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Additional Finite Elements and Additional Loads for Analysis of Systems with Several Nonlinear Properties

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Abstract

This paper dwells upon some problems of formation of additional finite elements and additional loads for analysis of systems having several nonlinear properties that are distinct in nature and time of manifestation. These problems emerge when the Additional Finite Element Method (AFEM) is used. The designed AFEM is a variant of Finite Element Method (FEM) the several units of Method of Ultimate Equilibrium and Method of Elastic Decisions (Method of Additional Loads) to the solution of nonlinear analysis. At the same time, the AFEM develops A.A. Ilyushin’s ideas of Elastic Decisions Method for analysis of systems with one nonlinear property and solves the problem of analysis of systems with several nonlinear properties.

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1. Introduction

Worked out by the author of this paper Additional Finite Element Method [1, 2] develops the ideas of Method of Elastic Decisions for analysis of systems with several nonlinear properties.

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2. Method of Elastic Decisions

Nowadays the Method of Elastic Decisions is the most prevalent and efficient method of analysis of systems with plastic properties. It was suggested by A.A. Ilyushin [3] for solving of deformational problems of plastic theory.

According to this method in place of elastic-plastic body an ideal elastic one with the same strains but with additional loads is taken. Therefore this method is called the Method of Additional Loads. Scheme of the actions such additional loads for the allowance of the first nonlinear property (plasticity) is

\[
\text{Plastic body (body with the first nonlinear property)} = \text{Elastic body with the same geometric size} + \text{Additional loads providing the strains of plastic body}
\]

In nonlinear (plastic) problem the main set of equations has next form:

\[
K_{\text{non},1}V = P \tag{1}
\]

where \( K_{\text{non},1} = \) stiffness matrix of structure with the first nonlinear property (plasticity) which is changed in accordance with the degree of influence of this property;

\( V = \) matrix-column of unknowns;

\( P = \) matrix-column of constant terms (external loads).

The next step is based on separation from the stiffness matrix of its linear component [4]:

\[
K_{\text{non},1} = K + \Delta K_{\text{non},1} \tag{2}
\]

where \( K = \) the linear part of stiffness matrix \( K_{\text{non}} \); 

\( \Delta K_{\text{non},1} = \) the nonlinear part of stiffness matrix \( K_{\text{non},1} \).

If we substitute the expression (2) in the formula (1), remove the parentheses, transfer the second term in right hand side, we may get the next expression:

\[
KV = P - \Delta K_{\text{non},1}V \tag{3}
\]

In result we may determine the value of additional load \((-\Delta K_{\text{non},1}V)\) in carrying out of the process of iteration. In this case the values of unknowns are obtained from the previous iteration. This method allows to solve the system of linear equations (3) with constant coefficients in the right hand side, i.e. by obtaining of inverse matrix \( K^{-1} \) only once. The main advantage is contained in its simplicity. The Method of Elastic Decisions is closely connected with two other methods: Method of Initial (Additional) Deformation and Method of Initial (Additional) stresses [5].


For the solving of linear problem Finite Element Method (FEM) uses the set of algebraic equations:

\[
KV = P \tag{4}
\]

where \( K = \) stiffness matrix with linear properties is formed from the stiffness matrices of the separate finite elements (FE) of the initial design diagram;

\( V = \) matrix-column of unknown node displacements;

\( P = \) matrix-column of external load.

In nonlinear (plastic) problem the set of equations has analogous form (1), where \( K_{\text{non},1} = \) stiffness matrix of structure with nonlinear properties which is changed in accordance with the degree of influence of these properties. Therefore this set of equations ought to be solved by iterative method with the help of Method of Elastic Decisions.
[6]. When Finite Element Method and Method of Elastic Decisions are used together the scheme of the actions such additional loads to FE for the allowance of the first nonlinear property (plasticity) is

\[ FE \text{ with plastic property (body with the first nonlinear property)} = \]

\[ = \text{Elastic FE with the same geometric size} + \text{Additional loads providing the node displacements} \]

The main properties of concrete triangular deep-beam finite element with the first nonlinear property (plasticity) and the way of formation of additional load corresponding to this finite element were determined by A.A. Karjakin [7] on the basis of the theory of plasticity of G.A. Geniev [8]. The research supervisor of this scientific work was professor A.A. Oatul. In a basis of this way the triangular deep-beam finite element with conditional crack and corresponding additional load were created by author of this paper under the same leadership [9, 10].

4. Main problem

The process of analysis ought to be formed in order to turn step-by-step matrix \( K \) into matrix \( K_{\text{nonl,1}} \). In the presence of several (\( n \)) physical nonlinear properties the transformation of set of equations (4) into set of equations (1) is connected with definite difficulties. Especially these difficulties are exhibited step-by-step design when nonlinear properties manifest gradually. The solution of this problem demands theoretic models of material behavior [11, 12] and new numerical method. Such method is the subject of the next part of this paper.

5. Additional Finite Element Method and Method of Elastic Decisions

Additional Finite Element Method (AFEM) was suggested by author of this paper and was developed as variant of FEM destined for design of reinforced concrete structures at limit states [1, 2, 13, 14]. For solving of the problem AFEM adds to design the units of Method of Ultimate Equilibrium and Method of Elastic Decisions (Method of Additional Loads) and uses additional design diagrams (ADD) consisting of additional finite elements (AFE) [15]. It makes possible the gradual transformation of set of equations (1). Analysis of its use in design of physical nonlinear reinforced deep-beams allows to presuppose that it may be used in design of other \( n \)-nonlinear systems [16].

5.1. Additional finite elements and additional loads

Each FE of the design diagram is the separate small part of it with its own physical nonlinear properties. In general the number of these properties coincides with the number of nonlinear properties of the whole considered system and equals \( n \).

Additional finite elements (AFE) are destined for gradual transformation of the initial finite elements into the same elements with nonlinear properties corresponding to the reached stage of behavior. Each AFE takes into account the only nonlinear property.

If FE has \( n \) nonlinear properties then \( n \) AFEs describes its characteristics.

The next relationship between node reactions \( R_e \) and node displacements \( V \) is correct for each FE:

\[ K_i V = R_e \]

where \( K_i = \text{stiffness matrix of this FE} \).

Namely on the basis of node reactions \( R_e \) of separate finite elements the equilibrium equations of the main decision system FEM (4) are formed. In order to change these equations in accordance of influence of nonlinear properties the values of node reactions ought to be changed by means of AFEs.

Stiffness matrix \( \Delta K_{i,e} \) of the \( i \)-th AFE (\( i \) changes from 1 to \( n \)) destined for the allowance of influence of the \( i \)-th nonlinear property is determined analogously to formula (13):
\[ \Delta K_{i,e} = K_{i,e} - K_{(i-1),e} \]  

(6)

where \( K_{i,e} \) = stiffness matrix of FE with the \( i \)-th nonlinear property;
\( K_{(i-1),e} \) = stiffness matrix of the same FE with out the \( i \)-th property, i.e. with \((i-1)\) nonlinear properties.

For example, if \( K_{i,e} = 0.5K_e \), then \( \Delta K_{i,e} = -0.5K_e \). If some the \( i \)-th out of \( n \) nonlinear properties of the considered FE is not exhibited the stiffness matrix of the corresponding \( i \)-th AFE \( \Delta K_{i,e} = 0 \).

At any the \( i \)-th AFE relation between node reactions \( \Delta R_{i,e} \) and node displacements \( V \) has the form:

\[ \Delta K_{i,e} V = \Delta R_{i,e} \]  

(7)

Thus node reactions of the \( i \)-th AFE influence node reactions of the main FE which in its turn influence the equilibrium equations with units of stiffness matrices of this FE.

Additional load taking account of the \( i \)-th the nonlinear property of the main FE is \((-\Delta R_{i,e})\).

When Additional Finite Element Method and method of elastic decisions are used together the scheme of the actions of additional load \((-\Delta R_{1,e})\) for the allowance of the first nonlinear property (plasticity) is

\[ \text{FE with the first nonlinear property and stiffness matrix } K_{1,e} = \]

\[ = \text{Elastic FE with stiffness matrix } K_e + \text{Additional loads } (-\Delta R_{1,e}) \]

where \( \Delta R_{1,e} = \) node reactions in AFE for the allowance of the first nonlinear property:

\[ \Delta R_{1,e} = \Delta K_{1,e} V = (K_{1,e} - K_e)V \]  

(8)

Then Additional Finite Element Method develops this scheme for the allowance for the influence of the rest nonlinear properties.

The scheme of the actions of additional load \((-\Delta R_{2,e})\) for the allowance of the second nonlinear property is

\[ \text{FE with the second nonlinear property and stiffness matrix } K_{2,e} = \]

\[ = \text{FE with the first nonlinear property and stiffness matrix } K_{1,e} + \text{Additional loads } (-\Delta R_{2,e}) \]

where \( \Delta R_{2,e} = \) node reactions in AFE for the allowance of the first nonlinear property:

\[ \Delta R_{2,e} = \Delta K_{2,e} V = (K_{2,e} - K_{1,e})V \]  

(9)

The scheme of the actions of additional load \((-\Delta R_{i,e})\) for the allowance of the \( i \)-th nonlinear property is

\[ \text{FE with the } i\text{-th nonlinear property and stiffness matrix } K_{i,e} = \]

\[ = \text{FE with the } (i-1)\text{-th nonlinear properties and stiffness matrix } K_{(i-1),e} + \text{Additional loads } (-\Delta R_{i,e}) \]

where \( \Delta R_{i,e} = \) node reactions in AFE for the allowance of the \( i \)-th nonlinear property:

\[ \Delta R_{i,e} = \Delta K_{i,e} V = (K_{i,e} - K_{(i-1),e})V \]  

(10)

The scheme of the actions of additional load \((-\Delta R_{n,e})\) for the allowance of the last \( n \)-th nonlinear property is
FE with the n-th nonlinear property and stiffness matrix $K_{n,e} =$

$= FE with the (n –1)-th nonlinear properties and stiffness matrix $K_{(n-1),e} + Additional loads (–\Delta R_{n,e})$

where $\Delta R_{n,e} = node reactions in AFE for the allowance of the last n-th nonlinear property:

$$\Delta R_{n,e} = \Delta K_{n,e}V = (K_{n,e} - K_{(n-1),e})V$$  \hspace{1cm} (11)

On the basis of the values of additional loads of each FE the additional load taking into account of this nonlinear property is determined. The examples of formation of triangular additional finite elements and additional loads for analysis of concrete deep beams with four nonlinear properties are given in papers [17] and [18] respectively.

5.2. Set of equation of AFEM and additional design diagrams

AFEM suggests to include some operations of Method of Elastic Decisions and use additional design diagrams (ADD) consisting of additional finite elements (AFE). It makes possible the gradual transformation of set of equations (1).

Thus when the first nonlinear property is exhibited system (1) is written down in the next form

$$(K + \Delta K_{nol,1})V = P$$  \hspace{1cm} (12)

where $K =$ stiffness matrix with linear properties;

$\Delta K_{nol,1} =$ stiffness matrix of the first additional design diagram consisting of additional finite elements taking into account the first nonlinear property.

In this case the external load is situated in limits $0 \leq P < P_1$, where $P_1 =$ load when the second nonlinear property appears.

When others nonlinear properties appear, for example from 2 up to i-th, relationship (12) has the form:

$$(K + \Delta K_{nol,1} + \Delta K_{nol,2} + \ldots + \Delta K_{nol,i})V = P$$  \hspace{1cm} (13)

where $\Delta K_{nol,2} =$ stiffness matrix of the second additional design diagram consisting of additional finite elements taking into account the second nonlinear property;

$\Delta K_{nol,i} =$ stiffness matrix of the i-th additional design diagram consisting of additional finite elements taking into account the i-th nonlinear property.

At this case the value of external load changes at interval $0 \leq P < P_i$, where $P_i =$ load when (i+1)-th nonlinear property appears.

If all n non-linear properties, for example from 1 up to n-th, relationship (9) looks like:

$$(K + \Delta K_{nol,1} + \Delta K_{nol,2} + \ldots + \Delta K_{nol,n})V = P$$  \hspace{1cm} (14)

where $\Delta K_{nol,n} =$ stiffness matrix of the n-th additional design diagram consisting of additional finite elements taking into account the n-th nonlinear property.

Here the external load changes at interval $0 \leq P < P_{lim}$, where $P_{lim} =$ load under which the considered system with all n physical nonlinear properties collapses.

Thus in nonlinear design step-by-step the initial set of algebraic equations (1) takes the form (12), (13) and (14) provided taking into account the influence of each nonlinear property. Each additional design diagram (ADD) is a geometrical replica of the initial design diagram but it is destined for step-by-step transformation of the initial design diagram into design diagram with all n nonlinear properties. Additional design diagram may be compared with empty space imbedded in the initial design diagram and filled negative or positive stiffness. For gradual
deterioration of stiffness characteristics of the system additional design diagram with negative stiffness is used and for improvement of these characteristics additional design diagram with positive stiffness is used. Additional design diagram allows to use Method of Elastic Decisions (Method of Additional Loads) for solving set of equations (12), (13) or (14). Thus in solving of set (12) of equations the displacements correspond to displacements of system with the first nonlinear property under the action of the only external load \( P \). This set has the form (3).

The relationships (13) and (14) may be transformed in the same way:

\[
KV = P - \Delta K_{n0,1} - \Delta K_{n0,2} - \cdots - \Delta K_{n0,n} V
\]  
(15)

\[
KV = P - \Delta K_{n0,1} - \Delta K_{n0,2} - \cdots - \Delta K_{n0,n} V
\]  
(16)

In these relationships each value \(-\Delta K_{i}V\) determines the influence of \( i \)-th nonlinear property.

The example of composition of the set of equations for nonlinear analysis concrete deep beams with four nonlinear properties is given in paper [19].

5.3. Ideal failure model.

In design of system with \( n \) nonlinear properties it is necessary to determine the limit of its operating period in this condition. The criterion of collapse of the system ought to be determined. For reinforced concrete structures this criterion may be formed on basis of the theory of ultimate equilibrium [20]. Before collapse these structures reaches the state of ultimate equilibrium or ultimate limit state. For description of structure at limit state the ideal failure model is suggested which is the design diagram of this structure at state of ultimate equilibrium (ultimate limit state), i.e. at the moment previous to collapse. The point is that the initial design diagram ought to change step-by-step loading in accordance with nonlinear properties appeared as the ultimate limit state is reached. As a result a initial design diagram transforms into an ideal failure model of the considered structure.

6. Conclusions

The suggested AFEM with the help of additional finite elements and additional loads develops of A.A. Ilushin’s Method of Elastic Decisions and allows to solve the three main problems connected with design of \( n \)-nonlinear systems. It gives opportunity to realize step-by-step design of such systems with gradual taking into account of influence of physical nonlinear properties and determine the limits of their behavior. The efficiency of AFEM was proved by design of a number of plane stressed reinforced concrete deep beams [1, 2] and space shell [21].

References