DISCRETE SUBSPACES OF TOPOLOGICAL SPACES [1])

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§. 1. *Introduction*

Recently several papers appeared in the literature proving theorems of the following type. A topological space with "very many points" contains a discrete subspace with "many points". J. DE GROOT and B. A. EFIMOV proved in [2] and [4] that a Hausdorff space of power $> \exp \exp \exp m$ contains a discrete subspace of power $> m$. J. ISBELL proved in [3] a similar result for completely regular spaces.

J. de Groot proved as well that for regular spaces R the assumption $|R| > \exp \exp m$ is sufficient to imply the existence of a discrete subspace of potency $> m$. One of our main issues will be to improve this result and show that the same holds for Hausdorff spaces (see Theorems 2 and 3).

Our Theorem 1 states that a Hausdorff space of density $> \exp m$ contains a discrete subspace of power $> m$. We give two different proofs for the main result already mentioned. The proof outlined for Theorem 3 is a slight improvement of de Groot's proof. The proof given for Theorem 2 is of purely combinatorial character. We make use of the ideas and some theorems of the so called set-theoretical partition calculus developed by P. ERDŐS and R. RADO (see [5], [11]). Almost all the other results we prove are based on combinatorial theorems. For the convenience of the reader we always state these theorems in full detail.

Our Theorem 4 states that if m is a strong limit cardinal which is the sum of $\aleph_0$ smaller cardinals, then every Hausdorff space of power m contains a discrete subspace of power m. The problem if the same holds for all strong limit cardinals remains open.

At the end of § 4 using the generalized continuum hypothesis (G.C.H. in what follows) we give a discussion of the results and problems.

In § 5 we consider the problem of existence of large discrete subspaces under additional assumptions.

Theorem 5 states that a Hausdorff space of power $> 2^m$ contains $> m$ disjoint open sets provided the character of the space is at most m.

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[1]) A preliminary report containing the main results of this paper appeared in the Doklady Akad. Nauk. SSSR (see [1]).

Theorem 6 states that a T_1-space of power $> 2^m$ contains a discrete subspace of power $> m$, provided the pseudo-character of the space is at most m.

Theorem 7 states that a Hausdorff space R of power m, where m is a singular strong limit cardinal contains m pairwise disjoint open sets provided the character of the space is less than m.

As a corollary of Theorem 5 we obtain a result concerning characters of points of the Čech–Stone compactification of discrete spaces (see Corollary 2).

§ 2. *Notations, Definitions*

The cardinal of the set A is denoted by $|A|$. m, n, p, q denote cardinals. $m^{\underline{n}}$ (m to the weak power n) is defined as

$$\sum_{p<n} m^p.$$

We write sometimes exp n and exp $\underline{n}$ for 2^n or $2^{\underline{n}}$ respectively. m^+ denotes the smallest cardinal greater than m. m^* denotes the smallest cardinal n for which m is the sum of n cardinals smaller than m. The cardinal m is called *regular* or *singular* according to $m^* = m$ or $m^* < m$.

The infinite cardinal m is a *limit cardinal* if $m \neq n^+$ for any n. m is said to be a *strong limit cardinal* if $2^n < m$ for every $n < m$. A regular strong limit cardinal is said to be *strongly inaccessible*. $\Omega(m)$ denotes the initial number of the cardinal m.

If $m = \aleph_\alpha$ then $m^+ = \aleph_{\alpha+1}$, $m^* = \aleph_{cf(\alpha)}$, $\Omega(m) = \omega_\alpha$. Note that for every infinite cardinal n and $m > 1$ we have $m^{\underline{n}^+} = m^n$.

For an arbitrary set A we put

$$[A]^n = \{B : B \subseteq A \text{ and } |B| = n\}.$$

If M is a set of cardinals put

$$\sup{}^* M = \min \{n : n > m, \ n \geqslant \aleph_0 \text{ for every } m \in M\}.$$

Let R be a topological space. A sequence $\{F_\xi\}_{\xi < \alpha}$ of non-empty closed subsets of R is said to be a *tower of length* $|\alpha|$ if $F_\xi \supseteq F_\eta$, $F_\xi \neq F_\eta$ for every pair $\xi < \eta < \alpha$.

$D(R)$, $W(R)$, $S(R)$, $H(R)$ denote the density, the weight, the spread and the height of the topological space R respectively. $D(R)$, $W(R)$ are defined as usual.

$$D(R) = \max \{\aleph_0, \min \{|S| : S \subseteq R, \ \bar{S} = R\}\},$$
$$W(R) = \max \{\aleph_0, \min \{|\mathfrak{B}| : \text{for open bases } \mathfrak{B} \text{ of } R\}\}.$$

We define the spread and the height slightly differently from [2].

$$S(R) = \sup{}^* \{|D| : D \subseteq R, \ D \text{ discrete}\}$$
$$H(R) = \sup{}^* \{|\alpha| : \text{for the length } |\alpha| \text{ of towers of } R\};$$

de Groot's problem if there always exists a discrete subspace of maximal power or a tower of maximal length is equivalent to the problem if $S(R)$ and $H(R)$ are never limit cardinals respectively.

We denote by $\mathscr{T}_i$ ($i = 0, \ldots, 5$) the class of all T_i-spaces.

Let $\mathscr{C}$ be a class of topological spaces and let $m \geqslant n$ be infinite cardinals. To have a brief notation we sometimes write

$$(\mathscr{C}, m) \to n$$

to denote that the following statement is true:

For every $R \in \mathscr{C}$, $|R| = m$ there exists a discrete subspace $D \subseteq R$ with $|D| = n$; $(\mathscr{C}, m) \nrightarrow n$ denotes that the statement is false.

Let $R = \{x_\xi\}_{\xi < \alpha}$ be a well-ordering of the topological space R. We say that this *well-ordering separates R from the left (from the right)* respectively if there exists a sequence $\{U_\xi\}_{\xi < \alpha}$ of open sets such that $x_\xi \in U_\xi$ and $x_\eta \notin U_\xi$ for $\xi < \eta$ ($x_\zeta \notin U_\xi$ for $\zeta < \xi$) respectively. R is said to be *separated from the left (from the right)* if there exist well orderings separating R from the left (from the right) respectively. If the same well-ordering separates R from the left and from the right R is obviously discrete.

Let R be a topological space. The *character* of a *point* x is the least cardinal p for which x has a base of neighbourhoods of power p. If $R \in \mathscr{T}_1$ the *pseudo character* of the point x is the least cardinal p for which there exists a system $\mathscr{U}$ of power p of open subsets with the property $\cap \mathscr{U} = \{x\}$.

The *character or the pseudo character of the space* is the supremum of the characters or the pseudo characters of the points respectively.

§ 3. *Preliminaries*

3.1. $|R| \leqslant 2^{W(R)}$ for $R \in \mathscr{T}_0$.

3.2. $D(R) \leqslant |R| \leqslant \exp \exp D(R)$ for $R \in \mathscr{T}_2$, $|R| \geqq \aleph_0$.

3.3. $D(R) \leqslant W(R) \leqslant 2^{D(R)}$ for $R \in \mathscr{T}_3$.

These lemmas are stated in [2].

3.4. Let R be an infinite topological space. Then R contains a subspace T separated from the right such that $|T| \geqslant D(R)$.

See e.g. [6], Theorem II. We obviously have

3.5. If R is separated from the left or from the right then $|R| \leqslant W(R)$.

We prove the following

Lemma 1. Let R be an infinite topological space which is separated from the left and from the right. Then R contains a discrete subspace $T \subseteq R$ which has the same potency as R.

Proof. Let $R = \{x_\xi\}_{\xi < \alpha} = \{y_\eta\}_{\eta < \beta}$ be two well-orderings of R separating it from the left and from the right respectively. Put $|R| = m$ and let $<_1$, $<_2$ briefly denote the above well-orderings. Split the set of all two-

element-subsets of R $\{x, y\}$, $x \neq y$, $x, y \in R$, i.e. $[R]^2$ into two classes I and II as follows

$$\text{if } x <_1 y \text{ and } x <_2 y \text{ then } \{x, y\} \in \text{I} \; ,$$
$$\text{if } x <_1 y \text{ and } y <_2 x \text{ then } \{x, y\} \in \text{II}.$$

Considering that a well-ordered set has no decreasing infinite subsets it follows that R does not contain an infinite subset all whose two-element-subsets belong to the second class. Then by a theorem of P. ERDÖS (see [10], Theorem 5.22 on p. 606), there exists a subset $T \subseteq R$ of potency m all whose pairs belong to the first class. That means the two well-orderings are the same on T hence the same well-ordering separates T from the left and from the right, and so T is discrete.

In DE GROOT's paper [2] the following lemma is stated.

If $R \in \mathscr{T}_2$, and $|R| > 2^m$ then R contains a tower of length $> m$. We restate this lemma in a slightly stronger form.

Lemma 2. Assume $|R| > 2^{\underline{m}}$ and $R \in \mathscr{T}_2$, $m \geq \aleph_0$. Then R contains a tower of length $\geq m$.

The proof can be carried out literally in the same way as in [2]:

Using the fact that every T_2 space consisting of more than one point is the union of two closed proper subsets one defines the closed proper subsets $R_0 \subseteq R$, $R_1 \subseteq R$ on such a way that $R_0 \cup R_1 = R$. One can build up a so called ramification system be repeating this procedure transfinitely. One concludes as in [2] that this procedure associates in a one-to-one way to every point x of R a labelling

$$(\varepsilon_\xi)_{\xi < \alpha}$$

where in the above sequence $\varepsilon_\xi = 0$ or 1 for every $\xi < \alpha$. It is also obvious from the construction that corresponding to every point x labelled with a sequence of length α R contains a tower of length $|\alpha|$.

Considering that the set of all possible labellings of length $\alpha < \Omega(m)$ has power

$$\sum_{\alpha < \Omega(m)} 2^{|\alpha|} = \sum_{p < m} 2^p = 2^{\underline{m}},$$

the assumption $|R| > 2^{\underline{m}}$ implies the existence of a tower of length at least m. The "ramification argument" used in this proof is stated in a very general form in [5] (see p. 103).

Similarly as in [2] one can state the corollary that

$$H(R) \leqslant |R|^+ \text{ and } |R| \leqslant 2^{\underline{H(R)}}.$$

The first inequality is trivial and the second is a consequence of Lemma 2 considering that by the definition of $H(R)$ R never contains a tower of length $H(R)$.

Lemma 3. If R contains a tower of length m then R contains a subspace T of power m separated from the left.

Proof. Let $\{F_\xi\}_{\xi<\alpha}$ be a tower of length m. $T=\{x_\xi\}_{\xi<\alpha}$ with $x_\xi \in F_\xi - F_{\xi+1}$ obviously satisfies the requirement.

We now restate the theorem of [11] mentioned in the introduction.

Lemma 4. Let R be a set, $m \geqslant \aleph_0$, p be cardinals, $r \geqslant 2$ an integer. Assume $[R]^r = \bigcup\limits_{\nu < \Omega(p)} I_\nu$ is an arbitrary partition of the set $[R]^r$.

The conditions $|R| > \exp \underset{(r-1)}{\ldots} \exp m$, $p \leqslant m$ imply that there exists a subset $S \subseteq R$ and an index $\nu < \Omega(p)$ such that

$$|S| > m \text{ and } [S]^r \subseteq I_\nu.$$

Lemma 4 is a corollary of theorem 39 of [11].

§ 4. *Discrete subspaces of spaces of large potency*

Our first theorem establishes an inequality between the density and the spread of the space.

Theorem 1.
$$D(R) \leqslant \exp \underset{\smile}{S(R)}$$
for every $R \in \mathcal{T}_2$.

Proof. Let $R \in \mathcal{T}_2$, $D(R) > 2^m$. It is sufficient to see that R contains a discrete subspace of power m. By 3.4 R contains a subspace S, $|S| > 2^m$ separated from the right. By Lemma 2 S contains a tower of length m, hence by Lemma 3 it contains a subspace T of power m separated from the left. T being separated from both sides, by Lemma 1 T contains a discrete subspace of power m.

Remarks. De Groot stated the problem if the stronger inequality $D(R)^+ \leqslant S(R)$ holds for Hausdorff spaces. We would like to point out that this would trivially imply that every hereditary Lindelöf space is separable. Considering that this would be a positive solution of Souslin's problem a positive answer seems to be improbable. This shows that Theorem 1 is in a sense best possible.

On the other hand we have the following estimations.

4.1.
$$S(R) \leqslant (\exp \exp D(R))^+ \text{ for } R \in \mathcal{T}_2;$$
$$S(R) \leqslant (\exp D(R))^+ \quad \text{ for } R \in \mathcal{T}_3.$$

Proof. The first inequality follows from 3.2 using that $S(R) \leqslant |R|^+$ while the second follows from 3.3. considering that $S(R) \leqslant W(R)^+$.

The first inequality is best possible for Hausdorff spaces as it is shown by the following

Example 1. For every infinite cardinal m there exists a Hausdorff space R, $|R| = \exp \exp m$ with

$$D(R) = m, \ S(R) = (\exp \exp m)^+.$$

Proof. Let M be a set of power m. Let R denote the set of all ultra-filters of M. As it is well-known, $|R| = \exp\exp m$. For every $x \in M$ let $\hat{x}$ be the ultrafilter of all subsets of M containing x. Let $X^* = \{\hat{x} : x \in X\}$ for every subset $X \subseteq M$. We have $|M^*| = m$. Let the base $\mathfrak{B}$ of R consist of the sets $\{u\} \cup X^*$ for $u \in R$, $X \in u$.

Then R is a Hausdorff space, since two different ultrafilters always contain disjoint subsets, all the points of M^* are isolated in R and M^* is dense in R. Hence $D(R) = m$. Considering that $R - M^*$ is obviously a discrete subspace of R, $S(R) = (\exp\exp m)^+$ and R satisfies the requirements of the theorem.

To prove our main theorem we need the following

Lemma 5. Let R be a set and assume that for every $x \in R$ U_x is a set of subsets of R satisfying the conditions

a) For every pair $x \neq y \in R$ there exist sets $A \in U_x$, $B \in U_y$ such that $A \cap B = 0$,

b) For every $x \in R$, A, $B \in U_x$ there is a $C \in U_x$ such that $C \subseteq A \cap B$.

Assume $|R| > \exp\exp m$ for some infinite cardinal m. Then there exists a subset $S \subseteq R$, $|S| = m^+$ and a function $A(x)$ defined for $x \in S$ satisfying the conditions

$$A(x) \in U_x \text{ for } x \in S,$$

$$y \notin A(x) \text{ for every pair } x \neq y \ x, y \in S.$$

(Lemma 5 states in other words that a Fréchet Hausdorff space of power $> \exp\exp m$ contains a discrete subspace of power $> m$).

Proof. Let $\{x_\alpha\}_{\alpha < \varphi}$ be a well-ordering of type φ of the points of R, $|\varphi| > \exp\exp m$. We are going to use the special case $r = 3$ of Lemma 4.

Let $\alpha < \beta < \varphi$. By condition a) we can define the sets $A_0(x_\alpha, x_\beta)$, $A_1(x_\alpha, x_\beta)$ satisfying the conditions

(1) $$A_0(x_\alpha, x_\beta) \in U_{x_\alpha}, \qquad A_1(x_\alpha, x_\beta) \in U_{x_\beta};$$

(2) $$A_0(x_\alpha, x_\beta) \cap A_1(x_\alpha, x_\beta) = 0.$$

We define a partition $[R]^3 = \bigcup_{\substack{\varepsilon_i < 2 \\ i = 1, 2}} I_{(\varepsilon_1, \varepsilon_2)}$ of the set of the triplets as follows.

(3) Let $X = \{x_\alpha, x_\beta, x_\gamma\}$, $\alpha < \beta < \gamma$ be an arbitrary element of $[R]^3$. Put $X \in I_{(\varepsilon_1, \varepsilon_2)}$ iff $x_\gamma \notin A_{\varepsilon_1}(x_\alpha, x_\beta)$ and $x_\alpha \notin A_{\varepsilon_2}(x_\beta, x_\gamma)$. It follows from (2) that

$$[R]^3 = \bigcup_{\substack{\varepsilon_i < 2 \\ i = 1, 2}} I_{(\varepsilon_1, \varepsilon_2)}.$$

Applying Lemma 4 for this partition with $p = 4$, $r = 3$ it follows that there exist a subset $S \subseteq R$ and two numbers $\varepsilon_1, \varepsilon_2 < 2$ such that

(4) $$|S| = m^+, \qquad [S]^3 \subseteq I_{(\varepsilon_1 \varepsilon_2)}.$$

We can assume that S is of the form $S = \{x_{\alpha_\beta}\}_{\beta < \Omega(m^+)}$ where the sequence α_β is increasing. Put briefly x'_β for x_{α_β}.

Let $S_1 = \{x'_\beta : \beta < \Omega(m^+),\ \beta \text{ is even}\}$ or $S_1 = \{x'_\beta : \beta < \Omega(m^+),\ \beta \text{ is odd}\}$ if $\varepsilon_1 = 0$, or $\varepsilon_1 = 1$ respectively. Put further $A'(x'_\beta) = A_0(x'_\beta, x'_{\beta+1})$ if β is even, $A'(x'_\beta) = A_1(x'_{\beta-1}, x'_\beta)$ if β is odd.

It follows from (1), (3) and (4) that

$$(5) \qquad\qquad A'(x'_\beta) \in U_{x'_\beta} \text{ for } x'_\beta \in S_1$$

and

$$(6) \qquad\qquad x'_\gamma \notin A'(x'_\beta) \text{ for } \beta < \gamma,\ x'_\beta, x'_\gamma \in S_1,$$

since both β and γ are either odd or even. S_1 is of the form $\{x'_{\beta_\gamma}\}_{\gamma < \Omega(m^+)}$ where the sequence β_γ is increasing. Put briefly $x'_{\beta_\gamma} = x''_\gamma$.

Let $S_2 = \{x''_\gamma : \gamma < \Omega(m^+),\ \gamma \text{ is even}\}$ or $S_2 = \{x''_\gamma : \gamma < \Omega(m^+),\ \gamma \text{ is odd}\}$ if $\varepsilon_2 = 0$ or $\varepsilon_2 = 1$ respectively. Put further $A''(x''_\gamma) = A_0(x''_\gamma, x''_{\gamma+1})$ if γ is even, $A''(x''_\gamma) = A_1(x''_{\gamma-1}, x''_\gamma)$ if γ is odd.

It follows from (1), (3) and (4) that

$$(7) \qquad\qquad A''(x''_\gamma) \in U_{x''_\gamma} \text{ for } x''_\gamma \in S_2,$$

and

$$(8) \qquad\qquad x''_\beta \notin A''(x''_\gamma) \text{ for } \beta < \gamma,\ x''_\beta, x''_\gamma \in S_2.$$

It is obvious that $|S_2| = m^+$. By condition b) and by (5) and (7) for every $x''_\gamma \in S_2$ there is a set $A(x''_\gamma) \in U_{x''_\gamma}$ such that $A(x''_\gamma) \subseteq A'(x''_\gamma) \cap A''(x''_\gamma)$. S_2 and $A(x''_\gamma)$ satisfy the requirements of the theorem since by (6) and (8) $x''_\beta \notin A(x''_\gamma)$ for every pair $x''_\beta \neq x''_\gamma \in S_2$.

As a corollary of Lemma 5 we obtain

Theorem 2. Every Hausdorff space of power $> \exp\exp m$ contains a discrete subspace of power $> m$ $(m \geqslant \aleph_0)$.

I.e.

$$(\mathscr{T}_2, (\exp\exp m)^+) \to m^+.$$

Theorem 2 should be compared with Theorem 1 of [2]. We do not know if this result can still be improved with one exp. We could not improve the second part of the above theorem concerning regular spaces.

An improvement of Theorem 2 would be that

$$(\mathscr{T}_2, [\exp(\exp \underline{m})]^+) \to m$$

holds for every cardinal $m \geqslant \aleph_0$. Considering that

$$\sum_{p < m} \exp\exp p \leqslant \prod_{p < m} \exp\exp p = \exp(\exp \underline{m})$$

the following Theorem 3 is even a further improvement of Theorem 2 [1]).

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[1]) We could not deduce Theorem 3 directly from the results of [11] since a corresponding generalization of Lemma 4 is not true.

Theorem 3. Assume $R \in \mathscr{T}_2$, $m \geqslant \aleph_0$ and $|R| > \sum_{p < m} \exp \exp p$. Then R contains a discrete subspace of power m.

We only outline the proof.

One builds up a ramification system similarly as outlined in the proof of our Lemma 2 or in the proof of the Lemma in [2]. Put $n = \sum_{p < m} \exp \exp p$.

Let x_0, x_1 be two arbitrary elements of R, R_0 and R_1 be two closed subsets whose union is R, $x_0 \notin R_0$ and $x_1 \notin R_1$.

Let $\alpha < \Omega(m)$ and assume that for every $\xi < \alpha$ we have already defined the elements $x_{(\varepsilon_0, \ldots, \varepsilon_\xi)}$ as well as the closed subsets $R_{(\varepsilon_0, \ldots, \varepsilon_\xi)}$ for some sequences $(\varepsilon_0, \ldots, \varepsilon_\xi)$, where $\varepsilon_\eta = 0$ or 1 for $\eta \leqslant \xi$. Let now $(\varepsilon_0, \ldots, \varepsilon_\xi \ldots)_{\xi < \alpha}$ be a sequence of length α such that $R_{(\varepsilon_0, \ldots, \varepsilon_\xi)}$ and $x_{(\varepsilon_0, \ldots, \varepsilon_\xi)}$ are defined for every $\xi < \alpha$. If $\bigcap_{\xi < \alpha} R_{(\varepsilon_0, \ldots, \varepsilon_\xi)} = R'_{(\varepsilon_0, \ldots, \varepsilon_\xi \ldots)_{\xi < \alpha}}$ has power $\leqslant n$, $x_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ and $R_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ will not be defined. Assume $|R'_{(\varepsilon_0, \ldots, \varepsilon_\xi, \ldots)_{\xi < \alpha}}| > n$. It follows from 3.2 that the set $Q = R'_{(\varepsilon_0, \ldots, \varepsilon_\xi \ldots)_{\xi < \alpha}} - \overline{\bigcup_{\xi < \alpha} \{x_{(\varepsilon_0, \ldots, \varepsilon_\xi)}\}}$ has power $> n$. ($\overline{X}$ denotes the closure of the set X in R). Let $x_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ be two arbitrary elements of this set Q for $\varepsilon_\alpha = 0$ or 1. Considering that $R \in \mathscr{T}_2$, there exist two closed subsets $R_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ ($\varepsilon_\alpha = 0, 1$) of R whose union is R such that $x_{(\varepsilon_0, \ldots, \varepsilon_\alpha)} \notin R_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$. Thus the elements $x_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ and the sets $R_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ are defined by induction on α for every $\alpha < \Omega(m)$.

Let us omit now from R those sets $R'_{(\varepsilon_0, \ldots, \varepsilon_\xi, \ldots)_{\xi < \alpha}}$ which have power $\leqslant n$ and the elements $x_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ ($\alpha < \Omega(m)$). Considering that there are at most $\exp^m \leqslant n$ sequences $(\varepsilon_0, \ldots, \varepsilon_\xi \ldots)_{\xi < \alpha}$ of length $\alpha < \Omega(m)$ there must be an element x of R which is not omitted. Then x belongs to some $R_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ for every $\alpha < \Omega(m)$ and this implies that there exists a sequence $(\varepsilon_0, \ldots, \varepsilon_\alpha, \ldots)_{\alpha < \Omega(m)}$ such that $x \in R_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ for every $\alpha < \Omega(m)$. Let x'_α briefly denote $x_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ for this sequence. $S = \{x'_\alpha\}_{\alpha < \Omega(m)}$ is a discrete subspace of R. In fact it is separated from the right since $x'_\alpha \notin \overline{\bigcup_{\xi < \alpha} \{x'_\xi\}}$, it is separated from the left since by the construction $x'_\alpha \notin R_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ and $x_\beta \in R_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ for every $\beta > \alpha$ hence the complement of $R_{(\varepsilon_0, \ldots, \varepsilon_\alpha)}$ is an open set containing x'_α and not containing x'_β for $\beta > \alpha$. Since the given well-ordering separates S from both sides S is discrete.

From Theorem 3 we obtain

Corollary 1.
$$|R| \leqslant \exp \left(\exp \underline{\underline{s(R)}} \right) \text{ for } R \in \mathscr{T}_2.$$

It is well-known that Theorems 2 and 3 can not be improved by two "exp", since for every infinite cardinal n there are even compact T_4 spaces of power $\exp n$ with spread $\leqslant n^+$, i.e.

$$(\mathscr{T}_4, 2^n) \nrightarrow n^+ \text{ for every } n \geqslant \aleph_0.$$

There is still possible that a sharper result holds for spaces of power m where m is a strong limit cardinal. As a corollary of Theorem 3 we know

that $S(R) \geqslant |R| = m$ holds for these spaces but $S(R) = |R|^+ = m^+$ might be true as well. We can prove this in case $m^* = \aleph_0$ and the problem remains open in the general case.

Theorem 4. Assume $R \in \mathscr{T}_2$, $|R| = m$, m is a strong limit cardinal and $m^* = \aleph_0$. Then R contains a discrete subspace of power m, i.e. $S(R) = m^+$.

We postpone the proof of Theorem 4 to p. 354 since it will be an easy consequence of our Lemma 7.

Assuming G.C.H. we can summarize our results as follows.

4.1. A) $(\mathscr{T}_2, \aleph_{\alpha+2}) \to \aleph_\alpha$ for every α

   B) $(\mathscr{T}_2, \aleph_{\alpha+1}) \to \aleph_\alpha$ if $\aleph_\alpha$ is a limit cardinal

   C) $(\mathscr{T}_2, \aleph_\alpha) \to \aleph_\alpha$ if $\mathrm{cf}(\alpha) = 0$ and

   $(\mathscr{T}_2, \aleph_\alpha) \to \aleph_\beta$ for every $\beta < \alpha$ provided $\aleph_\alpha$ is a limit cardinal.

Proof. A) and B) follow from Theorem 3 considering that

$$\sum_{p < \aleph_\alpha} \exp \exp p \leqslant 2^{\aleph_\alpha}. \quad \aleph_\alpha = \aleph_{\alpha+1}$$

for every α and $\displaystyle\sum_{p < \aleph_\alpha} \exp \exp p \leqslant \aleph_\alpha$ if α is of the second kind provided G.C.H. holds. The first assertion of C) is a corollary of Theorem 4, while the second follows e.g. from A).

Considering the obvious negative result $(\mathscr{T}_4, \aleph_{\alpha+1}) \nrightarrow \aleph_{\alpha+1}$ there remains no unsolved problem in case B.

In cases A), C) the simplest and typical unsolved problems are the following.

Problem 1. Assume G.C.H.

$$(\mathscr{T}_2, \aleph_2) \to \aleph_1 ?$$

$$(\mathscr{T}_2, \aleph_{\omega_1}) \to \aleph_{\omega_1} ?$$

§ 5. *Existence of discrete subspaces under additional assumptions*

Theorem 5. Let $R \in \mathscr{T}_2$, $m \geqslant \aleph_0$. Assume that $R' \subseteq R$, $|R'| > 2^m$ and that the character of every point of R' in R is at most m. Then R contains a family $\mathscr{U}$ of power m^+ of pairwise disjoint open subsets.

Theorem 6. Let $R \in \mathscr{T}_1$, $m \geqslant \aleph_0$. Assume that $|R| > 2^m$ and the pseudo character of R is at most m. Then R contains a discrete subspace of power m^+.

Both theorems will be corollaries of our next combinatorial lemma. Thus we postpone the proofs.

Lemma 6. Let R be a set and assume that for every $x \in R$ $\mathscr{U}_x$ is a collection of sets satisfying the condition

   a) For every $x \in R$, $A, B \in \mathscr{U}_x$ there is a $C \in \mathscr{U}_x$ such that $C \subseteq A \cap B$.

Assume $|R| > 2^m$ and $|\mathcal{U}_x| \leqslant m$ for every $x \in R$. Let further $P_{xy}(A, B)$ with $x, y \in R$ be a property of pairs of sets satisfying the following conditions.

b) For every pair $x \neq y \in R$ there are $A \in \mathcal{U}_x$, $B \in \mathcal{U}_y$ for which $P_{xy}(A, B)$ holds.

c) For all sets $A' \subseteq A$, $B' \subseteq B$, $A, A' \in \mathcal{U}_x$, $B, B' \in \mathcal{U}_y$ $P_{xy}(A, B)$ implies $P_{xy}(A', B')$.

Then there exist a subset $S \subseteq R$, $|S| = m^+$ and a function $A(x)$ defined for every $x \in S$ such that $A(x) \in \mathcal{U}_x$, and $P_{xy}(A(x), A(y))$ holds for every pair $x \neq y \in S$.

Proof. Let $\{x_\alpha\}_{\alpha < \varphi} = R$ be an arbitrary well-ordering of the set R. For every $\alpha < \varphi$ let $\{A^\alpha_\beta\}_{\beta < \Omega(m)}$ be a sequence containing all the elements of $\mathcal{U}_{x_\alpha}$. We define a partition $[R]^2 = \bigcup_{\beta < \Omega(m)} \bigcup_{\gamma < \Omega(m)} I_{\beta\gamma}$ of the pairs of R as follows. Assume $\alpha_1 < \alpha_2 < \varphi$, then

$$\{x_{\alpha_1}, x_{\alpha_2}\} \in I_{\beta\gamma} \text{ iff } P_{x_{\alpha_1} x_{\alpha_2}}(A^{\alpha_1}_\beta, A^{\alpha_2}_\gamma)$$

holds. By condition b) every pair belongs to one of the classes.

By case $r = 2$ of Lemma 4 then there exist a subset $S \subseteq R$ and ordinals $\beta_0, \gamma_0 < \Omega(m)$ such that $|S| = m^+$ and $[S]^2 \subseteq I_{\beta_0 \gamma_0}$. By condition a) then there exists for every $x_\alpha \in S$ an $A(x_\alpha) \in \mathcal{U}_{x_\alpha}$ such that $A(x_\alpha) \subseteq A^\alpha_{\beta_0} \cap A^\alpha_{\gamma_0}$. By condition c) and by $[S]^2 \subseteq I_{\beta_0 \gamma_0}$ then $P_{xy}(A(x), A(y))$ holds for every pair $x \neq y \in S$.

Proof of Theorem 5. By the assumption there exists a base of neighbourhoods $\mathcal{U}_x$, $|\mathcal{U}_x| \leqslant m$ for every $x \in R'$. Let the property $P_{x,y}(A, B)$ be defined by the stipulation $A \in \mathcal{U}_x$, $B \in \mathcal{U}_y$, $A \cap B = 0$ for $x \neq y \in R'$. R being a Hausdorff space conditions b) and c) of Lemma 6 are fulfilled. It follows from Lemma 6 that there exists a subset $S \subseteq R'$, $|S| = m^+$ and a function $A(x)$ such that the open sets $\{A(x)\}_{x \in S}$ are pairwise disjoint.

Remarks. Well known examples show that Theorem 5 does not remain true if the assumption $|R'| > 2^m$ is replaced by the weaker assumption $|R'| \geqslant 2^m$. However there is another possible refinement. As the example of the topological product of p discrete spaces of power m $(m \geqslant p \geqslant \aleph_0)$ shows there is a space of power m^p with character p not containing discrete subspaces of power $> m$. A generalization of Theorem 5 would be that a Hausdorff space of power $> m^p$, with character $\leqq p$ contains a discrete subspace of power $> m$, or more than m disjoint open subsets. We do not know if this is true even in case $p = \aleph_0$ and assuming the G.C.H. We would like to point out the following even simpler.

Problem 2. Assume G.C.H. Let R be an ordered set of power $\aleph_2$ and of character $\aleph_0$. Does then R necessarily contain a system of power $\aleph_2$ of disjoint open intervals?

Note. The assumptions imply that R contains no dense set of power $\leqslant \aleph_1$, hence there is no hope to disprove this without disproving the generalized Souslin conjecture.

From Theorem 5 we obtain the following

Corollary 2. Let βN_m be the Čech–Stone compactification of the discrete space N_m of power m. Let $\chi(x)$ denote the character of x in βN_m. Then
$$|\{x \colon \exp \chi(x) < \exp \exp m\}| < \exp \exp m.$$
i.e. for almost every $x \in \beta N_m$ we have $\exp \chi(x) = \exp \exp m$.

Proof. Assume that $m \leqslant p < \exp m$. Then βN_m contains at most $\exp p$ points of character $\leqslant p$, since otherwise βN_m would contain $p^+ > m$ disjoint open subsets in contradiction to the fact that N_m is dense in βN_m. It follows that
$$|\{x \colon \exp \chi(x) < \exp \exp m\}| = | \bigcup_{m \leqslant p < \exp m, \, \exp p < \exp \exp m} \{x \colon \chi(x) \leqq p\}| < \exp \exp m$$
since $(\exp \exp m)^* > \exp m$.

This should be compared with a theorem of Pospišil [7] which states that βN_m contains $\exp \exp m$ points of character $\exp m$. Our result does not imply his theorem however under certain consistent conditions on the exp function it is even stronger. Assume e.g. $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and $2^{\aleph_2} = \aleph_3 = \exp \exp \aleph_0$. Then the cardinality of the set of points of character $\leqslant \aleph_1$ of $\beta N_{\aleph_0}$ is at most $\exp \aleph_1 = \aleph_2$ hence almost every point has character $\aleph_2 = 2^{\aleph_0}$.

Proof of Theorem 6. By the assumption there exists a system of open sets $\mathscr{U}_x$, of power $\leqq m$ such that $\cap \, \mathscr{U}_x = \{x\}$ for every $x \in R$. Moreover we may assume that $\mathscr{U}_x$ is closed with respect to the operation of finite intersection, hence condition a) of Lemma 6 is satisfied. Let $P_{xy}(A, B)$ be the property $A \in \mathscr{U}_x$, $B \in \mathscr{U}_y$, $y \notin A$, $x \notin B$. Considering $\cap \, \mathscr{U}_x = \{x\}$ conditions b), c) of Lemma 6 are satisfied as well. Lemma 6 implies the existence of a subset $S \subseteq R$, $|S| = m^+$ and a function $A(x)$ such that $P_{x,y}(A(x), A(y))$ holds for every $x \neq y \in S$. Then $y \notin A(x) \in \mathscr{U}_x$ and $x \notin A(y) \in \mathscr{U}_y$, where $A(x)$ is an open set, for every $x \neq y \in S$. Hence the subspace S is discrete.

Note that if R is a set of power m and the non empty open sets are the complements of finite sets, $R \in \mathscr{T}_1$ and R does not even contain a denumerable discrete subspace.

We prove the following

Lemma 7. Let $R \in \mathscr{T}_2$, $|R| = m \geqslant \aleph_0$. Assume m is a strong limit cardinal. Then one of the following conditions holds.

a) There is a sequence $\{H_\alpha\}_{\alpha < \Omega(m*)}$ of disjoint open subsets, such that
$$| \bigcup_{\alpha < \Omega(m*)} H_\alpha| = m, \quad \sup_{\alpha < \Omega(m*)} |H_\alpha| = m;$$

b)   There is an open subset $S \subseteq R$ such that $|R-S| < m$ and every non empty open subset of S has power m.

Proof.   Put

$$\mathscr{U} = \{G : G \subseteq R \text{ is open, } |G| < m\}.$$

We distinguish two cases. (i): $|\cup \mathscr{U}| = m$ (ii) $|\cup \mathscr{U}| < m$. We prove that in cases (i), (ii), a) and b) hold respectively.

If (i) holds, let $m = \sum_{\alpha < \Omega(m*)} m_\alpha$, $m_\alpha < m$, where $\sup_{\alpha < \Omega(m*)} m_\alpha = m$. We define the sequence H_α by transfinite induction on α as follows:

Assume that H_β is defined for every $\beta < \alpha$ for some $\alpha < \Omega(m*)$ on such a way that $|H_\beta| < m$ for every $\beta < \alpha$. Then by 3.2 we have

$$\left| \overline{\bigcup_{\beta < \alpha} H_\beta} \right| < m$$

since m is a strong limit cardinal and $|\alpha| < m*$. Then $\cup \mathscr{U} - \overline{\bigcup_{\beta < \alpha} H_\beta}$ is an open set of power m. By the definition of $\mathscr{U}$ it contains an open subset H_α such that

$$m_\alpha \leqslant |H_\alpha| < m, \text{ since } \sup \{|H| : H \in \mathscr{U}\} = m.$$

The H_α's obviously satisfy the requirements of a).

If (ii) holds then $S = R - \overline{\cup \mathscr{U}}$ satisfies b) since $|\overline{\cup \mathscr{U}}| < m$ by 3.2.

Proof of Theorem 4.   Considering that $m* = \aleph_0$ and m is a strong limit cardinal, we may assume $m = \sum_{l < \omega} m_l$, $m_l < m$, $\exp \exp m_l < m_{l+1}$. It follows from Lemma 7 that either there exists a sequence $\{H_l\}_{l < \omega}$ of disjoint open sets of R such that $|H_l| \geqslant m_l$ or there is an open subset $S \subseteq R$, $|S| = m$ all whose non empty open subsets have power m. Considering $R \in \mathscr{T}_2$, S contains a sequence $\{H_l\}_{l < \omega}$ of disjoint open sets. In both cases, by Theorem 3 H_{l+1} contains a discrete subspace D_l of potency $> m_l$. Thus $D = \bigcup_{l < \omega} D_l$ is a discrete subspace of power m of R.

Theorem 7.   Let $R \in \mathscr{T}_2$, $|R| = m$ where m is a singular strong limit cardinal. Assume further that the character of the space R is $n < m$. Then R contains a system of power m of disjoint open subsets.

Proof.   Let $m = \sum_{\varrho < \Omega(m*)} m_\varrho$ where $n \leqslant m_\varrho < m$. We apply Lemma 7 to prove that there exists a sequence $\{H_\varrho\}_{\varrho < \Omega(m*)}$ of disjoint open subsets, such that $|H_\varrho| > 2^{m_\varrho}$ for $\varrho < \Omega(m*)$. If condition a) of Lemma 7 holds, this is obvious. Thus we may assume that there is an open set $S \subseteq R$, $|S| = m$ such that all non empty open subsets of S have power m. Considering $m* < m$, by Theorem 5 S contains $m*$ disjoint open subsets $\{H_\varrho\}_{\varrho < \Omega(m*)}$ which obviously satisfy the requirement. Then again by Theorem 5 H_ϱ contains a family $\mathscr{U}_\varrho$ of power $> m_\varrho$ of disjoint open subsets.

Hence $\mathscr{U} = \bigcup_{\varrho < \Omega(m*)} \mathscr{U}_\varrho$ is a family of power m of disjoint open subsets of R.

Remarks. Let $R \in \mathscr{T}_2$, $|R| = m$ and assume that R has a relatively small character. The case not covered by our results Theorem 5, 7 and by the trivial counterexamples is when m is strongly inaccessible. In this case the properties of strongly inaccessible cardinals recently considered in the literature are involved, see e.g. [8], [9].

The proof given for Theorem 6 shows that if m is a measurable strongly inaccessible cardinal or even a cardinal which does not belong to the class C_0 of [9] then even a space $R \in \mathscr{T}_1$, $|R| = m$ with pseudo character $< m$ contains a discrete subspace of power m. On the other hand the content of the results mentioned is that a very large section of strongly inaccessible cardinals belongs to class C_0. We can not prove that Theorem 7 is false for strongly inaccessible cardinals of this type.

Finally we give some remarks on T_1 spaces. First we give a general construction to obtain T_1 spaces. Let X be an infinite set, and let $\mathscr{A}$ be the system of the complements of finite subsets of X. Let $\{T_x\}_{x \in X}$ be a system of disjoint sets. Put $R = \bigcup_{x \in X} T_x$.

We define the open sets of R. $G \subseteq R$ is open if either G is empty or there exists an $A \in \mathscr{A}$ such that $\bigcup_{x \in A} T_x \subseteq G$. Then $R \in \mathscr{T}_1$.

It is obvious that for every $u \in R$, $u \in T_x$ the sets $\{u\} \cup \bigcup_{y \in A} T_y$ for $x \notin A \in \mathscr{A}$ form a base of neighbourhoods of x in R. Hence the character of R is $|\mathscr{A}| = |X|$. Considering that each set which intersects infinitely many T_x's is dense in R we have $D(R) = \aleph_0$.

Example 2. Let m be a singular cardinal, $m = \sum_{\varrho < \Omega(m*)} m_\varrho$, $m_\varrho < m$. Put $X = \{\varrho\}_{\varrho < \Omega(m*)}$, let $|T_\varrho| = m_\varrho$. Then the space $R \in \mathscr{T}_1$ has power m, $D(R) = \aleph_0$ and the character of the space as well as of each of its points is $m*$. On the other hand R does not contain a discrete subspace of power m since each discrete subspace meets less than $\aleph_0$ T_ϱ's.

This example shows that a theorem corresponding to Theorem 7 fails for T_1 spaces. We can not solve the following.

Problem 3. Let m be a singular strong limit cardinal. Let $R \in \mathscr{T}_1$ and assume that the character of R is less than $m*$. Does than R contain a discrete subspace of power m?

Example 3. Let m be an infinite cardinal, $|X| = \aleph_0$ and $|T_x| = m$ for every $x \in X$.

Then $R \in \mathscr{T}_1$, $D(R) = \aleph_0$ and R has character $\aleph_0$.

However R contains discrete subspaces of power m.

On the other hand we have

Example 4. For every regular cardinal m there is an $R \in \mathscr{T}_1$ such that $|R| = m$, $D(R) = m$ and $S(R) = \aleph_0$.

Let $R = \{\varrho\}_{\varrho < \Omega(m)}$ and let the non empty open sets be of the form

$$\{\sigma\}_{\sigma \geqslant \varrho} - \{\varrho_1, \ldots, \varrho_s\} \text{ for } \varrho, \varrho_1, \ldots, \varrho_s < \Omega(m), \, s < \omega.$$

Added in proof (February 1967)

Theorem 7. is an immediate consequence of Theorem 5. and a result of P. Erdös and A. Tarski, On mutually exclusive sets, Annals of Math., 44 (1943), pp. 315-329.

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