It was proved recently [3, 13] that the following two classes of algebras over algebraically closed fields coincide:

1. The class of representation-finite algebras $A$ with $\beta(A) \leq 2$, where $\beta(A)$ is the maximal number of nonprojective indecomposable summands in the middle term of an Auslander–Reiten sequence of $A$-modules.

2. The class of representation-finite biserial algebras; here biserial means that the radical of each nonuniserial indecomposable projective (left or right) $A$-module is the sum of two uniserial submodules with simple or trivial intersection.

A similar result is suspected if we drop the property “representation-finite” in classes 1 and 2, and we will start an investigation of biserial algebras of infinite representation type.

The key to the result for representation-finite algebras was to observe that each representation-finite biserial algebra is special [13] (see also the basic notations below), which means that it can be presented by a quiver with “nice” relations. Special algebras play an important role in the modular representation theory of finite groups. Namely, each representation-finite block of a group algebra and some of the tame blocks (they occur only in characteristic 2) are special [7, 10, 4]. Moreover in the complex representation theory of the Lorentz group the so-called Harish–Chandra modules are defined over (tame) special algebras [6].

In this paper we show that $\beta(A) \leq 2$ for any special algebra $A$ and that special algebras of infinite representation type are tame.

Our paper consists of four parts. First we prove that each special algebra is a factor of a special symmetric algebra (Theorem 1.5). For the latter the methods of Gelfand and Ponomarev used in the classification of the indecomposable Harish–Chandra modules of the Lorentz group apply and furnish a complete list of indecomposable finitely generated modules over special algebras (Proposition 2.3) presented in Section 2. From this list we
see that special algebras of infinite representation type are tame in the sense of Ringel [11]. We proceed with a calculation of the inverse $\tau^{-1} = \text{Tr} \, D$ of the Auslander translate in Section 3 and of the endomorphism rings of the indecomposable modules in Section 4. With this information on hand we can describe the Auslander–Reiten sequences over special algebras (Theorem 4.1).

### Basic Concepts

A quiver $Q$ consists of a set $Q_0$ of vertices and a set $Q_1$ of arrows connecting vertices; thus each arrow $\alpha$ has a domain $d(\alpha)$ and a range $r(\alpha)$, which we indicate by $\alpha = d(\alpha) \to r(\alpha)$ or $d(\alpha) \to^\alpha r(\alpha)$.

Throughout we denote by $L$ a connected linear quiver and, if not labeled otherwise, we assume

$$L = 1 \xrightarrow{\lambda_1} 2 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_{r-1}} r \xrightarrow{\lambda_r} r + 1, \quad r \geq 0,$$

where for all $i, 1 \leq i \leq r$, $\lambda_i$ is an arrow with either orientation. We define a function $\varepsilon: L_1 \to \{+1, -1\}$ by $\varepsilon(\lambda_i) = 1$ if $\lambda_i = i \to i + 1$, and $\varepsilon(\lambda_i) = -1$ if $\lambda_i = i \leftarrow i + 1$.

Similarly we denote by $Z$ a connected cyclic quiver and, if not labeled otherwise, we assume

$$Z = \overbrace{1 \xrightarrow{\zeta_1} 2 \xrightarrow{\zeta_2} \cdots \xrightarrow{\zeta_{r-1}} r}^r, \quad r \geq 2,$$

where for all $i, 1 \leq i \leq r$, $\zeta_i$ is an arrow with either orientation. Again we define $\varepsilon: Z_1 \to \{+1, -1\}$ by $\varepsilon(\zeta_i) = 1$ if $\zeta_i = i \to i + r$, $\varepsilon(\zeta_i) = -1$ if $\zeta_i = i \leftarrow i + r$; here $i$ denotes the congruence class of $i$ modulo $r$.

If $Q$ is an arbitrary quiver, a quiver homomorphism $w: L \to Q$ is called a walk of length $r$ in $Q$ from $w(1)$, the start-point, to $w(r + 1)$, the endpoint of $w$. A walk $w$ is called a path if $\varepsilon(\lambda_i) = 1$ for all $i$. In the latter case we shall use also notation $w = \alpha_1 \alpha_2 \cdots \alpha_n$, where $\alpha_i = w(\lambda_i) \in Q_1$. Similarly a quiver homomorphism $w: Z \to Q$ is called a tour in $Q$, and a circuit if $\varepsilon(\zeta_i) = 1$ for all $i$. If $v$ is a walk (respectively tour) in $Q$, we say that each arrow $v(\lambda_i)$ (respectively $v(\zeta_i)$) occurs in $v$ and $v$ runs through each vertex $v(i)$ ($v(i)$).

For each connected linear subquiver $L'$ of $L$ (respectively $Z$) the restriction of $v$ to $L'$ is called a subwalk of $v$, or else a subpath if $\varepsilon(\alpha) = 1$ for $\alpha \in L'$. A walk (respectively path) $v: L \to Q$ is closed if $v(1) = v(r + 1)$.

Throughout the paper we fix an algebraically closed field $K$ and use the

We recall that any finite-dimensional $K$-algebra $A$ is Morita equivalent to a bound quiver algebra $K[Q, I] := K[Q]/I$, where $Q$ is the ordinary quiver of $A$ and $I$ an appropriate ideal of the path algebra $K[Q]$ of $Q$ over $K$. Hence the category $\text{mod } A$ of finitely generated left $A$-modules is equivalent to the category $\text{mod}_{K}(Q, I)$ of finite-dimensional bound representations of $(Q, I)$ over $K$. As for our aim it is enough to consider basic, connected algebras, and we can restrict our discussions to bound quiver algebras $A = K[Q, I]$, where $Q$ is a finite, connected quiver.

We recall from [13] the following

(0.1) **DEFINITION.** An algebra $A$ is called special if it is isomorphic to a bound quiver algebra $K[Q, I]$ where the bound quiver $(Q, I)$ satisfies:

(i) The numbers of arrows in $Q$ with a fixed domain, respectively, range, are bounded by 2.

(ii) For each $\alpha \in Q_1$ there is at most one $\beta \in Q_1$ and at most one $\gamma \in Q_1$ such that $\alpha \beta$ and $\gamma \alpha$ do not belong to $I$.

In this case we also call $(Q, I)$ a special bound quiver. It was proved in [13] that each special algebra is biserial and each representation-finite biserial algebra is special. But there are simple examples of nonspecial biserial algebras [13].

1. **THE STRUCTURE OF SPECIAL ALGEBRAS**

Our first aim is to analyse the structure of special algebras. We shall show that each special algebra is a factor of a special symmetric algebra.

We start by extending the notion of tracks introduced earlier in the case of representation-finite, biserial selfinjective algebras [12]: A triple $T = (Q, I, v)$ consisting in a finite bound quiver $(Q, I)$ and a path (respectively a circuit) $v$ in $Q$ is called a linear (respectively cyclic) track, if $K[Q, I]$ is finite dimensional and if it satisfies

(1.1) (i) Each $\alpha \in Q_1$ occurs exactly once in $v$, and $v$ runs at most twice through each vertex of $Q$.

(ii) For any two composable arrows $\alpha, \beta \in Q_1$ the composition $\alpha \beta$ belongs to $I$ if and only if $\alpha \beta$ is not a subpath of $v$.

(1.2) **PROPOSITION.** The bound quiver algebra $K[Q, I]$ of a (linear or cyclic) track $T = (Q, I, v)$ is special. Any bound quiver algebra $A$ of a special
bound quiver \((Q, I)\) determines a unique set of tracks \(T_\alpha = (Q_\alpha, I_\alpha, v_\alpha)\), \(\alpha \in \Omega \subseteq Q_1\), where \((Q_\alpha, I_\alpha)\) is a bound subquiver of \((Q, I)\) (i.e., \(Q_\alpha\) is a subquiver of \(Q\) and \(I_\alpha = K[Q_\alpha] \cap I\)), such that each arrow of \(Q\) belongs to exactly one \(Q_\alpha\).

Proof. The first assertion is clear by the definition of a track. In a special bound quiver \((Q, I)\) each arrow \(\alpha\) determines either a unique circuit or a unique maximal path \(v_\alpha\) with the property that the composition of two consecutive arrows of \(v_\alpha\) does not belong to \(I\). Let \(Q_\alpha\) be the underlying quiver of \(v_\alpha\), \(I_\alpha = K[Q_\alpha] \cap I\). Then \(T_\alpha := (Q_\alpha, I_\alpha, v_\alpha)\) is a track and we have \(T_\alpha = T_\beta\) if and only if \(\beta\) belongs to \(Q_\alpha\). Take now a set \(\Omega \subseteq Q_1\) representing the different tracks \(T_\alpha\) in \((Q, I)\).

A special bound quiver \((Q, I)\) can be reconstructed from its set of tracks \(T_\alpha, \alpha \in \Omega\), if one knows some additional coefficients relating certain pairs of paths with the same startpoint and endpoint and maximal in the set of paths outside of \(I\).

Assume there are given two special bound quivers \((Q, I)\) and \((P, J)\) and vertices \(a_1, \ldots, a_n\) in \(Q\), \(b_1, \ldots, b_n\) in \(P\), \(n \geq 1\), such that for all \(i\), \(1 \leq i \leq n\), the number of arrows with domain (respectively range) \(a_i\) or \(b_i\) is bounded by 2. Identifying in the disjoint union of \(Q\) and \(P\) the vertices \(a_i = b_i, 1 \leq i \leq n\), we get a new quiver \(R\). Let \(X\) be a set of pairs \((u, v)\), where \(u\) and \(v\) are paths in \(Q\) and \(P\), respectively, \(u\) from \(a_i\) to \(a_j\), \(v\) from \(b_i\) to \(b_j\), and both are maximal in the set of paths outside of \(I\) respectively \(J\). For each \((u, v) \in X\) we choose an element \(\kappa_{u,v} \in K\) and consider the ideal \(H\) of \(K[R]\) generated by the union of the following subsets:

(i) \(I \cup J\);
(ii) \(\{\alpha \beta | \alpha \in Q_1, \beta \in P_1 \text{ or } \alpha \in P_1, \beta \in Q_1\}\);
(iii) \(\{u - \kappa_{u,v} v | (u, v) \in X\}\).

We call \((R, H)\) a gluing of \((Q, I)\) and \((P, J)\) along \(a_1, \ldots, a_n, b_1, \ldots, b_n\).

(1.3) Proposition. The bound quiver \((R, H)\) constructed above is special and its set of tracks is the disjoint union of those of \((Q, I)\) and \((P, J)\). Any special bound quiver \((Q, I)\) is obtained from its set of tracks \(T_\alpha, \alpha \in \Omega\), by gluing step by step with the bound quivers \((Q_\alpha, I_\alpha)\) in some appropriate way.

Proof. Since \(I \cup J \subseteq H\) and \(\alpha \beta \in H\) whenever \(\alpha \in Q_1, \beta \in P_1 \text{ or } \alpha \in P_1, \beta \in Q_1\), \((R, H)\) is special and any track in \((R, H)\) is one in \((Q, I)\) or \((P, J)\). The last assertion follows easily by induction on the number of tracks. Observe that in any case \(I\) is generated by zero relations and by relations of the form \(u - \kappa_{u,v} v\) where \(u\) and \(v\) have the same startpoint and endpoint and are maximal in the set of paths outside of \(I\).
(1.4) Theorem. Any special algebra $A$ is a factor of a special symmetric algebra $A_{\alpha}$.

Proof. We shall construct $A_{\alpha}$ from $A$ in some natural way but we point out that it is far from being uniquely determined by $A$. Let $A = K[Q, I]$ where $(Q, I)$ is a special bound quiver and $\mathcal{I} = \{T_{\alpha} = (Q_{\alpha}, I_{\alpha}, v_{\alpha}), \alpha \in \Omega\}$ the set of tracks of $(Q, I)$. We fix some enumeration of tracks, say $\mathcal{I} = \{T_1, \ldots, T_t\}$, $T_i = (Q_i, I_i, v_i)$ for $1 \leq i \leq t$, such that for some $0 \leq s < t$ $v_i$ is a circuit or a closed path for $1 \leq i \leq s$, whereas $v_i$ is not closed for $s + 1 \leq i \leq t$.

Our first step is to present $A$ as a factor of a special algebra $A'$ the tracks of which all contain circuits or closed paths. We do this by induction on the number $n = t - s$ of not closed paths $v_i$, $n = n(A)$.

Let $a_i$ and $b_i$ be respectively the startpoint and the endpoint of $v_i$, $s + 1 \leq i \leq t$.

Case 1. There are composable paths, say, $v_{r+1}, v_{r+2}, \ldots, v_{r+m}, m \geq 2$. If the composition $v$ of them is a closed path we define a new algebra $A' = K[Q, I']$ where $I'$ is obtained from $I$ by deleting the relations $p_{r+m+1}, r+m < r+1$. Then $A'$ is special, $A$ is a factor of $A'$ and $n(A') = t - (s + m) < n(A)$. If $v$ is not closed, we may assume that no other path $v_i$, $i > r + m$, is composable with $v$. This implies that at most one arrow of $Q$ has domain $b_{r+m}$ and at most one arrow of $Q$ has range $a_{r+m}$. Then we can add one new arrow $y: b_{r+m} \rightarrow a_{r+m}$ and with appropriate relations $I'$ on the new quiver $Q'$ we again get a special algebra $A'$ such that $A$ is a factor of $A'$ and $n(A') = t - (s + m) < n(A)$.

Case 2. The paths $v_i$, $s + 1 \leq i \leq t$, are all noncomposable. Again we can add one new arrow $y: b_{s+m} \rightarrow a_{s+m+1}$ and with appropriate relations $I'$ on the new quiver $Q'$ we again get a special algebra $A'$ such that $A$ is a factor of $A'$ and $n(A') = t - (s + m) < n(A)$.

We can now assume that all paths $v_{\alpha}$ in the tracks $T_{\alpha}, \alpha \in \Omega$, of $A$ are circuits or closed paths. We put $v_{\beta} = v_{\alpha}$ for each arrow $\beta$ occurring in $v_{\alpha}$ and denote by $m$ the nilpotence index of $\text{rad} A$. For each vertex $a$ of $Q$ and each arrow $\alpha$ with domain $a$ we denote by $v_{a,\alpha}$ the closed path starting in $a$ with $\alpha$, running $m$ times around $v_{\alpha}$ and ending in $a$. Consider the ideal $I_s$ of $K[Q]$ generated by all elements of one of the following forms:

(i) $\beta \alpha$, where $\beta \alpha$ is not a subpath of $v_{\alpha}$;
(ii) all paths properly containing a path $v_{a,\alpha}$;
(iii) all differences $v_{a,\alpha} - v_{a,\beta}$, where $\alpha \neq \beta$ are arrows with the same domain $a$. 

It is clear that the algebra $A_z := K[Q, I_z]$ is special and has $A$ as a factor algebra. We claim that $A_z$ is also symmetric. For this we look at the linear form $\varphi: A_z \to K$ given on the residue classes of paths $v$ by

$$\varphi(v + I_z) = 1 \quad \text{if } v = v_{a,\alpha} \text{ for some } \alpha = a \to b \in Q_1,$$

$$= 0 \quad \text{else.}$$

$\varphi$ vanishes on no left or right ideal different from 0 but on all commutators, because for two composable paths $u, v$ in $Q$ we have $u \cdot v = v_{a,\alpha}$ for some arrow $\alpha = a \to b$ if and only if $v \cdot u = v_{c,\gamma}$ for some arrow $\gamma = c \to d$.

2. INDECOMPOSABLE MODULES

Using methods of Gelfand and Ponomarev [6] or else their functorial interpretation by Gabriel (see [10]), applied by Ringel [10] and Donovan and Freislich [4] to certain special blocks of group algebras, one can determine the finitely generated indecomposable modules over any special symmetric algebra, hence by Proposition 1.5 those over arbitrary special algebras. For later use we will describe them in some appropriate language.

We start recalling from [13] the notions of $V$-sequences and primitive $V$-sequences in a special bound quiver $(Q, I)$. We may assume without loss of generality that each indecomposable injective-projective $A$-module is uniserial, where $A = K[Q, I]$. Otherwise, if $A = P_1 \oplus P_2$ is a decomposition of $A$, as a left module, where $P_1$ is a sum of indecomposable nonuniserial injective-projective summands of $A$ and where $P_2$ has not any such summand, $S := \text{soc} P_1$ is a two-sided ideal of $A$ [3, Lemma 4.3], annihilating each nonprojective indecomposable left or right $A$-module, and $A/S$ is a special algebra with the wanted property.

(2.1) DEFINITION. A walk $v: L \to Q$ in $Q$ is called a $V$-sequence in $(Q, I)$ if it satisfies:

(i) each subpath of $v$ does not belong to $I$;
(ii) $v(\lambda_i) \neq v(\lambda_{i+1})$ whenever $e(\lambda_i) \neq e(\lambda_{i+1})$.

(2.2) DEFINITION. A tour $u: Z \to Q$ in $Q$ is called a primitive $V$-sequence in $(Q, I)$ if it satisfies:

(i) $u$ is not a circuit and each subpath of $u$ does not belong to $I$;
(ii) $u(\xi_i) \neq u(\xi_{i+1})$ whenever $e(\xi_i) \neq e(\xi_{i+1})$;
(iii) There is no automorphism $\sigma \neq \text{id}$ of $Z$ permutting the vertices
cyclicly and such that \( u = u \circ \sigma \). (Hence by (ii) there is no automorphism \( \sigma \neq \text{id} \) at all with \( u = u \circ \sigma \).)

Associated with any quiver homomorphism \( h: X \to Q \) we have a functor

\[
F_h: \mod_K X \to \mod_K Q
\]

as follows (compare [14]): For a representation

\[
V = (V_x, V_\rho; x \in X_0, \rho \in X_1)
\]

of \( X \) define

\[
(F_h V)_a = \bigoplus_{h(x) = a} V_x \quad \text{for each } a \in Q_0
\]

\[
(F_h V)_a = \bigoplus_{h(x) = a} V_x \xrightarrow{(g_{yx})} \bigoplus_{h(y) = b} V_y \quad \text{for each } a = a \to b \in Q_1
\]

where \( g_{yx} = \sum_{\rho = x \to y, \rho \in X_1} V_\rho \). For a morphism

\[
\varphi = (\varphi_x: V_x \to W_x, x \in X_0): V \to W
\]

in \( \mod_K X \) define

\[
F_h \varphi = \left( \bigoplus_{h(x) = a} \varphi_x : (F_h(V))_a \to (F_h(W))_a; a \in Q_0 \right).
\]

For the quiver \( L = 1 \xrightarrow{\zeta_1} 2 \xrightarrow{\zeta_2} \cdots \xrightarrow{\zeta_r} r \xrightarrow{\zeta_{r+1}} L \) let \( L \) be the representation \( K \xrightarrow{\text{id}} K \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} K \xrightarrow{\text{id}} K \) of \( L \) (observe that each edge represents an arrow).

For the quiver

\[
Z = \begin{array}{c}
1 \xrightarrow{\zeta_1} 2 \xrightarrow{\zeta_2} \cdots \xrightarrow{\zeta_{r-1}} r
\end{array}
\]

a positive integer \( n \) and an element \( \kappa \in K^* := K \setminus \{0\} \) let \( Z(n, \kappa) \) be the representation

\[
\begin{array}{cccc}
K^n & \xrightarrow{\text{id}} & K^n & \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} K^n
\end{array}
\]

\[
J_n(\kappa)
\]
of \( Z \), where \( J_n(\kappa) \) is the Jordan block of rank \( n \) with eigenvalue \( \kappa \);
\[
J_n(\kappa) = \begin{pmatrix}
\kappa & 1 & 0 \\
& \ddots & \ddots \\
0 & & 1 & \kappa
\end{pmatrix}.
\]

If now \( v : L \to Q \) is a \( V \)-sequence in \((Q, I)\), the representation
\[
M(v) := F_v(L)
\]
is a bound representation of \((Q, I)\), called a representation of the first kind.

Similarly, if \( u : Z \to Q \) is a primitive \( V \)-sequence in \((Q, I)\), the representation
\[
M(u, n, \kappa) := F_u(Z(n, \kappa))
\]
is a bound representation of \((Q, I)\) for each positive integer \( n \) and each \( \kappa \in K^* \), called a representation of the second kind.

**(2.3) Proposition.** The representations \( M(v) \) and \( M(u, n, \kappa) \) are all indecomposable and each finite-dimensional indecomposable bound representation of \((Q, I)\) is isomorphic to one of them. Moreover no representation of the first kind is isomorphic to a representation of the second kind, \( M(v) \) is isomorphic to \( M(v') \) for \( V \)-sequences \( v : L \to Q, v' : L' \to Q \) if and only if there is an isomorphism \( \sigma : L' \to L \) with \( v' = v \circ \sigma \), and \( M(u, n, \kappa) \) is isomorphic to \( M(u', n', \kappa') \) for primitive \( V \)-sequences \( u : Z \to Q \) and \( u' : Z' \to Q \) if and only if \( n = n' \), \( \kappa = \kappa' \) and there is an isomorphism \( \sigma : Z' \to Z \) with \( u' = u \circ \sigma \).

**Proof.** We know by Theorem 1.5 that \( A = K[Q, I] \) is a factor of a special symmetric biserial algebra \( A_s = K[Q_s, I_s] \). Hence we have to determine the indecomposable finitely generated \( A_s \)-modules which are \( A \)-modules. For the indecomposable \( A_s \)-modules the reader is referred to [4]: By the results there the indecomposable \( A_s \)-modules are described by \( V \)-sequences and primitive \( V \)-sequences in \((Q_s, I_s)\) and it is clear that such a module is an \( A \)-module if and only if the (primitive) \( V \)-sequence is one in \((Q, I)\).

**(2.4) Corollary.** Any special algebra \( A \) is either representation-finite or tame (in the sense of Ringel [11]).

**Proof.** Let \( K[T] \) be the polynom algebra in one indeterminate \( T \). For a primitive \( V \)-sequence \( u : Z \to Q \) in \((Q, I)\) we denote by \( G_u : \text{mod } K[T] \to \text{mod } A \) the functor which is the composition of \( F_u : \text{mod } A_s \to \text{mod } A \).
mod_k Z \to \text{mod}_k (Q, I)$ with the functor $G: \text{mod} \ K[T] \to \text{mod}_k Z$ given on objects by

\[
\text{mod} \ K[T] \ni V \mapsto \begin{array}{c}
V \\
\xrightarrow{id}
\end{array} V \\
\xrightarrow{id} \cdots \xrightarrow{id} V \\
\xleftarrow{T}
\]

with the obvious action on morphisms. The functors $G_u$ are all embeddings and for each positive integer $d$ there are finitely many primitive $V$-sequences $u_i$, $1 \leq i \leq s(d)$, such that all but a finite number of indecomposable $A$-modules of dimension $d$ are of the form $G_{u_i}(V)$ for some $i$ and some indecomposable $K[T]$-module $V$.

3. THE AUSLANDER TRANSLATE

Our aim is to describe the Auslander–Reiten sequences over special algebras $A$. Again we can assume without loss of generality that each indecomposable injective-projective $A$-module is uniserial [2].

We start with a description of the Auslander translate $\tau^{-1} = \text{Tr} \ D$; here $D$ denotes the duality $D = \text{Hom}_k(-, K)$ and $\text{Tr} \ M$ is the cokernel of $\text{Hom}_A(f, A)$ where

\[
P_1 \xrightarrow{f} P_0 \to M \to 0
\]

is a minimal projective presentation of the $A$-module $M$.

We denote by $Q^{\text{op}}$ the quiver obtained from $Q$ by reversing all arrows. Any path (respectively walk) $v$ in $Q$ determines a unique path (respectively walk) $v^{\text{op}}$ in $Q^{\text{op}}$ and the map $v \mapsto v^{\text{op}}$ extends to an algebra antiisomorphism $K[Q] \to K[Q^{\text{op}}]$. Let $P^{\text{op}}$ be the image of $I$ under this map; then the algebra $A^{\text{op}} := K[Q^{\text{op}}, I^{\text{op}}]$ is the opposite algebra of $A$. Trivially $A$ is special if and only if $A^{\text{op}}$ is so, and a walk $v$ (respectively a tour $u$) is a $V$-sequence (respectively a primitive $V$-sequence) in $(Q, I)$ if and only if $v^{\text{op}}$ (respectively $u^{\text{op}}$) is a $V$-sequence (respectively a primitive $V$-sequence) in $(Q^{\text{op}}, I^{\text{op}})$.

(3.1) **LEMMA.** (1) $DM(v) = M(v^{\text{op}})$ for each $V$-sequence $v$ in $(Q, I)$.

(2) $DM(u, n, \kappa) = M(u^{\text{op}}, n, \kappa)$ for each primitive $V$-sequence $u$ in $(Q, I)$, each positive integer $n$ and each $\kappa \in K^*$. 

**Proof.** Assertion (1) is clear. For (2) we observe that the $K$-dual of the map $K^n \to K^n$ given by the Jordan block $J_n(\kappa)$ is represented with respect to
the dual basis of $DK^n$ by the transpose of $J_n(\kappa)$. But reversing the dual basis we get again $J_n(\kappa)$.

For a description of $\text{Tr } M(v)$ let $v : L \to Q$ be a $V$-sequence in $(Q, I)$. Then

$$L = 1 \leftarrow 2 \leftarrow \cdots \leftarrow s_1 \to \cdots \to s_2 \leftarrow \cdots \cdots \leftarrow s_m \to \cdots \to r + 1$$

with $1 \leq s_1 < s_2 < \cdots s_m \leq r + 1$. For each $p, q$ where $q$ is odd, $1 \leq q \leq m$, and $p = q + 1$ for $q < m$ or $p = q - 1$ for $q > 1$ we put

$$x_{pq} = v(s_p \to \cdots \to s_q) + I \in A.$$ 

If the restriction of $v$ to $L = 1 \leftarrow \cdots \leftarrow s_1 \to s_1 + 1$ is extendable to a $V$-sequence $v' : s_0 = 0 \leftarrow 1 \leftarrow \cdots \leftarrow s_1 \to s_1 + 1$ we put

$$x_{01} = v'(0 \leftarrow 1 \leftarrow \cdots \leftarrow s_1) + I \in A,$$

else $x_{01} = 0$. If the restriction of $v$ to $s_m - 1 \leftarrow s_m \to \cdots \to r + 1$ is extendable to a $V$-sequence $v' : s_m - 1 \leftarrow s_m \to \cdots \to r + 1 \to s_{m+1} = r + 2$ put

$$x_{m+1,m} = v'(s_m \to \cdots \to r + 1 \to r + 2) + I \in A,$$

else $x_{m+1,m} = 0$. We denote be $e_i$ the idempotent of $A$ corresponding to the vertex $v(s_i) \in Q_0$, $v \leq i \leq m + 1$, hence $x_{pq} = e_p x_{pq} e_q$. Then the minimal projective presentation of $M(v)$ has the form

$$
\begin{array}{c}
(Ae_0 \oplus) Ae_2 + Ae_4 \oplus \cdots \oplus Ae_{m-1} \oplus Ae_{m+1} \\
\longrightarrow f \longrightarrow Ae_1 \oplus Ae_3 \oplus \cdots \oplus Ae_m \longrightarrow M(v) \longrightarrow 0,
\end{array}
$$

where $f$ is given by multiplication from the right with the matrix

$$B(v) = \begin{pmatrix}
(x_{01}) & x_{21} & x_{23} & \cdots & 0 \\
x_{21} & x_{43} & \cdots & 0 & \cdots \\
x_{23} & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
x_{m-1,m-2} & \cdots & \cdots & \cdots & x_{m-1,m} \\
x_{m-1,m} & \cdots & \cdots & \cdots & (x_{m+1,m})
\end{pmatrix}.$$ 

Here the brackets $(Ae_0 \oplus), (\oplus Ae_{m+1}), (x_{01}), (x_{m+1,m})$ indicate that these terms should be omitted if $x_{01} = 0$ respectively $x_{m+1,m} = 0$. Hence $\text{Tr } M(v)$ is the cokernel of the map

$$e_1 A \oplus e_3 A \oplus \cdots \oplus e_mA \\
\to (e_0 A \oplus) e_2 A \oplus e_4 A \oplus \cdots e_{m-1} A (\oplus e_{m+1} A).$$
given by multiplication from the left with the matrix $B(v)$. Assume now that $v: L \to Q$ is a $V$-sequence of length $\geq 1$ in $(Q, I)$ (hence $L$ has at least one arrow). Then $L$ can be drawn as

$$L = 1 \to \cdots \to s_1 \leftarrow \cdots \leftarrow s_2 \to \cdots \to s_m \leftarrow \cdots \leftarrow r + 1$$

$$1 \leq s_1 < s_2 < \cdots < s_m \leq r + 1, \ m \geq 1.$$ 

If there is a $V$-sequence $v': L' \to Q$ which is an extension of $v$ to a quiver $L'$ of the form

$$L' = \underbrace{1 \to \cdots \to s_1 \leftarrow \cdots \leftarrow s_m \leftarrow \cdots \leftarrow r + 1 \leftarrow r + 2 \to \cdots \to t + 1},$$

where $t \geq r + 2$, let $v_1$ be the maximal extension of this form. Otherwise let $v_1$ be the restriction of $v$ to the subquiver

$$1 \to \cdots \to s_2 \leftarrow \cdots \leftarrow s_m \leftarrow \cdots \leftarrow (s_m - 1),$$

which may be empty and then $v_1 := \emptyset$. Similarly, if there is a $V$-sequence $v'': L'' \to Q$ which is an extension of $v$ to a quiver $L''$ of the form

$$L'' = \underbrace{t \leftarrow \cdots \leftarrow 0 \to 1 \to \cdots \to s_1 \leftarrow \cdots \leftarrow s_m \leftarrow \cdots \leftarrow r + 1},$$

where $t \leq 0$, let $v_2$ be a maximal extension of this form or else let $v_2$ be the restriction of $v$ to the subquiver

$$s_1 + 1 \leftarrow \cdots \leftarrow s_2 \to \cdots \to s_m \leftarrow \cdots \leftarrow r + 1,$$

which may be empty and then $v_2 := \emptyset$.

For a $V$-sequence $v$ of length 0 in $(Q, I)$, which is just a vertex $a$ of $Q$, we have at most two arrows $\alpha, \beta \in Q$ with range $a$. Accordingly we have at most two different maximal $V$-sequences $v_1$, respectively, $v_2$, which are maps $L \to Q$ for

$$L = 1 \leftarrow \hat{\lambda}_1 \ 2 \leftarrow \hat{\lambda}_2 \to \cdots \leftarrow \hat{\lambda}_i \to t + 1$$

with $v_1(\hat{\lambda}_1) = \alpha$, respectively $v_2(\hat{\lambda}_1) = \beta$. If there is only one such arrow $\alpha \in Q$, we define $v_2 := \emptyset$; if there is not any arrow with range $a$, we define $v_1 := v_2 := \emptyset$. Moreover we put $M(\emptyset) := 0$.

(3.2) LEMMA. If $v$ is a $V$-sequence in $(Q, I)$ such that $M(v)$ is not injective, then:
One of the $V$-sequences $v_1, v_2$ is not empty, and if both are then $(v_1)_2 = (v_2)_1$.

(2) Put $\tilde{v} := (v_1)_2$ or $\tilde{v} := (v_2)_1$ whatever is defined. Then $\text{Tr } DM(v) = M(\tilde{v})$.

Proof: (1) If $v_1$ is empty, $L$ has the form

$$L = 1 \leftarrow \cdots \leftarrow r + 1, \quad r \geq 0,$$

and there is no arrow with range $v(r + 1)$. Similarly, if $v_2$ is empty, $L$ has the form

$$L = 1 \rightarrow \cdots \rightarrow r + 1, \quad r \geq 0,$$

and there is no arrow with range $v(1)$. Hence if both $v_2$ and $v_1$ are empty, we need $r = 0$ and there is no arrow with range $v(1)$. But this means that $M(v)$ is simple injective [4]. It is clear that $(v_1)_2 = (v_2)_1$ if both are defined. The proof of (2) is a straightforward interpretation of the description of $D$ and $\text{Tr}$. But one has to distinguish several cases and the details are left to the reader (compare also [8, 7]).

Let now $u : Z \rightarrow Q$ be a primitive $V$-sequence in $(Q, I)$. By Proposition 2 we can assume that

$$Z = (\tilde{s}_1 = \tilde{l}) \leftarrow \cdots \leftarrow \tilde{s}_2 \rightarrow \cdots \rightarrow \tilde{s}_3 \leftarrow \cdots \leftarrow \tilde{s}_m \rightarrow \cdots \rightarrow \tilde{r}.$$

For each $p, q$, where $q$ is even, $2 \leq q \leq m$, and $p = q \pm 1$ for $q < m$, $p = m - 1$ or $p = 1$ for $q = m$ we put

$$x_{pq} - u(s_q \rightarrow \cdots \rightarrow s_p) + I \in A.$$

Again we denote by $e_i$ the idempotent of $A$ corresponding to the vertex $u(s_i) \in Q_0$. Then the minimal projective presentation of the module $M(u, n, \kappa)$ has the form

$$(Ae_1)^n \oplus (Ae_2)^n \oplus \cdots \oplus (Ae_{m-1})^n \xrightarrow{f} (Ae_2)^n \oplus (Ae_4)^n \oplus \cdots \oplus (Ae_m)^n \rightarrow M(u, n, \kappa) \rightarrow 0,$$

where $f$ is given by multiplication from the right with the block matrix

$$B(u, n, \kappa) = \begin{pmatrix}
    x_{12} \cdot E_n & x_{32} \cdot E_n & \cdots & 0 \\
    0 & x_{34} \cdot E_n & \cdots & x_{m-1} \cdot E_n \\
    x_{1m} \cdot J_n(\kappa) & \cdots & \cdots & x_{m-1,m-2} \cdot E_n \\
    & & \cdots & \cdots & x_{m-1,m} \cdot E_n
\end{pmatrix}. $$
Here $E_n$ is the unit matrix of rank $n$. Hence $\text{Tr } M(u, n, \kappa)$ is the cokernel of the map

$$(e_2 A)^n \oplus (e_4 A)^n \oplus \cdots \oplus (e_m A)^n$$

$$\longrightarrow (e_1 A)^n \oplus (e_3 A)^n \oplus \cdots \oplus (e_{m-1} A)^n$$

given by multiplication from the left with the matrix $B(u, n, \kappa)$.

(3.3) **Corollary.** $\text{Tr } DM(u, n, \kappa) \cong M(u, n, \kappa)$ for each primitive $V$-sequence $u$, each positive integer $n$ and each $\kappa \in K^*$.  

**Proof:** The assertion follows immediately from the description of $D$ and $\text{Tr}$. Observe that $DM(u, n, \kappa) = M(u^op, n, \kappa)$ is a right $A$-module, hence we have to take the left-right symmetric description for $\text{Tr}$.

4. **Auslander–Reiten Sequences**

Assume that $v$ is a $V$-sequence in $(Q, I)$ and $v_1$, $v_2$, $\bar{v}$ are as in Lemma 3.2. By construction the $V$-sequences $v_1$ and $v_2$ are extensions or restrictions of the given $V$-sequence $v$ and there are canonical monomorphisms or epimorphisms $f_1 : M(v) \to M(v_1)$ and $f_2 : M(v) \to M(v_2)$. Also $\bar{v}$ is an extension or a restriction of both $v_1$ and $v_2$ and there are canonical monomorphisms or epimorphisms $g_1 : M(v_1) \to M(\bar{v})$, $g_2 : M(v_2) \to M(\bar{v})$.

For a primitive $V$-sequence $u$ in $(Q, I)$ there are sequences of canonical monomorphisms $f_i$, and epimorphisms $g_i$, respectively, $i \geq 0$,

$$0 = M(u, 0, \kappa) \xrightarrow{f_0} M(u, 1, \kappa) \xrightarrow{f_1} M(u, 2, \kappa) \xrightarrow{f_2} \cdots$$

$$0 = M(u, 0, \kappa) \xleftarrow{g_0} M(u, 1, \kappa) \xleftarrow{g_1} M(u, 2, \kappa) \xleftarrow{g_2} \cdots$$

With these notations we can now state our main result.

(4.1) **Theorem.** (1) Let $v$ be a $V$-sequence in $(Q, I)$ such that $M(v)$ is noninjective. Then

$0 \to M(v) \xrightarrow{(f_1, f_2)} M(v_1) \oplus M(v_2) \xrightarrow{(g_1, g_2)} M(\bar{v}) \to 0$

is an Auslander–Reiten sequence in $\text{mod}_k(Q, I)$.

(2) Let $u$ be a primitive $V$-sequence in $(Q, I)$, $n$ a positive integer and $\kappa \in K^*$. Then
is an Auslander–Reiten sequence in $\text{mod}_k(Q, I)$.

The proof of this theorem will occupy the rest of this paper. Our aim is to verify the conditions in the definition of Auslander–Reiten sequences in [1] which are called there “almost split sequences.” The first step is to determine the endomorphisms of the modules $M(u)$ and $M(u, n, \kappa)$.

Let $v: L \to Q$ be a $V$-sequence in $(Q, I)$, where

$$L = 1 \to 2 \to \cdots \to r \to r + 1.$$ 

Then $M(v)$, as a vector space, is decomposed into a direct sum of one-dimensional spaces $V_i = K$:

$$M(v) = \bigoplus_{i=1}^{r+1} V_i.$$ 

With respect to this decomposition each endomorphism $f$ of $M(v)$ is given by a matrix $F = (f_{ij})$, $f_{ij} \in K$ for all $1 \leq i, j \leq r + 1$.

(4.2) **Lemma.** Let $f$ be an endomorphism of $M(v)$ with matrix $F = (f_{ij})$. Then, for all $i \neq j, f_{ij} \neq 0$ implies:

1. $v(i) = v(j)$ and $f_{ij} = 0$;
2. for each path $x$ from $i$ to $h$ in $L$ there is a path $y$ from $j$ to $k$ such that $v(y) = v(x)$;
3. for each path $y$ from $k$ to $j$ in $L$ there is a path $x$ from $h$ to $j$ such that $v(y) = v(x)$;
4. $f_{hk} = f_{ij}$ for the vertices $h, k$ in (2) and (3).

Moreover, if $f$ is nilpotent, $f_{ii} = 0$ for all $i$.

**Proof.** For each arrow $\alpha \in Q_1$ which belongs to the image of $v$ the linear map $[M(v)]_\alpha$ is given with respect to the above decomposition of $M(v)$ by a matrix $S(\alpha) = (s_{ij})$ given by $s_{ij} = 1$ if there is an arrow $\lambda = j \to i$ in $L$ satisfying $v(\lambda) = \alpha$ and $s_{ij} = 0$ otherwise.

For all $\alpha \in Q_1$ which do not belong to $v(L)$, $S(\alpha) = 0$. The endomorphism $f$ of $M(v)$ has to commute with the linear maps $[M(v)]_\alpha$, $\alpha \in Q_1$, hence $F = (f_{ij})$ commutes with the matrices $S(\alpha)$. This gives us immediately the conditions (2), (3) and (4). Also the first part of (1) is clear. Assume now that $f_{ij}$ and $f_{ji}$ are nonzero, so that we can apply (2) and (3) in both direc-
tions. But this means that there is an automorphism $\sigma$ of $L$ with $v = v \circ \sigma$ and $\sigma(i) = j$. By the definition (2.1) of a $V$-sequence $\sigma$ cannot be a reflection. Hence $\sigma$ is the identity and $i = j$. It remains to prove the last assertion. By (1) in the diagonal of the matrix of $f^2$ we have the elements $f_{ii}^2$ and inductively any power $G = f^{2n}$ has diagonal elements $g_{ii} = f_{ii}^{2n}$. Hence if $f$ is nilpotent, $f_{ii} = 0$ for all $i$.

Let $u: Z \to Q$ be a primitive $V$-sequence in $(Q, I)$, where

$$ Z = 1 \xrightarrow{\xi_1} 2 \xrightarrow{\xi_2} \cdots \xrightarrow{\xi_{r-1}} r. $$

For a positive integer $n$ and $\kappa \in K^*$ the module $M(u, n, \kappa)$ is then decomposed as a vector space into a direct sum of $n$-dimensional spaces $V_i = K^n$:

$$ M := M(u, n, \kappa) = \bigoplus_{i=1}^r V_i. $$

Accordingly an endomorphism $f$ of $M$ is given by an $r \times r$ block matrix $F = (F_{ij})$, where each $F_{ij}$ is an $n \times n$ matrix over $K$.

(4.3) **Lemma.** A block matrix $F = (F_{ij})$ as above is the matrix of an endomorphism $f$ of $M$ if and only if the following conditions are satisfied:

(i) $F_{11} = F_{22} = \cdots = F_{rr} = \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_n \\ \vdots & \ddots & \vdots & \vdots \\ \rho_2 & \cdots & \rho_1 \end{pmatrix},\quad \rho_i \in K.$

(ii) For all $i \neq j$, $F_{ij} \neq 0$ implies

(1) $u(i) = u(j)$ and $F_{ij} = 0$.

(2) For each path $x$ from $j$ to $h$ in $Z$ there is a path $y$ from $j$ to $k$ such that $u(y) = u(x)$.

(3) For each path $y$ from $k$ to $j$ in $Z$ there is a path $x$ from $h$ to $i$ such that $u(y) = u(x)$.

(iii) If $v, w: L \to Z$ are two walks in $Z$ maximal with respect to $u \circ v = u \circ w$ and $v \neq w$, we have:

(1) $F_{v(k),w(k)} = F_{v(k+1),w(k+1)}$ if $v(\lambda_k) \neq \xi_r \neq w(\lambda_k)$.

(2) $F_{v(k),w(k)} \cdot J_n(\kappa) = F_{v(k+1),w(k+1)}$ if $w(\lambda_k) = \xi_r$ and $v(\lambda_k) = 1,$

$F_{v(k),w(k)} = F_{v(k+1),w(k+1)} \cdot J_n(\kappa)$ if $w(\lambda_k) = \xi_r$ and $v(\lambda_k) = -1.$
(3) \( J_n(\kappa) \cdot F_{v(k),w(k)} = F_{v(k+1),w(k+1)} \) if \( v(\lambda_k) = \zeta, \) and \( e(\lambda_k) = 1, \)

\[ F_{v(k),w(k)} = J_n(\kappa) F_{v(k+1),w(k+1)} \] if \( v(\lambda_k) = \zeta, \) and \( e(\lambda_k) = -1. \)

Moreover, if \( f \) is nilpotent, \( p_1 = 0 \) in (i).

Proof. For each arrow \( \alpha \in \mathcal{Q}_1 \) which belongs to the image of \( u, \) the linear map \( M_\alpha \) is given with respect to the above decomposition of \( M \) by a block matrix \( S(\alpha) = (S_{ij}) \) where each \( S_{ij} \) is an \( n \times n \) matrix of \( K \) satisfying

\[
S_{ij} = E_n \quad \text{if there is an arrow } \xi: j \to i, \xi \neq \zeta, \text{ in } Z \text{ with } u(\xi) = \alpha
\]

\[
= J_n(\kappa) \quad \text{if } i = 1, j = r \text{ and } u(\zeta, r) = \alpha
\]

\[
= 0 \quad \text{else.}
\]

For each \( \alpha \in \mathcal{Q}_1 \) which does not belong to \( u(Z) \) we have \( S(\alpha) = 0. \) The sum \( S \) of the \( S(\alpha), \alpha \in \mathcal{Q}_1, \) is a block matrix \( S = (\Sigma_{ij}) \) reflecting the quiver \( Z \) in the sense that

\[
\Sigma_{ij} = E_n \quad \text{if there is an arrow } \xi: j \to i, \xi \neq \zeta, \text{ in } Z
\]

\[
= J_n(\kappa) \quad \text{if } i = 1, j = r
\]

\[
= 0 \quad \text{else.}
\]

Assume first that \( F = (F_{ij}) \) is the matrix of an endomorphism \( f \) of \( M. \) Then \( F_{ij} \neq 0 \) implies \( u(i) = u(j) \) and \( F \) commutes with all matrices \( S(\alpha), \alpha \in \mathcal{Q}_1. \)

Especially \( F \) commutes with \( S = (\Sigma_{ij}) \) and we get the identities

\[
J_n(\kappa) \cdot F_{rr} = F_{11} \cdot J_n(\kappa), \quad F_{11} = F_{22} = \cdots = F_{rr}.
\]

Hence \( J_n(\kappa) \cdot F_{11} = F_{11} \cdot J_n(\kappa) \) and (i) is proved. The property that \( F \) commutes with all \( S(\alpha) \) implies the conditions (ii)(2), (3) and (iii).

If now \( F_{ij} \) and \( F_{ji} \) are both nonzero we can apply (ii)(2) and (3) in both directions. This means that there is an automorphism \( \sigma \) of \( Z \) with \( u = u \circ \sigma \) and \( \sigma(i) = j. \) But by the definition (2.2) of a primitive \( V \)-sequence \( \sigma \) has to be the identity and \( i = j. \) The last assertion follows with the arguments used in the corresponding part of the proof of Lemma 4.2.

Assume now conversely that \( F = (F_{ij}) \) satisfies the conditions (i), (ii) and (iii) of the Lemma. Then by (ii)(1) \( f(M_a) \subseteq M_a \) for each vertex \( a \in Q_0. \) Moreover by (i), (ii)(2), (3) and (iii) \( f \) commutes with the linear maps \( M_a, \)

\[
a \in Q_1. \text{ Hence } f \text{ is an endomorphism of } M.
\]

We can now enter the proof of our Theorem 4.1. We look first at the sequence

\[
0 \to M(v) \xrightarrow{(f_1, f_2)} M(v_1) \oplus M(v_2) \xrightarrow{(g_1, g_2)} M(v) \to 0
\]
which is obviously exact but does not split. We have to show that each nilpotent endomorphism $f$ of $M(v)$ factors over $(f_1, f_2)$. For this we distinguish several cases.

Recall that $v: L \to Q$, where

$$L = 1 \leftarrow \cdots \leftarrow s_1 \to \cdots \to s_2 \leftarrow \cdots \leftarrow s_m \to \cdots \to r + 1,$$

$$1 \leq s_1 < s_2 < \cdots < s_m \leq r + 1,$$

and

$$M(v) = \bigoplus_{i=1}^{r+1} V_i, \quad V_i = K \text{ for all } i.$$

**Case 1.** $s_1 \neq 1$ and $s_m \neq r + 1$.

In this case we obtain from Lemma 4.2 for the matrix $F = (f_{iv})$ of $f$ the properties

$$f_{ii} = 0 \quad \text{for } i \neq r + 1,$$

$$f_{ir+1} = 0 \quad \text{for } i \neq 1.$$

Hence, again using Lemma 4.2, $f_{ii} = 0$ for all $i$ or $f_{ir+1} = 0$ for all $i$. Consequently $\ker f$ contains $V_1$ or $V_{r+1}$ and $f$ factors over $f_2$ or over $f_1$, respectively.

**Case 2.** $s_1 \neq 1$, $s_m = r + 1$.

In this case we get immediately from Lemma 4.2 $f_{ii} = 0$ for all $i$. Hence $V_1 \subseteq \ker f$ and $f$ factors over $f_2$.

**Case 3.** $s_1 = 1$, $s_m \neq r + 1$.

This case is symmetric to case 2.

**Case 4.** $s_1 = 1$, $s_m = r + 1$.

This case is dual to case 1.

Next we look at the sequence

$$0 \to M_n \xrightarrow{(g_{n-1}, f_n)} M_{n-1} \oplus M_{n+1} \xrightarrow{(f_{n-1})} M_n \to 0,$$

where $M_t := M(u, t, \kappa)$, $t \in \mathbb{N}$, for some fixed primitive $V$-sequence $u: Z \to Q$ and some fixed $\kappa \in K^*$. Again it is obvious that this sequence is exact and not split. Hence it remains to prove that each nilpotent endomorphism $f$ of $M_n$ factors over $(g_{n-1}, f_n)$. But this follows immediately from

(4.4) **Lemma.** Each nilpotent endomorphism $f$ of $M_n$ can be extended to an endomorphism $\overline{f}$ of $M_{n+1}$ such that $\operatorname{im} \overline{f}$ is contained in $M_n$. Here we identify $M_n$ with its image in $M_{n+1}$ under $f_n$. 
Proof. Let $F=(F_{ij})$ be the matrix of $f$ as before. We proceed to show that we can extend the $n \times n$-matrices $F_{ij}$ to $(n+1) \times (n+1)$ matrices $\tilde{F}_{ij}$ of the form

$$\tilde{F}_{ij} = \begin{pmatrix} F_{ij} & \ast \\ \vdots & \ast \\ 0 & \ldots & 0 \end{pmatrix},$$

such that the block matrix $\tilde{F}=(\tilde{F}_{ij})$ satisfies conditions (i), (ii) and (iii) of Lemma 4.3.

Let $W$ be the set of all pairs $(u, w)$ of walks $v, w: L \rightarrow Z$ of length $t = t(v, w)$ as in Lemma 4.3 (iii). We observe that by the definition of primitive $V$-sequences the walks $v$ and $w$ are not surjective maps, because otherwise they define an automorphism $\sigma$ of $Z$ with $u = u \circ \sigma$, which is impossible! Especially there is at most one arrow $\lambda \in L_1$ with $w(\lambda) = \zeta_r$. The relation on $W$ defined by

$$(v, w) \sim (v', w')$$

if and only if there is an isomorphism $\sigma: L' \rightarrow L$

with $v' = v \circ \sigma$, $w' = w \circ \sigma$

is an equivalence relation and we choose from each equivalence class one representative $(v, w)$ satisfying the condition

$$e(\lambda) = -1 \quad \text{if} \quad w(\lambda) = \zeta_r.$$

Then for each pair $(i, j), 1 \leq i \neq j \leq r$, with $F_{ij} \neq 0$, there is exactly one representative $(v, w)$ such that

$$i = v(k), \quad j = w(k) \quad \text{for some} \quad k \in L_0.$$

For such a pair $(i, j)$ we can now define $\tilde{F}_{ij}$ along the following distinguished cases:

Case 1. $w(\lambda) \neq \zeta_r$ for all $\lambda \in L_1$.

In this case we put

$$\tilde{F}_{ij} = \begin{pmatrix} F_{ij} & 0 \\ \vdots & \vdots \\ 0 & \ldots & 0 \end{pmatrix} \quad \text{for all} \quad (i, j) = (v(k), w(k)), \quad k \in L_0.$$

Case 2. $w(\lambda_s) = \zeta_r$ for some $s \in L_0$ and $v(\lambda) \neq \zeta_r$ for all $\lambda \in L_1$. 
In this case $\varepsilon(\lambda_s) = -1$ by our choice of the pairs $(v, w)$ and we put for $(i, j) = (v(k), w(k)), k \in L_0$,

$$\tilde{F}_{ij} = \begin{pmatrix} F_{ij} & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \quad \text{if } k \leq s$$

$$\tilde{F}_{ij} = \begin{pmatrix} F_{ij} & x_1 \\ \vdots & \vdots \\ x_n & 0 \end{pmatrix} \quad \text{if } k > s$$

where $(x_1, x_2, \ldots, x_n)$ is the last column of $F_{v(s), w(s)}$.

**Case 3.** $w(\lambda_s) = \zeta_r = v(\lambda_p)$ for $s, p \in L_0$.

We claim that in this case $p \neq s$ and $\varepsilon(\lambda_p) = \varepsilon(\lambda_s) = -1$. Indeed, if $\varepsilon(\lambda_p) = -\varepsilon(\lambda_s)$ we get a reflection $\sigma$ of the subquiver $v(L) \cap w(L) = L'$ of $Z$ such that $u|_{L'} = u|_{L'} \circ \sigma$ which is impossible as we remarked in 2.2. Especially if $p = s$ we get $v = w$. If now $p < s$, we define $\tilde{F}_{ij}$ as in the case 2. But for $p > s$ we define for $(i, j) = (v(k), w(k)), k \in L_0$,

$$\tilde{F}_{ij} = \begin{pmatrix} F_{ij} & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \quad \text{if } k \leq s$$

$$\tilde{F}_{ij} = \begin{pmatrix} F_{ij} & y_1 \\ \vdots & \vdots \\ y_n & 0 \end{pmatrix} \quad \text{if } s < k \leq p$$

$$\tilde{F}_{ij} = \begin{pmatrix} F_{ij} & y_1 \\ \vdots & \vdots \\ y_n & 0 \end{pmatrix} \quad \text{if } p < k$$

where $(x_1, x_2, \ldots, x_n)$ is the last column of $F_{v(s), w(s)}$ and where $y_1, y_2, \ldots, y_n$ are determined recursively by

$$x_m = \kappa \circ y_m + y_{m+1}, \quad 1 \leq m \leq n,$$

$$y_{n+1} = 0$$
Finally, we define the matrices $F_i$ by

$$F_{11} = F_{22} = \cdots = F_{rr} = \begin{pmatrix} 0 & \rho_2 & \cdots & \rho_n & 0 \\
 & 0 & \cdots & \cdots & \rho_n \\
 & & \ddots & \ddots & \ddots \\
 & & & 0 & \rho_n \\
 & & & & 0 \
\end{pmatrix}$$

where

$$F_i = \begin{pmatrix} 0 & \rho_2 & \cdots & \rho_n \\
 & 0 & \cdots & \cdots \\
 & & \ddots & \ddots \\
 & & & 0 
\end{pmatrix}$$

for all $i$.

Obviously our choice is arranged such that the matrix $F = (F_i)$ satisfies the conditions (i), (ii) and (iii) of Lemma 4.3, and $\text{im} F \subset M_{n+1}$ for the corresponding endomorphism of $M_{n+1}$. For the latter observe that the embedding $M = \bigoplus_{i=1}^n (K^n)_i \hookrightarrow M_{n+1} = \bigoplus_{i=1}^n (K^{n+1})_i$ is such that, for each $i$, $(K^n)_i$ is mapped identically into the first $n$ summands of $(K^{n+1})_i$.

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