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Calculus on the Sierpinski gasket I: polynomials, exponentials and power series

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Abstract

We study the analog of power series expansions on the Sierpinski gasket, for analysis based on the Kigami Laplacian. The analog of polynomials are multiharmonic functions, which have previously been studied in connection with Taylor approximations and splines. Here the main technical result is an estimate of the size of the monomials analogous to $x^n/n!$. We propose a definition of entire analytic functions as functions represented by power series whose coefficients satisfy exponential growth conditions that are stronger than what is required to guarantee uniform convergence. We present a characterization of these functions in terms of exponential growth conditions on powers of the Laplacian of the function. These entire analytic functions enjoy properties, such as rearrangement and unique determination by infinite jets, that one would expect. However, not all exponential functions (eigenfunctions of the Laplacian) are entire analytic, and also many other natural candidates, such as the heat kernel, do not belong to this class. Nevertheless, we are able to use spectral decimation to

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study exponentials, and in particular to create exponentially decaying functions for negative eigenvalues.

1. Introduction

Ordinary calculus is such a remarkable subject because it combines both a general conceptual framework and a detailed understanding of basic functions. For example, the theory of power series expansions hinges on the elementary observation that the function $f_n(x) = x^n/n!$ on $[0, 1]$ is bounded by $1/n!$. (Stated this way, it seems almost a tautology, so perhaps it is better to say that f_n is the polynomial characterized by the conditions $f_n^{(m)}(0) = \delta_{nm}$.) Another example: among all linear combinations of $\cosh x$ and $\sinh x$ there is one, $e^{-x} = \cosh x - \sinh x$, that decays as $x \to \infty$; moreover its rate of decay is the reciprocal of the growth rate of $\cosh x$ and $\sinh x$.

The goal of this paper is to understand analogous facts about basic functions on the Sierpinski gasket (SG), which should be regarded as the simplest nontrivial example of a fractal supporting a theory of differential calculus based on a Laplacian. Standard references are the books of Barlow [Ba] and Kigami [Ki2], and the expository paper [S2]. The references to this paper, and the more extensive bibliography in [Ki2], indicate an intensive development of the subject since Kigami's original paper [Ki1] giving a direct analytic definition of the Laplacian on SG.

Recall that SG is the attractor of the iterated functions system (IFS) consisting of three contractions in the plane $F_i(x) = \frac{1}{2}(x + q_i)$, $i = 0, 1, 2$ where q_i are the vertices of an equilateral triangle. In other words $SG = \bigcup_{i=0}^{2} F_i(SG)$, and we refer to the sets $F_i(SG)$ as *cells of order* 1. More generally, we write $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ for a word $w = (w_1, \dots, w_m)$ of length $|w| = m$, each $w_j = 0, 1$ or 2, and call $F_w(SG)$ a *cell of level* m. We regard SG as the limit of a sequence of graphs Γ_m (with vertices V_m and edge relation $x \sim_m y$) defined inductively as follows: Γ_0 is the complete graph on $V_0 = \{q_0, q_1, q_2\}$, and $V_m = \bigcup_{i=0}^{2} F_i V_{m-1}$ with $x \sim_m y$ if x and y belong to the same cell of level m. Then $V_* = \bigcup_{m=1}^{\infty} V_m$, the set of all *vertices*, the analog of the dyadic points in the unit interval, is dense in SG. We consider V_0 the set of boundary points of SG, and $V_* \backslash V_0$ is the set of *junction points*. Note that every junction point in V_m has exactly 4 neighbors in the graph Γ_m. The graph Laplacian Δ_m on Γ_m is defined by

$$\Delta_m u(x) = \sum_{y \sim_m x} (u(y) - u(x)) \text{ for } x \in V_m \backslash V_0. \tag{1.1}$$

The Laplacian Δ on SG is defined as the renormalized limit

$$\Delta u(x) = \lim_{m \to \infty} \frac{3}{2} 5^m \Delta_m u(x). \tag{1.2}$$

More precisely, $u \in dom\,\Delta$ and $\Delta u = f$ means u and f are continuous functions and the limit on the right side of (1.2) converges to f uniformly on $V_* \backslash V_0$. The Laplacian plays the role of the second derivative on the unit interval (although it is shown in [S5] that it does not behave like a second order operator). Thus we will define a *polynomial* P to be any solution of $\Delta^j P = 0$ for some j. More precisely, if we let $\mathscr{H}_j$ denote the space of solutions of $\Delta^{j+1} u = 0$, then $\mathscr{H}_j$ is a space of dimension $3j+3$, and it has an "easy" basis $\{f_{nk}\}$ for $0 \leqslant n \leqslant j$ and $k = 0, 1, 2$ characterized by

$$\Delta^\ell f_{nk}(q_{k'}) = \delta_{\ell n} \delta_{kk'}. \tag{1.3}$$

In [SU] a different basis was constructed in order to develop a theory of splines. Here we will consider yet another basis, implicitly used in [S3] in conjunction with Taylor expansions, to define power series.

The Laplacian is basically an interior operator, as (1.2) is not defined at the boundary (although $\Delta u = f$ makes sense at boundary points by continuity). There are also boundary derivatives. The *normal derivative*

$$\partial_n u(q_j) = \lim_{m \to \infty} \left(\frac{5}{3}\right)^m (2u(q_j) - u(F_j^m q_{j+1}) - u(F_j^m q_{j-1})) \tag{1.4}$$

(cyclic notation $q_{j+3} = q_j$) exists for every $u \in dom\,\Delta$ and plays a crucial role in the theory, especially in the analog of the Gauss–Green theorem:

$$\int_{SG} (u\Delta v - v\Delta u)\, d\mu = \sum_{i=0}^{2} (u(q_i)\partial_n v(q_i) - \partial_n u(q_i) v(q_i)). \tag{1.5}$$

Here μ is the natural probability measure that assigns weight 3^{-m} to each cell of order m. The normal derivative may be localized to boundary points of any cell, and there is also a localized version of (1.5). At a junction point there are two different normal derivatives with respect to the cells on either side. For $u \in dom\,\Delta$ we have the *matching condition* that the two normal derivatives sum to zero. This leads to the *gluing property*: if u and f are continuous functions and $\Delta u = f$ on each cell of order m (meaning $\Delta(u \circ F_w) = 5^{-m} f \circ F_w$ for all words w of length m), then $\Delta u = f$ on SG if and only if the matching conditions hold at every junction point in V_m.

There are also *tangential derivatives*

$$\partial_T u(q_j) = \lim_{m \to \infty} 5^m (u(F_0^m q_{j+1}) - u(F_0^m q_{j-1})) \tag{1.6}$$

that exist if $u \in dom\,\Delta$ and $\Delta^n u$ satisfies a Holder condition, and may be localized to boundary points of cells. In this case there are no matching conditions for $u \in dom\,\Delta$. However, we will show in Section 5 that there are matching conditions involving infinite series of tangential and normal derivatives valid for polynomials and analytic functions. Tangential derivatives were introduced in [S3]. Their true significance is still somewhat elusive. In this paper we will show that for polynomials and analytic functions the sum of the tangential derivatives over the three boundary points of any cell must vanish. In [S3] and [T2] the idea of creating a *gradient* of a function out of the normal and tangential derivatives is discussed. Here we will extend this to the idea of a *jet*. For simplicity we deal with a boundary point q_ℓ, but the definition can be localized to boundary points of any cell.

Definition 1.1. For $u \in dom\, \Delta^n$ with Δ^n satisfying a Holder condition, the *n-jet* of u at q_ℓ is the $(3n+3)$-tuple of values $(\Delta^j u(q_\ell), \partial_n \Delta^j u(q_\ell), \partial_T \Delta^j u(q_\ell))$ for $0 \leqslant j \leqslant n$. For $u \in dom\, \Delta^\infty$, the *jet* of u at q_ℓ is the infinite set of the same values for all $j \geqslant 0$.

Fix a boundary point q_ℓ. We define polynomials $P_{jk}^{(\ell)}$ by requiring that the j-jet at q_ℓ vanish except for one term, $\Delta^j P_{j1}^{(\ell)}(q_\ell) = 1$, $\partial_n \Delta^j P_{j2}^{(\ell)}(q_\ell) = 1$ and $\partial_T P_{j3}^{(\ell)}(q_\ell) = 1$, respectively. We refer to these functions as *monomials*. It is clear that the monomials $P_{jk}^{(\ell)}$ for $0 \leqslant j \leqslant n$ form a basis of $\mathscr{H}_n$. It is shown in [S3] that they exhibit a prescribed decay rate in neighborhoods of q_ℓ, but the estimates established there were not uniform in j. The first goal of this paper is to obtain sharp estimates for $||P_{jk}^{(\ell)}||_\infty$. For $P_{j1}^{(\ell)}$ and $P_{j3}^{(\ell)}$ we prove decay estimates faster than any exponential. For $P_{j2}^{(\ell)}$ the situation is different; we prove an exponential decay of order λ_2^{-j} for the specific value λ_2 equal to the second nonzero Neumann eigenvalue. This result is sharp. In fact we show that $(-\lambda_2)^j P_{j2}^{(\ell)}$ converges to a certain λ_2-eigenfunction of Δ. This result has no analog in ordinary calculus.

We define a power series about q_ℓ as an infinite linear combination of the monomials $P_{jk}^{(\ell)}$ with coefficients $\{c_{jk}\}$. We find growth conditions on the coefficients to guarantee convergence. We study the rearrangement problem: given a convergent power series about one boundary point, does the function also have a convergent power series about the other boundary points? Surprisingly, we find that it is necessary to assume a stronger growth restriction on the coefficients in order for this to be the case, namely

$$|c_{jk}| = O(R^j) \text{ for some } R < \lambda_2. \tag{1.7}$$

We end up defining an *entire analytic function* to be a function represented by a power series with coefficients satisfying (1.7). We then prove that rearrangement is possible at all boundary points, and in fact local power series expansions exist on all cells, with the estimate (1.7) preserved (in fact the same R value). This choice of definition means that there are some convergent power series that do not yield analytic functions. It also means that eigenfunctions of the Laplacian cannot be entire analytic functions unless the eigenvalue satisfies $|\lambda| < \lambda_2$. On the other hand it is easy to see that there are λ_2-eigenfunctions that cannot be represented by convergent power series, so the definition seems to be close to best possible. We then are able to characterize the class of entire analytic functions in $dom\, \Delta^\infty$ by the growth conditions

$$||\Delta^j u||_\infty = O(R^j) \text{ for some } R < \lambda_2 \tag{1.8}$$

(one could also use L^2 norms).

Our definition of entire analytic function means that a basic principle of unique analytic continuation holds. If we have a function defined on a cell and satisfying (1.8) there, it has a unique extension to an entire analytic function on the whole space. In fact its jet at any boundary point of the cell satisfies (1.7), and uniquely determines the function. This implies that a nonzero entire analytic function cannot vanish to infinite order at any junction point. We could also define local analytic functions on a cell of order m by relaxing the condition $R < \lambda_2$ in (1.7) and (1.8) to $R < 5^m \lambda_2$. One could hope to have a notion of analytic continuation that would allow such local analytic functions to extend to larger domains. However, we have not been able to find any interesting examples, so we will not pursue the matter here.

It is easy to extend the notion of entire analytic function to infinite blow-ups of SG [S1,T1]. The simplest of these is

$$SG_\infty = \bigcup_{n=1}^{\infty} F_0^{-n}(SG), \tag{1.9}$$

but more generally we could consider

$$\bigcup_{n=1}^{\infty} F_{j_1}^{-1} F_{j_2}^{-1} \cdots F_{j_n}^{-1}(SG) \tag{1.10}$$

for any choice of $j_1, j_2, j_3, \ldots$. A function on SG satisfying (1.8) for all $R > 0$ extends to an entire analytic function on any blow-up (1.10). It is not clear at present which, if any, of these functions will come to play the role of special functions (hypergeometric, Bessel functions, etc.) in real analysis. On the other hand it is very easy to construct many such functions simply by taking a power series with bounded or sub-exponential growing coefficients. The negative results of [BST] mean that none of these spaces of analytic functions is closed under multiplication, so this precludes using many standard techniques for ordinary power series.

Although none of the eigenfunctions of the Laplacian are entire analytic functions on the blow-ups, it is still important to understand their global behavior. In Section 6 we study this problem for the simplest example SG_∞ and negative eigenvalues. It is easy enough to define the analogs of the functions $\cosh \sqrt{\lambda} x$ and $\sinh \sqrt{\lambda} x$. In fact there are three, which we call $C_\lambda(x)$, $S_\lambda(x)$ and $Q_\lambda(x)$, characterized among $(-\lambda)$-eigenfunctions by their 0-jet at q_0, or equivalently by power series involving just $P_{j1}^{(0)}$, $P_{j2}^{(0)}$, or $P_{j3}^{(0)}$ terms, respectively. The power series for $C_\lambda(x)$ and $Q_\lambda(x)$ converge on all of SG_∞, while the power series for $S_\lambda(x)$ is only convergent on a neighborhood of q_0 (depending on λ). Fortunately, there is another method available to study these eigenfunctions, called *spectral decimation* [FS,DSV,T1]. Using this method we are able to show that they exhibit an exponential growth as $x \to \infty$ (or as $\lambda \to \infty$), and there is one linear combination, $E_\lambda(x) = C_\lambda(x) - S_\lambda(x)$ for the appropriate normalization, that decays as $x \to \infty$ at the reciprocal rate. Thus $E_\lambda(x)$ is the analog of $e^{-\sqrt{\lambda} x}$. It is not clear if there is any analog of $e^{\sqrt{\lambda} x}$.

Although we do not use power series in our study of properties of eigenfunctions, we can turn the tables and use facts about eigenfunctions to obtain information about power series. In particular, we are able to construct specific power series that are divergent, or power series that are convergent but not rearrangeable. We can also give an explanation for why the recursion relations for the size of monomials are unstable.

It is interesting to speculate on possible future extensions and developments of our results. It is important to understand all eigenfunctions, including those with positive eigenvalues, on all blow-ups (1.10). There should be some sort of Liouville-type theorem precluding nonconstant bounded entire analytic functions on blow-ups without boundary.

What is the behavior of an entire analytic function in a neighborhood of a generic point? Is there any notion of power series there? Are there interesting examples of local analytic functions with a natural domain that is not just a single cell? Is there a meaningful notion of analytic functions on fractafolds based on SG [S4]?

We have seen that there is no restriction on the jet of an analytic function other than the growth condition (1.7). For the larger class *dom* Δ^{∞}, is there an analog of Borel's theorem that an arbitrary jet may be specified at one (or all three) boundary points?

In [OSY], the structure of level sets of harmonic functions on SG was elucidated, with the remark that certain eigenfunctions of the Laplacian have level sets of an entirely different nature. It is clear now that these eigenfunctions are not analytic, so it is reasonable to ask if anything interesting can be said about level sets of entire analytic functions. Another remark from that paper is that harmonic functions enjoy a principle called "geography is destiny." Roughly speaking, this says that the restriction to a small cell of a harmonic function is essentially dictated (up to two parameters) by the location of the cell, rather than the specific harmonic function, in a certain generic sense. This holds because restrictions of harmonic functions are governed by long products of matrices, so the theory of products of random matrices makes generic predictions. For analytic functions, there is a similar description of the transformation of jets, except that the matrices are now infinite. So if we go to a small cell, while all jets satisfying (1.7) are possible, some may be very unlikely for a generic analytic function. Is there some way to make this precise?

A sequel to this paper, [BSSY], will discuss functions with point singularities, exponential functions on general blow-ups, and estimates for normal derivatives of Dirichlet eigenfunctions and heat kernels.

The website www.mathlab.cornell.edu/˜nman/ contains more numerical and graphical data, as well as the programs used to generate them.

2. Polynomials

The space $\mathscr{H}_j$ of $(j+1)$-harmonic functions (solutions of $\Delta^{j+1}u = 0$) has dimension $3(j+1)$ and plays the role of the space of polynomials of degree at most $2j+1$ on the unit interval. Several different bases for $\mathscr{H}_j$ are known. In [SU], in order to develop a theory of spline spaces, bases based on the behavior at all three boundary points were used. In this section we will discuss properties of yet another basis, based on the behavior at a single boundary point, that is more suited to the work on power series to follow. The polynomials in this basis are analogous to the monomials $x^n/n!$ on the unit interval. These functions were introduced in [S3], but not much was done there to describe their behavior.

Definition 2.1. Fix a boundary point q_ℓ. The *monomials* $P_{jk}^{(\ell)}$ for $k = 1,2,3$ and $j = 0,1,2,\dots$ are defined to be the functions in $\mathscr{H}_j$ satisfying

$$\Delta^m P_{jk}^{(\ell)}(q_\ell) = \delta_{mj}\delta_{k1}, \tag{2.1}$$

$$\partial_n \Delta^m P_{jk}^{(\ell)}(q_\ell) = \delta_{mj}\delta_{k2}, \tag{2.2}$$

$$\partial_T \Delta^m P_{jk}^{(\ell)}(q_\ell) = \delta_{mj}\delta_{k3}. \tag{2.3}$$

When $\ell = 0$ we will sometimes delete the upper exponent and just write P_{jk}.

Note that we only need to consider $m \leqslant j$ in (2.1)–(2.3), since $\Delta^m P_{jk}^{(\ell)}$ vanishes identically otherwise. Thus there are $3(j+1)$ conditions in all, and it follows from [S3] that there is a unique solution, and the monomials $P_{jk}^{(\ell)}$ for fixed ℓ and all $j \leqslant j_1$ form a basis for $\mathscr{H}_{j_1}$. We have the self-similar identities

$$P_{j1}^{(\ell)}(F_\ell^m x) = 5^{-jm} P_{j1}^{(\ell)}(x), \tag{2.4}$$

$$P_{j2}^{(\ell)}(F_\ell^m x) = \left(\frac{3}{5}\right)^m 5^{-jm} P_{j2}^{(\ell)}(x), \tag{2.5}$$

$$P_{j3}^{(\ell)}(F_\ell^m x) = 5^{-(j+1)m} P_{j3}^{(\ell)}(x) \tag{2.6}$$

that describe the decay rate of these functions as $x \to q_\ell$ (of course $P_{01}^{(\ell)} \equiv 1$). It is easy to see that $P_{j1}^{(\ell)}$ and $P_{j2}^{(\ell)}$ are symmetric while $P_{j3}^{(\ell)}$ is skew-symmetric under the reflection that fixes q_ℓ and permutes the other two boundary points. It is easy to compute the values of monomials to any desired precision. Fig. 1 shows the graphs of some of them. Since we may obtain $P_{jk}^{(\ell)}$ from $P_{jk}^{(0)}$ by simply rotating the variable x, we will restrict our discussion to $\ell = 0$ from now on.

It is clear from the definition that powers of the Laplacian send monomials to monomials, simply reducing the j index:

$$\Delta^m P_{jk} = P_{(j-m)k}. \tag{2.7}$$

We could use this property to give an inductive definition. When $j = 0$ the monomials are explicit harmonic functions, $P_{01} \equiv 1$, P_{02} has boundary values $P_{02}(q_0) = 0$, $P_{02}(q_1) = P_{02}(q_2) = -1/2$ and P_{03} has boundary values $P_{03}(q_0) = 0$,

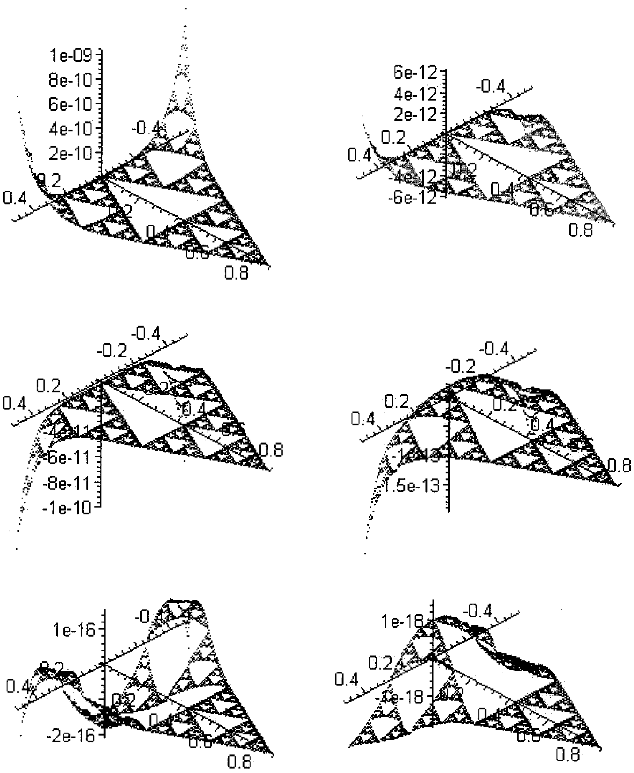

Fig. 1. The graphs of P_{jk} for some typical values. The graphs of P_{j1} are all qualitatively similar for $j \geqslant 1$, so we show only P_{51} (top left). Similarly for P_{j3} (top right). The nature of the graphs of P_{j2} changes drastically around $j = 5, 6, 7, 8$, so we display all of these. The graphs of P_{j2} for $j \geqslant 8$ are qualitatively similar to P_{82} (bottom right).

$P_{03}(q_1) = -P_{03}(q_2) = 1/2$. Then P_{jk} for $j > 0$ is the unique solution of $\Delta P_{jk} = P_{(j-1)k}$ with vanishing initial conditions

$$P_{jk}(q_0) = 0, \;\; \partial_n P_{jk}(q_0) = 0, \;\; \partial_T P_{jk}(q_0) = 0.$$

In [KSS] it is shown that P_{jk} may then be written as an integral operator (with explicit kernel) applied to $P_{(j-1)k}$. However, the kernel is quite singular, so we have not been able to extract any useful information out of this representation.

There are three main goals in this section: (1) to obtain sharp estimates for the size of the monomials, (2) to understand how to express monomials for one choice of ℓ in terms of monomials for another choice of ℓ, (3) to obtain certain universal identities that hold for all monomials. In pursuit of these goals we introduce some terminology.

Definition 2.2. For $j \geqslant 0$ let

$$\begin{cases} \alpha_j = P_{j1}(q_1), \ \beta_j = P_{j2}(q_1), \ \gamma_j = P_{j3}(q_1), \\ n_j = \partial_n P_{j1}(q_1), \ t_j = \partial_T P_{j2}(q_1). \end{cases} \tag{2.8}$$

Note that by symmetry we have $P_{j1}(q_2) = \alpha_j$, $P_{j2}(q_2) = \beta_j$ and $P_{j3}(q_2) = -\gamma_j$, so that all values of monomials at boundary points are expressible in terms of α's, β's and γ's. Soon we will see that the n's, t's and α's suffice to express all normal and tangential derivatives of monomials at boundary points.

Theorem 2.3. *The following recursion relations hold*:

$$\alpha_j = \frac{4}{5^j - 5} \sum_{\ell=1}^{j-1} \alpha_{j-\ell}\alpha_\ell \quad \textit{for} \quad j \geqslant 2, \tag{2.9}$$

$$\gamma_j = \frac{4}{5^{j+1} - 5} \sum_{\ell=0}^{j-1} \alpha_{j-\ell}\gamma_\ell \quad \textit{for} \quad j \geqslant 1, \tag{2.10}$$

$$\beta_j = \frac{1}{5^j - 1} \sum_{\ell=0}^{j-1} \left(\frac{2}{5} 5^{j-\ell} \alpha_{j-\ell}\beta_\ell - \frac{2}{3} \alpha_{j-\ell} 5^\ell \beta_\ell + \frac{4}{5} \alpha_{j-\ell}\beta_\ell \right) \quad \textit{for} \quad j \geqslant 1, \tag{2.11}$$

with initial data $\alpha_0 = 1$, $\alpha_1 = 1/6$, $\beta_0 = -1/2$, $\gamma_0 = 1/2$. *In particular*,

$$\gamma_j = 3\alpha_{j+1}. \tag{2.12}$$

Proof. It is convenient to work in matrix notation, with all matrices being infinite semi-circulant. For example, the matrix $\alpha = \{\alpha_{ij}\}_{i,j=0,1,2,\ldots}$ has $\alpha_{ij} = \alpha_{i-j}$ for $i \geqslant j$ and $\alpha_{ij} = 0$ for $i < j$. We consider two linear operators on such matrices, the shift σ and the dilation τ, given by

$$\sigma \begin{pmatrix} d_0 & 0 & \cdots & \\ d_1 & d_0 & 0 & \\ d_2 & d_1 & d_0 & 0 \\ \vdots & & & \end{pmatrix} = \begin{pmatrix} d_1 & 0 & \cdots & \\ d_2 & d_1 & 0 & \\ d_3 & d_2 & d_1 & 0 \\ \vdots & & & \end{pmatrix}$$

$$\tau \begin{pmatrix} d_0 & 0 & \cdots & & \\ d_1 & d_0 & 0 & \cdots & \\ d_2 & d_1 & d_0 & 0 & \cdots \\ \vdots & & & & \end{pmatrix} = \begin{pmatrix} d_0 & 0 & \cdots & \\ 5d_1 & d_0 & 0 & \\ 5^2 d_2 & 5d_1 & d_0 & 0 \\ \vdots & & & \end{pmatrix}.$$

Let $\{f_{j1}, f_{j2}, f_{j3}\}_{j=0}^{\infty}$ be the easy basis defined by (1.3). As in [SU] we let

$$a_{l-1} = \partial_n f_{lk}(q_k),$$

$$b_{l-1} = \partial_n f_{lk}(q_n) \quad n \neq k$$

for $l = 0, 1, 2, \ldots$. Then the Gauss–Green formula says for $l \geqslant 0$

$$\begin{aligned} a_l &= \partial_n f_{(l+1)1}(q_1) \\ &= \sum_{n=1}^{3} (f_{01}(q_n)\partial_n f_{(l+1)1}(q_n) - f_{(l+1)1}(q_n)\partial_n f_{01}(q_n)) \\ &= \int_{SG} (f_{01}\Delta f_{(l+1)1} - f_{(l+1)1}\Delta f_{01})\, d\mu \\ &= \int_{SG} f_{01} f_{l1}\, d\mu \end{aligned}$$

and

$$\begin{aligned} b_l &= \partial_n f_{(l+1)1}(q_2) \\ &= \sum_{n=1}^{3} (f_{02}(q_n)\partial_n f_{(l+1)1}(q_n) - f_{(l+1)1}(q_n)\partial_n f_{02}(q_n)) \\ &= \int_{SG} (f_{02}\Delta f_{(l+1)1} - f_{(l+1)1}\Delta f_{02})\, d\mu \\ &= \int_{SG} f_{02} f_{l1}\, d\mu. \end{aligned}$$

This shows that our definition is consistent with [SU]. It is easy to see that $a_{-1} = 2$, $b_{-1} = 1$.

We note here some typos from [SU]:

(i) in (5.4) the coefficient $\frac{47}{45}$ should be $\frac{47}{75}$;

(ii) in the first line of (5.7) the coefficients 2 of $a_{j-1-\ell}$ and $b_{j-1-\ell}$ should be deleted.

Now let p_j, q_j be defined by

$$p_j = 5^j f_{jk}(F_i q_k) \quad i \neq k,$$

$$q_j = 5^j f_{jk}(F_i q_\ell) \quad \text{for } i, j, \ell \text{ distinct}.$$

(Note that we are using the same symbol q_j for two different things, but it should be clear from context which is which.)

Then (5.7) of [SU] rearranged says

$$\sum_{l=0}^{j} (a_{j-l-1} + b_{j-l-1})(2p_l + q_l) + b_{j-1} = 0,$$

$$\sum_{l=0}^{j} (2a_{j-l-1} - b_{j-l-1})(p_l - q_l) + b_{j-1} = 0.$$

If we set

$$A = \begin{pmatrix} a_{-1} & 0 & & & \\ a_0 & a_{-1} & 0 & & \\ a_1 & a_0 & a_{-1} & 0 & \\ a_2 & a_1 & a_0 & a_{-1} & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} b_{-1} & 0 & & & \\ b_0 & b_{-1} & 0 & & \\ b_1 & b_0 & b_{-1} & 0 & \\ b_2 & b_1 & b_0 & b_{-1} & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix},$$

$$P = \begin{pmatrix} p_0 & 0 & & & \\ p_1 & p_0 & 0 & & \\ p_2 & p_1 & p_0 & 0 & \\ p_3 & p_2 & p_1 & p_0 & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix}, \quad Q = \begin{pmatrix} q_0 & 0 & & & \\ q_1 & q_0 & 0 & & \\ q_2 & q_1 & q_0 & 0 & \\ q_3 & q_2 & q_1 & q_0 & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix}.$$

Then in matrix notation this becomes

$$(A+B)(2P+Q) + B = 0, \quad (2A - B)(P - Q) + B = 0. \tag{2.13}$$

Now for $j \geqslant 0$,

$$\begin{cases} P_{j1} = f_{j0} + \sum_{l=0}^{j} \alpha_{j-l}(f_{l1} + f_{l2}), \\ P_{j2} = \sum_{l=0}^{j} \beta_{j-l}(f_{l1} + f_{l2}), \end{cases} \tag{2.14}$$

so taking normal derivatives at q_0, we have

$$a_{j-1} + 2\sum_{l=0}^{j} \alpha_{j-l} b_{l-1} = \partial_n P_{j1}(q_0) = 0,$$

$$2\sum_{l=0}^{j} \beta_{j-l} b_{l-1} = \partial_n P_{j2}(q_0) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In matrix notation this is

$$2\alpha B + A = 0, \quad 2\beta B = I,$$

i.e.

$$A = -\alpha\beta^{-1}, \quad B = \frac{1}{2}\beta^{-1} \tag{2.15}$$

Substituting (2.15) into (2.13), we get

$$2P + Q = -(A+B)^{-1}B = -\left[-\frac{1}{2}\beta^{-1}(2\alpha - I)\right]^{-1}\left[\frac{1}{2}\beta^{-1}\right] = (2\alpha - I)^{-1},$$

$$P - Q = -(2A - B)^{-1}B = -\left[-\frac{1}{2}\beta^{-1}(4\alpha + I)\right]^{-1}\left[\frac{1}{2}\beta^{-1}\right] = (4\alpha + I)^{-1}$$

so

$$(2\alpha - I)(2P + Q) = I = (4\alpha + I)(P - Q).$$

Expanding we get

$$4\alpha P + 2\alpha Q - 2P - Q = 4\alpha P - 4\alpha Q + P - Q,$$

i.e.

$$P = 2\alpha Q, \quad \text{and} \quad Q = (4\alpha + I)^{-1}(2\alpha - I)^{-1}. \tag{2.16}$$

Now evaluate (2.14) at $F_0 q_1$, noting that

$$P_{j1}(F_0 q_1) = 5^{-j} P_{j1}(q_1) = 5^{-j}\alpha_j,$$

$$P_{j2}(F_0 q_1) = \frac{3}{5} 5^{-j} P_{j1}(q_1) = \frac{3}{5} 5^{-j}\beta_j$$

by (2.4), (2.5) and

$$f_{l0}(F_0 q_1) = f_{l1}(F_0 q_1) = 5^{-l} p_l,$$

$$f_{l2}(F_0 q_1) = 5^{-l} q_l, \tag{2.17}$$

by the definitions of p_l's and q_l's. The result is

$$5^{-j}\alpha_j = 5^{-j} p_j + \sum_{l=0}^{j} \alpha_{j-l}\,(5^{-l} p_l + 5^{-l} q_l),$$

$$\frac{3}{5} 5^{-j}\beta_j = \sum_{l=0}^{j} \beta_{j-l}(5^{-l} p_l + 5^{-l} q_l)$$

so

$$\alpha_j = p_j + \sum_{l=0}^{j} 5^{j-l} \alpha_{j-l}(p_l + q_l)$$

and

$$\frac{3}{5}\beta_j = \sum_{l=0}^{j} 5^{j-l} \beta_{j-l}(p_l + q_l).$$

In matrix notation these read as

$$\alpha = P + \tau(\alpha)(P + Q)$$

and

$$\frac{3}{5}\beta = \tau(\beta)(P + Q).$$

From (2.16) we see that

$$\alpha = [2\alpha + \tau(\alpha)(2\alpha + I)]Q$$

and

$$\frac{3}{5}\beta = \tau(\beta)(2\alpha + I)Q.$$

Hence

$$\tau(\alpha) = 4\alpha^2 - 3\alpha$$

and

$$\frac{3}{5}\beta(2\alpha - I)(4\alpha + I) = \tau(\beta)(2\alpha + I),$$

from which (2.9) and (2.11) follow.

Finally

$$P_{j3} = \sum_{l=0}^{j} \gamma_{j-l}(f_{l1} - f_{l2}),$$

$$P_{j3}(F_0 q_1) = 5^{-(j+1)} P_{j3}(q_1) = 5^{-(j+1)} \gamma_j$$

and so by (2.17) we have

$$5^{-(j+1)}\gamma_j = \sum_{l=0}^{j} \gamma_{j-l}(5^{-l} p_l - 5^{-l} q_l),$$

i.e.

$$\frac{1}{5}\gamma_j = \sum_{l=0}^{j} 5^{j-l}\gamma_{j-l}(p_l - q_l),$$

or in matrix notation

$$\frac{1}{5}\gamma = \tau(\gamma)(P - Q).$$

Thus $\tau(\gamma) = \frac{1}{5}(4\alpha + I)\gamma$ from which (2.10) follows.

The values of α_0, β_0 and γ_0 are easy to check. Then (2.12) follows from (2.9) and (2.10) since α_j and γ_{j-1} satisfy the same recursion relation. □

Theorem 2.4. *For all* $j \geqslant 0$ *we have*

$$P_{j3}^{(0)}(x) + P_{j3}^{(1)}(x) + P_{j3}^{(2)}(x) = 0 \tag{2.18}$$

and

$$P_{j3}^{(0)}(x) = 3(P_{(j+1)1}^{(2)}(x) - P_{(j+1)1}^{(1)}(x)). \tag{2.19}$$

Proof. We prove (2.18) by induction. For $j = 0$ the left side is a harmonic function that vanishes on the boundary (because of the skew-symmetry of each term). Such a function must be zero. For the induction step, assume it is true for $j - 1$. Then

$$\Delta(P_{j3}^{(0)} + P_{j3}^{(1)} + P_{j3}^{(2)}) = P_{(j-1)3}^{(0)} + P_{(j-1)3}^{(1)} + P_{(j-1)3}^{(2)} = 0$$

by the induction hypothesis. Once again the left side is a harmonic function, and it vanishes on the boundary by skew symmetry.

To prove (2.19) we use

$$P_{j3}^{(0)} = \sum_{\ell=0}^{j} \gamma_{j-\ell}(f_{\ell 1} - f_{\ell 2}). \tag{2.20}$$

On the other hand, we have

$$P_{(j+1)1}^{(2)} = f_{(j+1)2} + \sum_{\ell=0}^{j+1} \alpha_{j-\ell+1}(f_{\ell 0} + f_{\ell 1}),$$

$$P_{(j+1)1}^{(1)} = f_{(j+1)1} + \sum_{\ell=0}^{j+1} \alpha_{j-\ell+1}(f_{\ell 0} + f_{\ell 2})$$

so that

$$\begin{aligned} P^{(2)}_{(j+1)1} - P^{(1)}_{(j+1)1} &= f_{(j+1)2} - f_{(j+1)1} + \sum_{\ell=0}^{j+1} \alpha_{j-\ell+1}(f_{\ell 1} - f_{\ell 2}) \\ &= \sum_{\ell=0}^{j} \alpha_{j-\ell+1}(f_{\ell 1} - f_{\ell 2}) \end{aligned}$$

since $\alpha_0 = 1$. The result follows from (2.12). □

The dihedral-3 symmetry group D_3 of SG consists of reflections ρ_0, ρ_1, ρ_2, where ρ_j preserves q_j and permutes the other two boundary points, and rotations I, R_1, $R_2 = (R_1)^2$ where $R_1 q_j = q_{j+1}$ (cyclic notation).

Theorem 2.5. *Any polynomial P satisfies the identity*

$$P(x) + P(R_1 x) + P(R_2 x) = P(\rho_0 x) + P(\rho_1 x) + P(\rho_2 x), \tag{2.21}$$

and more generally the local versions

$$P(x_0) + P(x_1) + P(x_2) = P(y_1) + P(y_2) + P(y_3) \tag{2.22}$$

for any sextuplet of points such that

$$\begin{cases} x_0 = F_w x, \;\; x_1 = F_w R_1 x, \;\; x_2 = F_w R_2 x, \\ y_0 = F_w \rho_0 x, \;\; y_1 = F_w \rho_1 x, \;\; y_2 = F_w \rho_2 x \end{cases} \tag{2.23}$$

for some $x \in SG$ and some word w.

Proof. The local version follows from (2.21) because $P \circ F_w$ is also a polynomial. To prove (2.21) it suffices to show it holds for all monomials. Now we claim that (2.21) is trivially true for any function that is symmetric with respect to one of the reflections ρ_j. Say $P(x) = P(\rho_0 x)$ for all x. Then $P(R_1 x) = P(\rho_1 x)$ and $P(R_2 x) = P(\rho_2 x)$ because $\rho_0 R_1 = \rho_1$ and $\rho_0 R_2 = \rho_2$. In particular, (2.21) holds for all $P^{(\ell)}_{j1}$ and $P^{(\ell)}_{j2}$. It follows from (2.19) that it also holds for $P^{(\ell)}_{j3}$. □

The same result holds for uniform limits of polynomials; in particular, the convergent power series discussed in the next section. Note that Kigami [Ki2] Theorem 4.3.6 has characterized the space of L^2 limits of polynomials by the condition of orthogonality to all joint Dirichlet and Neumann eigenfunctions. It is not hard to see that (2.22) implies the orthogonality to some of these eigenfunctions (those of the $\lambda^{(5)}$-type in [DSV]), but not others. On the other hand, it is not clear how these orthogonality conditions imply (2.22).

Corollary 2.6. *Any polynomial P satisfies*

$$\partial_T P(q_0) + \partial_T P(q_1) + \partial_T P(q_2) = 0, \tag{2.24}$$

and more generally the sum of tangential derivatives at the boundary points of any cell must vanish.

Proof. Taking $x = F_0^m q_1$ in (2.21), we find

$$(P(F_0^m q_1) - P(F_0^m q_2)) + (P(F_1^m q_2) - P(F_1^m q_0)) + (P(F_2^m q_0) - P(F_2^m q_1)) = 0 \tag{2.25}$$

because $R_1 F_0^m q_1 = F_1^m q_2$, $R_2 F_0^m q_1 = F_2^m q_0$, $\rho_0 F_0^m q_1 = F_0^m q_2$, $\rho_1 F_0^m q_1 = F_2^m q_1, \rho_2 F_0^m q_1 = F_1^m q_0$. Multiplying (2.25) by 5^m and taking the limit as $m \to \infty$ yields (2.24). The local form follows as before. □

Remark. As we observed in the proof of Theorem 2.5, any polynomial may be written as a sum of three polynomials, each symmetric with respect to one of the reflections ρ_j, $P = P^{(0)} + P^{(1)} + P^{(2)}$. It is easy to see that one way to do this explicitly is to take

$$P^{(j)}(x) = \frac{1}{3}(P(x) + P(\rho_j x)) - \frac{1}{9}(P(\rho_0 x) + P(\rho_1 x) + P(\rho_2 x)). \tag{2.26}$$

We consider next estimates for the size of α_j, β_j, γ_j. We show that α_j has rapid decay, which we believe is fairly sharp. This gives the same decay rate for γ_j.

Theorem 2.7. *There exists a constant c such that*

$$0 < \alpha_j < c(j!)^{-\log 5/\log 2} \quad \textit{for all} \quad j. \tag{2.27}$$

Proof. It is clear from (2.9) and the initial conditions that the α_j are positive. Let $\tilde{\alpha}_j = (j!)^{\log 5/\log 2}\alpha_j$. We need to show that the $\tilde{\alpha}_j$ are bounded, which we do by induction. If $\tilde{\alpha}_\ell \leqslant c$ for $\ell \leqslant j$, then (2.9) implies

$$\tilde{\alpha}_j \leqslant c^2 5^{1-j} \sum_{\ell=1}^{j-1} \binom{j}{\ell}^{\log 5/\log 2}.$$

It is well known that

$$\sum_{\ell=0}^{j} \binom{j}{\ell}^2 = \binom{2j}{j},$$

so by Stirling's formula and routine arguments we have

$$\sum_{\ell=1}^{j-1} \binom{j}{\ell}^{\log 5/\log 2} \leqslant M 5^j (j)^{-1/2}$$

Table 1

j	α_j	β_j	$(-\lambda_2)^j\beta_j$	$8^j(j!)^{\frac{\log(5)}{\log(2)}}\alpha_j$
0	1	−0.5000000000	−0.5000000000	1
1	0.1666666667	−0.04444444444	6.025427867	1.333333333
2	0.005555555556	−0.001008230453	−18.53107571	1.777777777
3	0.00006172839506	$-0.8554950809\times10^{-5}$	21.31713060	2.025658338
4	$0.3318730917\times10^{-6}$	$-0.3853047646\times10^{-7}$	−13.01625411	2.178127244
5	$0.1021147975\times10^{-8}$	$-0.9848282711\times10^{-10}$	4.510374011	2.250339083
6	$0.2007235906\times10^{-11}$	$-0.1933836698\times10^{-12}$	−1.200721414	2.268082964
7	$0.2713115918\times10^{-14}$	$-0.7720311754\times10^{-16}$	0.06498718216	2.248411184
8	$0.2656437390\times10^{-17}$	$-0.1187366658\times10^{-17}$	−0.1355027558	2.201440598
9	$0.1959165201\times10^{-20}$	$0.7232200062\times10^{-20}$	−0.1118933095	2.134277683
10	$0.1122370097\times10^{-23}$	$-0.5436238235\times10^{-22}$	−0.1140256558	2.052740417
11	$0.5120236416\times10^{-27}$	$0.4004514705\times10^{-24}$	−0.1138739539	1.961629028
12	$0.1898528071\times10^{-30}$	$-0.2954013973\times10^{-26}$	−0.1138826233	1.864726441
13	$0.5820142006\times10^{-34}$	$0.2178916451\times10^{-28}$	−0.1138822148	1.764891613
14	$0.1496625756\times10^{-37}$	$-0.1607201123\times10^{-30}$	−0.1138822304	1.664234594
15	$0.3268360869\times10^{-41}$	$0.1185495242\times10^{-32}$	−0.1138822298	1.564302197
16	$0.6126918156\times10^{-45}$	$-0.8744387717\times10^{-35}$	−0.1138822298	1.466232140
17	$0.9952451630\times10^{-49}$	$0.6449989323\times10^{-37}$	−0.1138822298	1.370864839
18	$0.1412543698\times10^{-52}$	$-0.4757607235\times10^{-39}$	−0.1138822298	1.278818576
19	$0.1764707126\times10^{-56}$	$0.3509281252\times10^{-41}$	−0.1138822298	1.190538877
20	$0.1953558627\times10^{-60}$	$-0.2588497599\times10^{-43}$	−0.1138822298	1.106332006

for all $j\geqslant 2$ for a small constant M, so $\tilde{\alpha}_j\leqslant c^2 5M(j)^{-1/2}$. It is easy to choose c and j_0 so that $\tilde{\alpha}_\ell\leqslant c$ for $\ell<j_0$ and $c\leqslant(j_0)^{1/2}/5M$. $\square$

Table 1 presents numerical computations of α_j and β_j.

It appears that $8^j(j!)^{\log 5/\log 2}\alpha_j$ remains bounded (8 is by no means the best constant, and perhaps it could be replaced by an arbitrary positive number). It also appears that $(-\lambda_2)^j\beta_j$ converges to the constant -0.1138822298, where $\lambda_2 = 135.572126995788\ldots$ is the second nonzero Neumann eigenvalue. It is easy to see that λ_2 is the largest value for which such an estimate could hold, because

$$\sum_{j=0}^{\infty}\beta_j(-\lambda_2)^j \text{ diverges.}$$

Indeed, if we did not have divergence then

$$\sum_{j=0}^{\infty}(-\lambda_2)^j P_{j2}(x)$$

would be a solution to the eigenvalue equation $-\Delta u=\lambda_2 u$ satisfying $\partial_n u(q_0)=1$. But, since λ_2 is not a Dirichlet eigenvalue, the space of eigenfunctions has dimension three, whereas the multiplicity of the λ_2-Neumann eigenspace is also three, so every eigenfunction automatically satisfies $\partial_n u(q_0) = 0$.

We note that the computation of β_j, carried out using the recursion relation (2.11), was done using exact rational arithmetic (the reported values are reported as decimal approximations, of course). This is significant because this solution of (2.11) is highly unstable. For example, if we take $\beta_0 = \frac{1}{2}$ and $\beta_1 = 0.044444444$ or 0.04444445 (the correct value being $2/45$) and then use (2.11) for $j \geqslant 2$, we find the ratio β_j/β_{j+1} approaching $-84.0799\ldots$ (this is $-5\lambda_1^D$, where $\lambda_1^D = 16.815999\ldots$ is the first Dirichlet eigenvalue). In Section 6 we will give an explanation for this phenomenon.

Next we will establish estimates for $||P_{jk}||_\infty$. To do this we will study the operator

$$Af(x) = Gf(x) - (\partial_n(Gf)(q_0))P_{02}, \tag{2.28}$$

where $Gf(x) = \int G(x,y)f(y)\,d\mu(y)$ is the Green's operator, satisfying $-\Delta Gf = f$ and $Gf(q_i) = 0,\ i = 0,1,2$. Note that A is a compact linear operator, but is not self-adjoint. Thus the spectrum of A consists of isolated eigenvalues of finite multiplicity, and zero. Note that we have

$$-\Delta Af = f,\ \ Af(q_0) = 0 \ \text{and}\ \partial_n Af(q_0) = 0. \tag{2.29}$$

In particular, this implies

$$AP_{jk} = -P_{(j+1)k} \ \text{for}\ k = 1,2. \tag{2.30}$$

Write A_0 for the restriction of A to the ρ_0-symmetric functions, where ρ_0 is the reflection preserving q_0.

Lemma 2.8. (a) *f is an eigenfunction of A_0 ($A_0 f = \lambda f$) if and only if f is a symmetric λ^{-1}-eigenfunction of Δ satisfying $f(q_0) = \partial_n f(q_0) = 0$.* (b) *$f$ is an eigenfunction of A_0 if and only if f is a symmetric λ^{-1}-Neumann eigenfunction of Δ satisfying $f(q_0) = 0$.* (c) *The Jordan block of A_0 associated to any eigenvalue is diagonal.*

Proof. (a) By (2.29), any eigenfunction of A is a λ^{-1}-eigenfunction of Δ satisfying $f(q_0) = \partial_n f(q_0) = 0$. For the converse, let $v = Af - \lambda f$. Then

$$\Delta v = \Delta Af - \lambda\Delta f = \Delta(Gf - \partial_n(Gf)P_2) + f = -f + f = 0$$

so v is harmonic. But v is symmetric with $v(q_0) = \partial_n v(q_0) = 0$, and this implies $v = 0$.

(b) The only new assertion here is that f in part (a) also satisfies $\partial_n f(q_1) = \partial_n f(q_2) = 0$. This requires a rather detailed knowledge of the description of eigenfunctions of Δ by spectral decimination. First we observe that if $|\lambda^{-1}|$ is small enough (less than the first Dirichlet eigenvalue), then a symmetric λ^{-1}-eigenfunction is uniquely determined by $f(q_0)$ and $\partial_n f(q_0)$. This implies that f vanishes identically on a cell $F_0^n(SG)$ for n large enough. But an eigenfunction can vanish on a cell only if the space of eigenfunctions has dimension greater than three, and that happens only

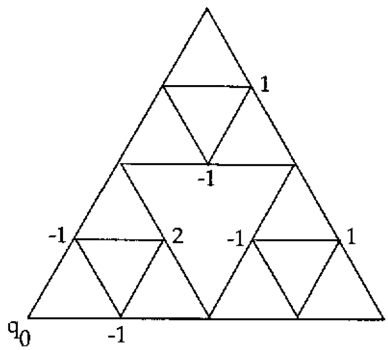

Fig. 2.

if λ^{-1} is a joint Dirichlet–Neumann eigenvalue. That means its restriction to the graph Γ_m for some value of m is either a 5-eigenfunction or a 6-eigenfunction. In the 6-eigenfunction case there is nothing to prove, since all eigenfunctions are Neumann eigenfunctions. In the 5-eigenfunction case this is not true, but the Neumann eigenfunctions have codimension two in the space of all eigenfunctions. When we impose the ρ_0-symmetry condition the codimension drops to one. We know exactly what this one function looks like (see Fig. 2 for the case $m = 2$). In particular, it does not vanish identically in any small cell $F_0^m(SG)$. Since f does (and so do all symmetric joint Dirichlet–Neumann eigenfunctions), it follows that f must be Neumann eigenfunction (in the 5-eigenfunction case it is also a Dirichlet eigenfunction, but not necessarily in the 6-eigenfunction case).

(c) Suppose λ is an eigenvalue of A_0, and $(A_0 - \lambda)^2 g = 0$. Then λ^{-1} is a Neumann eigenvalue of Δ, and $(\Delta + \lambda^{-1})^2 g = 0$. Also g is symmetric and satisfies $g(q_0) = \partial_n g(q_0) = 0$. By similar reasoning as before, g is a Neumann eigenfunction of Δ, hence the Jordan block associated with λ is diagonal. □

Theorem 2.9. (*a*) *For any $r < \infty$ there exists c_r such that*

$$||P_{j1}||_\infty \leqslant c_r r^{-j}, \tag{2.31}$$

or more precisely

$$\lim_{j\to\infty} \frac{1}{j} \log ||P_{j1}||_\infty = -\infty. \tag{2.32}$$

(*b*) *There exists c such that*

$$||P_{j2}||_\infty \leqslant c\lambda_2^{-j}, \tag{2.33}$$

and

$$\lim_{j\to\infty} (-\lambda_2)^j P_{j2} = \varphi, \tag{2.34}$$

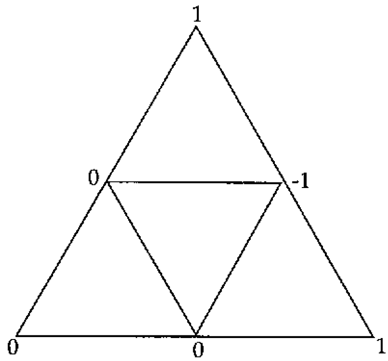

Fig. 3.

where φ is a λ_2-Neumann eigenfunction of Δ which is ρ_0-symmetric and vanishes on $F_0(SG)$ (a multiple of the eigenfunction shown in Fig. 3 on Γ_1), the limit existing uniformly and in energy.

Proof. (a) Consider the norm

$$||f|| = (||f||_2^2 + \mathscr{E}(f,f))^{1/2} \tag{2.35}$$

and define $\mathscr{L}_1$ and $\mathscr{L}_2$ as the closures in this norm of the spans of $\{P_{j1}\}$ and $\{P_{j2}\}$, respectively. By (2.30), A_0 preserves both spaces. Denote by A_1 and A_2 the restriction of A_0 to $\mathscr{L}_1$ and $\mathscr{L}_2$. We claim $\sigma(A_1) = \{0\}$. Indeed, otherwise A_1 would have to have a nonzero eigenvalue λ because A_1 is compact. Since this would also be an eigenvalue of A_0, by Lemma 2.8 λ^{-1} would have to be a Neumann eigenvalue of Δ. So $\lambda > 0$, and we may choose it to be the largest eigenvalue of A_1. Then $\lambda^{-j}A_1^j$ converges to a projection (not necessarily orthogonal) B_λ onto the finite dimensional λ-eigenspace of A_1. Note that $B_\lambda P_{01}$ cannot be the zero function, because that would imply $B_\lambda P_{j1} = 0$ for all j, contradicting the fact that B_λ is nonzero. But then $\lambda^{-j}A_1^jP_{01} = \lambda^{-j}P_{j1}$ would converge to a nonzero eigenfunction of A_1. By Theorem 2.7 this eigenfunction would vanish at q_1 and q_2, and of course it vanishes at q_0, since P_{j1} does for $j \geqslant 1$. So it would have to be a joint Dirichlet–Neumann eigenfunction of Δ. But Theorem 4.3.6 of [Ki2] asserts that all P_{jk} are orthogonal to all joint Dirichlet–Neumann eigenfunctions.

Thus we have shown that $\sigma(A_1) = \{0\}$, so the spectral radius of A_1 is zero,

$$\lim_{j\to\infty} ||A_1^j||^{1/j} = 0.$$

Applying this to P_{01} we obtain (2.32) (the norm (2.35) dominates the L^∞ norm), which implies (2.31).

(b) The result of Kigami used above moreover says that $\mathscr{L} = \mathscr{L}_1 \oplus \mathscr{L}_2$ contains all ρ_0-symmetric Neumann eigenfunctions of Δ that are orthogonal to all joint Dirichlet–Neumann eigenfunctions (note that Kigami uses the L^2 norm rather than (2.35), but the same argument applies). In particular, it contains the λ_2-eigenfunction shown in Fig. 3 (this is a Neumann eigenfunction, so it is orthogonal to all Neumann eigenfunctions with different eigenvalues, and there are no joint Dirichlet–Neumann eigenfunctions with the same eigenvalue). By Lemma 2.8 and the explicit description of Neumann eigenfunctions, λ_2^{-1} is the largest eigenvalue of A_0, and φ spans this multiplicity one eigenspace. Thus, as before, $\lambda_2^j A^j$ converges to a one-dimensional projection operator $B_{\lambda_2^{-1}}$, and $B_{\lambda_2^{-1}} P_{01} = 0$. That means $B_{\lambda_2^{-1}} P_{02} \neq 0$, for otherwise $B_{\lambda_2^{-1}} = 0$. So

$$\lim_{j \to \infty} (-\lambda_2)^j P_{j2} = \lim_{j \to \infty} \lambda_2^j A^j P_{02} = B_{\lambda_2^{-1}} P_{02}$$

which is (2.34). This implies (2.33). □

The estimate (2.33) is sharp, but (2.32) falls short of what we would have if we knew $||P_{j1}||_\infty = \alpha_j$, in view of (2.27). One approach to establish this would be to prove the following conjecture:

Conjecture 2.10. *For all $x \neq q_0$ and all j,*

$$P_{j1}(x) > 0. \tag{2.36}$$

We have numerical evidence for this conjecture for moderate values of j. To show that (2.36) implies $||P_{j1}||_\infty = \alpha_j$ is easy using the following well-known fact (we provide a proof since it does not appear explicitly in the literature).

Proposition 2.11. *If $u \in \operatorname{dom} \Delta$, $\Delta u(x_0) > 0$ and x_0 is not a boundary point, then u does not achieve its maximum value at x_0.*

Proof. If x_0 is a vertex in V_* the result follows immediately from the pointwise definition of $\Delta u(x_0)$. If not, then we can find a cell $F_w K$ such that x_0 is in the interior of $F_w K$ and $\Delta u > 0$ on $F_w K$. Let $v = u \circ F_w$. Then $\Delta v > 0$, and we have

$$v(x) = h(x) - \int_K G(x,y) \Delta v(y)\, dy$$

where G is the Dirichlet Green's function and $h(x)$ is the harmonic function with the same boundary values as $v(x)$. Since the Green's function is positive in the interior, we have $v(x) < h(x)$ in the interior. Since h attains its maximum on the boundary, it follows that v cannot attain its maximum in the interior, so $u(x_0)$ is not a maximum. □

Next we study the normal and tangential derivatives of monomials at boundary points.

Theorem 2.12. *We have initial values* $n_0 = 0$, $t_0 = -1/2$, *and recursion relations*

$$n_j = \frac{5^j + 1}{2}\alpha_j + 2\sum_{\ell=0}^{j-1} n_\ell \beta_{j-\ell} \quad \textit{for} \quad j \geqslant 1, \tag{2.37}$$

$$t_j = \beta_j - 6\sum_{\ell=0}^{j-1} \alpha_{j+1-\ell} t_\ell \quad \textit{for} \quad j \geqslant 1. \tag{2.38}$$

Moreover, we have

$$\partial_n P_{j2}(q_1) = \partial_n P_{j2}(q_2) = \begin{cases} \frac{1}{2} - \alpha_0 & \textit{if } j = 0, \\ -\alpha_j & \textit{if } j \geqslant 1, \end{cases} \tag{2.39}$$

$$\partial_n P_{j3}(q_1) = -\partial_n P_{j3}(q_2) = 3n_{j+1}, \tag{2.40}$$

$$\partial_T P_{j1}(q_1) = -\partial_T P_{j1}(q_2) = \begin{cases} \frac{1}{6} & \textit{if } j = 1, \\ 0 & \textit{if } j \neq 1, \end{cases} \tag{2.41}$$

$$\partial_T P_{j3}(q_1) = -\partial_T P_{j3}(q_2) = \begin{cases} -\frac{1}{2} & \textit{if } j = 0, \\ 0 & \textit{if } j \geqslant 1. \end{cases} \tag{2.42}$$

Proof. As in the proof of Theorem 2.3 we introduce matrices n, $\tilde{n}$ and t, where $\tilde{n}_j = \partial_n P_{j2}(q_1)$. When we evaluate the normal derivatives on both sides of (2.14) at q_1, we see that

$$n_j = b_{j-1} + \sum_{l=0}^{j} \alpha_{j-l}(a_{l-1} + b_{l-1}) \quad \text{for all } j,$$

or in matrix notations

$$n = B + \alpha(A + B).$$

Using (2.15) this yields

$$n = \frac{1}{2}\beta^{-1}(I + 2\alpha)(I - \alpha) = \frac{1}{4}\beta^{-1}(2I - \tau(\alpha) - \alpha) \tag{2.43}$$

which implies (2.37).

By the same reasoning

$$\tilde{n}_j = \sum_{l=0}^{j} \beta_{j-l}(a_{l-1} + b_{l-1}) \text{ for all } j.$$

Then

$$\tilde{n} = \beta(A + B)$$

and hence by (2.15) we obtain

$$\tilde{n} = \frac{1}{2} I - \alpha, \tag{2.44}$$

which implies (2.39).

Finally, the same reasoning shows

$$t_j = \sum_{l=0}^{j} \beta_{j-\ell} T_l \quad \text{for all } j,$$

where $T_l = \partial_T f_{l2}(q_1)$. Now $P_{j3} = \sum_{l=0}^{j} \gamma_{j-l}(f_{l1} - f_{l2})$, so taking tangential derivatives at q_0 we get

$$2 \sum_{l=0}^{j} \gamma_{j-l} T_l = \partial_T P_{j3}(q_0) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In matrix notations these become

$$t = \beta T,$$

$$\gamma T = \frac{1}{2} I.$$

Together we have

$$\beta = 2\gamma t = 6\sigma(\alpha)t, \tag{2.45}$$

where the last equality follows from (2.12).

This proves (2.38). The initial values of n_0, $\tilde{n}_0$ and t_0 are easy to check.

Note that the skew-symmetry implies $\partial_T P_{j3}(q_1) = \partial_T P_{j3}(q_2)$, so (2.2) implies $\partial_T P_{j3}(q_0) + 2\partial_T P_{j3}(q_1) = 0$, which yields (2.42). Then (2.41) follows from (2.19) and (2.42), and similarly (2.19) implies (2.40). □

Theorem 2.13. *For any $r < \infty$ there exists c_r such that, for all $j \geqslant 1$,*

$$|n_j| \leqslant c_r r^{-j}. \tag{2.46}$$

Also

$$|t_j| \leqslant c\lambda_2^{-j}. \tag{2.47}$$

Proof. From the Gauss–Green formula we have

$$\int \Delta u \, d\mu = \sum_{i=0}^{2} \partial_n u(q_i).$$

We apply this to $u = P_{j1}^{(0)}$, noting that $\partial_n P_{j1}^{(0)}(q_0) = 0$ and $\partial_n P_{j1}^{(0)}(q_1) = \partial_n P_{j1}^{(0)}(q_2) = n_j$. It follows that

$$n_j = \frac{1}{2} \int P_{(j-1)1}^{(0)} \, d\mu, \tag{2.48}$$

and (2.46) follows from (2.31).

Similarly, (2.47) will follow from (2.33) and the estimate (taking $u = P_{j2}$)

$$|\partial_T u(q_i)| \leqslant c(||u||_\infty + ||\Delta u||_\infty + ||\Delta^2 u||_\infty). \tag{2.49}$$

In [S3] it is shown that $\partial_T u(q_i)$ exists if $u \in dom\,\Delta$ and Δu satisfies a Hölder condition, and (2.49) is just a quantitative version of this fact. For the convenience of the reader we outline the argument. For simplicity take $i = 0$. Let g_m (see Fig. 4 for $m = 2$) denote the level m piecewise harmonic function satisfying $g_m(q_0) = 0$ and $g_m(F_0^k q_1) = 3^k$ and $g_m(F_0^k q_2) = -3^k$ for all $k \leqslant m$. Then

$$\int g_m \Delta u \, d\mu = \frac{14}{3} 5^m (u(F_0^m q_1) - u(F_0^m q_2)) - 5(u(q_1) - u(q_2)) \tag{2.50}$$

by the Gauss–Green formula, since the sum of the normal derivatives of g_m at $F_0^m q_1$ is $(14/3)5^m$ (there are no terms involving normal derivatives of u at $F_0^m q_i$ because u

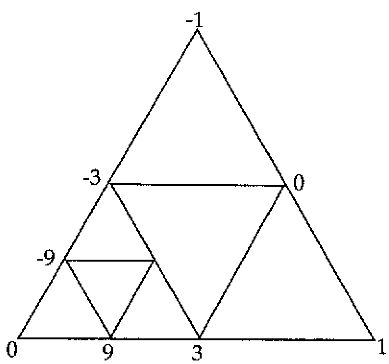

Fig. 4.

Table 2

j	n_j	t_j	$\frac{n_j}{j\alpha_j}$	$(-\lambda_2)^j t_j$	$\frac{\beta_j}{t_{j+1}}$
0	0	−0.50000000	∞	−0.50000000	18
1	0.50000000	−0.027777778	3	3.7658925	−432.0
2	0.027777778	0.00010288066	2.5000000	1.8909261	1439.0526
3	0.00041152263	$-0.70062097\ 10^{-6}$	2.2222222	1.7457996	−1679.0103
4	$0.27287343 \times 10^{-5}$	$0.50952342 \times 10^{-8}$	2.0555556	1.7212575	1027.9833
5	$0.98752993 \times 10^{-8}$	$-0.37481616 \times 10^{-10}$	1.9341564	1.7166051	−356.40392
6	$0.22167060 \times 10^{-10}$	$0.27632364 \times 10^{-12}$	1.8405958	1.7156968	94.889369
7	$0.33533009 \times 10^{-13}$	$-0.20379909 \times 10^{-14}$	1.7656562	1.7155176	−5.1358463
8	$0.36203261 \times 10^{-16}$	$0.15032210 \times 10^{-16}$	1.7035627	1.7154821	10.708638
9	$0.29106143 \times 10^{-19}$	$-0.11087934 \times 10^{-18}$	1.6507112	1.7154750	8.8428158
10	$0.18012308 \times 10^{-22}$	$0.81786167 \times 10^{-21}$	1.6048457	1.7154736	9.0113344
11	$0.88115370 \times 10^{-26}$	$-0.60326673 \times 10^{-23}$	1.5644762	1.7154734	8.9993459
12	$0.34823920 \times 10^{-29}$	$0.44497842 \times 10^{-25}$	1.5285491	1.7154734	9.0000311
13	$0.11321107 \times 10^{-32}$	$-0.32822264 \times 10^{-27}$	1.4962768	1.7154734	8.9999988
14	$0.30738762 \times 10^{-36}$	$0.24210186 \times 10^{-29}$	1.4670507	1.7154734	9.0000000
15	$0.70615767 \times 10^{-40}$	$-0.17857790 \times 10^{-31}$	1.4403911	1.7154735	9.0000000
16	$0.13880322 \times 10^{-43}$	$0.13172169 \times 10^{-33}$	1.4159159	1.7154735	9.0000000
17	$0.23573795 \times 10^{-47}$	$-0.97159864 \times 10^{-36}$	1.3933188	1.7154736	9.0000000
18	$0.34893132 \times 10^{-51}$	$0.71666548 \times 10^{-38}$	1.3723521	1.7154736	9.0000000
19	$0.45359082 \times 10^{-55}$	$-0.52862303 \times 10^{-40}$	1.3528138	1.7154736	9.0000000
20	$0.52141937 \times 10^{-59}$	$0.38992014 \times 10^{-42}$	1.3345373	1.7154737	9.0000000

satisfies matching conditions). Let $u_1 = \Delta u$. Note that g_m is odd, so only the odd part of u_1 contributes to the integral in (2.50). So (2.49) will follow from (2.50) and the estimate

$$\left| \int g_m(u_1 - u_1 \circ \rho_0)\, d\mu \right| \leqslant c(||u_1||_\infty + ||\Delta u_1||_\infty). \tag{2.52}$$

But (2.52) is routine, because on the cells $F_0^k F_1(SG)$ and $F_0^k F_2(SG)$ $(0 \leqslant k \leqslant m)$ of measure 3^{-k-1}, the function g_m is of size 3^k, and $u_1 - u_1 \circ \rho_0$ can be estimated by $(\frac{3}{5})^k ||\Delta u_1||_\infty$. $\square$

In Table 2 we display the results of solving the recursion relations for n_j and t_j. The data suggests that $(-\lambda_2)^j t_j$ converges, in fact quite a bit faster than for β_j, and $\lim_{j\to\infty} \beta_j / t_{j+1} = 9$. Moreover n_j is always positive and satisfies

$$n_j \leqslant cj\alpha_j. \tag{2.53}$$

If Conjecture 2.10 holds, then $||P_{(j-1)1}||_\infty = \alpha_{j-1}$ so (2.48) implies $n_j \leq \frac{1}{2}\alpha_{j-1}$, which is only slightly weaker than (2.53).

We also have found that the recursion relation for n_j is unstable, and any slight perturbation produces a decay rate $O((\lambda_1^D)^{-j})$, which is even slower than the decay rate for β_j and t_j. Also a slight perturbation of the t_j recursion relation produces a decay rate of $O((\lambda_2^D)^{-j})$. We will explain this in Section 6.

Next we describe the change of basis formula to pass between $\{P_{jk}^{(\ell)}\}$ for different values of ℓ, an immediate consequence of Theorem 2.12.

Corollary 2.14. *We have*

$$\begin{pmatrix} P_{j1}^{(\ell)} \\ P_{j2}^{(\ell)} \\ P_{j3}^{(\ell)} \end{pmatrix} = \sum_{k=0}^{j} M_{j-k} \begin{pmatrix} P_{k1}^{(\ell+1)} \\ P_{k2}^{(\ell+1)} \\ P_{k3}^{(\ell+1)} \end{pmatrix} \tag{2.54}$$

for matrices M_j given by

$$\begin{cases} M_j = \begin{pmatrix} \alpha_j & n_j & 0 \\ \beta_j & -\alpha_j & t_j \\ 3\alpha_{j+1} & 3n_{j+1} & 0 \end{pmatrix} \quad \textit{for} \quad j \geqslant 2, \\ M_1 = \begin{pmatrix} \alpha_1 & n_1 & \frac{1}{6} \\ \beta_1 & -\alpha_1 & t_1 \\ 3\alpha_2 & 3n_2 & 0 \end{pmatrix}, \quad M_0 = \begin{pmatrix} \alpha_0 & n_0 & 0 \\ \beta_0 & \frac{1}{2} - \alpha_0 & t_0 \\ 3\alpha_1 & 3n_1 & -\frac{1}{2} \end{pmatrix}. \end{cases} \tag{2.55}$$

Similarly

$$\begin{pmatrix} P_{j1}^{(\ell)} \\ P_{j2}^{(\ell)} \\ P_{j3}^{(\ell)} \end{pmatrix} = \sum_{k=0}^{j} \widetilde{M}_{j-k} \begin{pmatrix} P_{k1}^{(\ell-1)} \\ P_{k2}^{(\ell-1)} \\ P_{k3}^{(\ell-1)} \end{pmatrix} \tag{2.56}$$

for

$$\begin{cases} \widetilde{M}_j = \begin{pmatrix} \alpha_j & n_j & 0 \\ \beta_j & -\alpha_j & -t_j \\ -3\alpha_{j+1} & -3n_{j+1} & 0 \end{pmatrix} \quad \textit{for} \quad j \geqslant 2, \\ \widetilde{M}_1 = \begin{pmatrix} \alpha_1 & n_1 & -\frac{1}{6} \\ \beta_1 & -\alpha_1 & -t_1 \\ -3\alpha_2 & -3n_2 & 0 \end{pmatrix}, \quad \widetilde{M}_0 = \begin{pmatrix} \alpha_0 & n_0 & 0 \\ \beta_0 & \frac{1}{2} - \alpha_0 & -t_0 \\ -3\alpha_1 & -3n_1 & -\frac{1}{2} \end{pmatrix}. \end{cases} \tag{2.57}$$

3. Power series

A formal power series about q_ℓ is an expression of the form

$$\sum_{k=1}^{3} \sum_{j=0}^{\infty} c_{jk} P_{jk}^{(\ell)}(x). \tag{3.1}$$

We call $\{c_{jk}\}$ the *coefficients*, and we seek growth conditions on the coefficients that will make (3.1) converge nicely.

Theorem 3.1. *If the coefficients satisfy*

$$|c_{j1}| \text{ and } |c_{j3}| = O((j!)^r) \text{ for some } r < \log 5/\log 2, \tag{3.2}$$

and

$$|c_{j2}| = O(R^j) \text{ for some } R < \lambda_2 \tag{3.3}$$

then (3.1) *converges uniformly and absolutely to a function* $u \in dom\,(\Delta^\infty)$, *and* (3.1) *may be* "*differentiated term-by-term*",

$$\Delta^n u(x) = \sum_{k=1}^{3} \sum_{j=n}^{\infty} c_{jk} P_{(j-n)k}^{(\ell)}(x). \tag{3.4}$$

Moreover, the coefficients are given by the infinite jet of u at q_ℓ:

$$\begin{cases} c_{j1} = \Delta^j u(q_\ell), \\ c_{j2} = \partial_n \Delta^j u(q_\ell), \\ c_{j3} = \partial_T \Delta^j u(q_\ell). \end{cases} \tag{3.5}$$

Proof. The estimates in Theorem 2.9 conspire with the growth rates (3.2) and (3.3) to make (3.1) converge uniformly and absolutely. Call the limit u. Note that the right side (3.4) is also a formal power series, in fact

$$\sum_{k=1}^{3} \sum_{j=0}^{\infty} c_{(j+n)k} P_{jk}^{(\ell)}(x)$$

whose coefficients also satisfy the growth rate conditions (3.2) and (3.3). So the right side of (3.4) converges uniformly and absolutely. By terminating the sums at $j = N$ and letting $N \to \infty$ we obtain the equality in (3.4) by a routine argument using the Green's function [Ki2].

It suffices to prove the jet formulas (3.5) when $j = 0$ in view of (3.4), and for this it suffices to show that if $c_{01} = c_{02} = c_{03} = 0$ then $u(q_\ell) = \partial_n u(q_\ell) = \partial_T u(q_\ell) = 0$. Of course $u(q_\ell) = 0$ directly from (3.1). For simplicity put $\ell = 0$. Then (since $u(q_0) = 0$)

$$\partial_n u(q_0) = -\lim_{m\to\infty} \left(\frac{5}{3}\right)^m (u(F_0^m q_1) + u(F_0^m q_2)).$$

But we have

$$u(F_0^m x) = \sum_{j=1}^{\infty} c_{j1} 5^{-mj} P_{j1}(x) + c_{j2} \left(\frac{3}{5} 5^{-j}\right)^m P_{j2}(x) + c_{j3} 5^{-m(j+1)} P_{j3}(x). \quad (3.6)$$

Using the estimates for the coefficients and monomials we see that

$$u(F_0^m x) = O(5^{-m}), \quad (3.7)$$

and this suffices to prove $\partial_n u(q_0) = 0$. This by itself does not suffice for the tangential derivative, which has a factor of 5^m. However, for the tangential derivative we can restrict attention to the skew-symmetric part

$$\tilde{u}(x) = \frac{1}{2}(u(x) - u(\rho_0 x)) = \sum_{j=1}^{\infty} c_{j3} P_{j3}(x), \quad (3.8)$$

so the analog of (3.6) shows

$$\tilde{u}(F_0^m x) = O(5^{-2m}), \quad (3.9)$$

which implies $\partial_T u(q_0) = 0$. $\square$

As a corollary of the proof we can characterize rates of vanishing of power series.

Definition 3.2. A function f is said to *vanish to order* r (any positive real) at q_ℓ provided

$$\|f \circ F_\ell^m\|_\infty = O(5^{-mr}). \quad (3.10)$$

If (3.10) holds for all r then we say f *vanishes to infinite order* at q_ℓ.

Corollary 3.3. *If u is represented by a power series* (3.1) *with coefficients satisfying growth conditions* (3.2) *and* (3.3), *then u vanishes to order N* (*a positive integer*) *at* q_ℓ *if and only if* $c_{jk} = 0$ *for all* $j < N$. *In that case* $\Delta^\ell u$ *vanishes to order* $N - \ell$ *for all* $\ell < N$. *Moreover, the odd part* $\tilde{u}$ *vanishes to order* $N + 1$. *In particular, if u is not identically zero then it cannot vanish to infinite order.*

Next we consider rearrangement of power series, moving from one boundary point q_ℓ to another. It turns out that we need to make stronger assumptions on the coefficients, requiring c_{j1} and c_{j3} to satisfy the same exponential growth rate as c_{j2}.

Theorem 3.4. *Suppose the coefficients of a power series* (3.1) *about one boundary point* q_ℓ *satisfy*

$$|c_{jk}| = O(R^j) \text{ for some } R<\lambda_2, \ k = 1,2,3. \tag{3.11}$$

Then the function may also be represented by power series about the other boundary points with coefficients also satisfying (3.11). *More precisely, the coefficients at* $q_{\ell+1}$ *are given by*

$$(c'_{j'1} \ c'_{j'2} \ c'_{j'3}) = \sum_{j=0}^{\infty} (c_{(j+j')1} \ c_{(j+j')2} \ c_{(j+j')3})M_j \tag{3.12}$$

and similarly at $q_{\ell-1}$ *with* M_j *replaced by* $\widetilde{M}_j$ (*see* (2.55) *and* (2.57)).

Proof. The key observation is that the right side of (3.12) converges absolutely and the new coefficients again satisfy (3.11) (in fact with the same value of R) because the entries in M_j are $O(\lambda_2^{-j})$ by Theorem 2.13. Of course (3.11) is exactly what we get if we substitute (2.54) into (3.1) and interchange the order of summation, which is easily justified using the estimates of Theorem 2.9. □

Note that we could not allow slower growth rates like (3.2) for the c_{j1} and c_{j3} coefficients and still rearrange, because the second column of M_j has positive entries. In Section 6 we will present an example to show that rearrangement fails when $c_{j1} = O(\lambda_2^j)$. However, condition (3.11) is not sharp. We could replace it by

$$\sum_{j=0}^{\infty} \lambda_2^{-j}|c_{jk}|<\infty, \tag{3.13}$$

and the rearranged coefficients would satisfy the same growth condition. However, not all subsequent results would be valid under this hypothesis.

Definition 3.5. An *entire analytic function* is a function given by a power series (3.1) with coefficients satisfying (3.11).

We can also consider local power series expansions on any cell $F_w(SG)$ with respect to a boundary point $F_w q_\ell$ of the cell, namely

$$\sum_{j=0}^{\infty} \left(5^{-mj} c_{j1} P_{j1}^{(\ell)}(F_w^{-1}x) + \left(\frac{3}{5}5^{-j}\right)^m c_{j2} P_{j2}^{(\ell)}(F_w^{-1}x) \right.$$
$$\left. + 5^{-(j+1)m} c_{j3} P_{j3}^{(\ell)}(F_w^{-1}x) \right) \tag{3.14}$$

where $m = |w|$.

Theorem 3.6. *An entire analytic function has a local power series expansion* (3.14) *for any* w *and* ℓ *with coefficients satisfying* (3.11). *Conversely, suppose* $u(x)$ *is a function* *defined on* $F_w(SG)$ *given by a local power series expansion* (3.14) *with coefficients satisfying* (3.11). *Then* u *has a unique extension to an entire analytic function.*

Proof. Suppose first that $m=1$, say $w=(0)$. If $\ell=0$ then the local and global power series are identical, with identical coefficients. Moreover, $u \circ F_w$ is an entire analytic function with coefficients satisfying (3.11) (in fact with $R < \lambda_2/5$). The rearrangement for $u \circ F_w$ about q_1 and q_2 guaranteed by Theorem 3.4 gives the local power series of u in $F_0(SG)$ about $F_0 q_1$ and $F_0 q_2$, with the same coefficient estimates. We may then iterate this argument to get local power series about any boundary point in any cell.

Conversely, suppose u is given in $F_w(SG)$ by a local power series about $F_w q_\ell$, with coefficients satisfying (3.11). Write $w=(w', w_m)$ with $|w'| = m-1$. If $w_m \neq \ell$ then use Theorem 3.4 to rearrange the power series of $u \circ F_w$ about q_{w_m}. So we end up with a local power series of u about $F_{w'} F_\ell q_\ell$ in the cell $F_{w'} F_\ell(SG)$. But $F_{w'} F_\ell q_\ell = F_{w'} q_\ell$ and the power series makes sense in the cell $F_{w'}(SG)$. Use this power series to extend the definition of u. By iterating the argument, we obtain the desired extension. Note that the estimates (3.11) on the coefficients are reproduced in each extension or rearrangement step. It is clear that the extension is unique because the rearranged coefficients are determined by (3.12). $\square$

By the same reasoning, if a local power series has coefficients satisfying

$$c_{jk} = O(R^j) \text{ for some } R < 5^{m_0} \lambda_2, \tag{3.15}$$

then the function can be also represented by a power series on a level m_0 cell. One might hope that this "analytic continuation" might extend somewhat beyond the cell, with the domain of analyticity growing as R decreases toward $5^{m_0-1}\lambda_2$. However, the experimental evidence we have seen does not support this at all. On the contrary, we will see in Section 6 that there are power series (3.1) with coefficients $O(\lambda_2^j)$ where we have divergence outside $F_\ell(SG)$. We might describe this as a "quantized radius of convergence." Of course, this does not rule out a different type of behavior for special classes of power series.

Theorem 3.7. *An entire analytic function satisfies the estimate*

$$\|\Delta^n u\|_\infty = O(R^n) \text{ for some } R < \lambda_2. \tag{3.16}$$

Proof. We have

$$\Delta^n u = \sum_{k=1}^{3} \sum_{j=n}^{\infty} c_{jk} P^{(\ell)}_{(j-n)k}$$

so

$$||\Delta^n u||_\infty \leqslant M \sum_{k=1}^{3} \sum_{j=n}^{\infty} R^j ||P_{(j-n)k}^{(\ell)}||_\infty \leqslant M \sum_{j=n}^{\infty} R^j \lambda_2^{n-j} = O(R^n)$$

for R in (3.11). □

Condition (3.16) obviously implies the same estimate in L^2 norm:

$$||\Delta^n u||_2 = O(R^n) \text{ for some } R < \lambda_2. \tag{3.17}$$

But conversely, (3.17) implies (3.16), because $||f||_\infty \leqslant c(||f||_2 + ||\Delta f||_2)$. Estimate (3.17) is technically more convenient, since we can compute L^2 norms exactly from eigenfunction expansions.

It follows immediately from the definition that an eigenfunction of Δ is an entire analytic function if and only if the eigenvalue satisfies $|\lambda| < \lambda_2$. Theorem 3.7 shows us that many other functions that we might believe to be entire analytic functions are not. Indeed, suppose u is represented by a Dirichlet (or Neumann) eigenfunction expansion

$$u(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x), \tag{3.18}$$

where $\{\varphi_k\}$ is an orthonormal basis of Dirichlet (or Neumann) eigenfunctions. If the coefficients are rapidly decreasing,

$$a_k = O(k^{-n}) \text{ for all } n, \tag{3.19}$$

then we may differentiate term-by-term,

$$\Delta^n u(x) = \sum_{k=1}^{\infty} (\lambda_k^D)^n a_k \varphi_k(x). \tag{3.20}$$

It follows that

$$||\Delta^n u||_2 = \left(\sum_{k=1}^{\infty} (\lambda_k^D)^{2n} |a_k|^2 \right)^{1/2}. \tag{3.21}$$

If (3.18) is non-trivial in the sense that an infinite number of coefficients are non-zero, then not only does (3.17) fail to hold, but the estimate cannot hold for any finite R. So u cannot be represented by a local power series with (3.14) holding on any cell. In particular this applies to the heat kernel.

This observation stands in striking contrast to the situation on the unit interval, where analyticity properties of a function may be characterized by decay properties of the coefficients of its Fourier series expansion.

4. Characterization of analytic functions

The main purpose of this section is to prove the following theorem.

Theorem 4.1. *u is an entire analytic function if and only if $u \in dom(\Delta^{\infty})$ and* (3.16) (*or equivalently* (3.17)) *holds.*

We first consider the case when u is even with respect to ρ_0. In that case we would like a Taylor expansion with remainder about q_0,

$$u(x) = T_k u(x) + R_k(x) \tag{4.1}$$

for

$$T_k u(x) = \sum_{j=0}^{k-1} \Delta^j u(q_0) P_{j1}(x) + (\partial_n \Delta^j u(q_0)) P_{j2}(x) \tag{4.2}$$

and $R_k(x)$ the remainder term. While we can use (4.1) to define the remainder, to be useful we need some explicit expression for it. We are only able to do this for $x = q_1$ (or q_2).

Lemma 4.2. *Let v_k be a function in $\mathscr{H}_{k-1}$ that is even with respect to ρ_0 satisfying*

$$\Delta^j v_k(q_1) = 0 \text{ for } j \leqslant k-1, \tag{4.3}$$

$$\partial_n \Delta^j v_k(q_1) = \begin{cases} 0 & \text{for } j \leqslant k-2, \\ -\frac{1}{2} & \text{for } j = k-1. \end{cases} \tag{4.4}$$

Then

$$R_k(q_1) = R_k(q_2) = \int_{SG} v_k \Delta^k u \, d\mu \tag{4.5}$$

for even functions $u \in dom(\Delta^k)$.

Proof. Note that $\Delta^k u = \Delta^k (u - T_k u) = \Delta^k R_k$. We apply the Gauss–Green formula k times to obtain

$$\begin{aligned}\int v_k \Delta^k u \, d\mu &= \int v_k \Delta^k R_k \, d\mu \\ &= 2 \sum_{j=0}^{k-1} (\Delta^j v_k(q_1) \partial_n \Delta^{k-j-1} R_k(q_1) - \partial_n \Delta^j v_k(q_1) \Delta^{k-j-1} R_k(q_1))\end{aligned}$$

since $\Delta^{k-j-1}R_k(q_0) = \partial_n\Delta^{k-j-1}R_k(q_0) = 0$. By (4.3) and (4.4) all terms vanish except when $j = k-1$ and we obtain exactly $R_k(q_1)$. □

Lemma 4.3. *The function*

$$v_k = \sum_{\ell=0}^{k-1}(-\beta_{k-\ell-1}P^{(0)}_{\ell 1} + \alpha_{k-\ell-1}P^{(0)}_{\ell 2}) \tag{4.6}$$

satisfies the conditions of Lemma 4.2.

Proof. Clearly $v_k \in \mathscr{H}_{k-1}$ and is even. Since

$$\Delta^j v_k = \sum_{\ell=0}^{k-j-1} -\beta_{k-j-1-\ell}P_{\ell 1} + \alpha_{k-j-1-\ell}P_{\ell 2}$$

we obtain

$$\Delta^j v_k(q_1) = \sum_{\ell=0}^{k-j-1} (-\beta_{k-j-1-\ell}\alpha_\ell + \alpha_{k-j-1-\ell}\beta_\ell) = 0$$

which is (4.3). Similarly

$$\partial_n\Delta^j v_k(q_1) = \sum_{\ell=0}^{k-j-1}\left(-\beta_{k-j-1-\ell}n_\ell - \alpha_{k-j-1-\ell}\alpha_\ell\right) + \frac{1}{2}\alpha_{k-j-1}$$

by (2.35). When $j = k-1$ this is just

$$\partial_n\Delta^{k-1}v_k(q_1) = -\beta_0 n_0 - \alpha_0^2 + \frac{1}{2}\alpha_0 = -\frac{1}{2}.$$

For $j \leqslant k-2$ we have

$$\sum_{\ell=0}^{k-j-1} \beta_{k-j-1-\ell}n_\ell = -\left(\frac{5^{k-j-1}+1}{4}\right)\alpha_{k-j-1}$$

by (2.37) (this uses $k-j-1 \geqslant 1$), and

$$\sum_{\ell=0}^{k-j-1} \alpha_{k-j-1-\ell}\alpha_\ell = \left(\frac{5^{k-j-1}+3}{4}\right)\alpha_{k-j-1}$$

by (2.9). Thus $\partial_n\Delta^j v_k(q_1) = 0$, proving (4.4). □

Lemma 4.4. *If u is an even function in dom* (Δ^k) *satisfying* (3.16), *and $\tilde{u}$ is the entire analytic function whose expansion about q_0 has coefficients $c_{j1} = \Delta^j u(q_0)$, $c_{j2} = \partial_n\Delta^j u(q_0)$ and $c_{j3} = 0$, then $u(q_1) = \tilde{u}(q_1)$ and $u(q_2) = \tilde{u}(q_2)$.*

Proof. First we observe that (3.16) implies the coefficients of $\tilde{u}$ satisfy (3.11). This is obvious for c_{k1} and c_{k3}, but it follows for c_{k2} because $\partial_n f(q_0) = \int hf\,d\mu$ for a fixed harmonic function h. Now apply Lemma 4.2 to the function $u - \tilde{u}$ to obtain

$$|u(q_1) - \tilde{u}(q_1)| = \left| \int v_k \Delta^k (u - \tilde{u})\, d\mu \right| \leqslant cR^k ||v_k||_\infty .$$

But we easily obtain $||v_k||_\infty = O(\lambda_2^{-k})$ from (4.6) and Theorem 2.9. Letting $k \to \infty$ we obtain $u(q_1) - \tilde{u}(q_1) = 0$. □

Proof of Theorem 4.1. We begin by proving $u = \tilde{u}$ under the assumption that u is even and $R < \lambda_1^D$. Since $\Delta^j u$ satisfies the same hypotheses as u, we conclude from Lemma 4.4 that $\Delta^j(u - \tilde{u})$ vanishes at all three boundary points, for any j. Let $G(x,y)$ denote the Green's function and $G^j(x,y)$ the j-fold iteration of G. The vanishing at boundary points means that

$$u(x) - \tilde{u}(x) = \int G^j(x,y) \Delta^j (u(y) - \tilde{u}(y))\, d\mu(y). \tag{4.7}$$

We have an explicit representation

$$G^j(x,y) = \sum_{k=1}^{\infty} (\lambda_k^D)^{-j} \varphi_k(x) \varphi_k(y) \tag{4.8}$$

for an orthonormal basis of Dirichlet eigenfunctions $\{\varphi_k\}$ with $-\Delta\varphi_k = \lambda_k^D \varphi_k$. This yields the estimate

$$\left(\int \int |G^j(x,y)|^2 \, d\mu(x)\, d\mu(y) \right)^{1/2} = \left(\sum_{k=1}^{\infty} (\lambda_k^D)^{-2j} \right)^{1/2} \leqslant c(\lambda_1^D)^{-j} \tag{4.9}$$

by the Weyl asymptotics of $\{\lambda_k^D\}$. Thus

$$||u - \tilde{u}||_2 \leqslant c(\lambda_1^D)^{-j} ||\Delta^j (u - \tilde{u})||_2 \leqslant c(\lambda_1^D)^{-j} R^j.$$

Letting $j \to \infty$ we obtain $||u - \tilde{u}||_2 = 0$ hence $u = \tilde{u}$ as desired.

Next we can remove the assumption that u be even by writing u as a sum of even functions about each of the three boundary points using (2.26). It is clear that the hypotheses on u are inherited by the three summands, and a sum of three entire analytic functions is entire analytic.

Finally, we need to relax the assumption that $R < \lambda_1^D$ to $R < \lambda_2$. To do this we consider $u \circ F_w$ for all words of length 2 (because $5^{-2}\lambda_2 < \lambda_1^D$). Then $u \circ F_w$ satisfies (3.16) with $R < \lambda_1^D$, so by the previous argument it is entire analytic. This means for each w there exists $\tilde{u}_w$ entire analytic with $u = \tilde{u}_w$ on $F_w(SG)$. Next we claim that $\tilde{u}_{00} = \tilde{u}_{01} = \tilde{u}_{02}$. To see this we may assume without loss of generality that $\tilde{u}_{00} = 0$ by replacing u by $u - \tilde{u}_{00}$. So u is assumed to vanish on $F_0^2(SG)$, and we need to show that it vanishes on $F_0(SG)$. By Lemma 4.4 we have $u(F_0q_1) = u(F_0q_2) = 0$, and more generally $\Delta^j u(F_0q_1) = \Delta^j u(F_0q_2) = 0$ by the same reasoning for $\Delta^j u$. Let us consider $\tilde{u}_{01}$ which equals u on $F_0F_1(SG)$. At the point $F_0^2q_1$ where the cells $F_0F_1(SG)$ and $F_0^2(SG)$ intersect, we have $\Delta^j u$ vanishing and also $\partial_n\Delta^j u$ vanishing (obvious for the normal derivative with respect to $F_0^2(SG)$, and then true with respect to $F_0F_1(SG)$ by the matching condition for normal derivatives). Thus the local power series expansion in $F_0F_1(SG)$ of $\tilde{u}_{01}$ about the point $F_0^2q_1$ contains only P_{j3} terms, so $\tilde{u}_{01}$ and more generally $\Delta^j\tilde{u}_{01}$ must be odd, so the vanishing of $\Delta^j\tilde{u}_{01}$ at the second boundary point F_0q_1 implies the vanishing at the third boundary point $F_0F_1q_2$. So our previous argument shows that $\tilde{u}_{01}$ is identically zero.

The same argument works in the other two cells of level one, so we now know that there exist entire analytic functions $\tilde{u}_0, \tilde{u}_1, \tilde{u}_2$ such that $u = \tilde{u}_j$ on $F_j(SG)$. We need to show $\tilde{u}_0 = \tilde{u}_1 = \tilde{u}_2$, and by subtracting $\tilde{u}_0$ we may assume without loss of generality that $\tilde{u}_0 = 0$. At this point we cannot simply repeat the argument of the previous paragraph because the cell $F_1(SG)$ is too big. Of course we can argue as before that $\tilde{u}_1$ and more generally $\Delta^j\tilde{u}_1$ vanishes on all three boundary points of $F_1(SG)$, and that it is odd about the vertex F_0q_1. It is this oddness that saves the argument. Instead of (4.7) for $\tilde{u}_1\circ F_1$ we have

$$\tilde{u}_1\circ F_1(x) = \int \tilde{G}^j(x,y)\Delta^j(\tilde{u}_1\circ F_1)(y)d\mu(y) \tag{4.10}$$

where $\tilde{G}^j$ denotes the j-fold iteration of the odd part of the Green's function. Instead of (4.8), $\tilde{G}^j$ has the same representation where the sum is restricted to the odd eigenfunctions. The eigenfunction associated to λ_1^D is even, so the smallest eigenvalue appearing is $\lambda_2^D \approx 55.8858\ldots$. Thus we obtain the estimate

$$||\tilde{u}_1\circ F_1||_2 \leqslant c(\lambda_2^D)^{-j}5^{-j}R^j,$$

and this shows $\tilde{u}_1 = 0$ because $\lambda_2 \leqslant 5\lambda_2^D$. □

It is interesting that the growth conditions (3.16) imply the specific identities (2.22). There is nothing analogous to this in the theory of real analytic functions. In some way it is reminiscent of the Cauchy integral formula for complex analytic functions. But we do not want to read too much into this, since (2.22) holds for nonanalytic functions as well.

Corollary 4.5. *If u is defined on a cell $F_w(SG)$ and satisfies*

$$||\Delta^j u||_{L^\infty(F_w(SG))} = O(R^j) \text{ for some } R < \lambda_2 \tag{4.11}$$

then u has a unique extension to an entire analytic function.

Proof. The theorem shows $u\circ F_w$ is entire analytic. Then apply Theorem 3.6. □

We can also consider entire analytic functions on any infinite blow-up of SG. The coefficients must satisfy (3.11) for all $R>0$, and the characterization requires the estimate (3.16) to hold locally for all $R>0$.

5. Expansions about junction points

A junction point is a boundary point of two cells, so an entire analytic function will have two different local power series (3.14) centered at the point, each valid in a different cell. Since each local power series determines the function, it also determines the other local power series. Since the coefficients of the local power series are just the jets at the point with respect to each cell, these jets determine each other. The first goal of this section is to make this determination explicit.

To be specific, consider the junction point $F_0q_1 = F_1q_0$. We will write $F_0q_1 = q_{01}$ and write

$$(\Delta^j u(q_{01}), \partial_n \Delta^j u(q_{01}), \partial_T \Delta^j u(q_{01})) \tag{5.1}$$

for the jet associated with the cell $F_0(SG)$, and $F_1q_0 = q_{10}$ and

$$(\Delta^j u(q_{10}), \partial_n \Delta^j u(q_{10}), \partial_T \Delta^j u(q_{10})) \tag{5.2}$$

for the jet associated with the cell $F_1(SG)$. We know some relationships between the jets (5.1) and (5.2), namely

$$\Delta^j u(q_{01}) = \Delta^j u(q_{10}) \text{ and } \partial_n \Delta^j u(q_{01}) = -\partial_n \Delta^j u(q_{10}). \tag{5.3}$$

Note that (5.3) is valid for all $u \in dom\, \Delta^\infty$, but there should be no connections between tangential derivatives without the assumption that u is an entire analytic function. On the other hand, for entire analytic functions, we expect an identity of the form

$$\partial_T u(q_{01}) + \partial_T u(q_{10}) = \sum_{\ell=0}^{\infty} Y_\ell \partial_n \Delta^\ell u(q_{01}) \tag{5.4}$$

to hold for certain coefficients Y_ℓ. Note that (5.4) applied to $\Delta^j u$ yields

$$\partial_T \Delta^j u(q_{01}) + \partial_T \Delta^j u(q_{10}) = \sum_{\ell=j}^{\infty} Y_{\ell-j} \partial_n \Delta^\ell u(q_{01}), \tag{5.5}$$

and (5.3) and (5.5) show how the jets (5.1) and (5.2) determine each other. We may also interpret (5.4) as a matching condition for tangential derivatives.

Our strategy for determining the Y coefficients will be to first consider the case when u is a polynomial, making the sum finite. It is convenient to consider the monomials $P_{jk}^{(2)}$, because the ρ_2 symmetry is also a symmetry about q_{01}. For even functions, both sides of (5.4) are zero regardless of the Y coefficients: the left side vanishes because of the oddness of the tangential derivative, and the right side because of the matching condition $\partial_n \Delta^\ell u(q_{01}) = -\partial_n \Delta^\ell u(q_{01})$ and the evenness of the normal derivative and Laplacian. Thus we need only check (5.4) for the monomials $P_{j3}^{(2)}$.

Lemma 5.1. *The matching condition* (5.4) *holds for all polynomials for the* Y *coefficients satisfying* $Y_0 = 4$ *and recursively*

$$Y_j = -\alpha_j - 18 \sum_{\ell=0}^{j} n_{j+1-\ell} \frac{t_\ell}{5^\ell} + \sum_{\ell=0}^{j-1} Y_\ell \left(\left(\frac{3}{2} - \frac{5^{\ell-j}}{2} \right) n_{j-\ell+1} + \sum_{k=0}^{j-\ell} \left(5\alpha_{j+1-\ell-k} n_k 5^{-k} - 3 n_{j+1-\ell-k} \alpha_k 5^{-k} \right) \right) \quad \text{for } j \geqslant 1. \tag{5.6}$$

Proof. When $j = 0$ we compute directly that $\partial_T P_{03}^{(2)}(q_{01}) + \partial_T P_{03}^{(2)}(q_{10}) = -4$ and $\partial_n P_{03}^{(2)}(q_{01}) = -1$, so $Y_0 = 4$. For $j \geqslant 1$ we use Corollary 2.14 to rearrange $P_{j3}^{(2)}$ around q_0. By (2.54) we obtain

$$P_{j3}^{(2)} = -\frac{1}{2} P_{j3}^{(0)} + 3 \sum_{\ell=0}^{j} \left(\alpha_{j+1-\ell} P_{\ell 1}^{(0)} + n_{j+1-\ell} P_{\ell 2}^{(0)} \right). \tag{5.7}$$

Because $P_{j3}^{(2)}$ is odd we have

$$\partial_T P_{j3}^{(2)}(q_{01}) + \partial_T P_{j3}^{(2)}(q_{10}) = 2\partial_T P_{j3}^{(2)}(q_{01}).$$

By (5.7) and Theorem 2.12 we have

$$2\partial_T P_{j3}^{(2)}(q_{01}) = \alpha_j + 18 \sum_{\ell=0}^{j} n_{j+1-\ell} \frac{t_\ell}{5^\ell} \tag{5.8}$$

and

$$\partial_n P_{j3}^{(2)}(q_{01}) = \left(\frac{3}{2} - \frac{1}{2} 5^{-j} \right) n_{j+1} + \sum_{k=0}^{j} \left(5\alpha_{j+1-k} n_k 5^{-k} - 3 n_{j+1-k} \alpha_k 5^{-k} \right). \tag{5.9}$$

Since $\Delta^\ell P_{j3}^{(2)} = P_{(j-\ell)3}^{(2)}$, we have that (5.4) for $u = P_{j3}^{(2)}$ yields

$$Y_j = \sum_{\ell=0}^{j-1} Y_\ell \partial_n P_{(j-\ell)3}^{(2)}(q_{01}) - 2\partial_T P_{j3}^{(2)}(q_{01}).$$

Substituting (5.8) and (5.9) yields (5.6). $\square$

Table 3

j	Y_j	$(-\lambda_2)^j Y_j$
0	-4	-4
1	-0.08888888889	12.05085573
2	0.0002304526749	4.235674447
3	$-0.1434871749 \times 10^{-5}$	3.575397353
4	$0.1023938272 \times 10^{-7}$	3.459038654
5	$-0.7503519662 \times 10^{-10}$	3.436505741
6	$0.5527533783 \times 10^{-12}$	3.432052039
7	$-0.4076138308 \times 10^{-14}$	3.431166398
8	$0.3006465014 \times 10^{-16}$	3.430989845
9	$-0.2217590148 \times 10^{-18}$	3.430954602
10	$0.1635723837 \times 10^{-20}$	3.430947563
11	$-0.1206533528 \times 10^{-22}$	3.430946155
12	$0.8899568485 \times 10^{-25}$	3.430945874
13	$-0.6564452839 \times 10^{-27}$	3.430945818
14	$0.4842037197 \times 10^{-29}$	3.430945807
15	$-0.3571558034 \times 10^{-31}$	3.430945805
16	$0.2634433871 \times 10^{-33}$	3.430945805
17	$-0.1943197270 \times 10^{-35}$	3.430945805
18	$0.1433330961 \times 10^{-37}$	3.430945805
19	$-0.1057246052 \times 10^{-39}$	3.430945805
20	$0.7798402782 \times 10^{-42}$	3.430945805

Conjecture 5.2. The coefficients Y_j satisfy

$$|Y_j| \leqslant c\lambda_2^{-j}. \tag{5.10}$$

The numerical evidence for Conjecture 5.2 is presented in Table 3.

Theorem 5.3. *Assume Conjecture* 5.2. *If u is any entire analytic function, then* (5.4) *and* (5.5) *hold for the Y coefficients given in Lemma* 5.1. *More generally, if x is any junction point in* $V_{m+1} \backslash V_m$, *then*

$$\partial_T \Delta^j u(x) + \partial_T^* \Delta^j u(x) = \sum_{\ell=j}^{\infty} 3^m 5^{-m(\ell-j)} Y_{\ell-j} \partial_n \Delta^\ell u(x), \tag{5.11}$$

where ∂_T *and* ∂_n *are derivatives with respect to the left cell at x and* ∂_T^* *is the derivative with respect to the right cell.*

Proof. Note that the right side of (5.4) converges absolutely. The issue is then whether the term–by–term differentiation of power series extends to normal and tangential derivatives at points other than the expansion point. For normal derivatives this is easy to see because of the integral representation. But in any case this follows by combining Theorem 3.4 (the explicit expression (3.12) for the rearranged coefficients) with Theorem 3.1 (the jet formula (3.5) at the expansion point). We then obtain (3.10) by applying (3.5) to the function $u \circ F_w$ for $|w| = m$. $\square$

Next we consider the question of what would be a natural notion of a power series expansion centered about a junction point. We will see that there is no completely satisfactory answer. Again to be specific we consider the point $q_{01} = q_{10}$. We would like to have at least the following four conditions holding:

(i) every entire analytic function has an expansion;

(ii) the expansion is valid in a neighborhood of q_{01}, perhaps $F_0(SG) \cup F_1(SG)$;

(iii) the individual terms are polynomials that vanish to higher and higher order near q_{01};

(iv) the rate of growth of the coefficients should be characterized for entire analytic functions.

The local power series with respect to one of the cells, say $F_0(SG)$, gives a satisfactory answer only on that cell, but if we continue those monomials around we will find that the vanishing rate near q_{10} is not satisfactory. In fact the tangential derivatives will have to be nonzero by Lemma 5.1. For this reason we consider carefully what it takes to meet condition (iii). We denote by $P_{jk}^{(01)}$ the monomials of the $F_0(SG)$ local power series about q_{01}, so that

$$\Delta^\ell P_{jk}^{(01)}(q_{01}) = \delta_{j\ell}\delta_{k1},$$

$$\partial_n \Delta^\ell P_{jk}^{(01)}(q_{01}) = \delta_{j\ell}\delta_{k2},$$

$$\partial_T \Delta^\ell P_{jk}^{(01)}(q_{01}) = \delta_{j\ell}\delta_{k3}$$

or more precisely

$$P_{j1}^{(01)}(x) = 5^{-j} P_{j1}^{(1)}(F_0^{-1}x),$$

$$P_{j2}^{(01)}(x) = \frac{3}{5} 5^{-j} P_{j2}^{(1)}(F_0^{-1}x),$$

$$P_{j3}^{(01)}(x) = 5^{-j-1} P_{j3}^{(1)}(F_0^{-1}x).$$

Note that $P_{j1}^{(01)}$ and $P_{j3}^{(01)}$ extend to even polynomials about q_{01}, so they will have the same vanishing rate on both cells. We want to replace $P_{j2}^{(01)}$ by a different polynomial $\tilde{P}_{j2}^{(01)}$ that will have the same j-jet (except for $\partial_T \Delta^j u(q_{01})$), but will extend to be odd. This will give it the correct order of vanishing, but in exchange we have to take a higher order polynomial. The lowest possible order is $2j$:

$$\tilde{P}_{j2}^{(01)} = \sum_{\ell=0}^{j} (a_{j(j-\ell)} P_{(j+\ell)2}^{(01)} + b_{j(j-\ell)} P_{(j+\ell)3}^{(01)}) \tag{5.12}$$

for the appropriate choice of constants. Note that we can exclude $P_{(j+\ell)1}^{(01)}$ terms because we want the possibility of odd extension. We will take $a_{jj} = 1$ in order to obtain the correct j-jet. The odd extension means $\partial_T \Delta^n \tilde{P}_{j2}^{(01)}(q_{01}) = \partial_T \Delta^n \tilde{P}_{j2}^{(01)}(q_{10})$, so we have $2j+1$ equations of the form (5.5) to satisfy, and these will determine the remaining $2j+1$ constants. The equations are

$$2\partial_T \Delta^n \tilde{P}_{j2}^{(01)}(q_{10}) = \sum_{k=n}^{2j} Y_{k-n} \partial_n \Delta^k \tilde{P}_{j2}^{(01)}(q_{02}), \tag{5.13}$$

and when $0 \leqslant n < j$ the left side is zero and we obtain

$$0 = \sum_{k=n}^{2j} Y_{k-n} \partial_n \Delta^k \tilde{P}_{j2}^{(01)}(q_{01}) = \sum_{k=j}^{2j} Y_{k-n} a_{j(2j-k)}$$

so

$$0 = \sum_{\ell=0}^{j} Y_{2j-\ell-n} a_{j\ell}. \tag{5.14}$$

We use these equations to solve for $a_{j\ell}$. When $n \leqslant j \leqslant 2j$ the left side of (5.13) is $2b_{j(2j-n)}$ so

$$2b_{j(2j-n)} = \sum_{k=n}^{2j} Y_{k-n} a_{j(2j-k)},$$

and by letting $\ell = 2j - n$ we have

$$b_{j\ell} = \frac{1}{2} \sum_{k=0}^{\ell} Y_k a_{j(\ell-k)} \quad \text{for } 0 \leqslant \ell \leqslant j. \tag{5.15}$$

In Table 4 we show the values of $a_{j\ell}$ and $b_{j\ell}$ for small values of j. It is difficult to discern a pattern in these results. We have obtained graphs of $\tilde{P}_{j2}^{(01)}$ for small values of j using (5.12), but it appears that round-off error becomes significant before any pattern emerges, so we are not able to offer any conjectures about the growth rate of these functions as $j \to \infty$.

Table 4

j	l	a_{jl}	b_{jl}	j	l	a_{jl}	b_{jl}
0	0	1	2	7	0	$0.1330959781 \times 10^{23}$	$0.2661919562 \times 10^{23}$
1	0	0.02252966406	0.04505932812	7	1	$0.6141913960 \times 10^{21}$	$0.7847295317 \times 10^{21}$
1	1	1	1.999249011	7	2	$0.6084736857 \times 10^{19}$	$0.1968365718 \times 10^{23}$
2	0	6461.417615	12922.83523	7	3	$0.2707503937 \times 10^{17}$	$0.1030137030 \times 10^{22}$
2	1	-39.86777272	-295.1161326	7	4	$0.4581523610 \times 10^{14}$	$0.1231428577 \times 10^{20}$
2	2	1	9563.195714	7	5	$0.2620127789 \times 10^{11}$	$0.5414059059 \times 10^{17}$
3	0	$0.1631072895 \times 10^{7}$	$0.3262145790 \times 10^{7}$	7	6	3880.162356	$0.9158266227 \times 10^{14}$
3	1	48581.69671	42794.29693	7	7	1	$0.5243561927 \times 10^{11}$
3	2	-109.6002902	$0.2411384099 \times 10^{7}$	8	0	$-0.2849367688 \times 10^{25}$	$-0.5698735375 \times 10^{25}$
3	3	1	86782.07999	8	1	$-0.1352864496 \times 10^{24}$	$0.1755939762 \times 10^{24}$
4	0	$-0.1623039023 \times 10^{10}$	$-0.3246078045 \times 10^{10}$	8	2	$-0.1478090302 \times 10^{22}$	$-0.4214174069 \times 10^{25}$
4	1	$-0.6442287860 \times 10^{8}$	$-0.7474445645 \times 10^{8}$	8	3	$-0.7540725789 \times 10^{19}$	$-0.2261525660 \times 10^{24}$
4	2	-299734.8354	$-0.2399788368 \times 10^{10}$	8	4	$-0.1760661536 \times 10^{17}$	$-0.2930516976 \times 10^{22}$
4	3	-347.4611669	$-0.1101312661 \times 10^{9}$	8	5	$-0.1895987908 \times 10^{14}$	$-0.1511819510 \times 10^{20}$
4	4	1	-751724.7199	8	6	$-0.7756675150 \times 10^{10}$	$-0.3520528934 \times 10^{17}$
5	0	$0.1010368178 \times 10^{14}$	$0.2020736356 \times 10^{14}$	8	7	-3618.462380	$-0.3790729379 \times 10^{14}$
5	1	$0.4380632964 \times 10^{12}$	$0.5393372002 \times 10^{12}$	8	8	1	$-0.1552676258 \times 10^{11}$
5	2	$0.3374174349 \times 10^{10}$	$0.1494085527 \times 10^{14}$	9	0	$0.4817483229 \times 10^{29}$	$0.9634966458 \times 10^{29}$
5	3	$0.1015644445 \times 10^{8}$	$0.7403235769 \times 10^{12}$	9	1	$0.2289760048 \times 10^{28}$	$0.2973692352 \times 10^{28}$
5	4	-909.3198857	$0.7249040413 \times 10^{10}$	9	2	$0.2513117964 \times 10^{26}$	$0.7125008828 \times 10^{29}$
5	5	1	$0.1921254540 \times 10^{8}$	9	3	$0.1299020030 \times 10^{24}$	$0.3827224251 \times 10^{28}$
6	0	$-0.1389829261 \times 10^{18}$	$-0.2779658521 \times 10^{18}$	9	4	$0.3154544064 \times 10^{21}$	$0.4977745865 \times 10^{26}$
6	1	$-0.6247328496 \times 10^{16}$	$-0.7861892790 \times 10^{16}$	9	5	$0.3859718201 \times 10^{18}$	$0.2600348690 \times 10^{24}$
6	2	$-0.5605362673 \times 10^{14}$	$-0.2055333917 \times 10^{18}$	9	6	$0.2325380299 \times 10^{15}$	$0.6310751388 \times 10^{21}$
6	3	$-0.2151475440 \times 10^{12}$	$-0.1051115464 \times 10^{17}$	9	7	$0.5539946952 \times 10^{11}$	$0.7717084596 \times 10^{18}$
6	4	$-0.2169919676 \times 10^{9}$	$-0.1159983908 \times 10^{15}$	9	8	-6592.977986	$0.4652032965 \times 10^{15}$
6	5	-1787.130925	$-0.4257054009 \times 10^{12}$	9	9	1	$0.1107495241 \times 10^{12}$
6	6	1	$-0.4383706038 \times 10^{9}$				

6. Exponentials

Eigenfunctions of the Laplacian give us a natural class of special functions on SG. Until now, most attention has been paid to eigenfunctions satisfying Dirichlet or Neumann boundary conditions, which forces the eigenvalue to be positive. In contrast, we will mainly explore negative eigenvalues in this section, so we are exploring the analog of the functions $\cosh \sqrt{\lambda} t$ and $\sinh \sqrt{\lambda} t$ on the unit interval and their extension to the positive real line. Of particular interest is the linear combination that yields $e^{-\sqrt{\lambda} t}$, the unique choice that exhibits exponential decay (either as $\lambda \to \infty$ or as $t \to \infty$) as opposed to exponential growth. It is embarrassing to note that the exponential $e^{\sqrt{\lambda} t}$ does not distinguish itself among linear combinations of $\cosh \sqrt{\lambda} t$ and $\sinh \sqrt{\lambda} t$, if one is forbidden to use odd order derivatives. So we have not been able to find its analog on SG.

The space of all eigenfunctions with a fixed eigenvalue has dimension three, as long as one avoids Dirichlet eigenvalues. For fixed $\lambda > 0$ we can choose a basis C_λ, S_λ, Q_λ for the space of solutions to

$$-\Delta u = -\lambda u \tag{6.1}$$

determined by the conditions that C_λ and S_λ are even and Q_λ is odd with respect to ρ_0, and

$$C_\lambda(q_0) = 1, \quad \partial_n C_\lambda(q_0) = 0, \tag{6.2}$$

$$S_\lambda(q_0) = 0, \quad \partial_n S_\lambda(q_0) = a_\lambda, \tag{6.3}$$

$$\partial_T Q_\lambda(q_0) = 1, \tag{6.4}$$

where the normalization factor a_λ will be chosen later. This means that we have global power series representation

$$C_\lambda(x) = \sum_{j=0}^{\infty} \lambda^j P_{j1}^{(0)}(x) \tag{6.5}$$

and

$$Q_\lambda(x) = \sum_{j=0}^{\infty} \lambda^j P_{j3}^{(0)}(x), \tag{6.6}$$

and a local power series representation

$$S_\lambda(x) = a_\lambda \sum_{j=0}^{\infty} \lambda^j P_{j2}^{(0)}(x) \tag{6.7}$$

valid on $F_0^n(SG)$ provided $\lambda < 5^n \lambda_2$. We may also use (6.5) and (6.6) on the blow-ups $F_0^{-n}(SG)$ for any n. Of course, none of these functions are entire analytic for $\lambda \geqslant \lambda_2$.

We will consider the infinite blow-up $SG_\infty = \bigcup_{n=0}^{\infty} F_0^{-n}(SG)$ to play the role of the positive reals vis-á-vis the unit interval. Of course there are uncountably many infinite blow-ups of SG. We have chosen the simplest one to study first. To understand the "behavior at infinity" of these functions it suffices to study the values at the points $x_n = F_0^n q_1$ as $n \to -\infty$, for we may then get the values at the points $y_n = F_0^n q_2$ by parity, and then fill in by spectral decimation.

For SG_∞ we have graphs Γ_n for any integer n. Since $-\lambda$ is negative we never encounter the exceptional eigenvalues 2, 5 and 6. Thus the method of spectral decimation says that u satisfies (6.1) on SG_∞ if and only if the restriction of u to Γ_n is a graph eigenfunction with eigenvalue λ_n, where $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of negative numbers characterized by

$$\lambda_{n-1} = \lambda_n(5 - \lambda_n) \tag{6.8}$$

and

$$-\lambda = \lim_{n\to\infty} \frac{3}{2} 5^n \lambda_n. \tag{6.9}$$

Note that $\lambda_n \to 0$ as $n\to\infty$ and $\lambda_n \to -\infty$ as $n \to -\infty$. It is easy to see that the sequence $\{\lambda_j\}$ is uniquely characterized by these conditions, and the values may be effectively computed to any desired accuracy by replacing the limit in (6.9) by the value for a fixed large n and then using (6.8) to run n down.

The fact that u restricted to Γ_n is a λ_n-eigenfunction means that if we take any cell of level $n-1$ with boundary points a, b, c, and if d is the midpoint between a and b, then

$$u(d) = \frac{(4-\lambda_n)(u(a)+u(b))+2u(c)}{(2-\lambda_n)(5-\lambda_n)} \tag{6.10}$$

(see [DSV, Algorithm 2.4]).

Lemma 6.1. *The recurrence relations*

$$C_\lambda(x_n) = \frac{(4-\lambda_n)+(6-\lambda_n)C_\lambda(x_{n-1})}{(2-\lambda_n)(5-\lambda_n)}, \tag{6.11}$$

$$S_\lambda(x_n) = \frac{(6-\lambda_n)S_\lambda(x_{n-1})}{(2-\lambda_n)(5-\lambda_n)} \tag{6.12}$$

and

$$Q_\lambda(x_n) = \frac{Q_\lambda(x_{n-1})}{5-\lambda_n} \tag{6.13}$$

hold for all integers n.

Proof. Apply (6.10) for $a = q_0$, $b = F_0^{n-1}(q_1)$, $c = F_0^{n-1}(q_2)$ and $d = F_0^n(q_1)$. □

Lemma 6.2. *The function C_λ is positive. The function S_λ, with the appropriate choice of a_λ, is positive everywhere except at q_0 where it vanishes. The function Q_λ vanishes on the symmetry line through q_0 and is positive on the q_1 half of the symmetry line.*

Proof. Because $\lambda_n < 0$ for all n, the coefficients in (6.10)–(6.13) are all positive. That means that if u is nonnegative on the boundary of a cell and strictly positive at one of the boundary points then it is strictly positive in the interior. Thus it suffices to show that $C_\lambda(x_n)$, $S_\lambda(x_n)$ and $Q_\lambda(x_n)$ are positive. For S_λ and Q_λ it suffices to show $S_\lambda(q_1)$ and $Q_\lambda(q_1)$ are positive, since we can solve (6.12) and (6.13) for $S_\lambda(x_{n-1})$ and $Q_\lambda(x_{n-1})$ with positive coefficients. But we can make $S_\lambda(q_1) > 0$ by the appropriate choice of sign (negative) for a_λ, and $Q_\lambda(q_1) > 0$ follows easily from $\partial_T Q_\lambda(q_0) = 1$. When we solve (6.11) we obtain

$$C_\lambda(x_{n-1}) = \frac{(2-\lambda_n)(5-\lambda_n)C_\lambda(x_n) - (4-\lambda_n)}{6-\lambda_n}, \tag{6.14}$$

which contains a negative coefficient. Nevertheless, if $C_\lambda(x_n) > 1$ then (6.14) implies

$$C_\lambda(x_{n-1}) > \frac{(2-\lambda_n)(5-\lambda_n) - (4-\lambda_n)}{6-\lambda_n} > 1,$$

so it suffices to show $C_\lambda(q_1) > 1$. This follows because the contrary assumption $C_\lambda(q_1) \leqslant 1$ and (6.11) would imply $\partial_n C_\lambda(q_0) > 0$. $\square$

Theorem 6.3. (a) *For all n we have*

$$C_\lambda(x_n) = 1 - \frac{\lambda_n}{4}. \tag{6.15}$$

(b) *For the appropriate choice of a_λ we have*

$$S_\lambda(x_n) = -\frac{\lambda_n}{4} \prod_{k=0}^{\infty} \left(1 + \frac{4}{2-\lambda_{n-k}}\right), \tag{6.16}$$

and hence

$$\lim_{n \to -\infty} S_\lambda(x_n)/C_\lambda(x_n) = 1. \tag{6.17}$$

(c) *For all $n < 0$ we have*

$$Q_\lambda(x_n) = -\frac{3}{4}\frac{\lambda_n}{\lambda} \tag{6.18}$$

and hence

$$\lim_{n \to -\infty} Q_\lambda(x_n)/C_\lambda(x_n) = \frac{3}{\lambda}. \tag{6.19}$$

Proof. (a) A direct calculation using (6.8) shows that $1 - \frac{\lambda_n}{4}$ satisfies the same recurrence relation (6.11) as $C_\lambda(x_n)$. Thus if we define $\tilde{C}_\lambda(x_n) = 1 - \frac{\lambda_n}{4}$, $\tilde{C}_\lambda(q_0) = 1$ and extend $\tilde{C}_\lambda$ to all of SG_∞ using (6.10), we will have an even λ-eigenfunction. But a direct computation shows

$$\partial_n \tilde{C}_\lambda(q_0) = \lim_{j \to \infty} \left(\frac{5}{3}\right)^j \frac{1}{2}\lambda_j = 0$$

because $\lambda_j = O(5^{-j})$ as $j \to \infty$. So $\tilde{C}_\lambda = C_\lambda$, proving (6.15).

(b) First we observe that the infinite product in (6.16) converges, because of the rapid growth of λ_n as $n \to -\infty$. Since (6.12) may be written (using (6.8))

$$\frac{S_\lambda(x_n)}{\lambda_n} = \left(1 + \frac{4}{2-\lambda_n}\right)\frac{S_\lambda(x_{n-1})}{\lambda_{n-1}}, \tag{6.20}$$

it follows that the right side of (6.16) satisfies (6.12). Since S_λ was only defined up to a multiplicative constant, we may choose a_λ to make (6.16) hold. Note that from (6.20) we obtain $S_\lambda(x_n) = O((\frac{3}{5})^n)$ as $n \to \infty$, which is consistent with $S_\lambda(q_0) = 0$ and $\partial_n S_\lambda(q_0) \neq 0$. Then (6.17) follows from (6.15) and (6.16) by inspection.

(c) We may rewrite (6.13) as

$$\frac{Q_\lambda(x_n)}{\lambda_n} = \frac{Q_\lambda(x_{n-1})}{\lambda_{n-1}}$$

using (6.8), hence $Q_\lambda(x_n) = \lambda_n Q_\lambda(x_0)$ for all n. But then

$$\begin{aligned} 1 = \partial_T Q_\lambda(q_0) &= \lim_{n\to\infty} 5^n(Q_\lambda(x_n) - Q_\lambda(y_n)) \\ &= 2Q_\lambda(x_0) \lim_{n\to\infty} 5^n \lambda_n \\ &= -\frac{4}{3}\lambda Q_\lambda(x_0). \end{aligned}$$

This proves (6.18), and then (6.19) follows by inspection. □

We can compute the value of $a_\lambda = \partial_n S_\lambda(q_0)$ exactly. From the definition and (6.16) we have

$$\begin{aligned} \partial_n S_\lambda(q_0) &= -2 \lim_{n\to\infty} \left(\frac{5}{3}\right)^n S_\lambda(x_n) \\ &= \lim_{n\to\infty} \frac{\lambda_n}{2}\left(\frac{5}{3}\right)^n \prod_{k=0}^{\infty}\left(1 + \frac{4}{2-\lambda_{n-k}}\right) \\ &= -\frac{1}{3}\lambda \lim_{n\to\infty} \frac{1}{3^n} \prod_{k=0}^{\infty}\left(1 + \frac{4}{2-\lambda_{n-k}}\right) \\ &= -\frac{1}{3}\lambda \prod_{j=0}^{\infty}\left(1 + \frac{4}{2-\lambda_{-j}}\right) \lim_{n\to\infty} \prod_{k=1}^{n}\left(\frac{6-\lambda_k}{6-3\lambda_k}\right) \\ &= -\frac{1}{3}\lambda \prod_{j=0}^{\infty}\left(1 + \frac{4}{2-\lambda_{-j}}\right) \prod_{k=1}^{\infty}\left(\frac{6-\lambda_k}{6-3\lambda_k}\right). \end{aligned} \tag{6.21}$$

Definition 6.4. For $\lambda < 0$ define the *decaying exponential* function E_λ by

$$E_\lambda(x) = C_\lambda(x) - S_\lambda(x). \tag{6.22}$$

Theorem 6.5. *$E_\lambda(x_n) = O(\lambda_n^{-1})$ as $n \to -\infty$. In fact*

$$\lim_{n\to-\infty} \lambda_n E_\lambda(x_n) = -1 \tag{6.23}$$

and

$$\lim_{n\to-\infty} C_\lambda(x_n)^2 - S_\lambda(x_n)^2 = \frac{1}{2}. \tag{6.24}$$

More precisely

$$E_\lambda(x_n) = \frac{2}{2-\lambda_n} + \frac{\lambda_n}{2-\lambda_{n-1}} + \frac{4\lambda_n}{(2-\lambda_n)(2-\lambda_{n-1})} + O(\lambda_n^{-3}). \tag{6.25}$$

Proof. From (6.16) we obtain

$$S_\lambda(x_n) = -\frac{\lambda_n}{4}\left(1 + \frac{4}{2-\lambda_n}\right)\left(1 + \frac{4}{2-\lambda_{n-1}}\right) + O(\lambda_n^{-3}) \tag{6.26}$$

because $\lambda_n/\lambda_{n-2} = O(\lambda_n^{-3})$. Substituting (6.26) into (6.22) and using (6.15) we obtain (6.25). Using (6.8) we see that the first two terms on the right side of (6.25) sum to

$$\frac{2}{2-\lambda_n} + \frac{\lambda_n}{2-5\lambda_n+\lambda_n^2} = -\frac{1}{\lambda_n} + O(\lambda_n^{-2}).$$

The third term is clearly $O(\lambda_n^{-2})$, so we obtain (6.23). From (6.26) we find $S_\lambda(x_n) = -\frac{\lambda_n}{4} + O(1)$ and this yields (6.24). $\square$

Note that (6.26) and (6.25) allow for the efficient computation of S_λ and E_λ for n sufficiently negative. On the other hand (6.22) is computationally unstable since it involves subtracting values that are large and nearly identical. In Table 5 we present some numerical computations of these functions.

Instead of fixing λ and taking the limit as $n \to -\infty$, we could look at values at x_0 and let $\lambda \to -\infty$. As long as $|\lambda_0|$ is large, (6.25) and (6.26) will be good estimates. Table 6 shows this behavior. We could also allow λ to be complex, as long as the real part is positive to avoid the exceptional values for λ_n.

We now turn our attention to eigenfunctions with positive eigenvalues, with the goal of using information gleaned from spectral decimation to shed some light on the recursion relations from Section 2. Keeping the same notation as before, we are

Table 5
Values of functions at x_{-j} for $\lambda = 10.70160380$

$-j$	λ_{-j}	$C_\lambda(x_{-j})$	$S_\lambda(x_{-j})$
0	-10	3.500000000	3.421641174
-1	-150.0	38.50000000	38.49346321
-2	-23250.0	5813.500000	5813.499957
-3	$-0.540678750 \times 10^{9}$	$0.1351696885 \times 10^{9}$	$0.1351696885 \times 10^{9}$
-4	$-0.2923335134 \times 10^{18}$	$0.7308337835 \times 10^{17}$	$0.7308337835 \times 10^{17}$
-5	$-0.8545888306 \times 10^{35}$	$0.2136472076 \times 10^{35}$	$0.2136472076 \times 10^{35}$
-6	$-0.7303220694 \times 10^{70}$	$0.1825805173 \times 10^{70}$	$0.1825805173 \times 10^{70}$
-7	$-0.5333703250 \times 10^{140}$	$0.1333425813 \times 10^{140}$	$0.1333425813 \times 10^{140}$
-8	$-0.2844839036 \times 10^{280}$	$0.7112097590 \times 10^{279}$	$0.7112097590 \times 10^{279}$
-9	$-0.8093109142 \times 10^{559}$	$0.2023277285 \times 10^{559}$	$0.2023277285 \times 10^{559}$
-10	$-0.6549841558 \times 10^{1118}$	$0.1637460389 \times 10^{1118}$	$0.1637460389 \times 10^{1118}$
$-j$	$Q_\lambda(x_{-j})$	$E_\lambda(x_{-j})$	$\lambda_{-j}E_\lambda(x_{-j})$
0	0.7008295323	0.07835882554	-0.7835882554
-1	10.51244298	0.006536787301	-0.9805180952
-2	1629.428662	0.00004300520387	-0.9998709899
-3	$0.3789236353 \times 10^{8}$	$0.1849527089 \times 10^{-8}$	-0.9999999945
-4	$0.2048759594 \times 10^{17}$	$0.3420750458 \times 10^{-17}$	-1.0000000000
-5	$0.5989210902 \times 10^{34}$	$0.1170153370 \times 10^{-34}$	-1.0000000000
-6	$0.5118312741 \times 10^{69}$	$0.1369258909 \times 10^{-69}$	-1.0000000000
-7	$0.3738016753 \times 10^{139}$	$0.1874869960 \times 10^{-139}$	-1.0000000000
-8	$0.1993747210 \times 10^{279}$	$0.3515137367 \times 10^{-279}$	-1.0000000000
-9	$0.5671889891 \times 10^{558}$	$0.1235619071 \times 10^{-558}$	-1.0000000000
-10	$0.4590322393 \times 10^{1117}$	$0.1526754489 \times 10^{-1117}$	-1.0000000000

Table 6
Values of functions at x_0 for various λ values

λ_0	λ	$E_\lambda(x_0)$	First 2 terms in (6.25)	First 3 terms in (6.25)
-100	44.19536761	0.009711493217	0.01008584733	0.009712435727
-500	87.71437197	0.001988095160	0.002003881410	0.001988103065
-1000	112.0105482	0.0009970119472	0.001000985089	0.0009970129413
-5000	182.0354932	0.0001998800959	0.0002000398801	0.0001998801039
-10000	218.2833208	0.00009997001199	0.0001000099850	0.00009997001299
-50000	317.2473555	0.00001999880010	0.00002000039988	0.00001999880010
λ_0	λ	$S_\lambda(x_0)$	First 2 factors in (6.26)	First 3 factors in (6.26)
-100	44.19536761	25.99028851	25.98039216	25.99028756
-500	87.71437197	125.9980119	125.9960159	125.9980119
-1000	112.0105482	250.9990030	250.9980040	250.9990030
-5000	182.0354932	1250.999800	1250.999600	1250.999800
-10000	218.2833208	2500.999900	2500.999800	2500.999900
-50000	317.2473555	12500.99998	12500.99996	12500.99998

interested in the function

$$C_{-\lambda}(x) = \sum_{j=0}^{\infty} (-\lambda)^j P_{j1}(x)$$

and its values at the special points $x_0 = q_1$ and $x_1 = F_0 q_1$. It is convenient to define λ_n (here we only care about $n \geqslant 0$) to satisfy (6.8) but to remove the minus sign in (6.9). For the Dirichlet and Neumann eigenfunctions we know exactly what these values are, and then we can use Theorem 6.3 (a) to conclude that $C_{-\lambda}(x_0) = 1 - \frac{\lambda_0}{4}$ and $C_{-\lambda}(x_1) = 1 - \frac{\lambda_1}{4}$. (Strictly speaking, we need to use an analytic continuation and limit argument to get this for the values we are interested in.) In particular, if $\lambda_0 = -6$ then $C_{-\lambda}(x_0) = 5/2$, or

$$\sum_{j=0}^{\infty} (-\lambda)^j P_{j1}(q_1) = \sum_{j=0}^{\infty} (-\lambda)^j \alpha_j = 5/2.$$

This happens when $\lambda = \lambda_2$, the second nonzero Neumann eigenvalue (not to be confused with the λ_2 in (6.8) and (6.9)). This allows us to compute the limit of β_j / t_{j+1} as $j \to \infty$. Indeed, from (2.45) we have

$$\frac{\beta_j}{t_{j+1}} = 6 \sum_{\ell=0}^{j} \alpha_{j+1-\ell} \left(\frac{t_\ell}{t_{j+1}} \right) = 6 \sum_{\ell=0}^{j+1} \alpha_\ell \left(\frac{t_{j+1-\ell}}{t_{j+1}} \right) - 6.$$

We expect to have

$$\frac{t_{j+1-\ell}}{t_{j+1}} \approx (-\lambda_2)^\ell$$

and so

$$\lim_{j \to \infty} \frac{\beta_j}{t_{j+1}} = 6 \sum_{\ell=0}^{\infty} \alpha_\ell (-\lambda_2)^\ell - 6 = 6 \cdot \frac{5}{2} - 6 = 9.$$

This is confirmed by the data in Table 2.

We are also interested in the solutions of the equation

$$\sum_{\ell=0}^{\infty} \alpha_\ell (-z)^\ell = -\frac{1}{2}. \tag{6.27}$$

This holds for $z = \lambda_2/5$, because in this case $\lambda_1 = 6$, and

$$C_{-\lambda}(x_1) = \sum_{\ell=0}^{\infty} \alpha_\ell (-\lambda_2/5)^\ell.$$

But it also holds for $z = \lambda_1^D$, because in this case $\lambda_0 = 6$. In fact it is easy to see that λ_1^D is the smallest solution of (6.27) (there are infinitely many other choices of λ with

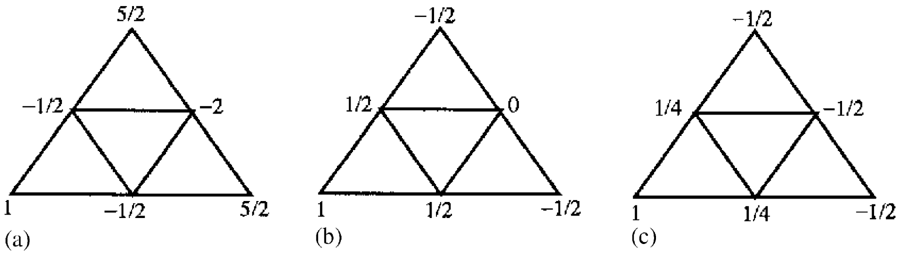

Fig. 5. The values of $C_{-\lambda}(x)$ on V_1 vertices for (a) $\lambda_0 = -6$ and $\lambda_1 = 6$, (b) $\lambda_0 = 6$ and $\lambda_1 = 2$, (c) $\lambda_0 = 6$ and $\lambda_1 = 3$.

either $\lambda_1 = 6$ or $\lambda_0 = 6$). Fig. 5 shows the values on V_1 of the function $C_{-\lambda}$ in these cases.

We can now explain why the recursion relation (2.11) for β_j is unstable. It is clear by inspection that the middle term on the right side of (2.11) is much larger than the other terms, so we would expect that a solution of (2.11) would be close to a solution of

$$\tilde{\beta}_j = -\frac{2}{3}\sum_{\ell=0}^{j-1} \alpha_{j-\ell} 5^{\ell-j} \tilde{\beta}_\ell,$$

which may be rewritten as

$$-\frac{1}{2} = \sum_{\ell=0}^{j} \alpha_\ell 5^{-\ell} \frac{\tilde{\beta}_{j-\ell}}{\tilde{\beta}_j}. \tag{6.28}$$

If we look for a solution of (6.28) of the form $\tilde{\beta}_j = (-5z)^{-j}$ then we obtain $\sum_{\ell=0}^{j} \alpha_\ell(-z)^\ell = -\frac{1}{2}$, which is very close to (6.27) in view of the very rapid decay of α_ℓ. The solution to (6.28) should thus be an infinite linear combination of exponential solutions with z a solution to (6.27). In the generic case the dominant term should correspond to the smallest solution of (6.27). Thus we expect the solution to (6.28) to behave like a multiple of $(-5\lambda_1^D)^{-j}$, and numerical computations confirm this. This pseudo-solution of (2.11) attracts any approximate solution of (2.11) that strays from the exact solution.

A related observation is that $\sum_{\ell=0}^{\infty} \alpha_\ell(-z)^\ell = 1$ holds for $z = \lambda_2^D \approx 55.885828\ldots$ by (6.15), since in this case $\lambda_0 = 0$ and $\lambda_1 = 5$. In the form $\sum_{\ell=1}^{\infty} \alpha_\ell(-\lambda_2^D)^\ell = 0$ this suggests that the entries of the matrix $\sigma(\alpha)^{-1}$, which are just $6T_j$, should decay like $(-\lambda_2^D)^{-j}$. The numerical data in Table 7 confirms this. This explains the instability in the recursion relation for $\{t_j\}$.

We also observe that the values of $C_{-\lambda_2}(x)$ given in Fig. 5 (a) show that the rearranged power series at q_1 does not converge to $C_{-\lambda_2}$ outside the cell $F_1(SG)$. Indeed, the even part of the power series about q_1, if it converged in SG, would have

Table 7

j	T_j	$(-\lambda_2^D)^j T_j$
0	1	1
1	−0.03333333333	1.862860915
2	0.0007407407407	2.313500526
3	−0.00001433691756	2.502423700
4	$0.2637965601 \times 10^{-6}$	2.573213790
5	$-0.4766054541 \times 10^{-8}$	2.598169232
6	$0.8556101104 \times 10^{-10}$	2.606669803
7	$-0.1532663873 \times 10^{-11}$	2.609508520
8	$0.2743475872 \times 10^{-13}$	2.610445492
9	$-0.4909650195 \times 10^{-15}$	2.610752605
10	$0.8785480907 \times 10^{-17}$	2.610852844
11	$-0.1572060595 \times 10^{-18}$	2.610885478
12	$0.2812997595 \times 10^{-20}$	2.610896085
13	$-0.5033478852 \times 10^{-22}$	2.610899530
14	$0.9006721805 \times 10^{-24}$	2.610900647
15	$-0.1611629185 \times 10^{-25}$	2.610901010
16	$0.2883788845 \times 10^{-27}$	2.610901127
17	$-0.5160143489 \times 10^{-29}$	2.610901165
18	$0.9233366935 \times 10^{-31}$	2.610901177
19	$-0.1652183992 \times 10^{-32}$	2.610901182
20	$0.2956355963 \times 10^{-34}$	2.610901182

to be $\frac{5}{2}\sum(-\lambda_2)^j P_{j1}^{(1)}(x)$, which gives the incorrect value of 25/4 for $\frac{1}{2}(C_{-\lambda_2}(q_0) + C_{-\lambda_2}(q_2)) = 7/4$.

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