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Non-cyclic phases for neutrino oscillations in quantum field theory

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ABSTRACT

We show the presence of non-cyclic phases for oscillating neutrinos in the context of quantum field theory. Such phases carry information about the non-perturbative vacuum structure associated with the field mixing. By subtracting the condensate contribution of the flavor vacuum, the previously studied quantum mechanics geometric phase is recovered.

1. Introduction

Much attention and study has been devoted in recent years to the phenomenon of neutrino mixing and oscillations since it offers the possibility to investigate new physics beyond the Standard Model of elementary particle physics, also involving hot issues in astro-particle physics and cosmology [1–3]. As a matter of fact, great experimental and theoretical achievements have been obtained and new horizons have been opened to be explored in future research. The mixing phenomenon also offers some features such as its connection with vacuum structure [4–6] and dark energy [7], which certainly deserve further study and attention due to their physical relevance. Among such specific features, the one of the geometric phase [8–10] characterizing mixed neutrino evolution has been pointed out in Ref. [11] in the quantum mechanical (QM) framework (Pontecorvo's formalism) of neutrino mixing [1–3].

In general, geometrical phases appear in many physical systems as an observable characterization of the system evolution. The phenomenological interest in the geometric phase in neutrino evolution arises since it is found [11] to be function only of the mixing angle which thus can be measured (at least in principle) independently from dynamical parameters such as masses and energies.

Other aspects of geometric phases associated to neutrino oscillations have been studied in Refs. [12,13]. The generalization [14–16] of geometric phase to non-cyclic evolution, such as the case of three and four flavor mixing, has been also recently analyzed by using the QM formalism [17,18]. Such a formalism is known to be an useful approximation of the quantum field theory (QFT) formalism which provides the correct theoretical setting for the study of particle mixing and oscillations [4–6,19–24].

Aim of the present Letter is to study the Aharonov–Anandan geometric invariant in neutrino evolution in such a QFT formalism. We show that the QFT condensate leads to a non-cyclic time evolution of the flavor states and we compute the non-cyclic phases for oscillating neutrinos. The QM geometric phase is recovered by subtracting from the Hamiltonian the contributions from the vacuum condensate. Some light on both, the condensate structure of the vacuum of QFT neutrino mixing, and its quantum mechanical approximation is thus shed. Here we consider the case of two-flavor Dirac neutrino fields, although the conclusions we reach can be extended to the case of three flavors [21] and Majorana neutrinos [23] and to the case of mixed bosons [24].

The Letter is organized as follows. In Section 2 we summarize the results on the geometric invariant obtained for oscillating neutrinos in the context of QM. We show that the geometric phase represents the distance along the evolution of the neutrino in the projective Hilbert space, as measured by the Fubini–Study metric. In Section 3 we study the geometric invariant and the non-cyclic phases for neutrino oscillations in the context of QFT and Section 4 is devoted to discussions and conclusions. A brief summary of the vacuum structure for Dirac neutrino mixing is presented in Appendix A. In Appendix B are reported useful formulas.

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2. Geometry of neutrino oscillations in quantum mechanics

We want to study the Aharonov–Anandan geometric invariant [10]

$$s = 2\int_0^t \Delta E \, dt' \tag{1}$$

in the case of neutrino mixing. In this section we summarize the results obtained in Ref. [11] in the QM formalism for two Dirac neutrinos (the case of three neutrinos is discussed in [11,17,18] and will be commented upon in the following). We also study the invariant s in terms of Fubini–Study metrics. The mixing transformations are:

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle, \tag{2}$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle. \tag{3}$$

We focus our attention on the electron neutrino. Same discussion applies to the muon neutrino. For simplicity of notation, we omit the momentum suffix $\mathbf{k}$, the helicity label r and use $\hbar = 1$ whenever no ambiguity arises. In the present case ΔE in Eq. (1) is given by

$$\Delta E \equiv \Delta E_{e,\mu} = \langle \nu_e(t)|H|\nu_\mu(t)\rangle = \langle \nu_\mu(t)|H|\nu_e(t)\rangle, \tag{4}$$

where $|\nu_e(t)\rangle$ and $|\nu_\mu(t)\rangle$ are the electron and the muon neutrino states at time t. $|\nu_e(t)\rangle$ is given by

$$|\nu_e(t)\rangle \equiv e^{-iHt}|\nu_e(0)\rangle = e^{-i\omega_1 t}\left(\cos\theta|\nu_1\rangle + e^{-i\Omega_- t}\sin\theta|\nu_2\rangle\right), \tag{5}$$

where $\Omega_- \equiv \omega_2 - \omega_1$, $H|\nu_i\rangle = \omega_i|\nu_i\rangle$ and ω_i are the energies associated with the mass eingestates $|\nu_i\rangle$, with $i = 1, 2$. Since at a time $T = 2\pi/\Omega_-$, the state is the same as the original one, apart from a phase factor:

$$|\nu_e(T)\rangle = e^{i\phi}|\nu_e(0)\rangle, \quad \phi = -2\pi\omega_1/\Omega_-, \tag{6}$$

one obtains [11] for $t = nT$

$$s = 2\int_0^{nT} \Delta E_{e,\mu}\, dt = 2n\pi \sin 2\theta, \tag{7}$$

which is a function of the mixing angle only. We remark that the same result is obtained by observing that the phase ϕ contains a dynamical part and a geometric part β_e [10]

$$\beta_e = \phi + \int_0^T \langle \nu_e(t)|i\partial_t|\nu_e(t)\rangle = 2\pi \sin^2\theta. \tag{8}$$

For muon neutrinos we get $\beta_\mu = 2\pi\cos^2\theta$. We have $\beta_e + \beta_\mu = 2\pi$. By considering the time interval $(0, nT)$, one thus sees that the geometric phase counts oscillations, i.e. after n oscillations the phase of the electron neutrino state is $2\pi n \sin^2\theta$.

Let us now analyze the invariant s in terms of the distance between states in the Hilbert space. The evolution of the Pontecorvo states $|\nu_\sigma(t)\rangle$, is governed by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\nu_\sigma(t)\rangle = H|\nu_\sigma(t)\rangle, \quad \sigma = e, \mu. \tag{9}$$

Expanding the state $|\nu_\sigma(t+dt)\rangle$ up to the second order in dt and considering that $\frac{d}{dt}H = 0$, we have

$$\langle \nu_\sigma(t)|\nu_\sigma(t+dt)\rangle = 1 - \frac{idt}{\hbar}\langle \nu_\sigma(t)|H|\nu_\sigma(t)\rangle - \frac{dt^2}{2\hbar^2}\langle \nu_\sigma(t)|H^2|\nu_\sigma(t)\rangle + O\left(dt^3\right), \tag{10}$$

and

$$\left|\langle \nu_\sigma(t)|\nu_\sigma(t+dt)\rangle\right|^2 = 1 - \frac{dt^2}{\hbar^2}\Delta E^2_{\sigma,\sigma} + O\left(dt^3\right), \tag{11}$$

where

$$\Delta E^2_{\sigma,\sigma} \equiv \langle \nu_\sigma(t)|H^2|\nu_\sigma(t)\rangle - \left(\langle \nu_\sigma(t)|H|\nu_\sigma(t)\rangle\right)^2 = (\Omega_-)^2\sin^2\theta\cos^2\theta, \quad \sigma = e, \mu. \tag{12}$$

Then, we obtain

$$\left|\langle \nu_\sigma(t)|\nu_\sigma(t+dt)\rangle\right|^2 = 1 - \frac{dt^2}{\hbar^2}(\Omega_-)^2\sin^2\theta\cos^2\theta + O\left(dt^3\right), \quad \sigma = e, \mu, \tag{13}$$

where we have used the equations

$$\langle \nu_e(t)|H|\nu_e(t)\rangle = \omega_1\cos^2\theta + \omega_2\sin^2\theta, \tag{14}$$

$$\langle \nu_\mu(t)|H|\nu_\mu(t)\rangle = \omega_1\sin^2\theta + \omega_2\cos^2\theta, \tag{15}$$

and

$$\langle\nu_e(t)|H^2|\nu_e(t)\rangle = \omega_1^2\cos^2\theta + \omega_2^2\sin^2\theta, \tag{16}$$

$$\langle\nu_\mu(t)|H^2|\nu_\mu(t)\rangle = \omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta. \tag{17}$$

From Eqs. (4) and (12) it follows:

$$\Delta E_{e,e} = \Delta E_{\mu,\mu} = \Delta E_{e,\mu} = \Delta E_{\mu,e}. \tag{18}$$

Eq. (18) implies that $\Delta E_{e,\mu}$ in Eq. (7) (see also Eq. (4)) is nothing but the energy uncertainty (variance) given in Eq. (12). We also have

$$\left|\langle\nu_e(t)|\nu_\mu(t+dt)\rangle\right|^2 = \left|\langle\nu_\mu(t)|\nu_e(t+dt)\rangle\right|^2 = \frac{dt^2}{\hbar}\Delta E_{e,\mu}^2 + O\big(dt^3\big). \tag{19}$$

The Fubini–Study metric [10] is defined as follows

$$ds^2 = 2g_{\mu\bar\nu}\,dZ^\mu\,d\bar Z^\nu = 4\big(1-\left|\langle\nu_\sigma(t)|\nu_\sigma(t+dt)\rangle\right|^2\big), \tag{20}$$

where Z^μ are coordinates in the projective Hilbert space $\mathcal{P}$, which is the set of rays of the Hilbert space $\mathcal{H}$. From Eqs. (11), (12) and (20), we have the infinitesimal geodetic distance between the points $\Pi(|\nu_e(t)\rangle)$ and $\Pi(|\nu_e(t+dt)\rangle)$ in the space $\mathcal{P}$

$$ds = 2\frac{\Delta E_{\sigma,\sigma}\,dt}{\hbar} = 2\frac{\Omega_-\sin\theta\cos\theta\,dt}{\hbar}. \tag{21}$$

In the case of the neutrino mixing, the above defined Fubini–Study metric is the usual metric on a sphere of unitary radius: $ds^2 = d\Theta^2 + \sin^2\Theta\,d\varphi^2$, with $\Theta = 2\theta$ ($\theta =$ mixing angle) and $\Theta\in[0,\pi]$. Since θ is constant, we have $ds = \sin 2\theta\,d\varphi$ and, by comparison with Eq. (21), $d\varphi = \frac{\Omega_-}{\hbar}dt$. We thus have

$$s = \int_0^{2n\pi}\sin 2\theta\,d\varphi = 2n\pi\sin 2\theta. \tag{22}$$

Eq. (22) coincides with Eq. (7), which thus represents the distance between neutrino evolution states, as measured by the Fubini–Study metric, in the projective Hilbert space $\mathcal{P}$.

The case of three and four flavor mixing has been considered in Refs. [17,18] where it has been shown that a generalization [14–16] of the geometric phase to non-cyclic evolution (non-cyclic phase or Pancharatnam phase) needs to be used in order to capture the geometric aspects of the neutrino phase in such cases. The definition for the non-cyclic phase adopted in Ref. [17] is

$$\beta = \mathrm{Arg}\left(\langle\nu_\sigma(0)|\exp\left[\frac{i}{\hbar}\int_0^t\langle E(t')\rangle dt'\right]|\nu_\sigma(t)\rangle\right), \tag{23}$$

where, for example, in the case of a three-flavor electron neutrino state, $\sigma = e$,

$$|\nu_e(t)\rangle = e^{-i\omega_1 t}\cos\theta_{12}\cos\theta_{13}|\nu_1\rangle + e^{-i\omega_2 t}\sin\theta_{12}\cos\theta_{13}|\nu_2\rangle + e^{-i\omega_3 t}e^{-i\delta}\sin\theta_{13}|\nu_3\rangle, \tag{24}$$

with θ_{12} and θ_{13} mixing angles; δ is the CP violating phase and $\langle E\rangle(t)$ is given by

$$\langle E\rangle(t) = \omega_1\cos^2\theta_{12}\cos^2\theta_{13} + \omega_2\sin^2\theta_{12}\cos^2\theta_{13} + \omega_2\sin^2\theta_{13}, \tag{25}$$

from which β_{ee} is calculated [17].

3. Non-cyclic phases for neutrino oscillations in QFT

We now study the Aharonov–Anandan geometric invariant in the context of QFT. For simplicity, we study only the case of two flavor mixing; three flavor mixing including CP violation will be analyzed elsewhere.

In a standard notation, the Dirac neutrino fields $\nu_1(x)$ and $\nu_2(x)$ with definite masses m_1 and m_2, respectively, are written as

$$\nu_i(x) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},r}\big[u^r_{\mathbf{k},i}\alpha^r_{\mathbf{k},i}(t) + v^r_{-\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},i}(t)\big]e^{i\mathbf{k}\cdot\mathbf{x}}, \quad i=1,2, \tag{26}$$

with $\alpha^r_{\mathbf{k},i}(t) = \alpha^r_{\mathbf{k},i}e^{-i\omega_{k,i}t}$, $\beta^{r\dagger}_{\mathbf{k},i}(t) = \beta^{r\dagger}_{\mathbf{k},i}e^{i\omega_{k,i}t}$, and $\omega_{k,i} = \sqrt{\mathbf{k}^2+m_i^2}$. The operator $\alpha^r_{\mathbf{k},i}$ and $\beta^r_{\mathbf{k},i}$, $i=1,2$, $r=1,2$, are the annihilator operators for the vacuum state $|0\rangle_{1,2}\equiv|0\rangle_1\otimes|0\rangle_2$: $\alpha^r_{\mathbf{k},i}|0\rangle_{12} = \beta^r_{\mathbf{k},i}|0\rangle_{12} = 0$. The above fields and wavefunctions satisfy standard anti-commutation, orthonormality and completeness relations (see Ref. [4]).

The field mixing relations are

$$\nu_e(x) = \cos\theta\,\nu_1(x) + \sin\theta\,\nu_2(x), \tag{27}$$

$$\nu_\mu(x) = -\sin\theta\,\nu_1(x) + \cos\theta\,\nu_2(x), \tag{28}$$

where $\nu_e(x)$ and $\nu_\mu(x)$ are the Dirac neutrino fields with definite flavors. The generator of these mixing transformations is given by [4]

$$G(\theta,t)=\exp\left[\theta\int d^3\mathbf{x}\big(\nu_1^\dagger(x)\nu_2(x)-\nu_2^\dagger(x)\nu_1(x)\big)\right], \tag{29}$$

$$\nu_e(x)=G^{-1}(\theta,t)\nu_1(x)G(\theta,t), \tag{30}$$

$$\nu_\mu(x)=G^{-1}(\theta,t)\nu_2(x)G(\theta,t). \tag{31}$$

At finite volume, $G(\theta,t)$ is an unitary operator, $G^{-1}(\theta,t)=G(-\theta,t)=G^\dagger(\theta,t)$, preserving the canonical anticommutation relations. The generator $G^{-1}(\theta,t)$ maps the Hilbert space $\mathcal{H}_{1,2}$ for ν_1,ν_2 fields to the Hilbert spaces for flavor fields $\mathcal{H}_{e,\mu}$ $:G^{-1}(\theta,t):\mathcal{H}_{1,2}\mapsto\mathcal{H}_{e,\mu}$. In particular, for the vacuum $|0\rangle_{1,2}$ we have, at finite volume V:

$$\big|0(t)\big\rangle_{e,\mu}=G^{-1}(\theta,t)|0\rangle_{1,2}. \tag{32}$$

$|0\rangle_{e,\mu}(t)$ is the vacuum for $\mathcal{H}_{e,\mu}$, which we will refer to as the flavor vacuum. It is annihilated by the annihilation operators of $\nu_e(x)$ and $\nu_\mu(x)$ neutrinos, $\alpha^r_{\mathbf{k},\sigma}(t)|0(t)\rangle_{e,\mu}=0=\beta^r_{\mathbf{k},\sigma}(t)|0(t)\rangle_{e,\mu}$, with $(\sigma,i)=(e,1),(\mu,2)$ and

$$\alpha^r_{\mathbf{k},\sigma}(t)\equiv G^{-1}(\theta,t)\alpha^r_{\mathbf{k},i}(t)G(\theta,t), \tag{33}$$

$$\beta^r_{\mathbf{k},\sigma}(t)\equiv G^{-1}(\theta,t)\beta^r_{\mathbf{k},i}(t)G(\theta,t). \tag{34}$$

The non-trivial structure of the flavor vacuum is such that even in the simplest two flavor case, flavor neutrino states have a multiparticle component which makes non-cyclic the time evolution associated to them. Indeed, at time t, the flavor states in the reference frame for which $\mathbf{k}=(0,0,|\mathbf{k}|)$ are:

$$\begin{aligned}\big|\nu^r_{\mathbf{k},e}(t)\big\rangle&\equiv\alpha^{r\dagger}_{\mathbf{k},e}(t)\big|0(t)\big\rangle_{e,\mu}=e^{-i:H:t}\big|\nu^r_{\mathbf{k},e}(0)\big\rangle\\&=e^{-i\omega_{k,1}t}\big[\cos\theta\,\alpha^{r\dagger}_{\mathbf{k},1}+|U_\mathbf{k}|e^{-i\Omega^k_-t}\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},2}-\epsilon^r|V_\mathbf{k}|e^{-i\Omega^k_+t}\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\big]G^{-1}_{\mathbf{k},s\neq r}(\theta,t)\prod_{\mathbf{p}\neq\mathbf{k}}G^{-1}_\mathbf{p}(\theta,t)|0\rangle_{1,2},\end{aligned} \tag{35}$$

$$\begin{aligned}\big|\nu^r_{\mathbf{k},\mu}(t)\big\rangle&\equiv\alpha^{r\dagger}_{\mathbf{k},\mu}(t)\big|0(t)\big\rangle_{e,\mu}=e^{-i:H:t}\big|\nu^r_{\mathbf{k},\mu}(0)\big\rangle\\&=e^{-i\omega_{k,2}t}\big[\cos\theta\,\alpha^{r\dagger}_{\mathbf{k},2}-|U_\mathbf{k}|e^{i\Omega^k_-t}\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},1}+\epsilon^r|V_\mathbf{k}|e^{-i\Omega^k_+t}\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}\big]G^{-1}_{\mathbf{k},s\neq r}(\theta,t)\prod_{\mathbf{p}\neq\mathbf{k}}G^{-1}_\mathbf{p}(\theta,t)|0\rangle_{1,2},\end{aligned} \tag{36}$$

where $\Omega^k_+\equiv\omega_{k,2}+\omega_{k,1}$, $\Omega^k_-\equiv\omega_{k,2}-\omega_{k,1}$, and

$$:H:=H-{}_{1,2}\langle 0|H|0\rangle_{1,2}=H+2\int d^3\mathbf{k}\,\Omega^k_+=\sum_i\sum_r\int d^3\mathbf{k}\,\omega_{k,i}\big[\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i}+\beta^{r\dagger}_{\mathbf{k},i}\beta^r_{\mathbf{k},i}\big], \tag{37}$$

is the Hamiltonian normal ordered with respect to the vacuum $|0\rangle_{1,2}$. It satisfies Eqs. (B.1)–(B.7) given in Appendix B, and $:H:|\nu_i\rangle=\omega_{k,i}|\nu_i\rangle$, with $i=1,2$. We have used the notation $G(\theta,t)=\prod_\mathbf{p}G_\mathbf{p}(\theta,t)=\prod_\mathbf{p}\prod_s G_{\mathbf{p},s}(\theta,t)$ (cf. Eq. (29)). Note that in the flavor states, the multiparticle components disappear in the relativistic limit $|\mathbf{k}|\gg\sqrt{m_1m_2}$, where $|U_\mathbf{k}|^2\to1$ and $|V_\mathbf{k}|^2\to0$ and the quantum mechanical Pontecorvo's states are recovered.

Eqs. (35), (36) show that the non-cyclic time evolution of mixed neutrino states is due to the presence of two oscillation frequencies, namely Ω_+ and Ω_-. Note however that the definition of the geometric phase given in Eq. (23) is not applicable in the QFT mixing formalism, since quantities like $\langle\nu_\sigma(t)|\nu_\sigma(t')\rangle$, with $t\neq t'$, are zero in the infinite volume limit [22]. On the other hand, the geometric invariant defined in Ref. [10] (see Eq. (7)) is suitable for the present case since it is well defined in the case of non-cyclic time evolution and does not involve products of states at different times. We thus consider the quantities

$$s_{\sigma,\tau}(t)=2\int_0^t\Delta E_{\sigma,\tau}\,dt, \tag{38}$$

where $\Delta E\equiv\Delta E^r_\mathbf{k}$ and σ,τ are labels specifying the states used in computing the uncertainties $\Delta E_{\sigma,\tau}$ in the integrals.

We first compute $\Delta E_{\sigma,\sigma}$ with $\sigma=e,\mu$ by using $:H:$. We have

$$\Delta E^2_{\sigma,\sigma}=\big\langle\nu^r_{\mathbf{k},\sigma}(t)\big|(:H:)^2\big|\nu^r_{\mathbf{k},\sigma}(t)\big\rangle-\big(\big\langle\nu^r_{\mathbf{k},\sigma}(t)\big|:H:\big|\nu^r_{\mathbf{k},\sigma}(t)\big\rangle\big)^2,\quad\sigma=e,\mu. \tag{39}$$

By using Eqs. (B.1), (B.3), and Eqs. (B.2), (B.4), we obtain

$$\Delta E^2_{e,e}=\sin^2\theta\cos^2\theta\left[\left(\Omega^k_-\right)^2+4\omega_{k,1}\omega_{k,2}|V_\mathbf{k}|^2\right]+4\omega^2_{k,1}\sin^4\theta\,|U_\mathbf{k}|^2|V_\mathbf{k}|^2, \tag{40}$$

$$\Delta E^2_{\mu,\mu}=\sin^2\theta\cos^2\theta\left[\left(\Omega^k_-\right)^2+4\omega_{k,1}\omega_{k,2}|V_\mathbf{k}|^2\right]+4\omega^2_{k,2}\sin^4\theta\,|U_\mathbf{k}|^2|V_\mathbf{k}|^2. \tag{41}$$

In analogy with Eq. (4) defined in QM, $\Delta E_{e,\mu}$ in QFT is given by

$$\Delta E_{e,\mu}=\big\langle\nu^r_{\mathbf{k},e}(t)\big|:H:\big|\nu^r_{\mathbf{k},\mu}(t)\big\rangle=\big\langle\nu^r_{\mathbf{k},\mu}(t)\big|:H:\big|\nu^r_{\mathbf{k},e}(t)\big\rangle=\Omega^k_-\sin\theta\cos\theta|U_\mathbf{k}|. \tag{42}$$

By defining, at time t, the multi-particle flavor states (their explicit expressions are given in Appendix A):

$$\big|\nu^r_{\mathbf{k},e\bar{e}\mu}(t)\big\rangle\equiv\alpha^{r\dagger}_{\mathbf{k},e}(t)\beta^{r\dagger}_{-\mathbf{k},e}(t)\alpha^{r\dagger}_{\mathbf{k},\mu}(t)\big|0(t)\big\rangle_{e,\mu}, \tag{43}$$

$$\big|\nu^r_{\mathbf{k},\mu\bar{\mu}e}(t)\big\rangle\equiv\alpha^{r\dagger}_{\mathbf{k},\mu}(t)\beta^{r\dagger}_{-\mathbf{k},\mu}(t)\alpha^{r\dagger}_{\mathbf{k},e}(t)\big|0(t)\big\rangle_{e,\mu}, \tag{44}$$

we have also the following non-zero expectation values:

$$\Delta E_{\mu\bar{e}e,e} = \langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|{:}H{:}|\nu^r_{\mathbf{k},e}(t)\rangle, \qquad \Delta E_{e\bar{\mu}\mu,e} = \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|{:}H{:}|\nu^r_{\mathbf{k},e}(t)\rangle, \tag{45}$$

$$\Delta E_{\mu\bar{e}e,\mu} = \langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|{:}H{:}|\nu^r_{\mathbf{k},\mu}(t)\rangle, \qquad \Delta E_{e\bar{\mu}\mu,\mu} = \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|{:}H{:}|\nu^r_{\mathbf{k},\mu}(t)\rangle, \tag{46}$$

whose explicit expressions are given in Appendix B.

Let us now note that $\Delta E^2_{e,e}$ and $\Delta E^2_{\mu,\mu}$ can be also obtained as follows:

$$\Delta E^2_{e,e} = \Delta E^2_{e,\mu} + \Delta E^2_{\mu\bar{e}e,e} + \Delta E^2_{e\bar{\mu}\mu,e}, \tag{47}$$

$$\Delta E^2_{\mu,\mu} = \Delta E^2_{e,\mu} + \Delta E^2_{\mu\bar{e}e,\mu} + \Delta E^2_{e\bar{\mu}\mu,\mu}. \tag{48}$$

Eqs. (47), (48) represent a generalization of the relation (18) to the case of QFT flavor states taking into account the multiparticle components due to the condensate structure of the flavor vacuum.

The explicit expressions for $s_{\sigma,\tau}$, with $\sigma, \tau = e, \mu, e\bar{\mu}\mu, \mu\bar{e}e$ are given by:

$$s_{e,e}(t) = 2t\sin\theta\sqrt{\cos^2\theta\left[\left(\Omega^k_-\right)^2 + 4\omega_{k,1}\omega_{k,2}|V_\mathbf{k}|^2\right] + 4\omega^2_{k,1}\sin^2\theta\,|U_\mathbf{k}|^2|V_\mathbf{k}|^2}, \tag{49}$$

$$s_{\mu,\mu}(t) = 2t\sin\theta\sqrt{\cos^2\theta\left[\left(\Omega^k_-\right)^2 + 4\omega_{k,1}\omega_{k,2}|V_\mathbf{k}|^2\right] + 4\omega^2_{k,2}\sin^2\theta\,|U_\mathbf{k}|^2|V_\mathbf{k}|^2}, \tag{50}$$

$$s_{e,\mu}(t) = \Omega^k_- t\sin 2\theta\,|U_\mathbf{k}|, \qquad s_{\mu\bar{e}e,e}(t) = s_{e\bar{\mu}\mu,\mu}(t) = \epsilon^r\Omega^k_+ t\sin 2\theta\,|V_\mathbf{k}|, \tag{51}$$

$$s_{e\bar{\mu}\mu,e}(t) = 4\epsilon^r\omega_{k,1}t\sin^2\theta\,|U_\mathbf{k}||V_\mathbf{k}|, \qquad s_{\mu\bar{e}e,\mu}(t) = -4\epsilon^r\omega_{k,2}t\sin^2\theta\,|U_\mathbf{k}||V_\mathbf{k}|. \tag{52}$$

From Eqs. (49)–(52) we see that in the relativistic limit, $\mathbf{k} \gg \sqrt{m_1 m_2}$, where $|V_\mathbf{k}| \to 0$, $|U_\mathbf{k}| \to 1$, we have $s_{\mu\bar{e}e,e} = s_{e\bar{\mu}\mu,e} = s_{\mu\bar{e}e,\mu} = s_{e\bar{\mu}\mu,\mu} = 0$. In such a limit, from Appendix B and Eqs. (40), (41), we have $\Delta E_{e,e} = \Delta E_{\mu,\mu} = \Delta E_{e,\mu} = \Omega^k_- \sin\theta\cos\theta$. In particular, if the time t is set $t = 2n\pi/\Omega^k_-$, the quantum mechanical result is consistently recovered and the geometric invariants $s_{e,e} = s_{\mu,\mu} = s_{e,\mu} = 2n\pi\sin 2\theta$ coincide with the one given in Eq. (7).

We point out that, since $|0\rangle_{1,2}$ and $|0\rangle_{e,\mu}$ are unitary inequivalent states in the infinite volume limit, two different normal orderings must be defined, respectively with respect to the vacuum $|0\rangle_{1,2}$ for fields with definite masses, as usual denoted by ${:}\cdots{:}$, and with respect to the vacuum for fields with definite flavor $|0\rangle_{e,\mu}$, denoted by ${::}\cdots{::}$. The uncertainties $\Delta E_{\sigma,\tau}$ can be then computed by using ${:}H{:}$ as done above or with ${::}H{::}$. The Hamiltonian normal ordered with respect to the vacuum $|0\rangle_{e,\mu}$ is given by

$${::}H{::} \equiv H - {}_{e,\mu}\langle 0|H|0\rangle_{e,\mu} = H + 2\int d^3\mathbf{k}\,\Omega^k_+\left(1 - 2|V_\mathbf{k}|^2\sin^2\theta\right). \tag{53}$$

Considering now the expectation values of ${::}H{::}$ on the flavor states given in Appendix B, we have

$$\Delta E_{e,\mu} = \langle \nu^r_{\mathbf{k},e}(t)|{:}H{:}|\nu^r_{\mathbf{k},\mu}(t)\rangle = \langle \nu^r_{\mathbf{k},e}(t)|{::}H{::}|\nu^r_{\mathbf{k},\mu}(t)\rangle. \tag{54}$$

On the other hand, defining the uncertainties $\Delta\tilde{E}_{\sigma,\sigma}$ as

$$\Delta\tilde{E}^2_{\sigma,\sigma} = \langle \nu^r_{\mathbf{k},\sigma}(t)|({::}H{::})^2|\nu^r_{\mathbf{k},\sigma}(t)\rangle - \left(\langle \nu^r_{\mathbf{k},\sigma}(t)|{::}H{::}|\nu^r_{\mathbf{k},\sigma}(t)\rangle\right)^2, \quad \sigma = e, \mu, \tag{55}$$

and by using the relations in Appendix B, we have $\Delta\tilde{E}^2_{e,e} = \Delta E^2_{e,e}$ and $\Delta\tilde{E}^2_{\mu,\mu} = \Delta E^2_{\mu,\mu}$, that is, $\Delta E^2_{\sigma,\sigma}$ are independent of the normal ordering used, ${:}H{:}$ or ${::}H{::}$. Moreover, by comparing the expectation values of ${:}H{:}$ and ${::}H{::}$ presented in Appendix B, we obtain that $\Delta E_{e,\mu}, \Delta E_{\mu\bar{e}e,e}, \Delta E_{e\bar{\mu}\mu,e}, \Delta E_{\mu\bar{e}e,\mu}, \Delta E_{e\bar{\mu}\mu,\mu}$ are also independent of the particular normal ordering used. This implies that the invariants of Eqs. (49)–(52) are independent of the normal ordering used.

4. Discussion and conclusions

Let us conclude the Letter with some further comments. It is interesting to define the operator $H'(t)$:

$$H'(t) \equiv \sum_r \int d^3\mathbf{k}\left[\omega_{ee}\left(\alpha^{r\dagger}_{\mathbf{k},e}(t)\alpha^r_{\mathbf{k},e}(t) + \beta^{r\dagger}_{-\mathbf{k},e}(t)\beta^r_{-\mathbf{k},e}(t)\right) + \omega_{\mu\mu}\left(\alpha^{r\dagger}_{\mathbf{k},\mu}(t)\alpha^r_{\mathbf{k},\mu}(t) + \beta^{r\dagger}_{-\mathbf{k},\mu}(t)\beta^r_{-\mathbf{k},\mu}(t)\right)\right.$$
$$\left. + \,\omega_{\mu e}\left(\alpha^{r\dagger}_{\mathbf{k},e}(t)\alpha^r_{\mathbf{k},\mu}(t) + \alpha^{r\dagger}_{\mathbf{k},\mu}(t)\alpha^r_{\mathbf{k},e}(t) + \beta^{r\dagger}_{-\mathbf{k},e}(t)\beta^r_{-\mathbf{k},\mu}(t) + \beta^{r\dagger}_{-\mathbf{k},\mu}(t)\beta^r_{-\mathbf{k},e}(t)\right)\right], \tag{56}$$

where $\omega_{ee} \equiv \omega_{k,1}\cos^2\theta + \omega_{k,2}\sin^2\theta$, $\omega_{\mu\mu} \equiv \omega_{k,1}\sin^2\theta + \omega_{k,2}\cos^2\theta$, $\omega_{\mu e} \equiv \Omega^k_-\sin\theta\cos\theta$. We have

$$\langle \nu^r_{\mathbf{k},e}(t)|H'(t)|\nu^r_{\mathbf{k},e}(t)\rangle = \omega_{k,1}\cos^2\theta + \omega_{k,2}\sin^2\theta, \tag{57}$$

$$\langle \nu^r_{\mathbf{k},\mu}(t)|H'(t)|\nu^r_{\mathbf{k},\mu}(t)\rangle = \omega_{k,2}\cos^2\theta + \omega_{k,1}\sin^2\theta, \tag{58}$$

$$\langle \nu^r_{\mathbf{k},e}(t)|H'(t)|\nu^r_{\mathbf{k},\mu}(t)\rangle = \Omega^k_-\sin\theta\cos\theta, \tag{59}$$

$$\langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|H'(t)|\nu^r_{\mathbf{k},e}(t)\rangle = \langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|H'(t)|\nu^r_{\mathbf{k},\mu}(t)\rangle = \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|H'(t)|\nu^r_{\mathbf{k},e}(t)\rangle = \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|H'(t)|\nu^r_{\mathbf{k},\mu}(t)\rangle = 0. \tag{60}$$

From the above expectation values, we see that contributions from the flavor vacuum condensate have been eliminated. Indeed, Eqs. (57)–(59) coincide with Eqs. (14), (15) and (4) derived in the QM case (see Section 2). Moreover, the uncertainties in the energy $H'(t)$ of the multi-particle states (43), (44) are zero such as in QM.

An invariant analogous to the one introduced in Section 2, can be then defined as

$$s'_e = s'_\mu = 2\int_0^{nT} \Delta E'\,dt = 2n\pi\,\sin 2\theta, \tag{61}$$

where $T = 2n\pi/\Omega_-^k$ and

$$\Delta E'^2_{e,e} = \Delta E'^2_{\mu,\mu} = \Delta E'^2_{e,\mu} = \langle \nu^r_{\mathbf{k},\sigma}(t)|H'^2(t)|\nu^r_{\mathbf{k},\sigma}(t)\rangle - \left(\langle \nu^r_{\mathbf{k},\sigma}(t)|H'(t)|\nu^r_{\mathbf{k},\sigma}(t)\rangle\right)^2$$
$$= \left(\langle \nu^r_{\mathbf{k},e}(t)|H'(t)|\nu^r_{\mathbf{k},\mu}(t)\rangle\right)^2 = \left(\Omega_-^k\right)^2 \sin^2\theta\cos^2\theta, \quad \sigma = e,\mu. \tag{62}$$

In summary, in this Letter we have calculated the non-cyclic phases for neutrino oscillations in the context of QFT, for the case of two flavors. In the relativistic limit, where the quantum mechanical approximation holds, the QM geometric phase is recovered. The above analysis is suitable for treatment of three flavor case (see Ref. [21]) where, however, differences due to the presence of CP violating phase are expected.

Questions not considered in the present Letter, like the extension of the present and previous results to wave-packet formalism, or the suggestion of experimental setups by means of which geometric phases associated to neutrino oscillations could be detected are certainly interesting and deserve a separate analysis.

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Appendix A. Flavor fields and QFT flavor states

By taking into account the relations Eqs. (26)–(33), the flavor fields can be written as:

$$\nu_\sigma(\mathbf{x},t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},r} e^{i\mathbf{k}.\mathbf{x}}\left[u^r_{\mathbf{k},i}\alpha^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\right], \quad (\sigma,i) = (e,1),(\mu,2). \tag{A.1}$$

In the reference frame such that $\mathbf{k} = (0,0,|\mathbf{k}|)$ the annihilation operators of $\nu_e(x)$ and $\nu_\mu(x)$ are explicitly given by

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1}(t) + \sin\theta\left(|U_\mathbf{k}|\alpha^r_{\mathbf{k},2}(t) + \epsilon^r|V_\mathbf{k}|\beta^{r\dagger}_{-\mathbf{k},2}(t)\right), \tag{A.2}$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2}(t) - \sin\theta\left(|U_\mathbf{k}|\alpha^r_{\mathbf{k},1}(t) - \epsilon^r|V_\mathbf{k}|\beta^{r\dagger}_{-\mathbf{k},1}(t)\right), \tag{A.3}$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1}(t) + \sin\theta\left(|U_\mathbf{k}|\beta^r_{-\mathbf{k},2}(t) - \epsilon^r|V_\mathbf{k}|\alpha^{r\dagger}_{\mathbf{k},2}(t)\right), \tag{A.4}$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2}(t) - \sin\theta\left(|U_\mathbf{k}|\beta^r_{-\mathbf{k},1}(t) + \epsilon^r|V_\mathbf{k}|\alpha^{r\dagger}_{\mathbf{k},1}(t)\right), \tag{A.5}$$

with $\epsilon^r = (-1)^r$ and

$$|U_\mathbf{k}| \equiv u^{r\dagger}_{\mathbf{k},i}u^r_{\mathbf{k},j} = v^{r\dagger}_{-\mathbf{k},i}v^r_{-\mathbf{k},j}, \qquad |V_\mathbf{k}| \equiv \epsilon^r u^{r\dagger}_{\mathbf{k},1}v^r_{-\mathbf{k},2} = -\epsilon^r u^{r\dagger}_{\mathbf{k},2}v^r_{-\mathbf{k},1},$$

where $i,j = 1,2$ and $i \neq j$. We have:

$$|U_\mathbf{k}| = \frac{|\mathbf{k}|^2 + (\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}{2\sqrt{\omega_{k,1}\omega_{k,2}(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}}, \qquad |V_\mathbf{k}| = \frac{(\omega_{k,1}+m_1)-(\omega_{k,2}+m_2)}{2\sqrt{\omega_{k,1}\omega_{k,2}(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}}|\mathbf{k}|, \tag{A.6}$$

$$|U_\mathbf{k}|^2 + |V_\mathbf{k}|^2 = 1. \tag{A.7}$$

The number of condensed neutrinos for each $\mathbf{k}$ is given by

$${}_{e,\mu}\langle 0|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i}|0\rangle_{e,\mu} = {}_{e,\mu}\langle 0|\beta^{r\dagger}_{\mathbf{k},i}\beta^r_{\mathbf{k},i}|0\rangle_{e,\mu} = \sin^2\theta\,|V_\mathbf{k}|^2, \quad i = 1,2. \tag{A.8}$$

The explicit expression for $|0\rangle_{e,\mu}$ at time $t = 0$ in the reference frame for which $\mathbf{k} = (0,0,|\mathbf{k}|)$ is

$$|0\rangle_{e,\mu} = \prod_{r,\mathbf{k}}\Big[\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right) - \epsilon^r\sin\theta\cos\theta\,|V_\mathbf{k}|\left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right)$$
$$+ \epsilon^r\sin^2\theta\,|V_\mathbf{k}||U_\mathbf{k}|\left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}\right) + \sin^2\theta\,|V_\mathbf{k}|^2\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]|0\rangle_{1,2}. \tag{A.9}$$

Eq. (A.9) exhibits the condensate structure of the flavor vacuum $|0\rangle_{e,\mu}$. The important point is that ${}_{1,2}\langle 0|0(t)\rangle_{e,\mu} \to 0$, for any t, in the infinite volume limit [4]. Thus, in such a limit the Hilbert spaces $\mathcal{H}_{1,2}$ and $\mathcal{H}_{e,\mu}$ turn out to be unitarily inequivalent spaces.

The explicit form of the multi-particle states defined in Eqs. (43), (44) is:

$$\left|\nu^r_{\mathbf{k},e\bar{e}\mu}(t)\right\rangle = -\Big[\cos\theta\,\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}e^{-i(2\omega_{k,1}+\omega_{k,2})t} + \epsilon^r|V_\mathbf{k}|\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},1}e^{-i\omega_{k,1}t}$$
$$+ |U_\mathbf{k}|\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}e^{-i(\omega_{k,1}+2\omega_{k,2})t}\Big]G^{-1}_{\mathbf{k},s\neq r}(\theta,t)\prod_{\mathbf{p}\neq\mathbf{k}}G^{-1}_\mathbf{p}(\theta,t)|0\rangle_{1,2}, \tag{A.10}$$

$$\left|\nu^r_{\mathbf{k},\mu\bar{\mu}e}(t)\right\rangle = \Big[\cos\theta\,\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}e^{-i(\omega_{k,1}+2\omega_{k,2})t} - \epsilon^r|V_\mathbf{k}|\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},2}e^{-i\omega_{k,2}t}$$
$$- |U_\mathbf{k}|\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}e^{-i(2\omega_{k,1}+\omega_{k,2})t}\Big]G^{-1}_{\mathbf{k},s\neq r}(\theta,t)\prod_{\mathbf{p}\neq\mathbf{k}}G^{-1}_\mathbf{p}(\theta,t)|0\rangle_{1,2}. \tag{A.11}$$

Appendix B. Expectation values of $:H:$ and $::H::$

The flavor states introduced in Appendix A are used in computing the following expectation values for the Hamiltonian $:H:$, $::H::$. We have:

$$\langle \nu^r_{\mathbf{k},e}(t)|{:}H{:}|\nu^r_{\mathbf{k},e}(t)\rangle = \omega_{k,1}\big(\cos^2\theta + 2\sin^2\theta\,|V_\mathbf{k}|^2\big) + \omega_{k,2}\sin^2\theta, \tag{B.1}$$

$$\langle \nu^r_{\mathbf{k},\mu}(t)|{:}H{:}|\nu^r_{\mathbf{k},\mu}(t)\rangle = \omega_{k,1}\sin^2\theta + \omega_{k,2}\big(\cos^2\theta + 2\sin^2\theta\,|V_\mathbf{k}|^2\big), \tag{B.2}$$

$$\langle \nu^r_{\mathbf{k},e}(t)|({:}H{:})^2|\nu^r_{\mathbf{k},e}(t)\rangle = \omega^2_{k,1}\big(\cos^2\theta + 4\sin^2\theta\,|V_\mathbf{k}|^2\big) + \omega^2_{k,2}\sin^2\theta + 4\omega_{k,1}\omega_{k,2}\sin^2\theta\,|V_\mathbf{k}|^2, \tag{B.3}$$

$$\langle \nu^r_{\mathbf{k},\mu}(t)|({:}H{:})^2|\nu^r_{\mathbf{k},\mu}(t)\rangle = \omega^2_{k,1}\sin^2\theta + \omega^2_{k,2}\big(\cos^2\theta + 4\sin^2\theta\,|V_\mathbf{k}|^2\big) + 4\omega_{k,1}\omega_{k,2}\sin^2\theta\,|V_\mathbf{k}|^2, \tag{B.4}$$

$$\langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|{:}H{:}|\nu^r_{\mathbf{k},e}(t)\rangle = 2\epsilon^r\omega_{k,1}\sin^2\theta\,|U_\mathbf{k}||V_\mathbf{k}|, \tag{B.5}$$

$$\langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|{:}H{:}|\nu^r_{\mathbf{k},\mu}(t)\rangle = -2\epsilon^r\omega_{k,2}\sin^2\theta\,|U_\mathbf{k}||V_\mathbf{k}|, \tag{B.6}$$

$$\langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|{:}H{:}|\nu^r_{\mathbf{k},\mu}(t)\rangle = \langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|{:}H{:}|\nu^r_{\mathbf{k},e}(t)\rangle = \epsilon^r\Omega^k_+\sin\theta\cos\theta\,|V_\mathbf{k}|. \tag{B.7}$$

The Hamiltonian normal ordered with respect to the flavor vacuum $::H::$ satisfies the following relations:

$$\langle \nu^r_{\mathbf{k},e}(t)|{::}H{::}|\nu^r_{\mathbf{k},e}(t)\rangle = \omega_{k,1}\cos^2\theta + \omega_{k,2}\sin^2\theta\,\big(1 - 2|V_\mathbf{k}|^2\big), \tag{B.8}$$

$$\langle \nu^r_{\mathbf{k},\mu}(t)|{::}H{::}|\nu^r_{\mathbf{k},\mu}(t)\rangle = \omega_{k,1}\sin^2\theta\big(1 - 2|V_\mathbf{k}|^2\big) + \omega_{k,2}\cos^2\theta, \tag{B.9}$$

$$\langle \nu^r_{\mathbf{k},e}(t)|{::}H{::}|\nu^r_{\mathbf{k},\mu}(t)\rangle = \langle \nu^r_{\mathbf{k},\mu}(t)|{::}H{::}|\nu^r_{\mathbf{k},e}(t)\rangle = \Omega^k_-\sin\theta\cos\theta\,|U_\mathbf{k}|, \tag{B.10}$$

$$\langle \nu^r_{\mathbf{k},e}(t)|({::}H{::})^2|\nu^r_{\mathbf{k},e}(t)\rangle = \omega^2_{k,1}\big(\cos^2\theta + 4\sin^4\theta\,|U_\mathbf{k}|^2|V_\mathbf{k}|^2\big) + \omega^2_{k,2}\sin^2\theta\big(1 - 4\sin^2\theta\,|U_\mathbf{k}|^2|V_\mathbf{k}|^2\big), \tag{B.11}$$

$$\langle \nu^r_{\mathbf{k},\mu}(t)|({::}H{::})^2|\nu^r_{\mathbf{k},\mu}(t)\rangle = \omega^2_{k,1}\sin^2\theta\big(1 - 4\sin^2\theta\,|U_\mathbf{k}|^2|V_\mathbf{k}|^2\big) + \omega^2_{k,2}\big(\cos^2\theta + 4\sin^4\theta\,|U_\mathbf{k}|^2|V_\mathbf{k}|^2\big). \tag{B.12}$$

Finally we have:

$$\langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|{::}H{::}|\nu^r_{\mathbf{k},e}(t)\rangle = \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|{:}H{:}|\nu^r_{\mathbf{k},e}(t)\rangle, \qquad \langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|{::}H{::}|\nu^r_{\mathbf{k},\mu}(t)\rangle = \langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|{:}H{:}|\nu^r_{\mathbf{k},\mu}(t)\rangle, \tag{B.13}$$

$$\langle \nu^r_{\mathbf{k},\mu\bar{e}e}(t)|{::}H{::}|\nu^r_{\mathbf{k},e}(t)\rangle = \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|{::}H{::}|\nu^r_{\mathbf{k},\mu}(t)\rangle = \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|{:}H{:}|\nu^r_{\mathbf{k},\mu}(t)\rangle = \langle \nu^r_{\mathbf{k},e\bar{\mu}\mu}(t)|{:}H{:}|\nu^r_{\mathbf{k},\mu}(t)\rangle. \tag{B.14}$$

References

[1] S.M. Bilenky, B. Pontecorvo, Phys. Rep. 41 (1978) 225.
[2] S.M. Bilenky, S.T. Petcov, Rev. Mod. Phys. 59 (1987) 671.
[3] R.N. Mohapatra, P.B. Pal, World Sci. Lect. Notes Phys. 41 (1991) 1;
J.N. Bahcall, Neutrino Astrophysics, Cambridge Univ. Press, Cambridge, 1989.
[4] M. Blasone, G. Vitiello, Ann. Phys. 244 (1995) 283.
[5] M. Blasone, P.A. Henning, G. Vitiello, Phys. Lett. B 451 (1999) 140.
[6] M. Blasone, P.P. Pacheco, H.W. Tseung, Phys. Rev. D 67 (2003) 073011.
[7] A. Capolupo, S. Capozziello, G. Vitiello, Phys. Lett. A 363 (2007) 53;
A. Capolupo, S. Capozziello, G. Vitiello, Phys. Lett. A 373 (2009) 601;
A. Capolupo, S. Capozziello, G. Vitiello, Int. J. Mod. Phys. A 23 (2008) 4979;
M. Blasone, A. Capolupo, S. Capozziello, S. Carloni, G. Vitiello, Phys. Lett. A 323 (2004) 182.
[8] M.V. Berry, Proc. R. Soc. London A 392 (1984) 45.
[9] Y. Aharonov, J. Anandan, Phys. Rev. Lett. 58 (1987) 1593.
[10] J. Anandan, Y. Aharonov, Phys. Rev. Lett. 65 (1990) 1697.
[11] M. Blasone, P.A. Henning, G. Vitiello, Phys. Lett. B 466 (1999) 262.
[12] S. Capozziello, G. Lambiase, Eur. Phys. J. C 16 (2000) 155;
S. Capozziello, G. Lambiase, Europhys. Lett. 52 (2000) 15.
[13] X.G. He, X.Q. Li, B.H.J. McKellar, Y. Zhang, Phys. Rev. D 72 (2005) 053012.
[14] S. Pancharatnam, in: A.D. Shapere, F. Wilczek (Eds.), Geometric Phases in Physics, World Scientific, Singapore, 1989.
[15] J. Samuel, R. Bhandari, Phys. Rev. Lett. 60 (1988) 2339.
[16] G.G. de Polavieja, Phys. Rev. Lett. 81 (1998) 1.
[17] X.B. Wang, L.C. Kwek, Y. Liu, C.H. Oh, Phys. Rev. D 63 (2001) 053003.
[18] Z.Y. Law, A.H. Chan, C.H. Oh, Phys. Lett. B 648 (2007) 289.
[19] K. Fujii, C. Habe, T. Yabuki, Phys. Rev. D 59 (1999) 113003;
K. Fujii, C. Habe, T. Yabuki, Phys. Rev. D 64 (2001) 013011.
[20] C.R. Ji, Y. Mishchenko, Phys. Rev. D 65 (2002) 096015.
[21] M. Blasone, A. Capolupo, G. Vitiello, Phys. Rev. D 66 (2002) 025033.
[22] M. Blasone, A. Capolupo, F. Terranova, G. Vitiello, Phys. Rev. D 72 (2005) 013003.
[23] M. Blasone, J. Palmer, Phys. Rev. D 69 (2004) 057301.
[24] M. Blasone, A. Capolupo, O. Romei, G. Vitiello, Phys. Rev. D 63 (2001) 125015;
A. Capolupo, C.R. Ji, Y. Mishchenko, G. Vitiello, M. Blasone, A. Capolupo, O. Romei, G. Vitiello, Phys. Lett. B 594 (2004) 135.