper circular arc graphs and have been recently introduced by Chudnovsky and Seymour as a fundamental subclass of claw-free graphs. In this paper, we provide a polynomial time algorithm for recognizing such graphs, and more importantly for building a suitable model for these graphs.

1. Introduction

A graph is *claw-free* if no vertex has three pairwise nonadjacent neighbors. (Graphs in this paper are undirected and simple; also they will have n vertices and m edges. Sometimes, for a graph G, we let $V(G)$ and $E(G)$ denote respectively the vertex set and the edge set.) Claw-free graphs are an important superclass of line graphs, sharing some of the same properties. For instance, in 1980 Minty [15,16] gave the first polynomial time algorithm for finding a maximum weighted stable set in a claw-free graph, generalizing the algorithm of Edmonds [10,9] to find a maximum weighted matching in a graph. Recently Chudnovsky and Seymour [4] shed some light on the structure of claw-free graphs by proposing a decomposition theorem where they describe how to compose all claw-free graphs from some basic classes.

The focus of this paper is on a class of claw-free graphs, that of fuzzy circular interval graphs, that, according to Chudnovsky and Seymour [5], is "one of the two principal basic classes of claw-free graphs". However, before stating a precise definition of *fuzzy* circular interval graphs, it is convenient to deal with the simpler class of circular interval graphs.

Definition 1.1 (*[4]*)**.** A *circular interval graph (CIG)* $G = (V, E)$ is defined by the following construction: take a circle $\mathcal{C}$ and a set of vertices V on the circle. Take a subset of closed and proper intervals $\mathcal{J}$ of $\mathcal{C}$, with no interval including another, and say that $u, v \in V$ are adjacent if $\{u, v\}$ is a subset of one of the intervals.

Circular interval graphs (see Fig. 1) are also known as *proper circular arc graphs*, i.e., they are equivalent to the intersection graphs of arcs of a circle with no containment between arcs [4]. Therefore, we may associate with a CIG both a representation based on intervals and a representation based on arcs. Proper circular arc graphs have been studied extensively in the last decades. For instance, given a graph G with n vertices and m edges, there are many polynomial time algorithms that recognize whether G is a proper circular arc graph (i.e., a CIG) and, in case, build an *arc* representation (see e.g., [1,8]). In this paper, we mainly refer to the $\mathcal{O}(n + m)$-time algorithm in [8], since it can be trivially adapted to build in $\mathcal{O}(n + m)$-time an *interval* representation for G, if any, with n intervals (see Proposition 2.6 in [8]).

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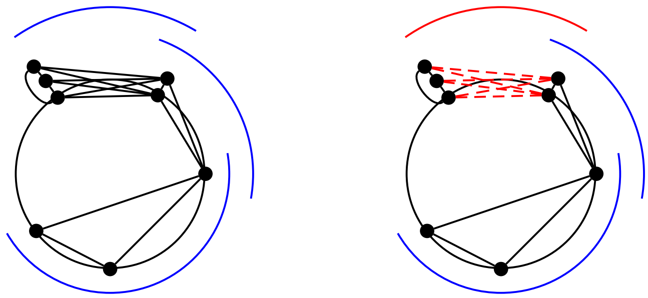

Fig. 1. A circular interval graph (on the left) and a fuzzy circular interval graph (on the right). Dashed lines represent fuzzy adjacencies.

Fuzzy circular interval graphs, or thickening of circular interval trigraphs (see e.g., [5]), have been introduced by Chudnovsky and Seymour as a generalization of circular interval graphs. A fuzzy circular interval graph with vertex set V can be defined by considering again a circle $\mathcal{C}$ and a subset $\mathcal{J}$ of closed and proper intervals of $\mathcal{C}$, with no interval including another and the additional property that different intervals do not share any extremity, and a mapping Φ of the vertices V on the circle. However, while adjacencies in a CIG are defined exactly by the set of intervals, i.e., $u, v \in V$ are adjacent if and only if $\{\Phi(u), \Phi(v)\}$ is a subset of one of the intervals, for a fuzzy circular interval graph additional flexibility is given to adjacencies between vertices at the extremities of an interval: they can be chosen arbitrarily. More formally, we have:

Definition 1.2 (*[4]*)**.** A graph $G = (V, E)$ is a *fuzzy circular interval graph (FCIG)* if the following conditions hold:

(i) There is a map Φ from V to a circle $\mathcal{C}$.
(ii) There is a set $\mathcal{J}$ of closed and proper intervals of $\mathcal{C}$, none including another, such that no point of $\mathcal{C}$ is the end of more than one interval and:
  (a) If two vertices u and v are adjacent, then $\Phi(u)$ and $\Phi(v)$ belong to a common interval.
  (b) If two vertices u and v belong to the same interval, which is not an interval with endpoints $\Phi(u)$ and $\Phi(v)$, then they are adjacent.

In other words, in a FCIG, adjacencies are completely described by the triple $(V, \Phi, \mathcal{J})$, except for vertices u and v such that one of the intervals with endpoints $\Phi(u)$ and $\Phi(v)$ belongs to $\mathcal{J}$. For these vertices adjacency is fuzzy (see Fig. 1), i.e., the adjacencies can be arbitrarily chosen.

In the following, we therefore refer to the triple $(V, \Phi, \mathcal{J})$ as an *interval model* (or simply a *model*) for G; note that while such a triple completely defines a CIG, this is not the case with a FCIG, because of fuzziness. Also, as we discussed above, there are polynomial time algorithms to build an interval model for a given CIG. In contrast, no such algorithm is available for FCIGs, which makes this question a natural open problem. In particular, while a recognition algorithm could be possibly derived from a characterization of FCIGs in terms of excluded subgraphs [2], no algorithm for constructing an interval model is available.

Note also that, while an interval model does not completely define a FCIG, its knowledge is crucial to the solution of some classical optimization problems on FCIGs, such as the coloring [11] and the maximum weighted stable set problem [18], as well as to providing a linear description of their stable set polytope [12]. Clearly, other problems on FCIGs might benefit from the knowledge of a polynomial time algorithm for recognizing a FCIG and getting an interval model for it.

In this paper, we devise a simple polynomial time algorithm to recognize whether a graph is a FCIG, and, in case, build an interval model. Building upon a few facts from the literature, this algorithm can be implemented as to run in $O(n^2m)$-time. It is worthwhile to mention that this algorithm outperforms the straightforward recognition algorithm that might be built from the forbidden subgraph characterization.

Our algorithm reduces the problem of recognizing and providing an interval model of a FCIG to the same problem on a suitable CIG. In the following, we illustrate the motivating intuitions and high-level ideas of our approach.

A crucial role is played by *homogeneous* pairs of cliques: a pair $\{K_1, K_2\}$ of (non necessarily maximal) disjoint cliques of a graph $G = (V, E)$ is homogeneous if every $v \in V \setminus (K_1 \cup K_2)$ is either complete or anti-complete to each of K_1 and K_2. A subset $S \subset V$, possibly made of a single vertex, is *complete* (respectively, *anti-complete*) to another subset $T \subset V$, with $S \cap T = \emptyset$, if each vertex in S is adjacent (respectively, *anti-adjacent*) to each vertex in T.

Suppose that G is a circular interval graph, for which we are given an interval model $(V, \Phi, \mathcal{J})$. Let $K(a)$ be a set of vertices that "sit" at a same point a of the circle C (i.e., $\Phi(v) = a$, for every $v \in K(a)$) and let $K(b)$ a set of vertices that sit at another point b; finally suppose that the interval $[a, b]$ belongs to $\mathcal{J}$. It follows that $K(a), K(b)$ are cliques. Now a vertex $v \in V \setminus (K(a) \cup K(b))$ sitting in c is either complete or anti-complete to $K(a)$, depending on whether an interval of $\mathcal{J}$ contains a and c. Similarly v is either complete or anti-complete to $K(b)$. Therefore, $\{K(a), K(b)\}$ is a homogeneous pair of cliques of G, and, moreover, $K(a)$ and $K(b)$ are complete to each other.

We now consider the same setting as above, but suppose that G' is a *fuzzy* circular interval graph. It is easy to see that $\{K(a), K(b)\}$ still forms a homogeneous pair of cliques, but now the adjacencies between $K(a)$ and $K(b)$ are arbitrary. However, without loss of generality, $\{K(a), K(b)\}$ is a *proper* pair of cliques: i.e., $K(a)$ and $K(b)$ are neither complete nor

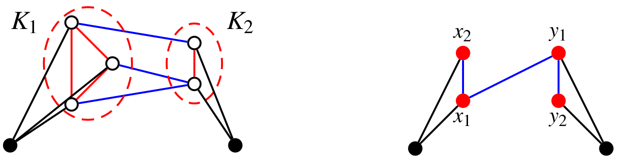

Fig. 2. A proper and homogeneous pair of cliques $\{K_1, K_2\}$ (on the left) and the reduction of the graph with respect to the pair $\{K_1, K_2\}$ (on the right). Note that $\{\{x_1, x_2\}, \{y_1, y_2\}\}$ defines an almost-proper and homogeneous pair of cliques for the reduced graph.

anti-complete to each other. Indeed, if there is a vertex in $K(a)$ complete to $K(b)$, we can move it a small distance into the interior of $[a, b]$ without changing the adjacencies, and if a vertex in $K(a)$ is anti-complete to $K(b)$ we can move it a small distance outside $[a, b]$ without changing the adjacencies (in this case one also needs to add to $\mathcal{J}$ an interval covering this vertex and all points in the interior of $[a, b]$). We can iterate this procedure: if the remaining sets of vertices sitting on a and b are non-empty, then they are a proper pair of cliques.

Note that because $\{K(a), K(b)\}$ is a proper pair of cliques, none of the vertices in $K(a) \cup K(b)$ can be moved (individually) from a and b without ruining the adjacencies. They can only sit in a or in b because they take advantage of the special role of pairs of endpoints of an interval in a FCIG. Thus proper and homogeneous pairs of cliques give rise to configurations that are realizable in FCIGs but obstructions in ordinary CIGs. This is the intuition behind the following lemma (see Section 2 for its proof):

Lemma 2.3. *Let G be a fuzzy circular interval graph. If G has no proper and homogeneous pairs of cliques, then G is a circular interval graph.*

(Note that proper and homogeneous pairs of cliques (see Fig. 2 (left)) are called non-trivial homogeneous pairs of cliques in [14].) Our recognition algorithm builds upon Lemma 2.3. The crucial operation of the algorithm is a "reduction" that replaces a proper and homogeneous pair of cliques $\{K_1, K_2\}$ by a pair of cliques, depicted in Fig. 2, that is still homogeneous but only *almost-proper*: a pair of non-empty vertex-disjoint cliques $\{K'_1, K'_2\}$ is almost-proper if every vertex in K'_1 (respectively, K'_2) is not complete to K'_2 (respectively, K'_1) and there exists $u \in K'_1$, $v \in K'_2$ that are adjacent.

We show in Theorem 3.3 that our reduction preserves the property of a graph of being (respectively, being not) a FCIG. Moreover, if we start with a graph G with m edges and iterate this reduction, in at most m steps we end up with a graph G' without proper and homogeneous pairs of cliques. Following Theorem 3.3, G is a FCIG if and only if G' is a FCIG and, by Lemma 2.3, G' is a FCIG if and only if it is a CIG. The latter fact may be easily tested by applying any existing algorithm for the recognition of CIGs. Even better, Theorem 3.3 also shows that, if G' is a CIG, we may easily extend an interval model for G' into an interval model for G.

The above description assumes that the graph is connected, but the algorithm can be applied separately to each connected component, see Lemma 4.3. However Lemma 4.3 calls for some more definitions. Namely, substituting *line* for *circle* in Definitions 1.1 *and* 1.2 allows to define *linear interval graphs* and *fuzzy linear interval graphs*. Linear interval graphs are also called *proper* (or *unit*) *interval graphs* and several algorithms are available for solving the recognition problem [6,7,17].

The paper is organized as follows. In Section 2, we give a sufficient condition for a fuzzy circular interval graph to be circular interval. Then in Section 3 we define a reduction operation for homogeneous pairs of cliques, and we prove that this reduction preserves the property of a graph to be a FCIG when the pair of cliques is proper, as well as it allows to extend an interval model for the reduced graph to an interval model for the original graph. Finally, in Section 4, we provide the recognition algorithm for FCIGs.

2. When a fuzzy circular interval graph is circular interval

The analysis of fuzzy circular interval graphs is often unpleasant, due to the fact that they are not completely defined by a model. Recall that in Section 1 we defined a (interval) model for a FCIG G to be a triple $(V, \Phi, \mathcal{J})$ that fits Definition 1.2. We also observed that, while a model completely defines a CIG, this is not the case with a FCIG, because the adjacencies between vertices u and v such that an interval with endpoints $\Phi(u)$ and $\Phi(v)$ belongs to $\mathcal{J}$ are fuzzy.

Note that, by definition, each interval of a model $(V, \Phi, \mathcal{J})$ of a FCIG has non-empty interior. It is also easy to see that, if we are given for some FCIG a model $(V, \Phi, \mathcal{J})$ such that $|\mathcal{J}| > n$, then there is some interval $I \in \mathcal{I}$ such that $(V, \Phi, \mathcal{J} \setminus I)$ is still a model for the same graph. Also, as we discussed above, with a trivial modification, the algorithm in [8] returns a model for a CIG with n intervals. Since our main result, an algorithm for recognizing and building a model for FCIGs, builds upon this latter algorithm, in this paper we assume the following:

Assumption 2.1. When we deal with a model $(V, \Phi, \mathcal{J})$ of a FCIG, we always assume that $|\mathcal{J}| \leq n$.

Suppose that we are given a model $(V, \Phi, \mathcal{J})$ for a fuzzy circular interval graph G. Let a and b be two points of $\mathcal{C}$. We denote by $[a, b]$ the interval of $\mathcal{C}$ that we span if we move clockwise from a to b. Similarly (a, b) denotes $[a, b] \setminus \{a, b\}$. Given a point p of $\mathcal{C}$, we denote by $\Phi^{-1}(p)$ the set $\{v \in V : \Phi(v) = p\}$ (note that $\Phi^{-1}(p)$ is a clique if the graph is connected), by $\Phi^{-1}([a, b])$ the set $\{v \in V : \Phi(v) \in [a, b]\}$, by $\Phi^{-1}((a, b))$ the set $\{v \in V : \Phi(v) \in (a, b)\}$, and so on. Sometimes, we abuse notation and we say $a \leq \Phi(v) \leq b$ for $\Phi(v) \in [a, b]$ and similarly $a < \Phi(v) < b$ for $\Phi(v) \in (a, b)$. If $[p, q]$ is an interval of $\mathcal{J}$ such that $\Phi^{-1}(p)$ and $\Phi^{-1}(q)$ are both non-empty, then we call $[p, q]$ a *fuzzy interval* and the cliques $(\Phi^{-1}(p), \Phi^{-1}(q))$ a *fuzzy pair* (note that both definitions are with respect to the given model $(V, \Phi, \mathcal{J})$).

The following lemmas link proper and homogeneous pairs of cliques to fuzzy circular interval graphs. The proofs are constructive and rely on the following fact: given a model $(V, \Phi, \mathcal{J})$ for a fuzzy circular interval graph G and a fuzzy pair $\{K_1, K_2\}$, if $v \in K_1$ is not proper to K_2, v is either anti-complete or complete to K_2 and one can slightly move $\Phi(v)$ inside or outside the fuzzy interval and recover adjacencies by adding suitable intervals.

Lemma 2.2 (*[12]*). *Let $(V, \Phi, \mathcal{J})$ be a model of a fuzzy circular interval graph G. One may build in $O(n^2)$-time a model for G for which each fuzzy pair of cliques is proper and homogeneous.*

Proof. Let $(\Phi^{-1}(p), \Phi^{-1}(q))$ be a fuzzy pair with respect to the given model $(V, \Phi, \mathcal{J})$. Trivially, it is homogeneous; suppose it is not proper. Then, there exists a vertex in $\Phi^{-1}(p)$ (respectively, $\Phi^{-1}(q)$) that is either complete or anti-complete to $\Phi^{-1}(q)$ (respectively, $\Phi^{-1}(p)$).

Suppose that $v \in \Phi^{-1}(p)$ is complete to $\Phi^{-1}(q)$. Then we can move $\Phi(v)$ by a small amount into the interior of the interval $[p, q]$. This operation yields a new model $(V, \Phi', \mathcal{J})$ of the graph G that does not introduce new fuzzy pairs and reduces the number of vertices which are contained in a fuzzy pair by one.

Similarly, if $v \in \Phi^{-1}(p)$ is anti-complete to $\Phi^{-1}(q)$, we can slightly move $\Phi(v)$ such that it is outside $[p, q]$. This operation yields a new mapping Φ'. In addition to that, we must add an interval I covering v and its neighbors in $[p, q]$. Since v is connected to every vertex which is mapped to the half-open interval $[p, q)$ and since $v \cup \Phi'^{-1}[p, q)$ is a clique, this interval I can be chosen such that both of its endpoints are not contained in $\Phi'(V) = \{\Phi'(v) : v \in V\}$. This new model $(V, \Phi', \mathcal{J} \cup \{I\})$ does not introduce new fuzzy pairs and reduces the number of vertices which are contained in a fuzzy pair by one.

We can iterate this process until we get a new model such that each fuzzy pair of cliques is proper and homogeneous. Also this model can be easily built in $O(n^2)$-time. □

We are ready for the main result of this section:

Lemma 2.3. *Let G be a fuzzy circular interval graph. If G has no proper and homogeneous pairs of cliques, then G is a circular linear interval graph.*

Proof. By Lemma 2.2, there exists a model $(V, \Phi, \mathcal{J})$ of G such that each fuzzy pair of clique is proper and homogeneous. But since G has no proper and homogeneous pair of cliques by hypothesis, it follows that there are no fuzzy pairs with respect to $(V, \Phi, \mathcal{J})$. That is, no interval $[p, q]$ of $\mathcal{J}$ is such that $\Phi^{-1}(p)$ and $\Phi^{-1}(q)$ are both non-empty, and thus G is a circular interval graph as the adjacencies between its vertices are completely defined by Φ and $\mathcal{J}$. □

We skip the proof of the next lemma as it goes along the same lines of that of Lemma 2.3.

Lemma 2.4. *Let G be a fuzzy linear interval graph. If G has no proper and homogeneous pairs of cliques, then G is a linear interval graph.*

3. A recursive characterization of fuzzy circular interval graphs

It follows from Lemma 2.3 that, in order to provide a recognition algorithm for a FCIG G, we have to deal with the case where G has a proper and homogenous pair of cliques $\{K_1, K_2\}$. In this case, we perform a crucial operation of *reduction* that replaces $\{K_1, K_2\}$ by a pair of cliques, depicted in Fig. 2.

Definition 3.1. Let $\{K_1, K_2\}$ be a homogenous pair of cliques of G. The *reduction* of G with respect to $\{K_1, K_2\}$ returns the graph $G|_{\{K_1,K_2\}}$ such that:

$V(G|_{\{K_1,K_2\}}) = V(G) \cup \{x_1, y_1, x_2, y_2\} \setminus (K_1 \cup K_2)$;
$E(G|_{\{K_1,K_2\}}) = \{uv : u, v \notin K_1 \cup K_2, uv \in E(G)\} \cup \{ux_1, ux_2 : u \notin K_1 \cup K_2, u \in \Gamma(K_1)\} \cup \{uy_1, uy_2 : u \notin K_1 \cup K_2, u \in \Gamma(K_2)\} \cup \{x_1x_2, y_1y_2, x_1y_1\}$, where $\Gamma(K_i)$ denotes the set of vertices that are complete to K_i, $i = 1, 2$.

We point out that the pair $\{\{x_1, x_2\}, \{y_1, y_2\}\}$ defines an almost-proper and homogeneous pair of cliques for the graph $G|_{\{K_1,K_2\}}$. We skip the proof of the following simple lemma:

Lemma 3.2. *Let G be a connected graph and $\{K_1, K_2\}$ a homogeneous pair of cliques. The graph $G|_{\{K_1,K_2\}}$ is connected.*

We are now ready to state the main result of this section, which shows that the reduction of proper and homogeneous pairs of cliques preserves the property of a graph to be fuzzy circular interval, and that an interval model for the reduced graph, if any, can be extended to an interval model for the original graph.

Theorem 3.3. *Let G be a connected graph and let $\{K_1, K_2\}$ be a proper and homogeneous pair of cliques. The graph $G|_{\{K_1,K_2\}}$ is connected. Moreover, G is a fuzzy circular interval graph if and only if $G|_{\{K_1,K_2\}}$ is a fuzzy circular interval graph and, from a model for G, one may build in $O(n^2)$-time a model for $G|_{\{K_1,K_2\}}$, and vice versa.*

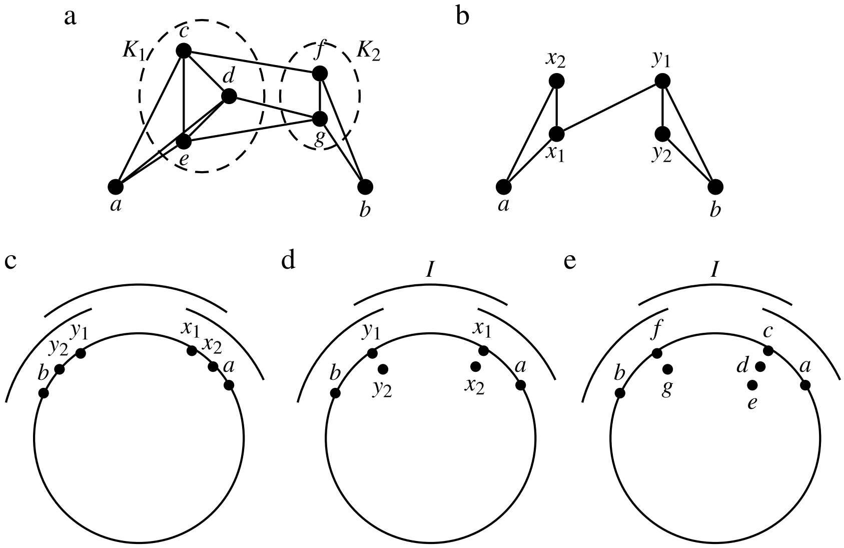

Fig. 3. The use of Lemma 3.4 in the recognition algorithm.

Our recognition algorithm builds upon Theorem 3.3. We sketch the algorithm here, while its formal statement is given in Section 4. Suppose that G has some proper and homogeneous pair of cliques. We iterate the reduction above so as to end up in at most m steps with a graph G' without proper and homogeneous pairs of cliques (m steps suffice because we will show that at each reduction we decrease the number of edges by at least one). Following Theorem 3.3, G is a FCIG if and only if G' is a FCIG and, by Lemma 2.3, G' is a FCIG if and only if it is a CIG. The latter fact may be easily tested by applying any existing algorithm for the recognition of CIGs. In case G' is a CIG, we can also build an interval model for it. Now observe that Theorem 3.3 also guarantees that an interval model for the reduced graph can be extended to an interval model for the original one; therefore, by induction, we may extend an interval model for G' into an interval model for G.

The proof of Theorem 3.3 builds upon a crucial lemma, Lemma 3.4 showing that, if we are given for a FCIG G' an interval model $(V, \Phi', \mathcal{J}')$ and an almost-proper (which of course can also be proper) and homogeneous pair of cliques $\{K_1', K_2'\}$, then we may build in polynomial time another model $(V, \Phi, \mathcal{J})$ for G', where the vertices of K_1' sit at one endpoint of an interval in $\mathcal{J}$ and the vertices of K_2' sit at the other endpoint. But once K_1' and K_2' sit at the endpoints of an interval of $\mathcal{J}$, it follows from the properties of FCIGs that, as long as we replace $\{K_1', K_2'\}$ with another homogeneous pair of cliques $\{K_1, K_2\}$ sitting at the same pair of points as K_1' and K_2', we still get a FCIG, as well as an interval model for it (trivially, the "replacement" should be such that a vertex $v \notin K_1 \cup K_2$ is complete/anti-complete to K_i' if and only if it is complete/anti-complete to K_i, for $i = 1, 2$).

Lemma 3.4. *Let $(V, \Phi, \mathcal{J})$ be a model of a fuzzy circular interval graph $G = (V, E)$ and let $\{K_1, K_2\}$ be an almost-proper and homogeneous pair of cliques. We can build in $O(n^2)$-time a model $(V, \Phi'', \mathcal{J}')$, such that: $\Phi''(v) = a$, for each $v \in K_1$; $\Phi''(v) = b$, for each $v \in K_2$; $[a, b]$ or $[b, a] \in \mathcal{J}'$, for some $a \neq b \in \mathcal{C}$.*

We prefer to postpone the rather long proof of Lemma 3.4 to Section 3.1, but we illustrate how we use this lemma in our recognition algorithm in Fig. 3. We are given a proper and homogeneous pair of cliques $\{K_1, K_2\}$ of a graph G (part a). We perform our reduction and get a new graph $G|_{\{K_1,K_2\}}$ (part b), with $\{\{x_1, x_2\}, \{y_1, y_2\}\}$ an almost-proper and homogeneous pair of cliques. The graph $G|_{\{K_1,K_2\}}$ happens to be circular interval, therefore we may build an interval model $(V(G|_{\{K_1,K_2\}}), \Phi, \mathcal{J})$ for it (part c). Following Lemma 3.4, we may derive from $(V(G|_{\{K_1,K_2\}}), \Phi, \mathcal{J})$ another model $(V(G|_{\{K_1,K_2\}}), \Phi'', \mathcal{J}')$ such that x_1, x_2 reside at one end of an interval $I \in \mathcal{J}'$ and y_1, y_2 reside at the other (part d). Once this has been done, $\{x_1, x_2\}$ and $\{y_1, y_2\}$ can be respectively replaced by K_1 and K_2, yielding an interval model of G (part e).

Proof of Theorem 3.3. From Lemma 3.2, $G|_{\{K_1,K_2\}}$ is connected. We now show that G is a FCIG if and only if $G|_{\{K_1,K_2\}}$ is a FCIG.

Necessity. From Lemma 3.4, we know that, from any model for G, we may build in $O(n^2)$-time another one $(V, \Phi, \mathcal{J})$ such that $\Phi(K_1) = a$, $\Phi(K_2) = b$ for some $a \neq b \in \mathcal{C}$ and, without loss of generality, $[a, b] \in \mathcal{J}$. Consider the following mapping Φ' for the vertices of $G|_{\{K_1,K_2\}}$ (cf. Definition 3.1):

- for $v \in V(G|_{\{K_1,K_2\}}) \setminus \{x_1, y_1, x_2, y_2\}$, $\Phi'(v) = \Phi(v)$;
- for $v \in \{x_1, x_2\}$, $\Phi'(v) = a$;
- for $v \in \{y_1, y_2\}$, $\Phi'(v) = b$.

We claim that $(V, \Phi', \mathcal{J})$ is a model for $G|_{\{K_1,K_2\}}$, i.e., that $(V, \Phi', \mathcal{J})$ is consistent with $E(G|_{\{K_1,K_2\}})$. First, consider u and $v \in \{x_1, y_1, x_2, y_2\}$. In this case, consistency holds since $[a, b]$ is an interval of $\mathcal{J}$. Now consider u and $v \notin \{x_1, y_1, x_2, y_2\}$. In this case, $uv \in E(G|_{\{K_1,K_2\}})$ if and only if $uv \in E(G)$: consistency follows since $\Phi'(v) = \Phi(v)$, $\Phi'(u) = \Phi(u)$ and we keep $\mathcal{J}$.

Finally consider u and v such that $u \in \{x_1, y_1, x_2, y_2\}$, e.g., $u \in \{x_1, x_2\}$, and $v \notin \{x_1, y_1, x_2, y_2\}$. In this case, $uv \in E(G|_{\{K_1,K_2\}})$ if and only if $v \in \Gamma(K_1)$: consistency follows since $\Phi'(v) = \Phi(v)$, $\Phi'(u) = \Phi(K_1)$ and we keep $\mathcal{J}$.

Sufficiency. Note that $G|_{\{K_1,K_2\}}$ satisfies the hypothesis of Lemma 3.4 with $(\{x_1, x_2\}, \{y_1, y_2\})$ being an almost-proper and homogeneous pair of cliques. Therefore, from any model for $G|_{\{K_1,K_2\}}$, we may build in $O(n^2)$-time another model $(V, \Phi, \mathcal{J})$ such that such that $\Phi(v) = a$ for $v \in \{x_1, x_2\}$, $\Phi(v) = b$ for $v \in \{y_1, y_2\}$ and without loss of generality $[a, b] \in \mathcal{J}$. In order to show that G is fuzzy circular interval too, we consider the triple $(V, \Phi', \mathcal{J})$, where Φ' is such that:

- for $v \in V(G) \setminus (K_1 \cup K_2)$, $\Phi'(v) = \Phi(v)$;
- for $v \in K_1$, $\Phi'(v) = a$;
- for $v \in K_2$, $\Phi'(v) = b$.

It is again easy to show that $(V, \Phi', \mathcal{J})$ is a model for G, we omit the details. □

We now take a long detour and prove Lemma 3.4. Note, however, that skipping this proof does not affect the comprehension of the rest of the paper.

3.1. The proof of Lemma 3.4

We here give a proof of Lemma 3.4. The proof is rather complex and builds upon a bunch of other results, that we describe in the following.

Definition 3.5. Three intervals I_1, I_2 and I_3 of $\mathcal{C}$ *cover* $\mathcal{C}$ if every point of $\mathcal{C}$ is in one of the intervals.

Definition 3.6. Let $(V, \Phi, \mathcal{J})$ be a model of a fuzzy circular interval graph $G = (V, E)$ and let $Q \subseteq V$. We say that an interval $I \in \mathcal{J}$ *covers* Q if $\bigcup_{v \in Q} \Phi(v) \subseteq I$.

Lemma 3.7. *Let $(V, \Phi, \mathcal{J})$ be a model of a fuzzy circular interval graph $G = (V, E)$ and let K be a clique of size two or more. Either there exists an interval $I \in \mathcal{J}$ covering K, or there exist three intervals I_1, I_2 and $I_3 \in \mathcal{J}$ covering the circle. In the latter case, no vertex of $V \setminus K$ is anti-complete to K.*

Proof. The proof is by induction on the size of K. If $|K| = 2$, there exists an interval $I \in \mathcal{J}$ covering K by definition of FCIGs. Now let K be such that $|K| > 2$ and $v \in K$. By induction, either there exists an interval $I_1 \in \mathcal{J}$ covering $K \setminus v$, or there exist three intervals covering the circle and no vertex of $(V \setminus K) \cup \{v\}$ is anti-complete to $K \setminus v$. In the latter case, the induction is trivial. Analogously, in the former case, the induction is trivial if $\Phi(v) \in I_1$. So suppose to the contrary that $\Phi(v) \notin I_1$, and assume that $I_1 = [a, b]$. Since v is adjacent to all vertices in $K \setminus v$, it easily follows that either there exists I_2 containing $(I_1 \cap \Phi(K_1)) \cup \{\Phi(v)\}$, and the result follows, or there must exist I_2 and I_3, such that $[\Phi(v), a] \subsetneq I_2$ and $[b, \Phi(v)] \subsetneq I_3$ (note that e.g., $[\Phi(v), a] \neq I_2$ because no point of $\mathcal{C}$ is the end of more than one interval of $\mathcal{J}$). Note that I_1, I_2 and I_3 cover $\mathcal{C}$. Finally, we are left with showing that, in this case, no vertex of $V \setminus K$ is anti-complete to K. The statement is trivial for any $u \in V \setminus K$ such that $\Phi(u) \in (a, b)$. So assume that $\Phi(u) \notin (a, b)$, and without loss of generality assume that $\Phi(u) \in [\Phi(v), a]$. Since $[\Phi(v), a] \subsetneq I_2$, it follows that u and v are adjacent, which is enough. □

The main tool for the proof of Lemma 3.4 is another lemma, Lemma 3.11. The proof of this latter lemma is easier, if we first deal with the case where G has small stability number. This fact motivates the following definition:

Definition 3.8. Let $G = (V, E)$ be a graph and $\{K_1, K_2\}$ a homogeneous pair of cliques. Let S_3 be the set of vertices that are complete to both K_1 and K_2, S_1 (respectively, S_2) the set of vertices complete to K_1 (respectively, K_2) and anti-complete to K_2 (respectively, K_1). We say that $\{K_1, K_2\}$ is a *fuzzy dominating pair* if $V = K_1 \cup K_2 \cup S_1 \cup S_2 \cup S_3$ and S_1 and S_2 are cliques that are complete to S_3 (all remaining adjacencies being possible).

Lemma 3.9. *Let $(V, \Phi, \mathcal{J})$ be a model of a fuzzy circular interval graph $G = (V, E)$ and let $\{K_1, K_2\}$ a homogeneous pair of cliques. In time $O(n^2)$ we can recognize whether $\{K_1, K_2\}$ is a fuzzy dominating pair and, in this case, build a model $(V, \Phi', \mathcal{J}')$, such that: $\Phi'(v) = a$, for each $v \in K_1$; $\Phi'(v) = b$, for each $v \in K_2$; $[a, b]$ or $[b, a] \in \mathcal{J}'$, for any $a \neq b$ on $\mathcal{C}$.*

Proof. Note that, since $\{K_1, K_2\}$ is a homogeneous pair, S_1, S_2 and S_3 can be built in time $O(n)$. In order to check that $\{K_1, K_2\}$ is a fuzzy dominating pair we then need to check that $V = K_1 \cup K_2 \cup S_1 \cup S_2 \cup S_3$, that S_1 and S_3 are cliques and that S_1, S_2 are complete to S_3. Trivially, that can be done in time $O(n^2)$.

Now suppose that $\{K_1, K_2\}$ is a fuzzy dominating pair. Every vertex in $K_1 \cup K_2$ is complete to S_3; therefore S_3 can be partitioned into two cliques S_4 and S_5 (every fuzzy circular interval graph is *quasi-line*, i.e., the neighborhood of any vertex can be partitioned into two cliques), that can be found in time $O(n^2)$. Now the sets $K_1' = S_1 \cup K_1 \cup S_4$ and $K_2' = S_2 \cup K_2 \cup S_5$ are cliques and we can therefore associate to G the model $(V, \Phi', \mathcal{J}')$ where $\Phi'(v) = a$, for each $v \in K_1'$, $\Phi'(v) = b$, for each $v \in K_2'$ and $\mathcal{J}' = \{[a, b]\}$ for any $a \neq b$ on the circle. □

Definition 3.10. Let $(V, \Phi, \mathcal{J})$ be a model of a fuzzy circular interval graph $G = (V, E)$ and let $\{K_1, K_2\}$ be an almost-proper and homogeneous pair of cliques. We say that $(V, \Phi, \mathcal{J})$ is *tight* with respect to $\{K_1, K_2\}$ if, for some $w \in K_1$ and $z \in K_2$, either $[\Phi(w), \Phi(z)]$ or $[\Phi(z), \Phi(w)]$ belongs to $\mathcal{J}$.

Lemma 3.11. *Let $(V, \Phi, \mathcal{I})$ be a model of a fuzzy circular interval graph $G = (V, E)$ and let $\{K_1, K_2\}$ be an almost-proper and homogeneous pair of cliques. In time $O(n^2)$ we can:*

- *either recognize that $(V, \Phi, \mathcal{I})$ is tight with respect to $\{K_1, K_2\}$;*
- *or build another model of G that is tight with respect to $\{K_1, K_2\}$.*

Proof. We can recognize whether $(V, \Phi, \mathcal{I})$ is tight with respect to $\{K_1, K_2\}$ in time $O(n^2)$ (recall that we are assuming that $|\mathcal{I}| \le n$). Also, following Lemma 3.9, in time $O(n^2)$ we can recognize whether $\{K_1, K_2\}$ is a fuzzy dominating pair, and in this case build a model $(V, \Phi', \mathcal{I}')$ that is tight with respect to $\{K_1, K_2\}$. In the following, we therefore assume that $(V, \Phi, \mathcal{I})$ is *not* tight with respect to $\{K_1, K_2\}$ and that $\{K_1, K_2\}$ is *not* a fuzzy dominating pair. We also assume that, for every fuzzy interval of $\mathcal{I}$, every vertex mapped at one of the extremities has an adjacent and a non-adjacent vertex mapped at the other extremity (see Lemma 2.2, the transformation obviously preserves $(V, \Phi, \mathcal{I})$ *not* tight with respect to $\{K_1, K_2\}$).

We first show that there exist intervals $I_1, I_2 \in \mathcal{I}$ such that I_1 covers K_1 or I_2 covers K_2. In fact, from Lemma 3.7, if no interval of $\mathcal{I}$ covers K_1, then no vertex of $V \setminus K_1$ is anti-complete to K_1; thus, by homogeneity, each vertex $z \in V \setminus (K_1 \cup K_2)$ is complete to K_1. Similarly, if there is no interval covering K_2, then each vertex $z \in V \setminus (K_1 \cup K_2)$ is complete to K_2. But then $\{K_1, K_2\}$ is a fuzzy dominating pair with $S_1 = S_2 = \emptyset$ and $S_3 = V \setminus (K_1 \cup K_2)$, a contradiction.

We can thus assume without loss of generality that there exists an interval $I_1 \in \mathcal{I}$ covering K_1. We also define $I'_1 := [a_1, b_1] \subseteq I_1$ to be the smallest interval of $\mathcal{C}$ covering K_1. (Notice that I'_1 might not be an interval of $\mathcal{I}$.) Observe that $a_1 \ne b_1$. Indeed, otherwise, since there exists $u \in K_1, v \in K_2$ such that $uv \in E$, it would follow that there is an interval $I \in \mathcal{I}$ covering K_1 and v. But because each vertex of K_2 is not complete to K_1, necessarily either $I = [a_1, \Phi(v)]$ or $I = [\Phi(v), a_1]$, and this contradicts the assumption that $(V, \Phi, \mathcal{I})$ is not tight with respect to $\{K_1, K_2\}$. Note that, since $I'_1 \subseteq I_1$, a similar argument shows that no vertex $v \in K_2$ is such that $\Phi(v) \in I'_1$. Therefore, we may define $I'_2 := [a_2, b_2]$ to be the smallest interval in $\mathcal{C} \setminus I'_1$ covering K_2; it follows that $I'_1 \cap I'_2 = \emptyset$. Also, by similar arguments as above, $a_2 \ne b_2$. Now there exists I_2 covering $[a_2, b_2]$ because otherwise there would exist an interval containing $[b_2, a_2]$ (K_2 is a clique and the vertices that map to a_2 and b_2 are adjacent) and thus $[a_1, b_1]$ would be in the interior of this interval and some vertices of K_2 (e.g., those that map to a_2 or b_2) would be complete to K_1, a contradiction.

It is convenient to summarize our results so far in the following:

Claim 3.12. *There exist intervals $I_1, I_2 \in \mathcal{I}$ such that I_1 covers K_1 and I_2 covers K_2 and with the property that, if we let $I'_1 := [a_1, b_1] \subseteq I_1$ be the smallest interval of $\mathcal{C}$ covering K_1 and $I'_2 := [a_2, b_2] \subseteq I_2$ be the smallest interval of $\mathcal{C}$ covering K_2, it follows that $I'_1 \cap I'_2 = \emptyset$. Note that, by definition, for $i = 1, 2$, $K_i \cap \Phi^{-1}(a_i) \ne \emptyset$ and $K_i \cap \Phi^{-1}(b_i) \ne \emptyset$.*

Claim 3.13. *For all $\bar{I}_1 \in \mathcal{I}$ such that $I'_1 \subseteq \bar{I}_1$, we have $\bar{I}_1 \cap I'_2 = \emptyset$, and, similarly, for all $\bar{I}_2 \in \mathcal{I}$ such that $I'_2 \subseteq \bar{I}_2$, we have $\bar{I}_2 \cap I'_1 = \emptyset$.*

Let us show that, for each $\bar{I}_2 \in \mathcal{I}$ with $I'_2 \subseteq \bar{I}_2$, we have $\bar{I}_2 \cap I'_1 = \emptyset$. Indeed, otherwise, there exists $v \in K_1 : \Phi(v) \in \bar{I}_2$. We can assume without loss of generality that $v \in \Phi^{-1}(b_1)$ and $[b_1, b_2] \subseteq \bar{I}_2$. But then either we have v or $[a_2, b_2]$ in the interior of $\bar{I}_2$ and in both cases v is complete to K_2, a contradiction, or we have $\bar{I}_2 = [b_1, b_2]$, and this contradicts the assumption that $(V, \Phi, \mathcal{I})$ is not tight with respect to $\{K_1, K_2\}$. □

Claim 3.14. *If there exist intervals $\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4$ of $\mathcal{I}$: $\bar{I}_1 \supseteq [a_1, b_1]$; $\bar{I}_2 \supseteq [a_2, b_2]$; $\bar{I}_3 \supsetneq [b_1, a_2]$; $\bar{I}_4 \supsetneq [b_2, a_1]$, then $\{K_1, K_2\}$ is a fuzzy dominating pair.*

We now show that, under the hypothesis of the claim, $\{K_1, K_2\}$ would be a fuzzy dominating pair, with $S_1 = \Phi^{-1}((a_1, b_1)) \setminus K_1$, $S_2 = \Phi^{-1}((a_2, b_2)) \setminus K_2$, $S_3 = \Phi^{-1}([b_1, a_2] \cup [b_2, a_1]) \setminus (K_1 \cup K_2)$, a contradiction. Indeed, because of $\bar{I}_4$, each vertex $v \notin K_1 \cup K_2$ such that $\Phi(v) \in [b_2, a_1]$ is adjacent to some vertex in $\Phi^{-1}(a_1) \cap K_1$ and to some vertex in $\Phi^{-1}(b_2) \cap K_2$, and therefore, by homogeneity, is complete to $K_1 \cup K_2$. Analogously, because of $\bar{I}_3$, each vertex $v \notin K_1 \cup K_2$ such that $\Phi(v) \in [b_1, a_2]$ is complete to $K_1 \cup K_2$. Therefore, the vertices of $V \setminus (K_1 \cup K_2)$ that are not complete to $K_1 \cup K_2$ are in $(a_1, b_1) \cup (a_2, b_2)$, and therefore in $S_1 \cup S_2$. Moreover, the vertices $v : \Phi(v) \in (a_1, b_1)$ are complete to each other (because they are in the interior of the interval $\bar{I}_1$) and similarly the vertices $v : \Phi(v) \in (a_2, b_2)$ are complete to each other: therefore, S_1 and S_2 are cliques. In order to show that $\{K_1, K_2\}$ is a fuzzy dominating pair, we are then left with proving that S_1 is complete to S_3 and S_2 is complete to S_3. Note that any vertex $v \notin K_1 \cup K_2$ with $\Phi(v) \in [b_2, a_1]$ is complete to $\Phi^{-1}(a_1)$, because of $\bar{I}_4$, and therefore, by homogeneity, it is adjacent to every vertex in $\Phi^{-1}(b_1) \cap K_1$. Hence, there must exist an interval containing $[\Phi(v), b_1]$ (and not $[b_1, \Phi(v)]$, as we would contradict Claim 3.13), and thus v is complete to S_1 by definition. Similarly v is complete to S_2. Using similar arguments, we can show that any vertex $v \notin K_1 \cup K_2$ with $\Phi(v) \in [b_1, a_2]$ is complete to $S_1 \cup S_2$. □

Claim 3.15. *For all $\bar{I}_1 \in \mathcal{I}$ such that $I'_1 \cap \bar{I}_1 \neq \emptyset$, we have $\Phi^{-1}(\bar{I}_1 \cap I'_2) \subseteq K_2$, and, similarly, for all $\bar{I}_2 \in \mathcal{I}$ such that $I'_2 \cap \bar{I}_2 \neq \emptyset$, we have $\Phi^{-1}(\bar{I}_2 \cap I'_1) \subseteq K_1$.*

Let us prove the first case. Suppose by contradiction that there exists an interval $\bar{I}_1 \in \mathcal{I} : \bar{I}_1 \cap I'_1 \neq \emptyset$ covering $z \in \Phi^{-1}(I'_2) \setminus K_2$. Without loss of generality, let us assume that $b_1 \in \bar{I}_1$ and that $[b_1, a_2] \subseteq [b_1, \Phi(z)] \subseteq \bar{I}_1$. In particular, $[b_1, a_2] \neq \bar{I}_1$, as otherwise we would contradict the assumption that $(V, \Phi, \mathcal{I})$ is not tight with respect to $\{K_1, K_2\}$.

First suppose that z is adjacent to some vertex in $K_1 \cap \Phi^{-1}(b_1)$. Then, by homogeneity, there must exist an interval $L \in \mathcal{I}$ such that either $[\Phi(z), a_1] \subseteq L$ or $[a_1, \Phi(z)] \subseteq L$, but this latter case is ruled out by Claim 3.13. Therefore, $[b_2, a_1] \subseteq [\Phi(z), a_1] \subseteq L$ and $L \neq [b_2, a_1]$, again by our assumptions. Summarizing, the following intervals belong to $\mathcal{I}$: $I_1 \supseteq [a_1, b_1]$; $I_2 \supseteq [a_2, b_2]$; $\bar{I}_1 \supsetneq [b_1, a_2]$; $L \supsetneq [b_2, a_1]$. But Claim 3.14 shows that this is a contradiction.

For the same reason, it follows that each vertex in $\Phi^{-1}(\Phi(z)) \setminus K_2$ is anti-complete to $K_1 \cap \Phi^{-1}(b_1)$; but then $\bar{I}_1 = [b_1, \Phi(z)]$ is a fuzzy interval. Now, because of our assumptions, there is no vertex $v \in K_2$ such that $\Phi(v) = \Phi(z)$. Therefore, each vertex $v \in K_1$ such that $\Phi(v) = b_1$ is anti-complete to $\Phi^{-1}(\Phi(z))$, a contradiction with the fact that each vertex at the extremity of a fuzzy interval is adjacent to some vertex at the other extremity. □

We are almost ready to build for G our alternative model $(V, \Phi', \mathcal{I}')$ that is tight with respect to $\{K_1, K_2\}$. Note that, since $\{K_1, K_2\}$ is an almost-proper pair of cliques, there exists an edge $uv \in E$ with $u \in K_1$, $v \in K_2$. Now, since u and v are adjacent, there is an interval $J^* \in \mathcal{I}$ covering u and v. We can assume without loss of generality that $[b_1, a_2] \subseteq [\Phi(u), \Phi(v)] \subseteq J^*$. Let $\mathcal{J} \subseteq \mathcal{I}$ be the family of all intervals containing $[b_1, a_2]$: observe that each interval in $\mathcal{J}$ intersects I'_1, I'_2 but is neither containing I'_1 nor I'_2 (by Claim 3.13) and does not cover any vertex $y \in \Phi^{-1}(I'_1) \cup \Phi^{-1}(I'_2)$ which is not in $K_1 \cup K_2$ (by Claim 3.15). We therefore define l to be the closest extremity to a_1 in I'_1 of all the intervals in $\mathcal{J}$ and r to be the closest extremity to b_2 in I'_2 of all the intervals in $\mathcal{J}$: note that $l \in (a_1, b_1]$ and $r \in [a_2, b_2)$ by Claim 3.13. By definition, each interval $J \in \mathcal{J}$ is such that $J \subseteq [l, r]$. It follows from Claim 3.15 that $\Phi^{-1}([l, b_1]) \subseteq K_1$, $\Phi^{-1}([a_2, r]) \subseteq K_2$; moreover, $\Phi^{-1}((b_1, a_2)) \cap (K_1 \cup K_2) = \emptyset$.

We then define $\mathcal{I}' = \mathcal{I} \setminus \mathcal{J} \cup [l, r]$; $\Phi'(x) = \Phi(x)$ for all $x \in V \setminus (K_1 \cup K_2)$, $\Phi'(x) = l$ for all $x \in K_1$ and $\Phi'(x) = r$ for all $x \in K_2$. We show in the following that the triple $(V, \Phi', \mathcal{I}')$ defines the same adjacencies as the triple $(V, \Phi, \mathcal{I})$ and therefore that $(V, \Phi', \mathcal{I}')$ is a model of G (note that no point of $\mathcal{C}$ is the end of more than one interval of $\mathcal{I}'$ and no interval of $\mathcal{I}'$ includes another: this follows by construction and because $(V, \Phi, \mathcal{I})$ holds this property). Moreover $(V, \Phi', \mathcal{I}')$ is tight with respect to $\{K_1, K_2\}$ by construction and, as it is easy to check, it can be built in time $O(n)$.

Claim 3.16. *For any vertex $v \in V \setminus (K_1 \cup K_2)$ such that there exists an interval of $\mathcal{I}$ containing either $\Phi(v)$ and a_1, or $\Phi(v)$ and b_1, there exists $I \in \mathcal{I}$ such that I contains $[a_1, b_1]$ and $\Phi(v)$.*

Indeed assume first that $\Phi(v)$ and a_1 are contained in an interval J of $\mathcal{I}$ and suppose the result does not hold. Observe that $\Phi(v) \notin [a_1, b_1]$ otherwise the statement would hold with $I = I_1$. Necessarily $[\Phi(v), a_1] \subseteq J$ (else the statement holds again trivially). But then $\Phi(v) \notin (b_1, a_2]$ because otherwise Claim 3.13 would be contradicted. But we have also $\Phi(v) \notin [a_2, r]$ as $\Phi^{-1}([a_2, r]) \subseteq K_2$. Thus Φ is in (r, a_1). If there exists $w \in \Phi^{-1}(\Phi(v)) \setminus K_2$ adjacent to $\Phi^{-1}(a_1) \cap K_1$, then by homogeneity, w is adjacent to $\Phi^{-1}(b_1) \cap K_1$ and since there does not exist an interval covering $[b_1, \Phi(v)]$ (it would contradict the definition of r or Claim 3.13), there is an interval covering $[\Phi(v), b_1]$, a contradiction. But then $\Phi^{-1}(a_1) \cap K_1$ is anti-complete to $\Phi^{-1}(\Phi(v)) \setminus K_2$ and, in particular, neither a_1 nor $\Phi(v)$ is in the interior of J and thus necessarily $J = [\Phi(v), a_1]$. But $\Phi^{-1}(\Phi(v)) \setminus K_2 = \Phi^{-1}(\Phi(v))$ because else we contradict our assumption that $(V, \Phi, \mathcal{I})$ is not tight with respect to $\{K_1, K_2\}$. But now this contradicts our assumption that for every fuzzy interval in $\mathcal{I}$, every vertex mapped at one of the extremities has an adjacent and a non-adjacent vertex mapped at the other extremity.

Suppose now that $\Phi(v)$ and b_1 belong to some interval J of $\mathcal{I}$ and suppose the result does not hold. Observe again that $\Phi(v) \notin [a_1, b_1]$ otherwise the statement would hold with $I = I_1$ and thus again necessarily $[b_1, \Phi(v)] \subseteq J$ (else the statement holds again trivially). But $\Phi(v) \notin [b_2, a_1)$ because of Claim 3.13 and $\Phi(v) \notin (r, b_2)$ by definition of r. We already observed that there is no vertex of $V \setminus (K_1 \cup K_2)$ in $[a_2, r]$ thus $\Phi(v) \in (b_1, a_2)$. Now, because of interval J^*, v is adjacent to K_1 and thus complete by homogeneity. Therefore there is an interval containing $a_1, \Phi(v)$ and this has to cover $[a_1, b_1]$ because otherwise this would contradict Claim 3.13. But this is a contradiction. □

We now show that the triple $(V, \Phi', \mathcal{I}')$ defines the same adjacencies as the triple $(V, \Phi, \mathcal{I})$. We split the analysis of adjacencies into 3 cases:

(i) Let v be a vertex of $V \setminus (K_1 \cup K_2)$, and first suppose it is complete to K_1. As a consequence of Claim 3.16, there exists an interval $\tilde{I} \in \mathcal{I}$ containing $\Phi(v)$, a_1 and b_1, and, since this interval is not in $\mathcal{J}$, it also belongs to $\mathcal{I}'$. Note that $l \in \tilde{I}$, and therefore $\Phi'(v)$ and $\Phi'(K_1)$ belong to $\tilde{I} \in \mathcal{I}'$, meaning that we can preserve v complete to K_1. Now suppose that v is anti-complete to K_1. Note that $\Phi(v) \in (r, a_1)$, therefore, if v is no more anti-complete to K_1, it is because l and $\Phi(v)$ belong to some interval $\tilde{I} \in \mathcal{I}$. It follows that either $[l, \Phi(v)] \subseteq \tilde{I}$, but this contradicts the definition of r, or $[\Phi(v), l] \subseteq \tilde{I}$, but then v is not anti-complete to K_1, as a_1 is in the interior of $\tilde{I}$. The same holds for adjacencies between vertices of $V \setminus (K_1 \cup K_2)$ and K_2. Thus adjacencies between vertices of $V \setminus (K_1 \cup K_2)$ and $K_1 \cup K_2$ are preserved.

(ii) Adjacencies between two vertices u, v in $V \setminus (K_1 \cup K_2)$ is unchanged because $\Phi'(u) = \Phi(u)$, $\Phi'(v) = \Phi(v)$ and $\Phi(u), \Phi(v) \notin [l, b_1] \cup [a_2, r]$. Indeed, if $uv \in E$, either there exists $J \notin \mathcal{J}$ covering u and v or $\Phi(u), \Phi(v) \in (b_1, a_2)$, and in both cases adjacencies are preserved. Similarly, if not adjacent, at least one of u or v is in (r, l) and thus no additional interval is added in $\mathcal{I}'$ that could add the adjacency between u and v.

(iii) Adjacencies between vertices in $K_1 \cup K_2$ can be made arbitrary thanks to fuzziness of interval $[l, r]$. □

Proof of Lemma 3.4. We know from Lemma 3.11 that in time $O(n^2)$ we may build for G a model $(V, \Phi', \mathcal{J}')$ that is tight with respect to $\{K_1, K_2\}$, i.e., is such that, for some $u \in K_1$ and $v \in K_2$, either $[\Phi(u), \Phi(v)]$ or $[\Phi(v), \Phi(u)]$ belongs to $\mathcal{J}'$. We now show that we can build in $O(n)$-time from Φ' another mapping Φ'' such that $(V, \Phi'', \mathcal{J}')$ gives another model for G and satisfies the properties in the statement. Namely, define Φ'' as follows: for every vertex $x \notin K_1 \cup K_2$, let $\Phi''(x) = \Phi'(x)$; for every vertex $x \in K_1$, let $\Phi''(x) = \Phi'(u)$; for every vertex $x \in K_2$, let $\Phi''(x) = \Phi'(v)$.

In order to prove that $(V, \Phi'', \mathcal{J}')$ is a model for G, it is enough to show that $(V, \Phi'', \mathcal{J}')$ and $(V, \Phi', \mathcal{J}')$ define the same adjacencies. In particular, it suffices to show that the neighborhood of every vertex x such that $\Phi''(x) \neq \Phi'(x)$ remains the same; observe that such a vertex must belong to $K_1 \cup K_2$. Without loss of generality choose $x \in K_1$. Now consider $y \in V \setminus x$. If $\Phi''(y) = \Phi'(v)$, then the adjacency between x and y is fuzzy in the new model and of course we can preserve it. If $\Phi''(y) = \Phi'(u)$, then according to the new model, y is adjacent to x. We now show this to be correct. First, $y \notin K_2$, since $\Phi''(y) \neq \Phi'(v)$. If $y \in K_1$, then adjacency between y and x follows from K_1 being a clique. If $y \notin K_1 \cup K_2$, then $\Phi'(y) = \Phi''(y) = \Phi'(u)$ and so $uy \in E$; adjacency between x and y follows then from homogeneity. Analogously, if $\Phi''(y) \notin \{\Phi'(v), \Phi'(u)\}$, then in particular $y \notin K_1 \cup K_2$ and thus the adjacency between x and y is the same as the adjacency between u and y (by homogeneity), which is preserved. Finally, Φ'' can be built in time $O(n)$ from Φ'. □

4. An algorithm to recognize fuzzy circular interval graphs

In order to provide our recognition algorithm for FCIGs, we need a couple of results from the literature. First, we have the following simple lemma, whose proof we skip.

Lemma 4.1 (*[3]*). *Let $\{K_1, K_2\}$ be a proper pair of non-empty cliques of a graph G. It follows that the subgraph of G induced by $K_1 \cup K_2$ contains C_4 (a chordless cycle of length 4) as an induced subgraph.*

Then we need a result from [13]. There an algorithm (Algorithm 2) to find and eliminate all proper and homogeneous pairs of cliques of a graph is presented. The algorithm makes use of a suitable reduction, generalizing the one in Definition 3.1. The reduction in [13] replaces a proper and homogenous pair of cliques $\{K_1, K_2\}$ of a graph G by *any* pair of cliques $\{A_1, A_2\}$ that are C_4-free (i.e., they are such that they induce a subgraph with no C_4) and therefore, according to Lemma 4.1, that form a *non-proper* pair of cliques. Note that the reduction in this paper clearly holds that property, if we set $A_1 = \{x_1, x_2\}$ and $A_2 = \{y_1, y_2\}$. We may therefore apply the main theorem in [13, Theorem 2] to conclude that, if we apply Algorithm 2 with respect to the reduction in this paper, in at most m iterations, and in time $O(n^2m)$, we end up with a graph G' without proper and homogeneous pairs of cliques (see [13] for more details).

We are now ready to state our algorithm for recognizing *connected* fuzzy circular interval graphs (we shall take care of non-connected graphs later). The algorithm receives as input a connected graph G and recognizes whether G is fuzzy circular interval and, in case, returns an interval model.

Algorithm 1 The recognition algorithm

Apply Algorithm 2 in [12], with respect to the reduction in Definition 3.1, as to build a sequence of graphs $G = G^0, G^1, \ldots, G^q$, with $q \leq m$, such that G^q has no proper and homogeneous pairs of cliques, and each G^i is equal to $G^{i-1}|_{\{X_i, Y_i\}}$, where $\{X_i, Y_i\}$ is a proper and homogeneous pair of cliques of G^i;
if G^q is not a CIG, **then** G is not a FCIG;
else for $i = q$ **down to** 1 compute an interval model for G^{i-1} from an interval model for G^i using Theorem 3.3.

Theorem 4.2. *Algorithm 1 is correct, terminates in at most m iterations and runs in $O(n^2m)$-time.*

Proof. The algorithm defines a sequence of graphs $G^0, \ldots, G^q$. We know from Theorem 2 in [13] that $q \leq m$ and that the sequence of graphs can be built in time $O(n^2m)$ (as for q being less than or equal to m it is enough to observe that each graph G^i will have at least one edge less than the graph G^{i-1}, as each proper and homogeneous pair of cliques contains a C_4, that has 4 edges, as an induced subgraph, while the gadget we use in our reduction operation has 3 edges).

The algorithm claims that $G = G^0$ is a FCIG if and only if G^q is a CIG. That is correct. In fact, on the one hand, Theorem 3.3 ensures that each graph in the sequence is a connected graph, that is a FCIG if and only if G is so. On the other hand, since G^q is a graph without proper and homogeneous pairs of cliques, from Lemma 2.3 it is a FCIG if and only if it is a CIG.

Moreover, if G is a FCIG, then the algorithm returns a model for it. In fact, in this case, G^q is a CIG and the algorithm computes an interval model for it; this model can be then extended onto models for $G^{q-1}, \ldots, G^0$ following Theorem 3.3.

We now analyze the complexity of Algorithm 1. As we recalled above, the sequence $G^0, \ldots, G^q$ can be built in time $O(n^2m)$. As we discussed in Section 2, we can recognize if a graph with n_1 vertices and m_1 edges is circular interval, and in case build an interval model, in $O(n_1 + m_1)$-time, with a trivial modification of the algorithm in [8]; therefore we can recognize in $O(n + m)$-time whether G^q is a circular interval graph (in fact, as we already discussed $|E(G^q)| < m$, moreover, $|V(G^q)| \leq n$). Each model for G^{i+1} can be extended into a model for G^i in time $O(n^2)$ (because of Theorem 3.3 and since $|V(G^i)| \leq n$). Since the number of iterations is bounded by m, it easily follows that Algorithm 1 can be indeed implemented as to run in $O(n^2m)$-time. □

We close the paper by discussing what to do when G is not connected. In this case, we have the following simple lemma, whose proof we omit.

Lemma 4.3. *Let G be a non-connected graph. G is a fuzzy circular interval graph if and only if each connected component is a fuzzy linear interval graph.*

Therefore, the problem of recognizing non-connected FCIGs reduces to the problem of recognizing (connected) fuzzy linear interval graphs. Also the latter problem can be solved by our reduction techniques; we have in fact the following:

Theorem 4.4. *Let G be a connected graph and let $\{K_1, K_2\}$ be a proper and homogeneous pair of cliques. G is a fuzzy linear interval graph if and only if $G|_{\{K_1,K_2\}}$ is a fuzzy linear interval graph and, from a model for G, one may build in $O(n^2)$-time a model for $G|_{\{K_1,K_2\}}$, and vice versa.*

The proof of Theorem 4.4 goes along the same lines as the proof for Theorem 3.3 so we skip it. Finally, Theorem 4.4 and Lemma 2.4 reduce the recognition of non-connected FCIGs to that of linear interval graphs. The latter problem can be easily solved [6,7,17].

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