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Polynomial invariants for a semisimple and cosemisimple Hopf algebra of finite dimension

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ABSTRACT

We introduce new polynomial invariants of a finite-dimensional semisimple and cosemisimple Hopf algebra A over a field $\boldsymbol{k}$ by using the braiding structures of A. The coefficients of polynomial invariants are integers if $\boldsymbol{k}$ is a finite Galois extension of $\mathbb{Q}$, and A is a scalar extension of some finite-dimensional semisimple Hopf algebra over $\mathbb{Q}$. Furthermore, we show that our polynomial invariants are indeed tensor invariants of the representation category of A, and recognize the difference between the representation category and the representation ring of A. Actually, by computing and comparing polynomial invariants, we find new examples of pairs of Hopf algebras whose representation rings are isomorphic, but whose representation categories are distinct.

1. Introduction

In representation theory of Hopf algebras over a field $\boldsymbol{k}$, it is a fundamental problem to know conditions for which the representation categories of two given Hopf algebras are equivalent as (abstract) $\boldsymbol{k}$-linear monoidal categories. A complete answer for this is given by Schauenburg [1,2]. He introduced the notion of bi-Galois extensions, and showed that the monoidal equivalences of comodule categories over Hopf algebras are classified by bi-Galois extensions of the base field $\boldsymbol{k}$. In the finite-dimensional case, they are also classified by cocycle deformations of Hopf algebras, which were introduced by Doi [3]. Many researchers have been successful in determining the bi-Galois objects and the cocycle deformations for various special families of Hopf algebras; see, for example, [4–7]. However, it is very difficult to do so in general.

In this paper we introduce a new family of invariants of a semisimple and cosemisimple Hopf algebra of finite dimension by using the braiding structures of it, and show that our invariants are useful for examining whether the representation categories of two such Hopf algebras are monoidal equivalent or not.

The basic idea of our method is to utilize quantum invariants of low-dimensional manifolds, which are topological invariants defined by using quantum groups, namely, Hopf algebras with braiding structures. In contrast to most current investigations on quantum invariants in which topological problems of low-dimensional manifolds are studied under a fixed Hopf algebra, in this research, we fix a framed knot or link, and study the representation categories of the Hopf algebras. In particular, in this paper, by use of quantum invariants of the unknot with $(+1)$-framing for a finite-dimensional semisimple and cosemisimple Hopf algebra A over $\boldsymbol{k}$, we introduce polynomials $P_A^{(d)}(x)$ $(d = 1, 2, \ldots)$ as invariants of A. For each positive integer d the polynomial $P_A^{(d)}(x)$ is defined by

$$P_A^{(d)}(x) = \prod_{i=1}^{t} \prod_{R:\text{braidings of } A} \left(x - \frac{\dim_R M_i}{\dim M_i} \right) \in \boldsymbol{k}[x],$$

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where $\{M_1, \ldots, M_t\}$ is a full set of non-isomorphic absolutely simple left A-modules of dimension d (so, $\dim M_i = d$ for all i), and $\underline{\dim}_R M_i \in \boldsymbol{k}$ is the quantum invariant of the unknot with $(+1)$-framing and colored by M_i. In algebraic language, $\underline{\dim}_R M_i$ is the category-theoretic rank of M_i in the left rigid braided monoidal category $({}_A\mathbb{M}^{\text{f.d.}}, c_R)$ [8], where ${}_A\mathbb{M}^{\text{f.d.}}$ is the monoidal category of finite-dimensional left A-modules and A-linear maps, and c_R is the braiding of ${}_A\mathbb{M}^{\text{f.d.}}$ determined by R.

Provided that the polynomial $P_A^{(d)}(x)$ is not a constant, all roots of $P_A^{(d)}(x)$ are n-th roots of unity for some positive integer n. Furthermore, the polynomial has an integer property in the following sense. All coefficients of the polynomial are integers if $\boldsymbol{k}$ is a finite Galois extension of the rational number field $\mathbb{Q}$, and A is a scalar extension of some finite-dimensional semisimple Hopf algebra over $\mathbb{Q}$.

It is more interesting to note that our polynomial invariants are indeed invariants of the representation categories of Hopf algebras, and recognize the difference between representation categories and representation rings of those algebras. In general, if the representation categories of two finite-dimensional semisimple Hopf algebras are equivalent as monoidal categories, then their representation rings are isomorphic. However, the converse is not true. For example, Tambara and Yamagami [9], and also Masuoka [7], proved that if the characteristic of $\boldsymbol{k}$ is 0 or $p > 2$, then three non-commutative and semisimple Hopf algebras $\boldsymbol{k}[D_8]$, $\boldsymbol{k}[Q_8]$, K_8 of dimension 8 have the same representation ring, but their representation categories are not mutually equivalent, where D_8 is the dihedral group of order 8, Q_8 is the quaternion group, and K_8 is the Kac–Paljutkin algebra [10,11]. This result is again confirmed by our polynomial invariants. Moreover, by computing and comparing polynomial invariants we find new examples of pairs of Hopf algebras, whose representation rings are the same, but whose representation categories are distinct.

This paper consists of six sections in total, and they are divided into two parts following this introduction: in Section 2 to Section 4 the definition and general properties of the polynomial invariants are discussed, and from Section 5 on, several concrete examples are computed, and applications are described. Detailed contents are as follows. In Section 2 we introduce the definition of our polynomial invariants of a semisimple and cosemisimple Hopf algebra of finite dimension. It is proved that the polynomial invariants are indeed invariants of the representation category of such a Hopf algebra. In Section 3 some basic properties of polynomial invariants are studied. It is shown that the polynomial invariants have a nice property such as the integer property. In Section 4, by dualizing the method of construction of our polynomial invariants, we state a formula to compute them in terms of coalgebraic and comodule-categorical language. In Section 5 we demonstrate computations of polynomial invariants for several Hopf algebras including the Hopf algebras $A_{Nn}^{+\lambda}$ (N is odd, and $\lambda = \pm 1$), which were introduced by Suzuki [12], and by comparing them we re-prove the result of Tambara, Yamagami and Masuoka as previously mentioned, and also find some pairs of Hopf algebras, whose representation rings are isomorphic, but whose representation categories are distinct. In the final section, as an Appendix, we determine the structures of representation rings of the Hopf algebras $A_{Nn}^{+\lambda}$, and determine when they are self-dual; this is used in Section 5.

Throughout this paper, we use the notation $\otimes$ instead of $\otimes_{\boldsymbol{k}}$, and denote by $\mathrm{ch}(\boldsymbol{k})$ the characteristic of the field $\boldsymbol{k}$. For a Hopf algebra A, denoted by Δ, ε and S are the coproduct, the counit, and the antipode of A, respectively, and $G(A)$ is the group consisting of the group-like elements in A, and A^{cop} is the resulting Hopf algebra obtained from A by replacing Δ by the opposite coproduct Δ^{cop}. We use the sigma notation, such as $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ for $x \in A$. We write ${}_A\mathbb{M}$ for the $\boldsymbol{k}$-linear monoidal category whose objects are left A-modules and morphisms are left A-linear maps, and write ${}^A\mathbb{M}$ for the $\boldsymbol{k}$-linear monoidal category whose objects are left A-comodules and morphisms are left A-colinear maps. For general references on Hopf algebras we refer to Abe's book [13], Montgomery's book [14] and Sweedler's book [15]. For general references on monoidal categories we refer to MacLane's book [16] and Joyal and Street's paper [17].

2. Definition of polynomial invariants

In this section we introduce a new family of invariants of a semisimple and cosemisimple Hopf algebra of finite dimension over an arbitrary field. They are given by polynomials derived from the quasitriangular structures of the Hopf algebra. By the method of construction of the polynomials they also become invariants under the monoidal equivalence of the representation categories of Hopf algebras.

Let us recall the definition of a quasitriangular Hopf algebra [18]. Let A be a Hopf algebra over a field $\boldsymbol{k}$, and $R \in A \otimes A$ be an invertible element. The pair (A, R) is said to be a *quasitriangular Hopf algebra*, and R is said to be a *universal R-matrix* of A, if the following three conditions are satisfied:

- $\Delta^{\text{cop}}(a) = R \cdot \Delta(a) \cdot R^{-1}$ for all $a \in A$,
- $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$,
- $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$.

Here $\Delta^{\text{cop}} = T \circ \Delta$, $T : A \otimes A \longrightarrow A \otimes A$, $T(a \otimes b) = b \otimes a$, and $R_{ij} \in A \otimes A \otimes A$ is given by $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, $R_{13} = (T \otimes \mathrm{id})(R_{23}) = (\mathrm{id} \otimes T)(R_{12})$.

If $R = \sum_i \alpha_i \otimes \beta_i$ is a universal R-matrix of A, then the element $u = \sum_i S(\beta_i)\alpha_i$ of A is invertible, and it has the following properties:

(i) $S^2(a) = uau^{-1}$ for all $a \in A$,
(ii) $S(u) = \sum_i \alpha_i S(\beta_i)$.

The above element u is called the *Drinfel'd element* associated to R. If A is semisimple and cosemisimple of finite dimension, then the Drinfel'd element u belongs to the center of A by property (i) and $S^2 = \mathrm{id}_A$ [19, Corollary 3.2(i)].

Let (A, R) be a quasitriangular Hopf algebra over a field $\boldsymbol{k}$ and u be the Drinfel'd element associated to R. For a finite-dimensional left A-module M, we denote by $\underline{\dim}_R M$ the trace of the left action of u on M, and call it the *R-dimension* of M. The R-dimension $\underline{\dim}_R M$ is a special case of the braided dimension of M in the left rigid braided monoidal category $({}_A\mathbb{M}^{\text{f.d.}}, c_R)$ (see Section 4 for the definition of braided dimensions).

To define polynomial invariants, we use the following result established by Etingof and Gelaki [19].

Theorem 2.1 (*Etingof and Gelaki*). *Let A be a cosemisimple Hopf algebra of finite dimension over a field $\boldsymbol{k}$. Then*

(1) *the set of universal R-matrices* $\underline{\mathrm{Braid}}(A)$ *is finite,*
(2) *provided that A is semisimple,* $(\dim M)1_{\boldsymbol{k}} \neq 0$ *for any absolutely simple left A-module M.*

Remark 2.2. The proof of Part (1) was given in [19, Corollary 1.5]. In an extra case such as characteristic 0 or positive characteristic with some additional assumptions, it was proved by Radford [20, Theorem 1]. The proof of Part (2) was given in [19, Corollary 3.2(ii)]. The Etingof and Gelaki proof is based on Larson's result [21, Theorem 2.8], which is the same as Part (2) with the assumption $S^2 = \mathrm{id}_A$.

Let A be a semisimple and cosemisimple Hopf algebra of finite dimension over a field $\boldsymbol{k}$. For a finite-dimensional left A-module M with $(\dim M)1_{\boldsymbol{k}} \neq 0$, we have a polynomial

$$P_{A,M}(x) := \prod_{R\in \underline{\mathrm{Braid}}(A)} \left(x - \frac{\underline{\dim}_R M}{\dim M} \right) \in \boldsymbol{k}[x].$$

Furthermore, for each positive integer d a polynomial $P_A^{(d)}(x)$ is defined by

$$P_A^{(d)}(x) := \prod_{i=1}^{t} P_{A,M_i}(x) \in \boldsymbol{k}[x],$$

where $\{M_1, \ldots, M_t\}$ is a full set of non-isomorphic absolutely simple left A-modules of dimension d. Here, if there is no absolutely simple left A-module of dimension d, then we set $P_A^{(d)}(x) = 1$.

Example 2.3. Let $G = C_m$ be the cyclic group of order m generated by g, and let $\boldsymbol{k}$ be a field whose characteristic does not divide m. Suppose that $\boldsymbol{k}$ contains a primitive m-th root of unity ω. Then any universal R-matrix of the group Hopf algebra $\boldsymbol{k}[C_m]$ is given by

$$R_d = \sum_{i,j=0}^{m-1} \omega^{dij} E_i \otimes E_j \quad (d = 0, 1, \ldots, m-1), \tag{2.1}$$

where $E_i = \frac{1}{m}\sum_{j=0}^{m-1} \omega^{-ij} g^j$ (see [22] for example). Let $M_j = \boldsymbol{k}$ be the (absolutely) simple left $\boldsymbol{k}[C_m]$-module equipped with the action $\chi_j(g^p) = \omega^{jp}$ $(p = 0, 1, \ldots, m-1)$. For each d and i, then $\underline{\dim}_{R_d} M_j = \omega^{-dj^2}$ since the Drinfel'd element u_d of R_d is given by $u_d = \sum_{i=0}^{m-1} \omega^{-di^2} E_i$. Thus we have

$$P^{(1)}_{\boldsymbol{k}[C_m]}(x) = \prod_{d,j=0}^{m-1} (x - \omega^{-dj^2}) = \prod_{j=0}^{m-1} (x^{\frac{m}{\gcd(j^2,m)}} - 1)^{\gcd(j^2,m)}.$$

Two Hopf algebras A and B over $\boldsymbol{k}$ are said to be *monoidally Morita equivalent* if the monoidal categories ${}_A\mathbb{M}$ and ${}_B\mathbb{M}$ are equivalent as $\boldsymbol{k}$-linear monoidal categories.

Lemma 2.4. *Let A and B be Hopf algebras of finite dimension over $\boldsymbol{k}$. If a $\boldsymbol{k}$-linear monoidal functor $F : {}_A\mathbb{M} \longrightarrow {}_B\mathbb{M}$ is an equivalence between monoidal categories, then* $\dim M = \dim F(M)$ *for any finite-dimensional left A-module M.*

Proof. For a left A-module M we have an A-module isomorphism

$$A \otimes M_0 \longrightarrow A \otimes M, \qquad a \otimes m \longmapsto \sum a_{(1)} \otimes a_{(2)} m,$$

where M_0 stands for the trivial A-module with underlying vector space M. Thus if M is finite-dimensional, we have an isomorphism

$$F(A) \otimes F(M) \cong F(A \otimes M) \cong F(A \otimes M_0) \cong F(A^{\oplus \dim M}) \cong F(A)^{\oplus \dim M}.$$

This implies that $\dim F(M) = \dim M$. □

Lemma 2.5. *Let A and B be Hopf algebras of finite dimension over $\boldsymbol{k}$. If a $\boldsymbol{k}$-linear monoidal functor $(F,\phi,\omega) : {}_A\mathbb{M} \longrightarrow {}_B\mathbb{M}$ is an equivalence between monoidal categories, then there is a bijection $\Phi : \underline{\mathrm{Braid}}(A) \longrightarrow \underline{\mathrm{Braid}}(B)$ such that for a finite-dimensional left A-module M and a universal R-matrix $R \in \underline{\mathrm{Braid}}(A)$,*

$$\underline{\dim}_R M = \underline{\dim}_{\Phi(R)} F(M).$$

Proof. Let $(G,\phi',\omega') : {}_B\mathbb{M} \longrightarrow {}_A\mathbb{M}$ be a quasi-inverse of (F,ϕ,ω). Then there are $\boldsymbol{k}$-linear monoidal natural transformations $\varphi : (F,\phi,\omega)\circ(G,\phi',\omega') \Longrightarrow 1_{{}_B\mathbb{M}}$ and $\psi : (G,\phi',\omega')\circ(F,\phi,\omega) \Longrightarrow 1_{{}_A\mathbb{M}}$, where 1_ν stands for the identity functor on $\nu = {}_A\mathbb{M}, {}_B\mathbb{M}$.

A universal R-matrix $R = \sum_i \alpha_i \otimes \beta_i$ of A defines a braiding $c = \{c_{M,N} : M\otimes N \longrightarrow N \otimes M\}_{M,N\in {}_A\mathbb{M}}$ consisting of A-linear isomorphisms

$$c_{M,N}(m\otimes n) = \sum_i \beta_i n \otimes \alpha_i m \quad (m\in M, n\in N).$$

The braiding c gives rise to a braiding c' of ${}_B\mathbb{M}$, which consists of B-linear isomorphisms $c'_{P,Q} : P\otimes Q \longrightarrow Q\otimes P$ $(P,Q\in {}_B\mathbb{M})$ such that the following diagram commutes.

$$\begin{array}{ccc}
P\otimes Q & \xrightarrow{\;c'_{P,Q}\;} & Q\otimes P \\
{\scriptstyle \varphi(P)\otimes\varphi(Q)}\uparrow & & \uparrow{\scriptstyle \varphi(Q)\otimes\varphi(P)} \\
FG(P)\otimes FG(Q) & & FG(Q)\otimes FG(P) \\
{\scriptstyle \phi_{G(P),G(Q)}}\downarrow & & \downarrow{\scriptstyle \phi_{G(Q),G(P)}} \\
F(G(P)\otimes G(Q)) & \xrightarrow{\;F(c_{G(P),G(Q)})\;} & F(G(Q)\otimes G(P))
\end{array}$$

Then $\Phi(R) := (T\circ c'_{B,B})(1)$ is a universal R-matrix of B, where $T : B\otimes B \longrightarrow B\otimes B$ is defined by $T(a\otimes b) = b\otimes a$ $(a,b\in B)$. It is easy to see from the definition that the map $\Phi : \underline{\mathrm{Braid}}(A) \longrightarrow \underline{\mathrm{Braid}}(B)$ defined as above is bijective.

Let M be a finite-dimensional left A-module, and e_M and n_M be the evaluation and coevaluation morphisms defined by

$$\begin{aligned}
&e_M : M^*\otimes M \longrightarrow \boldsymbol{k}, && e_M(f\otimes m) = f(m) \quad (f\in M^*, m\in M),\\
&n_M : \boldsymbol{k} \longrightarrow M\otimes M^*, && n_M(1) = \sum_i e_i \otimes e_i^* \quad (\text{the canonical element}).
\end{aligned}$$

Then $e_M\circ c_{M,M^*}\circ n_M = (\underline{\dim}_R M)\mathrm{id}_{\boldsymbol{k}}$. So, we may identify $\underline{\dim}_R M = e_M\circ c_{M,M^*}\circ n_M$. We set $\mathrm{e}'_{F(M)} := \omega^{-1}\circ F(e_M)\circ\phi_{M^*,M}$ and $n'_{F(M)} := \phi^{-1}_{M,M^*}\circ F(n_M)\circ\omega$. Then $(F(M^*), \mathrm{e}'_{F(M)}, n'_{F(M)})$ is a left dual for $F(M)$. Since the $\boldsymbol{k}$-linear monoidal functor (F,ϕ,ω) becomes a braided monoidal functor from $({}_A\mathbb{M}, c)$ to $({}_B\mathbb{M}, c')$, it follows that

$$\begin{aligned}
\underline{\dim}_{\Phi(R)} F(M) &= \mathrm{e}'_{F(M)}\circ c'_{F(M),F(M^*)}\circ n'_{F(M)}\\
&= \omega^{-1}\circ F(e_M\circ c_{M,M^*}\circ n_M)\circ\omega = (\underline{\dim}_R M)\omega^{-1}\circ F(\mathrm{id}_{\boldsymbol{k}})\circ\omega = \underline{\dim}_R M. \quad\square
\end{aligned}$$

Theorem 2.6. *Let A and B be semisimple and cosemisimple Hopf algebras of finite dimension over $\boldsymbol{k}$. If A and B are monoidally Morita equivalent, then $P_A^{(d)}(x) = P_B^{(d)}(x)$ for any positive integer d.*

Proof. Let $F : {}_A\mathbb{M} \longrightarrow {}_B\mathbb{M}$ be a $\boldsymbol{k}$-linear monoidal functor which gives an equivalence of monoidal categories, and let us consider the bijection $\Phi : \underline{\mathrm{Braid}}(A) \longrightarrow \underline{\mathrm{Braid}}(B)$ given as in the proof of Lemma 2.4.

Let M be an absolutely simple left A-module. Then $F(M)$ is also an absolutely simple left B-module, and by Lemmas 2.4 and 2.5 we have

$$P_{A,M}(x) = P_{B,F(M)}(x). \tag{2.2}$$

Let $\{M_1,\ldots,M_t\}$ be a full set of non-isomorphic absolutely simple left A-modules of dimension d. Then $\{F(M_1),\ldots,F(M_t)\}$ is also a full set of non-isomorphic absolutely simple left B-modules of dimension d. Applying Eq. (2.2) to $M = M_i$ $(i = 1,\ldots,t)$, and taking the product of them, we have $P_A^{(d)}(x) = P_B^{(d)}(x)$. $\square$

Remark 2.7. Our polynomial invariants are useful only if a semisimple and cosemisimple Hopf algebra has a quasitriangular structure. However, by considering the polynomial invariants of the Drinfel'd double of it we have monoidal invariants of the original (arbitrary) semisimple Hopf algebra of finite dimension.

3. Integer property of polynomial invariants

In this section we investigate basic properties of the polynomial invariants $P_A^{(d)}(x)$ $(d = 1, 2, \ldots)$ defined in Section 2. We prove that all coefficients of $P_A^{(d)}(x)$ are integers if $\boldsymbol{k}$ is a finite Galois extension of $\mathbb{Q}$, and A is a scalar extension of some finite-dimensional semisimple Hopf algebra over $\mathbb{Q}$.

First of all, we show that the coefficients of $P_A^{(d)}(x)$ lie in the integral closure of the prime ring of the base field of A. To do this we need the following lemma.

Lemma 3.1. *Let (A, R) be a quasitriangular Hopf algebra over $\boldsymbol{k}$ and u be the Drinfel'd element associated to R. If A is semisimple and cosemisimple, then $u^{(\dim A)^3} = 1$.*

Proof. Let us consider the subHopf algebras $B = \{(\alpha \otimes \mathrm{id})(R) \mid \alpha \in A^*\}$ and $H = \{(\mathrm{id} \otimes \alpha)(R) \mid \alpha \in A^*\}$ of A. By [23, Proposition 2], the Hopf algebra B is isomorphic to the Hopf algebra $H^{*\mathrm{cop}}$. Let $(D(H), \mathcal{R})$ be the Drinfel'd double of H. By [23, Theorem 2], there is a homomorphism $F : (D(H), \mathcal{R}) \longrightarrow (A, R)$ of quasitriangular Hopf algebras. It follows that the Drinfel'd element $\tilde{u}$ of $(D(H), \mathcal{R})$ satisfies $F(\tilde{u}) = u$. Since A is semisimple, the subHopf algebras H and $H^{*\mathrm{cop}} \cong B$ are also semisimple [24, Corollary 2.5]. Thus H is semisimple and cosemisimple. So, we have $\tilde{u}^{(\dim H)^3} = 1$ by [25, Theorem 2.5 & Theorem 4.3], and $u^{(\dim H)^3} = 1$. Since $\dim A$ is divided by $\dim H$ [23, Proposition 2], the equation $u^{(\dim A)^3} = 1$ is obtained. $\square$

For a field K, let Z_K denote the integral closure of the prime ring of K; that is, if the characteristic of K is 0, then Z_K is the ring of algebraic integers in K, and if the characteristic of K is $p > 0$, then Z_K is the algebraic closure of the prime field $\mathbb{F}_p$ in K.

Lemma 3.2. *Let (H, R) be a semisimple and cosemisimple quasitriangular Hopf algebra over a field K. If M is an absolutely simple left H-module, then (by Theorem 2.1 $(\dim M)1_K \neq 0$ and,)*

$$\left(\frac{\underline{\dim}_R M}{\dim M}\right)^{(\dim H)^3} = 1.$$

In particular, $\frac{\underline{\dim}_R M}{\dim M} \in Z_K$.

Proof. The Drinfel'd element u of (H, R) belongs to the center of H since H is semisimple and cosemisimple. Thus the left action $\underline{u}_M : M \longrightarrow M$ of u is a left H-endomorphism. Since M is absolutely simple, $\underline{u}_M$ is a scalar multiple of the identity morphism, so it can be written as $\underline{u}_M = \omega_M \mathrm{id}_M$ for some $\omega_M \in K$. Then by Lemma 3.1 we have $\dim M = \mathrm{Tr}(\underline{u}_M^{(\dim H)^3}) = \omega_M^{(\dim H)^3} \mathrm{Tr}(\mathrm{id}_M) = \omega_M^{(\dim H)^3} \dim M$. Thus $\omega_M^{(\dim H)^3} = 1$. This implies that $\omega_M = \frac{\underline{\dim}_R M}{\dim M}$ belongs to Z_K. $\square$

Form Lemma 3.2, we have the following immediately.

Proposition 3.3. *Let H be a semisimple and cosemisimple Hopf algebra of finite dimension over a field K. Then for any absolutely simple left H-module M, the coefficients of the polynomial $P_{H,M}(x)$ are in Z_K. Therefore, $P_H^{(d)}(x) \in Z_K[x]$ for any positive integer d.*

Now, we will examine relationship between polynomial invariants and Galois extensions of fields. Let $K/\boldsymbol{k}$ be a field extension, and let $\mathrm{Aut}(K/\boldsymbol{k})$ denote the automorphism group of $K/\boldsymbol{k}$. For a K-linear space M and $\sigma \in \mathrm{Aut}(K/\boldsymbol{k})$, a K-linear space ${}^{\sigma}M$ is defined as follows:

(i) ${}^{\sigma}M = M$ as additive groups,
(ii) the action $\star$ of K on ${}^{\sigma}M$ is given by

$$c \star m := \sigma(c) \cdot m \quad (c \in K, m \in M), \tag{3.1}$$

where $\cdot$ in the right-hand side stands for the original action of K on M.

For a K-linear map $f : M \longrightarrow N$ and $\sigma \in \mathrm{Aut}(K/\boldsymbol{k})$, we have

$$f(c \star m) = f(\sigma(c) \cdot m) = \sigma(c) \cdot f(m) = c \star f(m) \quad (c \in K, m \in M);$$

thus f can be regarded as a K-linear map from ${}^{\sigma}M$ to ${}^{\sigma}N$. We denote by ${}^{\sigma}f$ the K-linear map $f : {}^{\sigma}M \longrightarrow {}^{\sigma}N$.

The monoidal category ${}_K\mathbb{M}$ of K-linear spaces and K-linear maps has a canonical braiding, which is given by usual twist maps. For an automorphism $\sigma \in \mathrm{Aut}(K/\boldsymbol{k})$, the functor

$${}^{\sigma}F : {}_K\mathbb{M} \longrightarrow {}_K\mathbb{M}, \qquad M \longmapsto {}^{\sigma}M, \qquad f \longmapsto {}^{\sigma}f \tag{3.2}$$

gives a K-linear braided monoidal functor. Since ${}^{\sigma}F \circ {}^{\tau}F = {}^{\sigma\tau}F$ for all $\sigma, \tau \in \mathrm{Aut}(K/\boldsymbol{k})$, the functor ${}^{\sigma}F : {}_K\mathbb{M} \longrightarrow {}_K\mathbb{M}$ gives an isomorphism of K-linear braided monoidal categories. In general, a K-linear braided monoidal functor $F : {}_K\mathbb{M} \longrightarrow {}_K\mathbb{M}$ maps a Hopf algebra to a Hopf algebra. So, for a Hopf algebra H over K and an automorphism $\sigma \in \mathrm{Aut}(K/\boldsymbol{k})$, ${}^{\sigma}H$ is also a Hopf algebra over K. The Hopf algebra structure of ${}^{\sigma}H$ is the same as that of H with the exception that the action of K on ${}^{\sigma}H$ is given by (3.1), and the counit $\varepsilon_{{}^{\sigma}H}$ of ${}^{\sigma}H$ is given by $\varepsilon_{{}^{\sigma}H} = \sigma^{-1} \circ \varepsilon_H$.

Lemma 3.4. *Let $K/\boldsymbol{k}$ be a field extension, and H be a Hopf algebra over K. Let $R \in H \otimes_K H$ be a universal R-matrix of H, and $\sigma \in \mathrm{Aut}(K/\boldsymbol{k})$. Then*

(1) *R is also a universal R-matrix of ${}^{\sigma}H$. We write ${}^{\sigma}R$ for this universal R-matrix of ${}^{\sigma}H$.*
(2) *The Drinfel'd element of the quasitriangular Hopf algebra $({}^{\sigma}H, {}^{\sigma}R)$ coincides with the Drinfel'd element of (H, R).*
(3) *For a finite-dimensional left H-module M, we have $\underline{\dim}_{{}^{\sigma}R}{}^{\sigma}M = \sigma^{-1}(\underline{\dim}_R M)$.*

Proof. Parts (1) and (2) follow from the definition. We show Part (3). Let $u \in H$ be the Drinfel'd element of (H, R). By Part (2), u is also the Drinfel'd element of $({}^{\sigma}H, {}^{\sigma}R)$. Let $\{e_i\}_{i=1}^n$ be a basis of M over K, and write $u \cdot e_i = \sum_{j=1}^n a_{ji}e_j$ $(a_{ji} \in K)$. Then $u \cdot e_i = \sum_{j=1}^n \sigma^{-1}(a_{ji}) \star e_j$, and hence $\underline{\dim}_{{}^{\sigma}R}{}^{\sigma}M = \sum_{i=1}^n \sigma^{-1}(a_{ii}) = \sigma^{-1}(\sum_{i=1}^n a_{ii}) = \sigma^{-1}(\underline{\dim}_R M)$. □

For a field automorphism $\sigma : K \longrightarrow K$ and a polynomial $P(x) = c_0 + c_1x + \cdots + c_mx^m$ $(c_i \in K, i = 1, \ldots, m)$, we define $\sigma \cdot P(x) \in K[x]$ by $\sigma \cdot P(x) := \sigma(c_0) + \sigma(c_1)x + \cdots + \sigma(c_m)x^m$.

Lemma 3.5. *Let $K/\boldsymbol{k}$ be a field extension, and H be a semisimple and cosemisimple Hopf algebra over K of finite dimension. If M is a finite-dimensional left H-module such that $(\dim M)1_K \neq 0$, then for an automorphism $\sigma \in \mathrm{Aut}(K/\boldsymbol{k})$ we have $\sigma^{-1} \cdot P_{H,M}(x) = P_{{}^{\sigma}H,{}^{\sigma}M}(x)$.*

Proof. Setting $N = {}^{\sigma}M$, by Lemma 3.4(3) we have

$$\sigma^{-1} \cdot P_{H,M}(x) = \prod_{R \in \underline{\mathrm{Braid}}(H)} \left(x - \frac{\sigma^{-1}(\underline{\dim}_R M)}{\dim M}\right) = \prod_{R \in \underline{\mathrm{Braid}}(H)} \left(x - \frac{\underline{\dim}_{{}^{\sigma}R} N}{\dim N}\right) \underset{(*)}{=} P_{{}^{\sigma}H,N}(x).$$

Here, the last equation $(*)$ follows from the map $\underline{\mathrm{Braid}}(H) \longrightarrow \underline{\mathrm{Braid}}({}^{\sigma}H)$, $R \longmapsto {}^{\sigma}R$ being bijective by Lemma 3.4(1). □

Let A be a Hopf algebra over a field $\boldsymbol{k}$, and K be an extension field of $\boldsymbol{k}$. Then $A^K = A \otimes K$ becomes a Hopf algebra over K, and the automorphism group $\mathrm{Aut}(K/\boldsymbol{k})$ acts on A^K as follows:

$$\sigma \cdot (a \otimes c) = a \otimes \sigma(c) \quad (\sigma \in \mathrm{Aut}(K/\boldsymbol{k}), a \in A, c \in K). \tag{3.3}$$

We set $H = A^K$. Then for each $\sigma \in \mathrm{Aut}(K/\boldsymbol{k})$ the left action on H given by (3.3) defines a Hopf algebra isomorphism from H to ${}^{\sigma}H$, denoted by $\tilde{\sigma} : H \longrightarrow {}^{\sigma}H$. Furthermore, we see that, if R is a universal R-matrix of H, then $\tilde{\sigma}$ becomes a homomorphism of quasitriangular Hopf algebras from (H, R) to $({}^{\sigma}H, {}^{\sigma}R)$. Thus $\mathrm{Aut}(K/\boldsymbol{k})$ acts on $\underline{\mathrm{Braid}}(H)$ from the right by

$$R \in \underline{\mathrm{Braid}}(H) \longmapsto (\tilde{\sigma}^{-1} \otimes_K \tilde{\sigma}^{-1})({}^{\sigma}R) \in \underline{\mathrm{Braid}}(H).$$

Theorem 3.6. *Let $K/\boldsymbol{k}$ be a finite Galois extension of fields, and A be a semisimple and cosemisimple Hopf algebra over $\boldsymbol{k}$ of finite dimension. Then $P^{(d)}_{A^K}(x) \in (\boldsymbol{k} \cap Z_K)[x]$ for each positive integer d.*

Proof. First of all, let us check to see that the Hopf algebra $H = A^K$ is semisimple and cosemisimple. Since the Hopf algebra A is semisimple, it is separable (see [14, Corollary 2.2.2]). Thus H is a semisimple Hopf algebra over K of finite dimension. Applying the same argument to the dual Hopf algebra A^*, we see that H is a cosemisimple Hopf algebra over K. Eventually, we see that H is semisimple and cosemisimple.

Let $\mathrm{Irr}^{(d)}_0(H)$ denote the set of isomorphism classes $[M]$ of absolutely simple left H-modules M of dimension d, and set $t = \sharp\mathrm{Irr}^{(d)}_0(H)$. If $t = 0$, then $P^{(d)}_H(x) = 1$, and hence $P^{(d)}_H(x) \in (\boldsymbol{k} \cap Z_K)[x]$.

Hereinafter, we consider the case when $t > 0$. For an automorphism $\sigma \in \mathrm{Gal}(K/\boldsymbol{k})$, the map

$$\mathrm{Irr}^{(d)}_0(H) \longrightarrow \mathrm{Irr}^{(d)}_0({}^{\sigma}H), \quad [M] \longmapsto [{}^{\sigma}M]$$

is bijective. Here, by Lemma 3.5 we have $\sigma^{-1} \cdot P^{(d)}_H(x) = P^{(d)}_{{}^{\sigma}H}(x) = P^{(d)}_H(x)$, and we see that $P^{(d)}_H(x) \in \boldsymbol{k}[x]$. On the other hand, since $P^{(d)}_H(x) \in Z_K[x]$ by Proposition 3.3, it follows that $P^{(d)}_H(x) \in (\boldsymbol{k} \cap Z_K)[x]$. □

As applications of the above theorem we have two corollaries.

Corollary 3.7. *Let K be a finite Galois extension field of $\mathbb{Q}$, and A be a semisimple Hopf algebra over $\mathbb{Q}$ of finite dimension. Then $P^{(d)}_{A^K}(x) \in \mathbb{Z}[x]$ for any positive integer d, where $\mathbb{Z}$ denotes the rational integral ring.*

Proof. By [26], a finite-dimensional semisimple Hopf algebra over a field of characteristic 0 is cosemisimple. Thus the semisimple Hopf algebra A^K is cosemisimple. Since $\mathbb{Q} \cap Z_K = \mathbb{Z}$, by applying Theorem 3.6 we have $P^{(d)}_{A^K}(x) \in \mathbb{Z}[x]$. □

Corollary 3.8. *Let Γ be a finite group, and K be a finite Galois extension field of $\mathbb{Q}$. Then $P^{(d)}_{K[\Gamma]}(x) \in \mathbb{Z}[x]$ for any positive integer d.*

Proof. The group Hopf algebra $K[\Gamma]$ is isomorphic to the scalar extension of $\mathbb{Q}[\Gamma]$ by K. Since a group algebra over a field of characteristic 0 is semisimple, by Corollary 3.7 we have $P^{(d)}_{K[\Gamma]}(x) \in \mathbb{Z}[x]$ for any positive integer d. □

4. Dual formulas for polynomial invariants

In this section we give a formula to compute the polynomial invariants for a self-dual Hopf algebra of finite dimension in terms of the braidings of the dual Hopf algebra.

Let us recall the definition of a braiding of a Hopf algebra [3]. Let A be a Hopf algebra over a field $\boldsymbol{k}$, and let $\sigma : A\otimes A \longrightarrow \boldsymbol{k}$ be a $\boldsymbol{k}$-linear map that is invertible with respect to the convolution product. The pair (A, σ) is said to be a *braided Hopf algebra*, and σ is said to be a *braiding* of A, if the following conditions are satisfied: for all $x, y, z \in A$

(B1) $\sum \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)} = \sum \sigma(x_{(2)}, y_{(2)})y_{(1)}x_{(1)}$,
(B2) $\sigma(xy, z) = \sum \sigma(x, z_{(1)})\sigma(y, z_{(2)})$,
(B3) $\sigma(x, yz) = \sum \sigma(x_{(1)}, z)\sigma(x_{(2)}, y)$.

It is easy to see that any braiding σ of A satisfies

(B4) $\sigma(1_A, x) = \sigma(x, 1_A) = \varepsilon(x)$ for all $x \in A$.

Let (A, σ) be a braided Hopf algebra over $\boldsymbol{k}$. Then the braiding σ defines a braiding c of the monoidal category $\mathbb{M}^A$ consisting of right A-comodules and A-colinear maps as follows. For two right A-comodules V and W, a $\boldsymbol{k}$-linear isomorphism $c_{V,W} : V \otimes W \longrightarrow W \otimes V$ is defined by

$$c_{V,W}(v \otimes w) = \sum \sigma(v_{(1)}, w_{(1)})w_{(0)} \otimes v_{(0)} \quad (v \in V, w \in W),$$

where we use the notations $\rho_V(v) = \sum v_{(0)} \otimes v_{(1)}$ and $\rho_W(w) = \sum w_{(0)} \otimes w_{(1)}$ for the given right coactions ρ_V and ρ_W of V and W, respectively. From the axiom of braiding (B1)–(B3), we see that $c_{V,W}$ is a right A-comodule isomorphism, and the collection $c = \{c_{V,W} : V \otimes W \longrightarrow W \otimes V\}_{V,W\in\mathbb{M}^A}$ gives a braiding of $\mathbb{M}^A$.

Let us consider the element in the braided Hopf algebra which plays the role of the Drinfel'd element in a quasitriangular Hopf algebra.

Lemma 4.1 (*[3, Theorem 1.3] or [27, 3.3.2]*). *Let (A, σ) be a braided Hopf algebra over $\boldsymbol{k}$, and define $\mu \in A^*$ by*

$$\mu(a) = \sum \sigma(a_{(2)}, S(a_{(1)})), \quad a \in A. \tag{4.1}$$

Then μ is convolution-invertible, and the following equation holds for any element $a \in A$:

$$S^2(a) = \sum \mu(a_{(1)})\mu^{-1}(a_{(3)})a_{(2)}.$$

The $\boldsymbol{k}$-linear functional μ is called the (dual) Drinfel'd element of (A, σ).

Let $\mathcal{V} = (\mathcal{C}, \otimes, \mathbb{I}, a, r, l, c)$ be a left rigid braided monoidal category. For each object $X \in \mathcal{C}$ we choose a left dual X^* with an evaluation morphism $e_X : X^* \otimes X \longrightarrow \mathbb{I}$ and a coevaluation morphism $n_X : \mathbb{I} \longrightarrow X \otimes X^*$. Then for an endomorphism $f : X \longrightarrow X$ in $\mathcal{C}$, the braided trace of f in $\mathcal{V}$, denoted by $\underline{\mathrm{Tr}}_c f$, is defined by the composition

$$\mathbb{I} \xrightarrow{n_X} X \otimes X^* \xrightarrow{f\otimes \mathrm{id}} X \otimes X^* \xrightarrow{c_{X,X^*}} X^* \otimes X \xrightarrow{e_X} \mathbb{I}.$$

In particular, the braided trace of the identity morphism id_X is denoted by $\underline{\dim}_c X$, and is called the *braided dimension* of X in $\mathcal{V}$.

Applying this to the braided monoidal category $(\mathbb{M}^A, c)$ constructed from a braiding $\sigma \in (A\otimes A)^*$, we have the following.

Lemma 4.2. *Let (A, σ) be a braided Hopf algebra over $\boldsymbol{k}$, and c be the braiding of $\mathbb{M}^A$ constructed from σ. Then for a finite-dimensional right A-comodule V, the braided dimension $\underline{\dim}_c V$ is given by $\underline{\dim}_\sigma V = \mu(\chi_V)$, where μ is the Drinfel'd element of (A, σ), and χ_V is the character of the comodule V, which is defined by*

$$\chi_V := \sum_{i=1}^{n} (v_i^* \otimes \mathrm{id}_C)(\rho_V(v_i)) \in C$$

by use of dual bases $\{v_i\}_{i=1}^n$ and $\{v_i^*\}_{i=1}^n$.

Lemma 4.3. *Let A be a Hopf algebra over $\boldsymbol{k}$ of finite dimension, and $\iota : A^* \otimes A^* \longrightarrow (A \otimes A)^*$ be the canonical $\boldsymbol{k}$-linear isomorphism. Let σ be an element of $(A \otimes A)^*$ and set $R := \iota^{-1}(\sigma)$. Then*

(1) *σ is convolution-invertible if and only if R is invertible as an element of the algebra $A^* \otimes A^*$.*
(2) *σ is a braiding of A if and only if R is a universal R-matrix of the dual Hopf algebra A^*. In this case, the Drinfel'd element $\mu \in A^*$ of the quasitriangular Hopf algebra (A^*, R) is given by $\mu(a) = \sum \sigma(a_{(2)}, S(a_{(1)}))$ for all $a \in A$.*
(3) *For a finite-dimensional right A-comodule V, the equation $\underline{\dim}_R V = \underline{\dim}_\sigma V$ holds, where $\underline{\dim}_R V$ is the R-dimension of the left A^*-module V with the action $p \cdot v := \sum p(v_{(1)})v_{(0)}$ $(p \in A^*, v \in V)$.*

Let C be a coalgebra over a field $\boldsymbol{k}$. A right C-comodule V is said to be *absolutely simple* if the right C^K-comodule V^K is simple for an arbitrary field extension $K/\boldsymbol{k}$. This condition is equivalent to V being absolutely simple as a left C^*-module. We note that, if a right C-comodule is simple, then it is automatically finite-dimensional (see [14, Corollary 5.1.2]). So, there is a one-to-one correspondence between the absolutely simple right C-comodules and the absolutely simple left C^*-modules.

Let $\underline{\text{braid}}(A)$ denote the set of all braidings of a Hopf algebra A. Then by Part (2) of Lemma 4.3, the map $\underline{\text{braid}}(A) \longrightarrow \underline{\text{Braid}}(A^*)$ defined by $\sigma \longmapsto \iota^{-1}(\sigma)$ is bijective, and by Part (3) of the same lemma, the equation $\underline{\dim}_{\iota^{-1}(\sigma)} V = \underline{\dim}_{\sigma} V$ holds for a finite-dimensional right A-comodule V. Hence we have the following:

Lemma 4.4. *Let A be a semisimple and cosemisimple Hopf algebra over $\boldsymbol{k}$ of finite dimension.*

(1) *For an absolutely simple right A-comodule V,*

$$P_{A^*,V}(x) = \prod_{\sigma \in \underline{\text{braid}}(A)} \left(x - \frac{\underline{\dim}_{\sigma} V}{\dim V} \right),$$

where in the left-hand side $P_{A^*,V}(x)$ is the polynomial for V regarded as an left A^*-module by usual manner.

(2) *Let $\{V_1, \ldots, V_t\}$ be a full set of non-isomorphic absolutely simple right A-comodules of dimension d. Then*

$$P_{A^*}^{(d)}(x) = \prod_{i=1}^{t} P_{A^*,V_i}(x).$$

A Hopf algebra A over a field $\boldsymbol{k}$ of finite dimension is called *self-dual* if A is isomorphic to the dual Hopf algebra A^* as a Hopf algebra. Applying the above lemma to a self-dual Hopf algebra, we immediately obtain the following proposition.

Proposition 4.5. *Let A be a semisimple and cosemisimple Hopf algebra over a field $\boldsymbol{k}$ of finite dimension. If A is self-dual, then for a positive integer d*

$$P_A^{(d)}(x) = \prod_{i=1}^{t} \prod_{\sigma \in \underline{\text{braid}}(A)} \left(x - \frac{\underline{\dim}_{\sigma} V_i}{\dim V_i} \right),$$

where $\{V_1, \ldots, V_t\}$ is a full set of non-isomorphic absolutely simple right A-comodules of dimension d.

By using the above formula, we compute the self-dual Hopf algebras $A_{Nn}^{+\lambda}$ introduced by Suzuki [12] in the next section.

5. Examples

In this section we give several computational results of polynomial invariants of Hopf algebras. By comparing polynomial invariants one may find new examples of pairs of Hopf algebras such that their representation rings are isomorphic, but they are not monoidally Morita equivalent.

5.1. Eight-dimensional non-commutative semisimple Hopf algebras

By Masuoka [11], it is known that there are exactly three types of eight-dimensional non-commutative semisimple Hopf algebras over an algebraically closed field $\boldsymbol{k}$ of $\text{ch}(\boldsymbol{k}) \neq 2$. They are $\boldsymbol{k}[D_8]$, $\boldsymbol{k}[Q_8]$ and K_8, where D_8 and Q_8 are the dihedral group of order 8 and the quaternion group, respectively, and K_8 is the unique eight-dimensional semisimple Hopf algebra which is non-commutative and non-cocommutative, which is called the Kac–Paljutkin algebra [10]. Tambara and Yamagami [9] and also Masuoka [7] showed that their representation rings are isomorphic meanwhile their representation categories are not. In this subsection we derive this result by using our polynomial invariants. Throughout this subsection, we fix the following group presentation(s) of D_8 and Q_8:

$$D_8 = \langle s, t \mid s^4 = 1, t^2 = 1, st = ts^{-1} \rangle, \qquad Q_8 = \langle s, t \mid s^4 = 1, t^2 = s^2, st = ts^{-1} \rangle.$$

Let us start by determining the universal R-matrices of the group Hopf algebras $\boldsymbol{k}[D_8]$ and $\boldsymbol{k}[Q_8]$. For this the following proposition is useful.

Lemma 5.1. *Let G be a group, and $\boldsymbol{k}$ be a field. Then for a universal R-matrix R of $\boldsymbol{k}[G]$ there is a commutative and normal finite subgroup H such that $R \in \boldsymbol{k}[H] \otimes \boldsymbol{k}[H]$.*

Proof. We set $A = \boldsymbol{k}[G]$. By [23, Proposition 2(a)], $B = \{(\text{id} \otimes \alpha)(R) \mid \alpha \in A^*\}$, $H := \{(\alpha \otimes \text{id})(R) \mid \alpha \in A^*\}$ are finite-dimensional subHopf algebras of A. Since, for each $g \in G$, $\boldsymbol{k}g$ is a subcoalgebra of $A = \bigoplus_{g \in G} \boldsymbol{k}g$, by [15, Lemma 9.0.1(b)] the subcoalgebra B is written as $B = \bigoplus_{g \in G} B \cap \boldsymbol{k}g$. We set $K := B \cap G$. Then K is a subgroup of G, and $B \cap \boldsymbol{k}g \neq \{0\}$ if and only if $g \in K$. Thus $B = \bigoplus_{g \in K} B \cap \boldsymbol{k}g = \bigoplus_{g \in K} \boldsymbol{k}g = \boldsymbol{k}[K]$. Since B is finite-dimensional, K is a finite group. As a similar argument, we see that there is a finite subgroup L of G satisfying $H = \boldsymbol{k}[L]$. Since by [23, Proposition 2(c)] $\boldsymbol{k}[K]^{*\text{cop}} \cong \boldsymbol{k}[L]$ as Hopf algebras, L is commutative. Furthermore, L is a normal subgroup of G since $Lg \subset gH$, or equivalently $g^{-1}Lg \subset H = \boldsymbol{k}[L]$ by [23, Proposition 3]. Similarly, we see that K is a commutative and normal subgroup of G.

At this point, $R \in \boldsymbol{k}[K] \otimes \boldsymbol{k}[L] \subset \boldsymbol{k}[LK] \otimes \boldsymbol{k}[LK]$ is verified. To complete the proof it is sufficient to show that LK is a commutative and normal subgroup of G. Since L is normal, it follows immediately that LK is normal. To show that LK is commutative, we write R in the form $R = \sum_{k\in K} k \otimes X_k$ ($X_k \in \boldsymbol{k}[L]$). Since $\Delta^{\mathrm{cop}}(k') \cdot R = R \cdot \Delta(k')$ for all $k' \in K$, we have $X_{k'kk'^{-1}} = k'X_kk'^{-1}$ for all $k, k' \in K$. Since K is commutative, this condition is equivalent to $X_kk' = k'X_k$. It follows from $\boldsymbol{k}[L] = H = \mathrm{Span}\{X_k \mid k \in K\}$ that any element $l \in L$ is represented by a $\boldsymbol{k}$-linear combination of $\{X_k \mid k \in K\}$. Thus $lk' = k'l$ holds. This means that LK is commutative. □

For a cyclic group the universal R-matrices of the group Hopf algebra are given by (2.1) in Example 2.3 in Section 2. For a direct product of two cyclic groups the universal R-matrices of the group Hopf algebra are given as in the following lemma. The lemma can be verified by use of the same method as that used in the proof of Lemma 5.13 given later.

Lemma 5.2. *Let G be the direct product of the cyclic groups $C_m = \langle g\rangle$ and $C_n = \langle h\rangle$, and let ω be a primitive mn-th root of unity in a field $\boldsymbol{k}$ whose characteristic does not divide mn. We set $X(m, n) = \{d \in \{0, 1, \dots , m-1\} \mid dn \equiv 0 \pmod m\}$. Then any universal R-matrix of $\boldsymbol{k}[G]$ is given by the formula*

$$R^{\boldsymbol{k}[G]}_{pqrs} := \sum_{i,j=0}^{m-1}\sum_{k,l=0}^{n-1} \omega^{n(pij+rkj)+m(skl+qil)} E_{ik} \otimes E_{jl},$$

where $p \in X(m, m)$, $q \in X(n, m)$, $r \in X(m, n)$, $s \in X(n, n)$, and $E_{ik} = \frac{1}{mn}\sum_{j=0}^{m-1}\sum_{l=0}^{n-1} \omega^{-nij-mkl} g^jh^l$.

By using Lemmas 5.1 and 5.2, one can determine that the quasitriangular structures of $\boldsymbol{k}[D_8]$ and $\boldsymbol{k}[Q_8]$ as described in the following lemma [28].

Lemma 5.3. *Let $\boldsymbol{k}$ be a field of $\mathrm{ch}(\boldsymbol{k}) \neq 2$ that contains a primitive 4-th root of unity ζ.*

(1) *The universal R-matrices of $\boldsymbol{k}[D_8]$ are given by*

$$R^{\boldsymbol{k}[D_8]}_{d} := \frac{1}{4}\sum_{i,k=0}^{3} \zeta^{-ik} s^k \otimes s^{di} \quad (d = 0, 1, 2, 3),$$

$$R^{\boldsymbol{k}[D_8]}_{d+4} := \frac{1}{4}\sum_{i,j,k,l=0}^{1} (-1)^{-ij-kl} t^is^{2k} \otimes t^ls^{2(j+dl)} \quad (d = 0, 1),$$

$$R^{\boldsymbol{k}[D_8]}_{d+6} := \frac{1}{4}\sum_{i,j,k,l=0}^{1} (-1)^{-ij-kl} (ts)^is^{2k} \otimes (ts)^ls^{2(j+dl)} \quad (d = 0, 1).$$

The Drinfel'd element $u^{\boldsymbol{k}[D_8]}_d$ of $R^{\boldsymbol{k}[D_8]}_d$ is given by

$$u^{\boldsymbol{k}[D_8]}_d = \begin{cases} 1 & (d = 0, 4, 6), \\ \frac{1}{2}(1+\zeta^{-1}) + \frac{1}{2}(1+\zeta)s^2 & (d = 1), \\ s^2 & (d = 2, 5, 7), \\ \frac{1}{2}(1+\zeta) + \frac{1}{2}(1+\zeta^{-1})s^2 & (d = 3). \end{cases}$$

(2) *The universal R-matrices of $\boldsymbol{k}[Q_8]$ are given by*

$$R^{\boldsymbol{k}[Q_8]}_{d} := \frac{1}{4}\sum_{i,k=0}^{3} \zeta^{-ik} s^k \otimes s^{di} \quad (d = 0, 1, 2, 3),$$

$$R^{\boldsymbol{k}[Q_8]}_{d+4} := \frac{1}{4}\sum_{i,k=0}^{3} \zeta^{-ik} t^k \otimes t^{(2d+1)i} \quad (d = 0, 1),$$

$$R^{\boldsymbol{k}[Q_8]}_{d+6} := \frac{1}{4}\sum_{i,k=0}^{3} \zeta^{-ik} (ts)^k \otimes (ts)^{(2d+1)i} \quad (d = 0, 1).$$

The Drinfel'd element $u^{\boldsymbol{k}[Q_8]}_d$ of $R^{\boldsymbol{k}[Q_8]}_d$ is given by

$$u^{\boldsymbol{k}[Q_8]}_d = \begin{cases} 1 & (d = 0), \\ \frac{1}{2}(1+\zeta^{-1}) + \frac{1}{2}(1+\zeta)s^2 & (d = 1, 4, 6), \\ s^2 & (d = 2), \\ \frac{1}{2}(1+\zeta) + \frac{1}{2}(1+\zeta^{-1})s^2 & (d = 3, 5, 7). \end{cases}$$

Proof. In the case of either $\boldsymbol{k}[D_8]$ or $\boldsymbol{k}[Q_8]$, the proof of the lemma can be done by the same method. So, we will determine the universal R-matrices only in the case of $\boldsymbol{k}[Q_8]$ (for $\boldsymbol{k}[D_8]$ see also Proposition 5.15 given later).

It is easy to show that the maximal commutative and normal subgroups of Q_8 coincide with one of $H_1 = \langle s\rangle, H_2 = \langle t\rangle$ and $H_3 = \langle ts\rangle$. Therefore, by Lemma 5.1 any universal R-matrix of $\boldsymbol{k}[Q_8]$ is that of $\boldsymbol{k}[H_i]$ for some $i = 1, 2, 3$. Let R be a universal R-matrix of $\boldsymbol{k}[H_1]$. Then $\Delta^{\mathrm{cop}}(t)\cdot R = R\cdot\Delta(t)$ holds, and hence R is a universal R-matrix of $\boldsymbol{k}[Q_8]$. Similarly, we see that any universal R-matrix of $\boldsymbol{k}[H_i]$ for $i = 2, 3$ satisfies $\Delta^{\mathrm{cop}}(s)\cdot R = R\cdot\Delta(s)$, and hence it is a universal R-matrix of $\boldsymbol{k}[Q_8]$. □

Next, we describe the quasitriangular structures of the Kac–Paljutkin algebra K_8, which were determined by Suzuki [12]. As an algebra the Kac–Paljutkin algebra K_8 coincides with the group algebra $\boldsymbol{k}[D_8]$. Let us consider the primitive orthogonal idempotents $e_0 = \frac{1+s^2}{2}$, $e_1 = \frac{1-s^2}{2}$ in $K_8 = \boldsymbol{k}[D_8]$. Then the Hopf algebra structure of K_8 is described as follows [7]:

$$\begin{aligned}
&\Delta(t) = t\otimes e_0t + st\otimes e_1t, \qquad \Delta(s) = s\otimes e_0s + s^{-1}\otimes e_1s,\\
&\varepsilon(t) = 1, \qquad \varepsilon(s) = 1,\\
&S(t) = e_0t + e_1st, \qquad S(s) = e_0s^{-1} + e_1s.
\end{aligned}$$

We note that $\Delta(e_0) = e_0\otimes e_0 + e_1\otimes e_1$, $\Delta(e_1) = e_0\otimes e_1 + e_1\otimes e_0$, $\varepsilon(e_0) = 1$, $\varepsilon(e_1) = 0$, $S(e_0) = e_0$, $S(e_1) = e_1$, and therefore, $\boldsymbol{k}e_0 + \boldsymbol{k}e_1$ is a subHopf algebra of K_8 which is isomorphic to the group Hopf algebra $\boldsymbol{k}[C_2]$. Let $\zeta\in\boldsymbol{k}$ be a primitive 4-th root of unity. Then

$$g := \frac{1}{2}(1+\zeta)s + \frac{1}{2}(1-\zeta)s^{-1}, \qquad h := \frac{1}{2}(1-\zeta)s + \frac{1}{2}(1+\zeta)s^{-1}$$

satisfy $g^2 = h^2 = 1$, and $\boldsymbol{k}[\langle s\rangle] = \boldsymbol{k}1 + \boldsymbol{k}g + \boldsymbol{k}h + \boldsymbol{k}gh$ holds. Moreover, since g and h are group-like, the subHopf algebra $\boldsymbol{k}[\langle s\rangle]$ of K_8 is isomorphic to the group Hopf algebra $\boldsymbol{k}[C_2\times C_2]$.

Lemma 5.4. *Let $\boldsymbol{k}$ be a field of $\mathrm{ch}(\boldsymbol{k})\neq 2$ that contains a primitive 8-th root of unity ω. Then the universal R-matrices of K_8 are given as follows:*

(i) *universal R-matrices of $\boldsymbol{k}[\langle g,h\rangle]\cong\boldsymbol{k}[C_2\times C_2]$:*

$$R^{K_8}_{pq} := \frac{1}{4}\sum_{i,j,k,l=0}^{1}(-1)^{-(ij+kl)}g^ih^k\otimes g^{pj+(q+1)l}h^{qj+pl} \quad (p,q\in\{0,1\}),$$

(ii) *minimal universal R-matrices of K_8:*

$$R^{K_8}_{l} := \frac{1}{8}\sum_{i,j,p,q,r,s=0,1}\omega^{(2l+1)\frac{1-(-1)^i}{2}\cdot\frac{1-(-1)^j}{2}}(-1)^{jp+ir+(j(i+1)+lj+r+s)(li+p+q)}t^ig^ph^q\otimes t^jg^rh^s \quad (l=0,1,2,3).$$

The Drinfel'd elements $u^{K_8}_{p,q}$ and $u^{K_8}_{l}$ of $R^{K_8}_{p,q}$ and $R^{K_8}_{l}$, respectively, are given by

$$\begin{aligned}
u^{K_8}_{pq} &= \frac{1}{2}\sum_{i,l=0}^{1}(-1)^{(i+p)(l+p)}g^ih^l \quad (p,q\in\{0,1\}),\\
u^{K_8}_{l} &= \frac{\omega^{2l-1}}{2}(1-gh) + \frac{1}{2}(g+h) \quad (l=0,1,2,3).
\end{aligned}$$

Proof. Since the universal R-matrix $R^{K_8}_{pq}$ of $\boldsymbol{k}[\langle g,h\rangle]$ satisfies $\Delta^{\mathrm{cop}}(t)\cdot R^{K_8}_{pq} = R^{K_8}_{pq}\cdot\Delta(t)$, we see immediately that $R^{K_8}_{pq}$ is a universal R-matrix of K_8. It can be also verified straightforwardly that $R^{K_8}_{l}$ is indeed a universal R-matrix of K_8, although the proof is tedious. Since K_8 is isomorphic to the Suzuki Hopf algebra A^{+-}_{12} (for example see [29]), by [12, Proposition 3.10(ii)] the number of universal R-matrices of K_8 is 8. Thus there is no universal R-matrix of K_8 other than $R^{K_8}_{pq}$ $(p,q\in\{0,1\})$ and $R^{K_8}_{l}$ $(l=0,1,2,3)$. □

Let $\boldsymbol{k}$ be a field of $\mathrm{ch}(\boldsymbol{k})\neq 2$ that contains a primitive 4-th root of unity ζ. Then for any of the algebras $\boldsymbol{k}[D_8]$, $\boldsymbol{k}[Q_8]$, K_8 the number of isomorphism classes of (absolutely) simple modules is 5. They consist of four one-dimensional modules and one two-dimensional simple module. The one-dimensional modules of $\boldsymbol{k}[D_8]$, $\boldsymbol{k}[Q_8]$ and K_8 are given by $V_{ij} = \boldsymbol{k}$ $(i,j=0,1)$ equipped with the left actions ρ_{ij} defined by

$$\rho_{ij}(s) = (-1)^i, \quad \rho_{ij}(t) = (-1)^j.$$

For both $\boldsymbol{k}[D_8]$ and K_8 a two-dimensional simple module, which is unique up to isomorphism, is given by $V = \boldsymbol{k}\oplus\boldsymbol{k}$ equipped with the left action ρ defined by

$$\rho(s) = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \qquad \rho(t) = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},$$

and for $\boldsymbol{k}[Q_8]$ it is given by $V = \boldsymbol{k} \oplus \boldsymbol{k}$ equipped with the left action ρ defined by

$$\rho(s) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \qquad \rho(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Suppose that $\boldsymbol{k}$ contains a primitive 8-th root of unity ω. Then by the above two lemmas we see that the polynomial invariants of $\boldsymbol{k}[D_8]$, $\boldsymbol{k}[Q_8]$, K_8 are given by $P^{(1)}_{\boldsymbol{k}[D_8]}(x) = P^{(1)}_{\boldsymbol{k}[Q_8]}(x) = (x-1)^{32}$, $P^{(1)}_{K_8}(x) = (x-1)^{16}(x+1)^{16}$, and

$$P^{(1)}_{\boldsymbol{k}[D_8]}(x) = P_{\boldsymbol{k}[D_8],V}(x) = (x-1)^3(x+1)^3(x-\zeta)(x+\zeta) = x^8 - 2x^6 + 2x^2 - 1,$$
$$P^{(2)}_{\boldsymbol{k}[D_8]}(x) = P_{\boldsymbol{k}[Q_8],V}(x) = (x-1)(x+1)(x-\zeta)^3(x+\zeta)^3 = x^8 + 2x^6 - 2x^2 - 1,$$
$$P^{(2)}_{K_8}(x) = P_{K_8,V}(x) = \prod_{p,q=0,1}(x-(-1)^p) \cdot \prod_{l=0}^{3}(x-\omega^{2l-1}) = x^8 - 2x^6 + 2x^4 - 2x^2 + 1.$$

Since the polynomials $P^{(2)}_{\boldsymbol{k}[D_8]}(x)$, $P^{(2)}_{\boldsymbol{k}[Q_8]}(x)$, $P^{(2)}_{K_8}(x)$ are mutually distinct, we conclude that the Hopf algebras $\boldsymbol{k}[D_8]$, $\boldsymbol{k}[Q_8]$, K_8 are not mutually monoidally Morita equivalent by Theorem 2.6.

5.2. The Hopf algebra $A^{\nu\lambda}_{Nn}$

Suzuki introduced a family of cosemisimple Hopf algebras of finite dimension parameterized by ν, λ, N, n, where $\nu, \lambda = \pm 1$, and $N \geq 1$ and $n \geq 2$ are integers. This family includes not only the Kac–Paljutkin algebra K_8, but also Hopf algebras which can be regarded as a generalization of K_8. In this subsection we compute the polynomial invariants for Suzuki's Hopf algebras.

Let us recall the definition of Suzuki's Hopf algebras $A^{\nu\lambda}_{Nn}$ [12]. Let $\boldsymbol{k}$ be a field of $\mathrm{ch}(\boldsymbol{k}) \neq 2$, which contains a primitive $4nN$-th root of unity, and let C be the 2×2-matrix coalgebra over $\boldsymbol{k}$. By definition C has a basis $\{X_{11}, X_{12}, X_{21}, X_{22}\}$ which satisfies the equation $\Delta(X_{ij}) = X_{i1} \otimes X_{1j} + X_{i2} \otimes X_{2j}$ and $\varepsilon(X_{ij}) = \delta_{ij}$. Since C is a coalgebra, the tensor algebra $\mathcal{T}(C)$ of C has a bialgebra structure in a natural way. Let I be the coideal of $\mathcal{T}(C)$ defined by

$$I = \boldsymbol{k}(X_{11}^2 - X_{22}^2) + \boldsymbol{k}(X_{12}^2 - X_{21}^2) + \sum_{i-j \not\equiv l-m \pmod 2} \boldsymbol{k}(X_{ij}X_{lm}),$$

and consider the quotient bialgebra $B := \mathcal{T}(C)/\langle I \rangle$. We write x_{ij} for the image of X_{ij} under the natural projection $\mathcal{T}(C) \longrightarrow B$. We fix $N \geq 1$, $n \geq 2$ and $\nu, \lambda = \pm 1$, and use the following notations. For $m \geq 1$, we set

$$\chi_{11}^m := \overbrace{x_{11}x_{22}x_{11}\cdots\cdots}^{m}, \qquad \chi_{22}^m := \overbrace{x_{22}x_{11}x_{22}\cdots\cdots}^{m},$$
$$\chi_{12}^m := \overbrace{x_{12}x_{21}x_{12}\cdots\cdots}^{m}, \qquad \chi_{21}^m := \overbrace{x_{21}x_{12}x_{21}\cdots\cdots}^{m}.$$

Here,

$$\overbrace{x_{11}x_{22}x_{11}\cdots\cdots}^{m} = \begin{cases} (x_{11}x_{22})^{\frac{m}{2}} & \text{if } m \text{ is even}, \\ (x_{11}x_{22})^{\frac{m-1}{2}}x_{11} & \text{if } m \text{ is odd}, \end{cases}$$

and the other notation has the same meaning as that, too. Let $J^{\nu\lambda}_{Nn}$ be the following subspace of B:

$$J^{\nu\lambda}_{Nn} := \boldsymbol{k}(x_{11}^{2N} + \nu x_{12}^{2N} - 1) + \boldsymbol{k}(\chi_{11}^n - \chi_{22}^n) + \boldsymbol{k}(-\lambda\chi_{12}^n + \chi_{21}^n).$$

Since the subspace $J^{\nu\lambda}_{Nn}$ is a coideal of B, we obtain the quotient bialgebra $A^{\nu\lambda}_{Nn} := B/\langle J^{\nu\lambda}_{Nn}\rangle$. This bialgebra $A^{\nu\lambda}_{Nn}$ becomes a $4nN$-dimensional cosemisimple Hopf algebra over $\boldsymbol{k}$. For the image of x_{ij} under the natural projection $\pi : B \longrightarrow A^{\nu\lambda}_{Nn}$ we write x_{ij}, again. Then $A^{\nu\lambda}_{Nn}$ is equipped with the basis

$$\{x_{11}^s\chi_{22}^t, x_{12}^s\chi_{21}^t \mid 1 \leq s \leq 2N, 0 \leq t \leq n-1\}, \tag{5.1}$$

and the Hopf algebra structure is given by

$$\Delta(x_{ij}) = x_{i1} \otimes x_{1j} + x_{i2} \otimes x_{2j}, \qquad \varepsilon(x_{ij}) = \delta_{ij}, \qquad S(x_{ij}) = x_{ji}^{2N-1}.$$

So, the following equations hold:

$$\Delta(\chi_{ij}^m) = \chi_{i1}^m \otimes \chi_{1j}^m + \chi_{i2}^m \otimes \chi_{2j}^m \quad (m \geq 1, i, j = 1, 2).$$

Thus, for $s, t \geq 0$ with $s + t \geq 1$,

$$\begin{aligned} \Delta(x_{11}^s\chi_{22}^t) &= x_{11}^s\chi_{22}^t \otimes x_{11}^s\chi_{22}^t + x_{12}^s\chi_{21}^t \otimes x_{21}^s\chi_{12}^t, \\ \Delta(x_{12}^s\chi_{21}^t) &= x_{11}^s\chi_{22}^t \otimes x_{12}^s\chi_{21}^t + x_{12}^s\chi_{21}^t \otimes x_{22}^s\chi_{11}^t. \end{aligned} \tag{5.2}$$

Furthermore,

$$S(x_{11}^s\chi_{22}^t)=\begin{cases} x_{11}^{(2N-2)(t+s)+s}\chi_{11}^t & (s,t\text{ are even}),\\ x_{11}^{(2N-2)(t+s)+s}\chi_{22}^t & (s\text{ is odd, and }t\text{ is even}),\\ x_{11}^{(2N-2)(t+s)+s}\chi_{22}^t & (s\text{ is even, and }t\text{ is odd}),\\ x_{22}^{(2N-2)(t+s)+s}\chi_{11}^t & (s,t\text{ are odd}),\end{cases}$$
$$S(x_{12}^s\chi_{21}^t)=\begin{cases} x_{12}^{(2N-2)(t+s)+s}\chi_{21}^t & (s,t\text{ are even}),\\ x_{21}^{(2N-2)(t+s)+s}\chi_{12}^t & (s\text{ is odd, and }t\text{ is even}),\\ x_{12}^{(2N-2)(t+s)+s}\chi_{12}^t & (s\text{ is even, and }t\text{ is odd}),\\ x_{12}^{(2N-2)(t+s)+s}\chi_{21}^t & (s,t\text{ are odd}).\end{cases} \tag{5.3}$$

For each $i,j=1,2$, the square x_{ij}^2 is in the center of $A_{Nn}^{\nu\lambda}$, and the following equations hold in $A_{Nn}^{\nu\lambda}$.

- $x_{11}^2=x_{22}^2,\ x_{12}^2=x_{21}^2,\ x_{ij}x_{lm}=0\ (i-j\not\equiv l-m\ (\mathrm{mod}\ 2))$.
- If n is even, then $(x_{22}x_{11})^{\frac{n}{2}}=(x_{11}x_{22})^{\frac{n}{2}},\ (x_{21}x_{12})^{\frac{n}{2}}=\lambda(x_{12}x_{21})^{\frac{n}{2}}$.
- If n is odd, then $(x_{22}x_{11})^{\frac{n-1}{2}}x_{22}=(x_{11}x_{22})^{\frac{n-1}{2}}x_{11},\ (x_{21}x_{12})^{\frac{n-1}{2}}x_{21}=\lambda(x_{12}x_{21})^{\frac{n-1}{2}}x_{12}$.
- $x_{ii}^{2N+1}=x_{ii},\ x_{i\,i+1}^{2N+1}=\nu x_{i\,i+1}$.
- $x_{11}^{4N}+x_{12}^{4N}=1$.
- $(x_{11}x_{22})^n=x_{11}^{2n},\ (x_{21}x_{12})^n=\lambda x_{12}^{2n}$.

If $\mathrm{ch}(\boldsymbol{k})\nmid 2nN$, then $A_{Nn}^{\nu\lambda}$ is semisimple [12, Theorem 3.1(viii)]. Moreover, A_{12}^{+-} is isomorphic to the Kac–Paljutkin algebra K_8, and the Hopf algebras A_{1n}^{++} and A_{1n}^{+-} are isomorphic to the Hopf algebras $\mathcal{A}_{4n}$ and $\mathcal{B}_{4n}$, respectively, that were introduced by Masuoka [7,29].

Hereafter to the end of this subsection we suppose that $N\geq 1$, $n\geq 2$ and $\nu,\lambda=\pm 1$, and that $\boldsymbol{k}$ is a field of $\mathrm{ch}(\boldsymbol{k})\neq 2$, which contains a primitive $4nN$-th root of unity. The following proposition was proved by Suzuki.

Proposition 5.5 (*[12, Theorem 3.1]*).

(1) *The dimension of a simple subcoalgebra of $A_{Nn}^{\nu\lambda}$ is 1 or 4.*
(2) *The order of the group $G:=G(A_{Nn}^{\nu\lambda})$ is $4N$, and G is explicitly given by*

$$G=\{x_{11}^{2s}\pm x_{12}^{2s},\ x_{11}^{2s+1}\chi_{22}^{n-1}\pm\sqrt{\lambda}x_{12}^{2s+1}\chi_{21}^{n-1}\mid 1\leq s\leq N\}.$$

(3) *There are exactly $N(n-1)$ simple subcoalgebras of dimension 4, and they are given by*

$$C_{st}=\boldsymbol{k}x_{11}^{2s}\chi_{11}^t+\boldsymbol{k}x_{12}^{2s}\chi_{12}^t+\boldsymbol{k}x_{12}^{2s}\chi_{21}^t+\boldsymbol{k}x_{11}^{2s}\chi_{22}^t\quad (0\leq s\leq N-1,\ 1\leq t\leq n-1).$$

Therefore, the cosemisimple Hopf algebra $A_{Nn}^{\nu\lambda}$ is decomposed to the direct sum of simple subcoalgebras such as $A_{Nn}^{\nu\lambda}=\bigoplus_{g\in G}\boldsymbol{k}g\oplus\bigoplus_{\substack{0\leq s\leq N-1\\1\leq t\leq n-1}}C_{st}$.

Since $A_{Nn}^{\nu\lambda}$ is cosemisimple, a full set of non-isomorphic simple right $A_{Nn}^{\nu\lambda}$-comodules can be obtained by taking a simple right D-comodule for each simple subcoalgebra D of $A_{Nn}^{\nu\lambda}$ and by collecting them. So, we have:

Corollary 5.6. *The set $\{\boldsymbol{k}g\mid g\in G(A_{Nn}^{\nu\lambda})\}\cup\{\boldsymbol{k}x_{11}^{2s}\chi_{22}^t+\boldsymbol{k}x_{12}^{2s}\chi_{21}^t\mid 0\leq s\leq N-1,1\leq t\leq n-1\}$ is a full set of non-isomorphic (absolutely) simple right $A_{Nn}^{\nu\lambda}$-comodules, where the coactions of the comodules listed above are given by restrictions of the coproduct Δ.*

Let us explain on the braiding structures of $A_{Nn}^{\nu\lambda}$, which were determined by Suzuki [12]. Suppose that $\alpha,\beta\in\boldsymbol{k}$ satisfy $(\alpha\beta)^N=\nu$, $(\alpha\beta^{-1})^n=\lambda$. Then there is a braiding $\sigma_{\alpha\beta}$ of $A_{Nn}^{\nu\lambda}$ such that the values $\sigma_{\alpha\beta}(x_{ij},x_{kl})$ $(i,j,k,l=1,2)$ are given by the list below.

x	y			
	x_{11}	x_{12}	x_{21}	x_{22}
x_{11}	0	0	0	0
x_{12}	0	α	β	0
x_{21}	0	β	α	0
x_{22}	0	0	0	0

In fact, by using the braiding conditions (B2) and (B3), described in Section 4, repeatedly, we see that the values of $\sigma_{\alpha\beta}$ on the basis (5.1) are given as follows: for integers $s,s',t,t'\geq 0$ with $s+t\geq 1$ and $s'+t'\geq 1$, and $j'=1,2$,

$$\sigma_{\alpha\beta}(x_{11}^s\chi_{22}^t, x_{1j'}^{s'}\chi_{2,j'+1}^{t'}) = \begin{cases} \delta_{0,t'}\delta_{1+j',t}(\alpha\beta)^{\frac{t's+(s+t)s'}{2}}\beta^{tt'} & (s \text{ and } s' \text{ are even}), \\ \delta_{1,t'}\delta_{1+j',t}(\alpha\beta)^{\frac{t's+(s+t)s'-t}{2}}\alpha^{t(t'+1)} & (s \text{ is even, and } s' \text{ is odd}), \\ \delta_{0,t'}\delta_{j',t}(\alpha\beta)^{\frac{t's+(s+t)s'-t'}{2}}\alpha^{(t+1)t'} & (s \text{ is odd, and } s' \text{ is even}), \\ \delta_{1,t'}\delta_{j',t}(\alpha\beta)^{\frac{t's+(s+t)s'-(1+t+t')}{2}}\beta^{(t+1)(t'+1)} & (s \text{ and } s' \text{ are odd}), \end{cases}$$

$$\sigma_{\alpha\beta}(x_{12}^s\chi_{21}^t, x_{1j'}^{s'}\chi_{2,j'+1}^{t'}) = \begin{cases} \delta_{1,t'}\delta_{1+j',t}(\alpha\beta)^{\frac{t's+(s+t)s'}{2}}\alpha^{tt'} & (s \text{ and } s' \text{ are even}), \\ \delta_{0,t'}\delta_{1+j',t}(\alpha\beta)^{\frac{t's+(s+t)s'-t}{2}}\beta^{t(t'+1)} & (s \text{ is even, and } s' \text{ is odd}), \\ \delta_{1,t'}\delta_{j',t}(\alpha\beta)^{\frac{t's+(s+t)s'-t'}{2}}\beta^{(t+1)t'} & (s \text{ is odd, and } s' \text{ is even}), \\ \delta_{0,t'}\delta_{j',t}(\alpha\beta)^{\frac{t's+(s+t)s'-(1+t+t')}{2}}\alpha^{(t+1)(t'+1)} & (s \text{ and } s' \text{ are odd}), \end{cases}$$

where the indices of χ and δ are treated as modulo 2.

When $n = 2$, in addition to the above braidings $\sigma_{\alpha\beta}$, for $\gamma, \xi \in \boldsymbol{k}$ which satisfy $\gamma^2 = \xi^2$, $\gamma^{2N} = 1$ there is a braiding $\tau_{\gamma\xi}$ of $A_{N2}^{\nu\lambda}$ such that the values $\tau_{\gamma\xi}(x_{ij}, x_{kl})$ $(i, j, k, l = 1, 2)$ are given by the list below, where x_{ij} and x_{kl} correspond to a row and a column, respectively.

x	y			
	x_{11}	x_{12}	x_{21}	x_{22}
x_{11}	γ	0	0	ξ
x_{12}	0	0	0	0
x_{21}	0	0	0	0
x_{22}	$\lambda\xi$	0	0	γ

We see also that the values of $\tau_{\gamma\xi}$ on the basis (5.1) are given as follows: for integers $s, s', t, t' \geq 0$ with $s + t \geq 1$ and $s' + t' \geq 1$, and $j' = 1, 2$,

$$\tau_{\gamma\xi}(x_{11}^s\chi_{22}^t, x_{1j'}^{s'}\chi_{2,j'+1}^{t'}) = \begin{cases} \delta_{j'1}\gamma^{ss'}(\gamma\xi)^{\frac{tt'}{4}+\frac{st'}{2}}(\gamma\lambda\xi)^{\frac{tt'}{4}+\frac{ts'}{2}} & (t \text{ and } t' \text{ are even}), \\ \delta_{j'1}\gamma^{ss'}(\lambda\xi)^{s'}(\gamma\xi)^{\frac{(t-1)t'}{4}+\frac{st'}{2}}(\gamma\lambda\xi)^{\frac{(t+1)t'}{4}+\frac{(t-1)s'}{2}} & (t \text{ is odd, and } t' \text{ is even}), \\ \delta_{j'1}\gamma^{ss'}\xi^s(\gamma\xi)^{\frac{t(t'+1)}{4}+\frac{s(t'-1)}{2}}(\gamma\lambda\xi)^{\frac{t(t'-1)}{4}+\frac{ts'}{2}} & (t \text{ is even, and } t' \text{ is odd}), \\ \delta_{j'1}\gamma^{ss'+1}\xi^s(\lambda\xi)^{s'}(\gamma\xi)^{\frac{(t-1)(t'+1)}{4}+\frac{s(t'-1)}{2}}(\gamma\lambda\xi)^{\frac{(t+1)(t'-1)}{4}+\frac{(t-1)s'}{2}} & (t \text{ and } t' \text{ are odd}), \end{cases}$$

$$\tau_{\gamma\xi}(x_{12}^s\chi_{21}^t, x_{1j'}^{s'}\chi_{2,j'+1}^{t'}) = 0,$$

where the indices of χ and δ are treated as modulo 2.

Theorem 5.7 ([12, Proposition 3.10]). *If $n \geq 3$, then the braidings of $A_{Nn}^{\nu\lambda}$ are given by*

$$\{\sigma_{\alpha\beta} \mid \alpha, \beta \in \boldsymbol{k}, (\alpha\beta)^N = \nu, (\alpha\beta^{-1})^n = \lambda\},$$

and if $n = 2$, then they are given by

$$\{\sigma_{\alpha\beta} \mid \alpha, \beta \in \boldsymbol{k}, (\alpha\beta)^N = \nu, (\alpha\beta^{-1})^2 = \lambda\} \cup \{\tau_{\gamma\xi} \mid \gamma, \xi \in \boldsymbol{k}, \gamma^2 = \xi^2, \gamma^{2N} = 1\}.$$

From (4.1), (5.2) and (5.3) we have the following lemma.

Lemma 5.8. (1) *For $\alpha, \beta \in \boldsymbol{k}$ satisfying $(\alpha\beta)^N = \nu, (\alpha\beta^{-1})^n = \lambda$, the Drinfel'd element $\mu_{\alpha\beta}$ of the braided Hopf algebra $(A_{Nn}^{\nu\lambda}, \sigma_{\alpha\beta})$ is given as follows: for all integers $s, t \geq 0$ with $s + t \geq 1$,*

$$\mu_{\alpha\beta}(x_{11}^s\chi_{22}^t) = \begin{cases} \nu^t(\alpha\beta)^{\frac{-s^2}{2}-st-t^2}\alpha^{t^2} & (s \text{ is even}), \\ \nu^{t+1}(\alpha\beta)^{\frac{-1-s^2}{2}-t-st-t^2}\alpha^{(t+1)^2} & (s \text{ is odd}), \end{cases}$$

$$\mu_{\alpha\beta}(x_{12}^s\chi_{21}^t) = 0.$$

(2) *For $\gamma, \xi \in \boldsymbol{k}$ satisfying $\gamma^2 = \xi^2, \gamma^{2N} = 1$, the Drinfel'd element $\mu_{\gamma\xi}$ of the braided Hopf algebra $(A_{N2}^{\nu\lambda}, \tau_{\gamma\xi})$ is given as follows: for all integers $s, t \geq 0$ with $s + t \geq 1$,*

$$\mu_{\gamma\xi}(x_{11}^s\chi_{22}^t) = \begin{cases} \gamma^{-s^2-2st-t^2}\lambda^{\frac{t^2}{4}} & (s \text{ and } t \text{ are even}), \\ \gamma^{-s^2-2st-t^2}\lambda^{\frac{t^2-1}{4}} & (t \text{ is odd, and } s \text{ is even}), \\ \gamma^{-s^2-2st-t^2}\lambda^{\frac{t(t+2)}{4}} & (t \text{ is even, and } s \text{ is odd}), \\ \gamma^{-s^2-2st-t^2}\lambda^{\frac{(t+1)^2}{4}} & (s \text{ and } t \text{ are odd}), \end{cases}$$

$$\mu_{\gamma\xi}(x_{12}^s\chi_{21}^t) = 0.$$

By using Lemma 5.8, we know the braided dimensions of the simple right $A_{Nn}^{\nu\lambda}$-comodules.

Lemma 5.9. (1) *Let α, β be elements in $\boldsymbol{k}$ satisfying $(\alpha\beta)^N = \nu$ and $(\alpha\beta^{-1})^n = \lambda$.*

(i) *For an element $g \in G(A_{Nn}^{\nu\lambda})$ the character $\chi_g \in A_{Nn}^{\nu\lambda}$ of the simple right $A_{Nn}^{\nu\lambda}$-comodule $\boldsymbol{k}g$ is given by $\chi_g = g$, and the braided dimension $\underline{\dim}_{\sigma_{\alpha\beta}}\boldsymbol{k}g$ with respect to the braiding $\sigma_{\alpha\beta}$ is given by*

$$\underline{\dim}_{\sigma_{\alpha\beta}}\boldsymbol{k}g = \begin{cases} (\alpha\beta)^{-2s^2} & (g = x_{11}^{2s} \pm x_{12}^{2s}\ (1 \le s \le N)), \\ \nu^n(\alpha\beta)^{-2s^2-2sn-n^2}\alpha^{n^2} & (g = x_{11}^{2s+1}x_{22}^{n-1} \pm \sqrt{\lambda}x_{12}^{2s+1}x_{21}^{n-1}\ (1 \le s \le N)). \end{cases}$$

(ii) *For the simple right $A_{Nn}^{\nu\lambda}$-comodule $V_{st} = \boldsymbol{k}x_{11}^{2s}x_{22}^{t} + \boldsymbol{k}x_{12}^{2s}x_{21}^{t}$ $(0 \le s \le N-1,\ 1 \le t \le n-1)$, the character $\chi_{st} \in A_{Nn}^{\nu\lambda}$ of V_{st} is given by $\chi_{st} = x_{11}^{2s+1}x_{22}^{t-1} + x_{11}^{2s}x_{22}^{t}$, and the braided dimension $\underline{\dim}_{\sigma_{\alpha\beta}}V_{st}$ is given by $\underline{\dim}_{\sigma_{\alpha\beta}}V_{st} = 2\nu^t\alpha^{t^2}(\alpha\beta)^{-2s^2-2st-t^2}$.*

(2) *Let γ, ξ be elements in $\boldsymbol{k}$ such that $\gamma^2 = \xi^2$, $\gamma^{2N} = 1$.*

(i) *For an element $g \in G(A_{N2}^{\nu\lambda})$ the braided dimension $\underline{\dim}_{\tau_{\gamma\xi}}\boldsymbol{k}g$ is given by*

$$\underline{\dim}_{\tau_{\gamma\xi}}\boldsymbol{k}g = \begin{cases} \gamma^{-4s^2} & (g = x_{11}^{2s} \pm x_{12}^{2s}\ (1 \le s \le N)), \\ \gamma^{-4(s+1)^2}\lambda & (g = x_{11}^{2s+1}x_{22} \pm \sqrt{\lambda}x_{12}^{2s+1}x_{21}\ (1 \le s \le N)). \end{cases}$$

(ii) *For the simple right $A_{N2}^{\nu\lambda}$-comodule $V_{s1} = \boldsymbol{k}x_{11}^{2s}x_{22} + \boldsymbol{k}x_{12}^{2s}x_{21}$ $(0 \le s \le N-1)$, the braided dimension $\underline{\dim}_{\tau_{\gamma\xi}}V_{s1}$ is given by $\underline{\dim}_{\tau_{\gamma\xi}}V_{s1} = 2\gamma^{-(2s+1)^2}$.*

Let $\omega \in \boldsymbol{k}$ be a primitive $4nN$-th root of unity. The set $I_{\nu\lambda} = \{ (\alpha, \beta) \in \boldsymbol{k} \times \boldsymbol{k} \mid (\alpha\beta)^N = \nu, (\alpha\beta^{-1})^n = \lambda \}$ can be expressed as

$$I_{\nu\lambda} = \left\{ (\omega^{n(2i+\frac{1-\nu}{2})+N(2j+\frac{1-\lambda}{2})}, \omega^{n(2i+\frac{1-\nu}{2})-N(2j+\frac{1-\lambda}{2})}) \;\middle|\; \begin{array}{l} i = 0, 1, \ldots, N-1, \\ j = 0, 1, \ldots, n-1 \end{array} \right\}$$
$$\cup \left\{ (-\omega^{n(2i+\frac{1-\nu}{2})+N(2j+\frac{1-\lambda}{2})}, -\omega^{n(2i+\frac{1-\nu}{2})-N(2j+\frac{1-\lambda}{2})}) \;\middle|\; \begin{array}{l} i = 0, 1, \ldots, N-1, \\ j = 0, 1, \ldots, n-1 \end{array} \right\}.$$

Similarly, the set $J = \{ (\gamma, \xi) \in \boldsymbol{k} \times \boldsymbol{k} \mid \gamma^2 = \xi^2, \gamma^{2N} = 1 \}$ can be expressed as

$$J = \{(\omega^{4i}, \omega^{4i}) \mid i = 0, 1, \ldots, 2N-1\} \cup \{(\omega^{4i}, -\omega^{4i}) \mid i = 0, 1, \ldots, 2N-1 \}.$$

We put $A = A_{Nn}^{\nu\lambda}$, and compute the polynomial $P_{A^*,V}(x)$ for an absolutely simple right A-comodule V by using Lemmas 4.4 and 5.9.

Let g be an element of $G(A)$. In the case when $g = x_{11}^{2s} \pm x_{12}^{2s}$, if $n \ge 3$, then by Lemma 5.9(1)(i)

$$P_{A^*,\boldsymbol{k}g}(x) = \prod_{(\alpha,\beta)\in I_{\nu\lambda}} (x - (\alpha\beta)^{-2s^2}) = \prod_{i=0}^{N-1} (x - \omega^{-4n(2i+\frac{1-\nu}{2})s^2})^{2n},$$

and if $n = 2$, then by Lemma 5.9(2)(i)

$$P_{A^*,\boldsymbol{k}g}(x) = \prod_{(\alpha,\beta)\in I_{\nu\lambda}} (x - (\alpha\beta)^{-2s^2}) \cdot \prod_{(\gamma,\xi)\in J} (x - \gamma^{-4s^2}) = \prod_{i=0}^{N-1} (x - \omega^{-8(2i+\frac{1-\nu}{2})s^2})^4 \cdot \prod_{i=0}^{N-1} (x - \omega^{-16is^2})^4.$$

In the case when $g = x_{11}^{2s+1}x_{22}^{n-1} \pm \sqrt{\lambda}x_{12}^{2s+1}x_{21}^{n-1}$, if $n \ge 3$, then by Lemma 5.9(1)(i)

$$\begin{aligned} P_{A^*,\boldsymbol{k}g}(x) &= \prod_{(\alpha,\beta)\in I_{\nu\lambda}} (x - \nu^n(\alpha\beta)^{-2s^2-2sn-n^2}\alpha^{n^2}) \\ &= \begin{cases} \prod_{i=0}^{N-1} (x^2 - \omega^{-2n(2s+n)^2(2i+\frac{1-\nu}{2})}(-1)^{\frac{1-\lambda}{2}})^n & (n \text{ is odd}), \\ \prod_{i=0}^{N-1} (x - \omega^{-n(2s+n)^2(2i+\frac{1-\nu}{2})}(-1)^{\frac{n}{2}\frac{1-\lambda}{2}})^{2n} & (n \ge 4 \text{ is even}), \end{cases} \end{aligned}$$

and if $n = 2$, then by Lemma 5.9(2)(i)

$$\begin{aligned} P_{A^*,\boldsymbol{k}g}(x) &= \prod_{(\alpha,\beta)\in I_{\nu\lambda}} (x - (\alpha\beta)^{-2s^2-4s-4}\alpha^4) \cdot \prod_{(\gamma,\xi)\in J} (x - \gamma^{-4(s+1)^2}\lambda) \\ &= \prod_{i=0}^{N-1} (x - \omega^{-4(s+1)^2(2i+\frac{1-\nu}{2})}(-1)^{\frac{1-\lambda}{2}})^4 \cdot \prod_{i=0}^{N-1} (x - \omega^{-16i(s+1)^2}\lambda)^4. \end{aligned}$$

For the simple right A-comodule $V_{st} = \boldsymbol{k}x_{11}^{2s}x_{22}^{t} + \boldsymbol{k}x_{12}^{2s}x_{21}^{t}$ $(0 \le s \le N-1, 1 \le t \le n-1)$, the polynomial $P_{A^*,V_{st}}(x)$ is given as follows. If $n \ge 3$, then by Lemma 5.9(1)(ii),

$$P_{A^*,V_{st}}(x) = \prod_{(\alpha,\beta)\in I_{\nu\lambda}} (x - \nu^t\alpha^{t^2}(\alpha\beta)^{-2s^2-2st-t^2})$$
$$= \begin{cases} \prod_{i=0}^{N-1}\prod_{j=0}^{n-1}(x^2 - \omega^{-2n(2s+t)^2(2i+\frac{1-\nu}{2})-2Nt^2(2j+\frac{\lambda-1}{2})}) & (t \text{ is odd}) \\ \prod_{i=0}^{N-1}\prod_{j=0}^{n-1}(x - \omega^{-n(2s+t)^2(2i+\frac{1-\nu}{2})-Nt^2(2j+\frac{\lambda-1}{2})})^2 & (t \text{ is even}), \end{cases}$$

and if $n = 2$, then by Lemma 5.9(2)(ii),

$$P_{A^*,V_{s1}}(x) = \prod_{(\alpha,\beta)\in I_{\nu\lambda}} (x - \nu\alpha(\alpha\beta)^{-2s^2-2s-1}) \cdot \prod_{(\gamma,\xi)\in J} (x - \gamma^{-(2s+1)^2})$$
$$= \prod_{i=0}^{N-1}(x^4 - \omega^{-8(2i+\frac{1-\nu}{2})(2s+1)^2+2N(1-\lambda)}) \cdot \prod_{i=0}^{N-1}(x^2 - \omega^{-8i(2s+1)^2})^2.$$

In the case when N is odd and $\nu = +$, if λ and n satisfy the condition

(A) $\lambda = -1$, or
(B) $\lambda = 1$, and n is odd,

then the Hopf algebra $A_{Nn}^{+\lambda}$ is self-dual (see Corollary A.6 in the next section). Therefore, by using Proposition 4.5 we have:

Proposition 5.10. *Let $N \geq 1$ be an odd integer and $\omega \in \boldsymbol{k}$ a primitive $4nN$-th root of unity. Suppose that λ and n satisfy the above condition (A) or (B). Then*

$$P^{(1)}_{A_{Nn}^{+\lambda}}(x) = \begin{cases} \prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x - \omega^{-8nis^2})^{4n}(x^2 - \omega^{-4in(2s+1)^2}(-1)^{\frac{1-\lambda}{2}})^{2n} & (n \text{ is odd}), \\ \prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x - \omega^{-8nis^2})^{4n}(x - \omega^{-8ins^2}(-1)^{\frac{n}{2}})^{4n} & (n \geq 4 \text{ is even}), \\ \prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x - \omega^{-16is^2})^{16}(x + \omega^{-8is^2})^{8}(x + \omega^{-16is^2})^{8} & (n = 2), \end{cases}$$

$$P^{(2)}_{A_{Nn}^{+\lambda}}(x) = \begin{cases} \prod_{s=0}^{N-1}\prod_{t=1}^{\frac{n-\epsilon(n)}{2}}\prod_{i=0}^{N-1}\prod_{j=0}^{n-1}(x^2 - \omega^{-4in(2s+1)^2-2N(2t-1)^2(2j+\frac{\lambda-1}{2})}) \\ \quad \times \prod_{s=0}^{N-1}\prod_{t=1}^{\frac{n-2+\epsilon(n)}{2}}\prod_{i=0}^{N-1}\prod_{j=0}^{n-1}(x - \omega^{-8ins^2-4Nt^2(2j+\frac{\lambda-1}{2})})^2 & (n \geq 3), \\ \prod_{s=0}^{N-1}\prod_{i=0}^{N-1}(x^4 + \omega^{-16i(2s+1)^2})(x^2 - \omega^{-8i(2s+1)^2})^2 & (n = 2), \end{cases}$$

where $\epsilon(n) = 0$ if n is even, and $\epsilon(n) = 1$ if n is odd.

From the above proposition the polynomial invariants of the Kac–Paljutkin algebra $K_8 \cong A_{12}^{+-}$ are computed, again.

5.3. The group Hopf algebra $\boldsymbol{k}[G_{Nn}]$

If N is odd, $n \geq 2$, and $\lambda = \pm 1$, then $A_{Nn}^{+\lambda}$ is isomorphic to the group algebra of the finite group

$$G = \langle h, t, w \mid t^2 = h^{2N} = 1, w^n = h^{(n+\frac{\lambda-1}{2})N}, tw = w^{-1}t, ht = th, hw = wh\rangle$$

as an algebra (see the next Appendix A.1). The order of G is $4nN$. If $(n, \lambda) = (\text{even}, 1)$ or $(n, \lambda) = (\text{odd}, -1)$, then the group G is isomorphic to the direct product $D_{2n} \times C_{2N}$. If $(n, \lambda) = (\text{even}, -1)$ or $(n, \lambda) = (\text{odd}, 1)$, then G is isomorphic to the semidirect product of $H := \langle w, h \mid h^{2N} = 1, w^n = h^N, wh = hw\rangle$ and C_2, where the action of $C_2 = \langle t\rangle$ on H is given by $t \cdot w := w^{-1}$ and $t \cdot h := h$. In particular, when $N = 1$, the group G is the dihedral group of order $4n$.

We shall determine the universal R-matrices of the group Hopf algebra $\boldsymbol{k}[G]$, and calculate the polynomial invariants of it in the case when $(n, \lambda) = (\text{even}, -1)$ or $(n, \lambda) = (\text{odd}, 1)$. In this case, G coincides with the group

$$G_{Nn} = \langle h, t, w \mid t^2 = h^{2N} = 1, w^n = h^N, tw = w^{-1}t, ht = th, hw = wh\rangle.$$

Lemma 5.11. *Let N be an odd integer, and $n \geq 2$ be an integer.*

(1) *If $n \geq 3$, then the finite group G_{Nn} has a unique maximal commutative normal subgroup, which is given by $H = \langle h, w\rangle$.*
(2) *If $n = 2$, then the finite group G_{Nn} has exactly three maximal commutative normal subgroups, which are given by $H_1 = \langle h, w\rangle$, $H_2 = \langle h, t\rangle$ and $H_3 = \langle h, tw\rangle$.*

Proof. We set $G = G_{Nn}$. It is clear that H $(=H_1)$ is a commutative and normal subgroup of G, and it is not hard to show that H is maximal between commutative subgroups. Hence H is a maximal commutative normal subgroup of G. In the case when $n = 2$ we see that H_2 and H_3 are also commutative and normal subgroups of G, and maximal between commutative subgroups.

We show that the converse is true.

(1) Let K be a maximal commutative normal subgroup K of $G = G_{Nn}$. Suppose that $K \not\subset H$. Then $K \cap (G - H) \neq \emptyset$. So, if we take an element $tw^ih^j \in K \cap (G - H)$ $(0 \le i < n, 0 \le j < N)$, then $w(tw^ih^j)w^{-1} = tw^{i-2}h^j \in K$; that is, $tw^ih^j \in K$ implies that $h^{-j}w^{-i+2}t = (tw^{i-2}h^j)^{-1} \in K$. Since K is commutative, we have the equation $w^2 = h^{-j}w^{-i+2}t \cdot tw^ih^j = tw^ih^j \cdot h^{-j}w^{-i+2}t = w^{-2} = w^{2n-2} = w^{(n-2)+n} = w^{n-2}h^N$. This is a contradiction since $n \ge 3$. Thus $K \subset H$. This implies that $K = H$ from maximality of K.

(2) Let K be a maximal commutative normal subgroup K of $G = G_{Nn}$. Then $\langle h\rangle \subset K$ holds from maximality of K since $K\langle h\rangle$ is a commutative and normal subgroup of G. If $K \subset H_1$, then $K = H_1$ since H_1 is a maximal commutative normal subgroup of G. So, we suppose that $K \not\subset H_1$. Then $tw^ih^j \in K$ for some i, j $(i = 0, 1, j = 0, 1, \ldots, N - 1)$. Since $\langle h\rangle \subset K$, we have $tw^i \in K$. If $i = 0$, then $t \in K$, and hence $H_2 \subset K$. By maximality of H_2, we see that $K = H_2$. By the same argument, we see that, if $i = 1$, then $K = H_3$. Thus there is no maximal commutative normal subgroup of G except for H_1, H_2, H_3. □

In what follows, we assume that the characteristic ch($\boldsymbol{k}$) does not divide $2nN$. To determine the universal R-matrices of $\boldsymbol{k}[H]$, we use the basis consisting of the primitive idempotents of $\boldsymbol{k}[H]$.

Lemma 5.12. *Let $N \ge 1$ be an odd integer, and $n \ge 2$ be an integer. Let H be the commutative group of order $2nN$ defined by $H = \langle h, w \mid h^{2N} = 1, w^n = h^N, hw = wh\rangle$, and ω be a primitive $4nN$-th root of unity in $\boldsymbol{k}$. For $i, k \in \mathbb{Z}$, we set*

$$E_{ik} = \frac{1}{2nN}\sum_{j=0}^{n-1}\sum_{l=0}^{2N-1}\omega^{-2Nj(k+2i)-2nkl}w^jh^l \in \boldsymbol{k}[H].$$

Then $\{E_{ik} \mid i = 0, 1, \ldots, n-1, k = 0, 1, \ldots, 2N-1\}$ is the set of primitive idempotents of $\boldsymbol{k}[H]$, and the following equations hold: for all $i, j, k, l \in \mathbb{Z}$

$$E_{i+n,k} = E_{ik}, \qquad E_{i-N,k+2N} = E_{ik}, \tag{5.4}$$

$$E_{ik}E_{jl} = \delta_{kl}^{(2N)}\delta_{k+2i,l+2j}^{(2n)}E_{jl} = \delta_{kl}^{(2N)}\delta_{k+2i,l+2j}^{(2n)}E_{ik}, \tag{5.5}$$

where

$$\delta_{kl}^{(m)} = \begin{cases} 1 & (k \equiv l \ (\mathrm{mod}\ m)) \\ 0 & (k \not\equiv l \ (\mathrm{mod}\ m)) \end{cases}$$

for $m = 2N$ or $m = 2n$. Furthermore, the coproduct Δ, the counit ε, and the antipode S of the group Hopf algebra $\boldsymbol{k}[H]$ are given as follows:

$$\Delta(E_{ik}) = \sum_{\substack{0\le p,q\le 2N-1\\ p+q\equiv k \ (\mathrm{mod}\ 2N)}}\ \sum_{\substack{0\le a,b\le n-1\\ a+b+\frac{-k+p+q}{2}\equiv i \ (\mathrm{mod}\ n)}} E_{ap}\otimes E_{bq}, \tag{5.6}$$

$$\varepsilon(E_{ik}) = \delta_{i,0}\delta_{k,0}, \tag{5.7}$$

$$S(E_{ik}) = E_{-i,-k}. \tag{5.8}$$

Proof. Eqs. (5.4) and (5.7) are obtained immediately. For integers i and k, let χ_{jk} be the group homomorphism from H to $\boldsymbol{k}$ defined by $\chi_{ik}(w) = \omega^{2N(k+2i)}$, $\chi_{ik}(h) = \omega^{2nk}$. Then the set $\{\chi_{ik} \mid i = 0, 1, \ldots, n-1, k = 0, 1, \ldots, 2N-1\}$ consists of all irreducible characters of H. Eqs. (5.5), (5.6) and (5.8) follow from w^jh^l being written as $w^jh^l = \sum_{i=0}^{n-1}\sum_{k=0}^{2N-1}\omega^{2Nj(2i+k)+2nkl}E_{ik}$, which comes from the orthogonality of characters. □

Lemma 5.13. *Let $N \ge 1$ be an odd integer, and $n \ge 2$ be an integer. Let H be the commutative group of order $2nN$ defined in Lemma 5.12, and $\omega \in \boldsymbol{k}$ be a primitive $4nN$-th root of unity. Then any universal R-matrix R of $\boldsymbol{k}[H]$ is given by*

$$R = \sum_{i,j=0}^{n-1}\sum_{k,l=0}^{2N-1}\nu^{kl}\omega^{2aN(2i+k)(2j+l)+2n(qkl+2pjk+2rij)}E_{ik}\otimes E_{jl} \tag{5.9}$$

for some $\nu \in \{\pm 1\}$, $a \in \{0, 1, \ldots, n-1\}$ and $p, q, r \in \{0, 1, \ldots, N-1\}$ such that pn, rn are multiples of N, where $\{E_{ik}\}$ is the set of primitive idempotents of $\boldsymbol{k}[H]$ defined in Lemma 5.12. Conversely, R given above by (5.9) is a universal R-matrix of $\boldsymbol{k}[H]$.

Proof. Let R be an element of $\boldsymbol{k}[H]\otimes\boldsymbol{k}[H]$, and write R in the form

$$R = \sum_{i,j=0}^{n-1}\sum_{k,l=0}^{2N-1}R_{jl}^{ik}E_{ik}\otimes E_{jl} \quad (R_{jl}^{ik} \in \boldsymbol{k}).$$

We treat the indices i and j of R_{jl}^{ik} as modulo n; therefore R_{jl}^{ik} has a meaning for all integers i and j. For $0 \leq m \leq 4N-1$, we define $\delta(m)$ by

$$\delta(m) = \begin{cases} 0 & (0 \leq m \leq 2N-1), \\ 1 & (2N \leq m \leq 4N-1). \end{cases}$$

Then we have

$$(\Delta \otimes \mathrm{id})(R) = \sum_{j,a,b=0}^{n-1} \sum_{l,p,q=0}^{2N-1} R_{jl}^{a+b+N\delta(p+q),p+q-2N\delta(p+q)} E_{ap} \otimes E_{bq} \otimes E_{jl},$$

$$R_{13}R_{23} = \sum_{j,a,b=0}^{n-1} \sum_{l,p,q=0}^{2N-1} R_{jl}^{ap} R_{jl}^{bq} E_{ap} \otimes E_{bq} \otimes E_{jl}.$$

Hence $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$ if and only if

$$R_{jl}^{a+b+N\delta(p+q),p+q-2N\delta(p+q)} = R_{jl}^{ap} R_{jl}^{bq} \tag{5.10}$$

for all $j, a, b = 0, 1, \ldots, n-1$ and $l, p, q = 0, 1, \ldots, 2N-1$. Similarly, $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$ if and only if

$$R^{ik}_{a+b+N\delta(p+q),p+q-2N\delta(p+q)} = R^{ik}_{bq} R^{ik}_{ap} \tag{5.11}$$

for $i, a, b = 0, 1, \ldots, n-1$ and $k, p, q = 0, 1, \ldots, 2N-1$, and also we have $(\varepsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \varepsilon)(R) = 1$ if and only if

$$R_{jl}^{00} = R_{00}^{jl} = 1 \tag{5.12}$$

for $j = 0, 1, \ldots, n-1$ and $l = 0, 1, \ldots, 2N-1$. Thus R is a universal R-matrix of $\boldsymbol{k}[H]$ if and only if Eqs. (5.10)–(5.12) hold. From the Eqs. (5.10) and (5.11), R_{jl}^{ik} can be expressed in the form

$$R_{jl}^{ik} = (R_{01}^{01})^{kl} (R_{10}^{01})^{jk} (R_{01}^{10})^{il} (R_{10}^{10})^{ij}.$$

Since $(R_{01}^{10})^n = R_{01}^{n0} = R_{01}^{00} = 1$ and $(R_{10}^{01})^n = R_{n0}^{01} = R_{00}^{01} = 1$ by (5.10), we see that R_{01}^{10} and R_{10}^{01} can be written as

$$R_{01}^{10} = \omega^{4aN}, \qquad R_{10}^{01} = \omega^{4bN} \quad (0 \leq a, b \leq n-1). \tag{5.13}$$

By using Eqs. (5.10) and (5.11), repeatedly, we have

$$\begin{cases} (R_{01}^{01})^{2N} = (R_{01}^{01})^{2N-2} R_{01}^{02} = \cdots = R_{01}^{01} R_{01}^{0,2N-1} = R_{01}^{N0} = (R_{01}^{10})^N = \omega^{4aN^2}, \\ (R_{01}^{01})^{2N} = (R_{01}^{01})^{2N-2} R_{02}^{01} = \cdots = R_{01}^{01} R_{0,2N-1}^{01} = R_{N0}^{01} = (R_{10}^{01})^N = \omega^{4bN^2}. \end{cases}$$

Therefore $\omega^{4bN} = \omega^{4aN+4pn}$ must be required for some $0 \leq p \leq N-1$ such that pn is a multiple of N. From equation $(R_{01}^{01})^{2N} = \omega^{4aN^2}$, we may set $(R_{01}^{01})^2 = \omega^{4aN+4qn}$ $(0 \leq q \leq N-1)$, and hence

$$R_{01}^{01} = \pm \omega^{2aN+2qn}. \tag{5.14}$$

By using Eq. (5.10), repeatedly, we have $(R_{10}^{10})^N = R_{N0}^{10} = (R_{01}^{10})^{2N} = \omega^{8aN^2}$. So, R_{10}^{10} can be expressed as

$$R_{10}^{10} = \omega^{8aN+4rn} \tag{5.15}$$

for some $0 \leq r \leq N-1$ such that rn is a multiple of N. From (5.13)–(5.15) we see that a universal R-matrix R of $\boldsymbol{k}[H]$ is written in the form

$$R = \sum_{i,j=0}^{n-1} \sum_{k,l=0}^{2N-1} R_{jl}^{ik} E_{ik} \otimes E_{jl}, \quad R_{jl}^{ik} = \nu^{kl} \omega^{2aN(2i+k)(2j+l)+2n(qkl+2pjk+2rij)} \tag{5.16}$$

for some $\nu \in \{\pm 1\}$, $a \in \{0, 1, \ldots, n-1\}$ and $p, q, r \in \{0, 1, \ldots, N-1\}$ such that pn, rn are multiples of N.

Conversely, it is not hard to check that $R \in \boldsymbol{k}[H] \otimes \boldsymbol{k}[H]$ which is given by the form above is a universal R-matrix of $\boldsymbol{k}[H]$. □

By use of Lemmas 5.1 and 5.13 one can determine the universal R-matrices of $\boldsymbol{k}[G_{Nn}]$.

Proposition 5.14. *Let $N \geq 1$ be an odd integer, and $n \geq 2$ be an integer. Let ω be a primitive $4nN$-th root of unity in $\boldsymbol{k}$, and $\{E_{ik}\}$ be the set of primitive idempotent of $\boldsymbol{k}[H]$ defined in Lemma 5.12. Then for any $\nu \in \{\pm 1\}$, $a \in \{0, 1, \ldots, n-1\}$, $q \in \{0, 1, \ldots, N-1\}$,*

$$R_{aq\nu} := \sum_{i,j=0}^{n-1} \sum_{k,l=0}^{2N-1} \nu^{kl} \omega^{2aN(2i+k)(2j+l)+2qkln} E_{ik} \otimes E_{jl} \tag{5.17}$$

is a universal R-matrix of the group Hopf algebra $\boldsymbol{k}[G_{Nn}]$. Furthermore,

- *$R_{aq\nu}$ is a universal R-matrix of the group Hopf algebra $\boldsymbol{k}[\langle h\rangle]$ if and only if $a=0$, and*
- *if $n\geq 3$, then any universal R-matrix is given by the above form; therefore, the number of universal R-matrices of $\boldsymbol{k}[G_{Nn}]$ is $2nN$.*

Proof. Let R be a universal R-matrix of $\boldsymbol{k}[H]$, where H is the subgroup of G_{Nn} defined in Lemma 5.13, and write it in the form (5.16). By using $E_{ik}t=tE_{-i-k,k}$ we have

$$\Delta^{\mathrm{cop}}(t)\cdot R=\sum_{i,j=0}^{n-1}\sum_{k,l=0}^{2N-1}R_{jl}^{ik}tE_{ik}\otimes tE_{jl},\qquad R\cdot\Delta(t)=\sum_{i,j=0}^{n-1}\sum_{k,l=0}^{2N-1}R_{-j-l,l}^{-i-k,k}tE_{ik}\otimes tE_{jl}.$$

Hence R is a universal R-matrix of $\boldsymbol{k}[G_{Nn}]$ if and only if $R_{-j-l,l}^{-i-k,k}=R_{jl}^{ik}$ for all i,j,k,l. Therefore, we have

$$R_{-j-l,l}^{-i-k,k}=R_{jl}^{ik}\iff\omega^{2n(-2p(2j+l)k+2r(kj+il+kl))}=1.$$

Considering the equations in R.H.S. for $(i,j,k,l)=(1,0,0,1)$ and $(i,j,k,l)=(0,0,1,1)$, we see that R is a universal R-matrix of $\boldsymbol{k}[G_{Nn}]$ if and only if $\omega^{4pn}=\omega^{4rn}=1$. This condition is equivalent to both p and r being multiples of N. It follows from $0\leq p,r\leq N-1$ that $p=r=0$. □

Proposition 5.15. *Let $N\geq 1$ be an odd integer, and let ω be a primitive $8N$-th root of unity in a field $\boldsymbol{k}$ whose characteristic does not divide $2N$. Then the number of universal R-matrices of the group Hopf algebra $\boldsymbol{k}[G_{N2}]$ is $8N$, and they are given by the list below.*

- *Universal R-matrices of $\boldsymbol{k}[\langle h\rangle]$:*

$$R_d=\sum_{k,l=0}^{2N-1}\omega^{4dkl}E_k\otimes E_l\quad(d=0,1,\ldots,2N-1),$$

where $E_k=\frac{1}{2N}\sum_{l=0}^{2N-1}\omega^{-4kl}h^l$.

- *Universal R-matrices of $\boldsymbol{k}[H_1]$, where $H_1=\langle h,w\rangle$:*

$$R_{1q\nu}=\sum_{i,j=0,1}\sum_{k,l=0}^{2N-1}\nu^{kl}\omega^{2N(2i+k)(2j+l)+4qkl}E_{ik}\otimes E_{jl}\quad(q=0,1,\ldots,N-1,\ \nu=\pm 1),$$

where $E_{ik}=\frac{1}{4N}\sum_{j=0,1}\sum_{l=0}^{2N-1}(-1)^{ij}\omega^{-2Njk-4kl}w^jh^l$.

- *Universal R-matrices of $\boldsymbol{k}[H_2]$, where $H_2=\langle h,t\rangle$:*

$$R_d^{(1)}:=R_{0N1d}^{\boldsymbol{k}[H_2]}=\sum_{i,j=0,1}\sum_{k,l=0}^{2N-1}(-1)^{jk+il}\omega^{4dkl}E_{ik}\otimes E_{jl}\quad(d=0,1,\ldots,2N-1),\tag{5.18}$$

where $E_{ik}=\frac{1}{4N}\sum_{j=0,1}\sum_{l=0}^{2N-1}(-1)^{ij}\omega^{-4kl}t^jh^l$.

- *Universal R-matrices of $\boldsymbol{k}[H_3]$, where $H_3=\langle h,tw\rangle$:*

$$R_d^{(2)}:=R_{0N1d}^{\boldsymbol{k}[H_3]}=\sum_{i,j=0,1}\sum_{k,l=0}^{2N-1}(-1)^{jk+il}\omega^{4dkl}E_{ik}\otimes E_{jl}\quad(d=0,1,\ldots,2N-1),\tag{5.19}$$

where $E_{ik}=\frac{1}{4N}\sum_{j=0,1}\sum_{l=0}^{2N-1}(-1)^{ij}\omega^{-4kl}(tw)^jh^l$.

Proof. By Proposition 5.14, it is sufficient to determine the universal R-matrices of $\boldsymbol{k}[G_{N2}]$ which come from that of $\boldsymbol{k}[H_2]$ or $\boldsymbol{k}[H_3]$. By Lemma 5.2, it follows from $H_i\cong C_2\times C_{2N}$ for $i=2,3$ that a universal R-matrix of $\boldsymbol{k}[H_i]$ is given by

$$R_{pqrs}^{\boldsymbol{k}[H_i]}=\sum_{i,j=0}^{m-1}\sum_{k,l=0}^{n-1}(-1)^{(pi+rk)j}\omega^{2(skl+qil)}E_{ik}\otimes E_{jl},$$

where $p,r\in\{0,1\}$, $q\in\{0,N\}$, $s\in\{0,1,\ldots,2N-1\}$.

Let us consider the case when $i=2$ and $R=R_{pqrs}^{\boldsymbol{k}[H_2]}$. We set $R_{jl}^{ik}=(-1)^{j(pi+rk)}\omega^{2l(sk+qi)}$. Then, by using $E_{ik}w=wE_{i+k,k}$, we have

$$\begin{aligned}\Delta^{\mathrm{cop}}(w)\cdot R=R\cdot\Delta(w)&\iff R_{j-l,l}^{i-k,k}=R_{jl}^{ik}\quad\text{for all } i,j,k,l\\&\iff(-1)^{-(j-l)pk-l(pi+rk)}\omega^{-2lqk}=1\quad\text{for all } i,j,k,l\\&\iff(p,q,r)=(0,0,0)\quad\text{or } (p,q,r)=(0,N,1).\end{aligned}$$

Thus $R^{\boldsymbol{k}[H_2]}_{pqrs}$ is a universal R-matrix of $\boldsymbol{k}[G_{N2}]$ if and only if $(p, q, r) = (0, 0, 0)$ or $(p, q, r) = (0, N, 1)$. It is easily proved that $R^{\boldsymbol{k}[H_2]}_{000s}$ is a universal R-matrix of $\boldsymbol{k}[\langle h\rangle]$, and $R^{\boldsymbol{k}[H_2]}_{0N1s}$ is not. In a similar manner, it is shown that a universal R-matrix $R^{\boldsymbol{k}[H_3]}_{pqrs}$ of $\boldsymbol{k}[H_3]$ is a universal R-matrix of $\boldsymbol{k}[G]$ if and only if $(p, q, r) = (0, 0, 0), (0, N, 1)$, and $R^{\boldsymbol{k}[H_3]}_{pqrs}$ is that of $\boldsymbol{k}[\langle h\rangle]$ if and only if $(p, q, r) = (0, 0, 0)$. $\square$

Lemma 5.16. *Let $N \geq 1$ be an odd integer, and $n \geq 2$ be an integer.*

(1) *The Drinfel'd element $u_{aq\nu}$ associated to $R_{aq\nu}$ in Proposition 5.14 is given by*

$$u_{aq\nu} = \sum_{i=0}^{n-1}\sum_{k=0}^{2N-1} \nu^k \omega^{-2aN(2i+k)^2-2nqk^2} E_{ik},$$

where $\{E_{ik}\}$ is the set of primitive idempotents of $\boldsymbol{k}[H]$ defined in Lemma 5.12.

(2) *In the case when $n = 2$, for each $i = 1, 2$ the Drinfel'd element $u^{(i)}_d$ associated to $R^{(i)}_d$ given by (5.18) and (5.19) is given by*

$$u^{(i)}_d = \sum_{k=0}^{2N-1} \omega^{-4dk^2} E_k,$$

where $E_k = \frac{1}{2N}\sum_{l=0}^{2N-1} \omega^{-4kl} h^l$.

Proof. The proof follows from direct computations. $\square$

Let $\omega \in \boldsymbol{k}$ be a primitive $4nN$-th root of unity. Then a full set of non-isomorphic simple left $\boldsymbol{k}[G_{Nn}]$-modules is given by

$$\{V_{ijk} \mid i, j = 0, 1, k = 0, 2, \ldots, 2N-2\} \cup \{V_{jk} \mid k = 0, 1, \ldots, 2N-1, j = 1, 2, \ldots, n-1, j \equiv k \pmod 2\},$$

where the action χ_{ijk} of $\boldsymbol{k}[G_{Nn}]$ on $V_{ijk} = \boldsymbol{k}$ is given by

$$\chi_{ijk}(t) = (-1)^i, \qquad \chi_{ijk}(w) = (-1)^j, \qquad \chi_{ijk}(h) = \begin{cases} \omega^{2kn} & (n \text{ is even}), \\ \omega^{2(j+k)n} & (n \text{ is odd}), \end{cases} \tag{5.20}$$

and the left action ρ_{jk} of $\boldsymbol{k}[G_{Nn}]$ on $V_{jk} = \boldsymbol{k} \oplus \boldsymbol{k}$ is given by

$$\rho_{jk}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho_{jk}(w) = \begin{pmatrix} \omega^{2jN} & 0 \\ 0 & \omega^{-2jN} \end{pmatrix}, \qquad \rho_{jk}(h) = \begin{pmatrix} \omega^{2kn} & 0 \\ 0 & \omega^{2kn} \end{pmatrix}. \tag{5.21}$$

For each universal R-matrix R of $\boldsymbol{k}[G_{Nn}]$, the braided dimensions of the simple left $\boldsymbol{k}[G_{Nn}]$-modules V_{ijk} and V_{jk} are given as follows.

For $n \geq 2$,

$$\underline{\dim}_{R_{aq\nu}} V_{ijk} = \begin{cases} \omega^{-2nqk^2} & (n \text{ is even}), \\ \nu^j(-1)^{aj}\omega^{-2nq(j+k)^2} & (n \text{ is odd}), \end{cases} \qquad \underline{\dim}_{R_{aq\nu}} V_{jk} = 2\nu^k\omega^{-2nqk^2-2Naj^2}. \tag{5.22}$$

For $n = 2$,

$$\underline{\dim}_{R^{(1)}_d} V_{ijk} = \underline{\dim}_{R^{(2)}_d} V_{ijk} = \omega^{-4dk^2}, \qquad \underline{\dim}_{R^{(1)}_d} V_{1k} = \underline{\dim}_{R^{(2)}_d} V_{1k} = 2\omega^{-4dk^2}. \tag{5.23}$$

Combining the results in Proposition 5.14, Proposition 5.15 and Eqs. (5.22), (5.23), we have the following.

Proposition 5.17. *Let $N \geq 1$ be an odd integer, and $n \geq 2$ be an integer, and consider the group*

$$G_{Nn} = \langle h, t, w \mid t^2 = h^{2N} = 1, w^n = h^N, tw = w^{-1}t, ht = th, hw = wh\rangle.$$

Let ω be a primitive $4nN$-th root of unity in a field $\boldsymbol{k}$ whose characteristic does not divide $2nN$. Then the polynomial invariants of the group Hopf algebra $\boldsymbol{k}[G_{Nn}]$ are given by the following.

$$P^{(1)}_{\boldsymbol{k}[G_{Nn}]}(x) = \begin{cases} \displaystyle\prod_{s=0}^{N-1}\prod_{q=0}^{N-1}(x-\omega^{-8nqs^2})^{4n}(x^2-\omega^{-4nq(2s+1)^2})^{2n} & (n \text{ is odd}), \\ \displaystyle\prod_{s=0}^{N-1}\prod_{q=0}^{N-1}(x-\omega^{-8nqs^2})^{8n} & (n \geq 4 \text{ is even}), \\ \displaystyle\prod_{s=0}^{N-1}\prod_{q=0}^{N-1}(x-\omega^{-16qs^2})^{32} & (n = 2), \end{cases}$$

$$P^{(2)}_{\boldsymbol{k}[G_{Nn}]}(x) = \begin{cases} \prod_{s=0}^{N-1} \prod_{t=1}^{\frac{n-\epsilon(n)}{2}} \prod_{a=0}^{n-1} \prod_{q=0}^{N-1} (x^2 - \omega^{-4(nq(2s+1)^2+Na(2t-1)^2)}) \\ \quad \times \prod_{s=0}^{N-1} \prod_{t=1}^{\frac{n-2+\epsilon(n)}{2}} \prod_{a=0}^{n-1} \prod_{q=0}^{N-1} (x - \omega^{-8(nqs^2+Nat^2)})^2 & (n \geq 3), \\ \prod_{s=0}^{N-1} \prod_{q=0}^{N-1} (x^4 - \omega^{-16q(2s+1)^2})(x^2 - \omega^{-8q(2s+1)^2})^2 & (n = 2), \end{cases}$$

where $\epsilon(n) = 0$ if n is even, and $\epsilon(n) = 1$ if n is odd.

Proof. First, we consider the case when $n \geq 3$. By Proposition 5.14, any universal R-matrix of $\boldsymbol{k}[G_{Nn}]$ coincides with exactly one of $R_{aq\nu}$ $(a = 0, 1, \dots, n-1, q = 0, 1, \dots, N-1, \nu = \pm 1)$.

Suppose that n (≥ 4) is even. Then, by (5.22), the polynomial invariant $P^{(1)}_{\boldsymbol{k}[G_{Nn}]}(x)$ is given by

$$P^{(1)}_{\boldsymbol{k}[G_{Nn}]}(x) = \prod_{i,j=0}^{1} \prod_{s=0}^{N-1} P_{\boldsymbol{k}[G_{Nn}],V_{ij,2s}}(x) = \prod_{i,j=0}^{1} \prod_{s=0}^{N-1} \prod_{q=0}^{N-1} (x - \omega^{-8nqs^2})^{2n} = \prod_{s=0}^{N-1} \prod_{q=0}^{N-1} (x - \omega^{-8nqs^2})^{8n}.$$

By the same lemma, since $P_{\boldsymbol{k}[G_{Nn}],V_{jk}}(x)$ is given by $P_{\boldsymbol{k}[G_{Nn}],V_{jk}}(x) = \prod_{a=0}^{n-1} \prod_{q=0}^{N-1} \prod_{\nu=\pm 1} (x - \nu^k \omega^{-2(nqk^2+Naj^2)})$ for the simple left $\boldsymbol{k}[G_{Nn}]$-module V_{jk}, we have

$$\begin{aligned} P^{(2)}_{\boldsymbol{k}[G_{Nn}]}(x) &= \left(\prod_{s=0}^{N-1} \prod_{t=1}^{\frac{n}{2}} P_{\boldsymbol{k}[G_{Nn}],V_{2t-1,2s+1}}(x) \right) \cdot \left(\prod_{s=0}^{N-1} \prod_{t=1}^{\frac{n-2}{2}} P_{\boldsymbol{k}[G_{Nn}],V_{2t,2s}}(x) \right) \\ &= \prod_{s=0}^{N-1} \prod_{t=1}^{\frac{n}{2}} \prod_{a=0}^{n-1} \prod_{q=0}^{N-1} (x^2 - \omega^{-4(nq(2s+1)^2+Na(2t-1)^2)}) \cdot \prod_{s=0}^{N-1} \prod_{t=1}^{\frac{n-2}{2}} \prod_{a=0}^{n-1} \prod_{q=0}^{N-1} (x - \omega^{-8(nqs^2+Nat^2)})^2. \end{aligned}$$

By a quite similar consideration, we calculate the polynomial invariants of $\boldsymbol{k}[G_{Nn}]$ in the case when n (≥ 3) is odd.

Next, we consider the case when $n = 2$. Then, by Proposition 5.15, any universal R-matrix of $\boldsymbol{k}[G_{N2}]$ coincides with exactly one of $R_{aq\nu}$ $(a = 0, 1, q = 0, 1, \dots, N-1, \nu = \pm 1)$, $R^{(1)}_d, R^{(2)}_d$ $(d = 0, 1, \dots, 2N-1)$. Thus, by (5.22) and (5.23), the polynomial invariant $P^{(1)}_{\boldsymbol{k}[G_{N2}]}(x)$ is given by

$$P^{(1)}_{\boldsymbol{k}[G_{N2}]}(x) = \prod_{i,j=0,1} \prod_{s=0}^{N-1} \left(\prod_{q=0}^{N-1} (x - \omega^{-4q(2s)^2})^4 \cdot \prod_{d=0}^{2N-1} (x - \omega^{-4d(2s)^2})^2 \right) = \prod_{s=0}^{N-1} \prod_{q=0}^{N-1} (x - \omega^{-16qs^2})^{32}.$$

Similarly, by Proposition 5.15 and Eqs. (5.22), (5.23), we have

$$P^{(2)}_{\boldsymbol{k}[G_{N2}]}(x) = \prod_{s=0}^{N-1} \left(\prod_{a=0}^{1} \prod_{q=0}^{N-1} (x^2 - \omega^{-4aN-8q(2s+1)^2}) \cdot \prod_{d=0}^{2N-1} (x - \omega^{-4d(2s+1)^2})^2 \right).$$

Here, $\prod_{d=0}^{2N-1} (x - \omega^{-4d(2s+1)^2}) = \prod_{q=0}^{N-1} (x^2 - \omega^{-8q(2s+1)^2})$, and hence

$$\begin{aligned} P^{(2)}_{\boldsymbol{k}[G_{N2}]}(x) &= \prod_{s=0}^{N-1} \left(\prod_{q=0}^{N-1} (x^2 - \omega^{-4N-8q(2s+1)^2}) \cdot \prod_{q=0}^{N-1} (x^2 - \omega^{-8q(2s+1)^2})^3 \right) \\ &= \prod_{s=0}^{N-1} \prod_{q=0}^{N-1} (x^4 - \omega^{-16q(2s+1)^2})(x^2 - \omega^{-8q(2s+1)^2})^2. \quad \square \end{aligned}$$

For an odd integer $N \geq 1$ and an integer $n \geq 2$, we set

$$A_{Nn} = \begin{cases} A^{++}_{Nn} & \text{if } n \text{ is odd}, \\ [0.1\,cm] A^{+-}_{Nn} & \text{if } n \text{ is even}. \end{cases}$$

We see immediately that, if n is odd, then $P^{(d)}_{A_{Nn}}(x) = P^{(d)}_{\boldsymbol{k}[G_{Nn}]}(x)$ for $d = 1, 2$. So, our polynomial invariants do not detect the difference between the representation categories of A_{Nn} and $\boldsymbol{k}[G_{Nn}]$. However, for an even integer n we have:

Corollary 5.18. *Let $N \geq 1$ be an odd integer, and $n \geq 2$ be an even integer. Let ω be a primitive $4nN$-th root of unity in a field $\boldsymbol{k}$ whose characteristic does not divide $2nN$. Then two Hopf algebras A_{Nn} and $\boldsymbol{k}[G_{Nn}]$ are not monoidally Morita equivalent.*

Proof. First, let us consider the case when $n \geq 3$, and compare $P^{(2)}_{A_{Nn}}(x)$ and $P^{(2)}_{\boldsymbol{k}[G_{Nn}]}(x)$. By Proposition 5.10, we see that ω^{-N} is a root of the polynomial $P^{(2)}_{A_{Nn}}(x)$ since n is even. On the other hand, by Proposition 5.17, an arbitrary root of $P^{(2)}_{\boldsymbol{k}[G_{Nn}]}(x)$ is written in the form ω^{2k} for some integer k. If $P^{(2)}_{A_{Nn}}(x) = P^{(2)}_{\boldsymbol{k}[G_{Nn}]}(x)$, then $P^{(2)}_{\boldsymbol{k}[G_{Nn}]}(x)$ has to possess ω^{-N} as a root. Then $\omega^{-N} = \omega^{2k}$ for some k; that is, $N + 2k \equiv 0 \pmod{4nN}$. This leads to a contradiction such that N is even. So, $P^{(2)}_{A_{Nn}}(x) \neq P^{(2)}_{\boldsymbol{k}[G_{Nn}]}(x)$, and, by Theorem 2.6, ${}_{A_{Nn}}\mathbb{M}$ and ${}_{\boldsymbol{k}[G_{Nn}]}\mathbb{M}$ are not equivalent as $\boldsymbol{k}$-linear monoidal categories.

Next, let us consider the case when $n = 2$, and compare $P^{(1)}_{A_{N2}}(x)$ and $P^{(1)}_{\boldsymbol{k}[G_{N2}]}(x)$. By Proposition 5.10, we see that -1 is a root of the polynomial $P^{(1)}_{A_{Nn}}(x)$. However, by Proposition 5.17, an arbitrary root of $P^{(1)}_{\boldsymbol{k}[G_{N2}]}(x)$ is written in the form ω^{16k} for some integer k. By a similar argument to that above, we see that $-1 = \omega^{4N}$ is not a root of $P^{(1)}_{\boldsymbol{k}[G_{N2}]}(x)$. Thus $P^{(1)}_{A_{N2}}(x) \neq P^{(1)}_{\boldsymbol{k}[G_{N2}]}(x)$, and hence, by Theorem 2.6, ${}_{A_{N2}}\mathbb{M}$ and ${}_{\boldsymbol{k}[G_{N2}]}\mathbb{M}$ are not equivalent as $\boldsymbol{k}$-linear monoidal categories. □

Example 5.19. For a non-negative integer h, Φ_h denotes the h-th cyclotomic polynomial. Then, by using Maple12 software, we see that the polynomial invariants of Hopf algebras $\boldsymbol{k}[G_{Nn}]$ and A_{Nn} for $N = 1, 3, 5$ and $n = 2, 3, 4$ are given as in the following table.

A	$P^{(1)}_A(x)$	$P^{(2)}_A(x)$	A	$P^{(1)}_A(x)$	$P^{(2)}_A(x)$
$\boldsymbol{k}[G_{12}]$	Φ_1^{32}	$\Phi_4\Phi_2^3\Phi_1^3$	$\boldsymbol{k}[G_{53}]$	$\Phi_{10}^{24}\Phi_5^{72}\Phi_2^{54}\Phi_1^{162}$	$\Phi_{30}^4\Phi_{15}^{12}\Phi_{10}^4\Phi_5^{12}\Phi_6^9\Phi_3^{27}\Phi_2^9\Phi_1^{27}$
A_{12}	$\Phi_2^{16}\Phi_1^{16}$	$\Phi_8\Phi_2^2\Phi_1^2$	A_{53}		
$\boldsymbol{k}[G_{32}]$	$\Phi_3^{64}\Phi_1^{160}$	$\Phi_{12}^2\Phi_4^5\Phi_6^6\Phi_3^6\Phi_2^{15}\Phi_1^{15}$	$\boldsymbol{k}[G_{14}]$	Φ_1^{32}	$\Phi_8^2\Phi_4^2\Phi_2^6\Phi_1^6\Phi_{16}^2\Phi_4^4$
A_{32}	$\Phi_6^{32}\Phi_3^{32}\Phi_2^{80}\Phi_1^{80}$	$\Phi_{24}^2\Phi_8^5\Phi_6^4\Phi_3^4\Phi_2^{10}\Phi_1^{10}$	A_{14}		
$\boldsymbol{k}[G_{52}]$	$\Phi_5^{128}\Phi_1^{288}$	$\Phi_{20}^4\Phi_{10}^{12}\Phi_5^{12}\Phi_4^9\Phi_2^{27}\Phi_1^{27}$	$\boldsymbol{k}[G_{34}]$	$\Phi_3^{64}\Phi_1^{160}$	$\Phi_{24}^4\Phi_{12}^4\Phi_8^{10}\Phi_6^{12}\Phi_3^{12}\Phi_4^{10}\Phi_2^{30}\Phi_1^{30}\Phi_{48}^4\Phi_{16}^{10}\Phi_{12}^8\Phi_4^{20}$
A_{52}	$\Phi_{10}^{64}\Phi_5^{64}\Phi_2^{144}\Phi_1^{144}$	$\Phi_{40}^4\Phi_{10}^8\Phi_5^8\Phi_8^9\Phi_2^{18}\Phi_1^{18}$	A_{34}		
$\boldsymbol{k}[G_{13}]$	$\Phi_2^6\Phi_1^{18}$	$\Phi_6\Phi_3^3\Phi_2\Phi_1^3$	$\boldsymbol{k}[G_{54}]$	$\Phi_5^{128}\Phi_1^{288}$	$\Phi_{40}^8\Phi_{20}^8\Phi_{10}^{24}\Phi_5^{24}\Phi_8^{18}\Phi_4^{18}\Phi_2^{54}\Phi_1^{54}\Phi_{80}^8\Phi_{20}^{16}\Phi_{16}^{18}\Phi_4^{36}$
A_{13}			A_{54}		
$\boldsymbol{k}[G_{33}]$	$\Phi_6^{12}\Phi_3^{36}\Phi_2^{30}\Phi_1^{90}$	$\Phi_6^9\Phi_3^{27}\Phi_2^9\Phi_1^{27}$			
A_{33}					

We note that the representation rings of A_{Nn} and $\boldsymbol{k}[G_{Nn}]$ are isomorphic as rings with $*$-structure (for details see Proposition A.3 in the next section). Thus the pair of Hopf algebras A_{Nn} and $\boldsymbol{k}[G_{Nn}]$ gives an example of their representation rings being isomorphic, though their representation categories are not.

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Appendix. The representation ring and self-duality of $A^{+\lambda}_{Nn}$

In this Appendix, in the case when $N \geq 1$ is odd, by analyzing the algebraic structure of $A^{+\lambda}_{Nn}$ we introduce a "convenient" basis of $A^{+\lambda}_{Nn}$ to compute the braidings $\sigma_{\alpha\beta}$ given in Section 5, and determine the structure of the representation ring of it. As an application, we determine when $A^{+\lambda}_{Nn}$ is self-dual. As a further application, we show that, if n is even, then the representation ring of A^{++}_{Nn} is non-commutative. This means that the dual Hopf algebra of $A^{+\lambda}_{Nn}$ has no quasitriangular structure. These results have already been shown in [29] in the case when $N = 1$.

Throughout this section we assume that $N \geq 1$ is an odd integer, $n \geq 2$ is an integer, and $\lambda = \pm 1$.

A.1. The algebra structure of $A^{+\lambda}_{Nn}$

First of all, we determine the algebra structure of the Hopf algebra $A^{+\lambda}_{Nn}$. This was done by Masuoka [7] for the case of $N = 1$ (see also [29]). Let G be the finite group presented by

$$G = \langle h, t, w \mid t^2 = h^{2N} = 1,\ w^n = h^{(n+\frac{\lambda-1}{2})N},\ tw = w^{-1}t,\ ht = th,\ hw = wh \rangle. \qquad \text{(A.1)}$$

Then there is an algebra isomorphism $\varphi : \boldsymbol{k}[G] \longrightarrow A^{+\lambda}_{Nn}$ such that

$$\varphi(h) = x_{11}^2 - x_{12}^2, \tag{A.2}$$

$$\varphi(t) = x_{12}^N + x_{22}^N, \tag{A.3}$$

$$\varphi(w) = x_{11}^{2N-1}x_{22} - x_{21}^{2N-1}x_{12}. \tag{A.4}$$

Thus A_{Nn} is isomorphic to $\boldsymbol{k}[G_{Nn}]$ as algebras. Set $N = 2m + 1$, and consider the following elements in $A_{Nn}^{+\lambda}$:

$$h := x_{11}^2 - x_{12}^2, \qquad t := x_{12}^N + x_{22}^N, \qquad w := x_{11}^{2N-1}x_{22} - x_{21}^{2N-1}x_{12}.$$

If $\mathrm{ch}(\boldsymbol{k}) \neq 2$, then the following relations hold:

- $t^2 = h^{2N} = 1$, $w^n = h^{(n+\frac{\lambda-1}{2})N}$, $tw = w^{-1}t$, $ht = th$, $hw = wh$.
- $x_{22} = \frac{h^{-m(N+1)}+h^{m(N+1)+1}}{2}t$, $x_{12} = \frac{h^{-m(N+1)}-h^{m(N+1)+1}}{2}t$.
- $x_{11}^{2N-1} + x_{21}^{2N-1} = wth^{-m(N+1)-1}$.
- $\begin{cases} x_{11}^N = \dfrac{h^N+1}{2}wt, \\ x_{21}^N = \dfrac{h^N-1}{2}wt, \end{cases} \begin{cases} x_{12}^N = -\dfrac{h^N-1}{2}t, \\ x_{22}^N = \dfrac{h^N+1}{2}t. \end{cases}$
- $\begin{cases} x_{11}^{2N-1}x_{22} = \dfrac{1+h^N}{2}w, \\ x_{21}^{2N-1}x_{12} = \dfrac{-1+h^N}{2}w, \end{cases} \begin{cases} x_{12}^{2N-1}x_{21} = x_{12}x_{21}^{2N-1} = \dfrac{h^N-1}{2}w^{-1}, \\ x_{22}^{2N-1}x_{11} = x_{22}x_{11}^{2N-1} = \dfrac{h^N+1}{2}w^{-1}. \end{cases}$

  In particular, $w^{-1} = x_{22}^{2N-1}x_{11} - x_{12}^{2N-1}x_{21}$.

Proposition A.1. *Let G be the finite group given in (A.1). For the group algebra $\boldsymbol{k}[G]$ over $\boldsymbol{k}$ of $\mathrm{ch}(\boldsymbol{k}) \neq 2$, we define algebra maps $\Delta : \boldsymbol{k}[G] \otimes \boldsymbol{k}[G] \longrightarrow \boldsymbol{k}[G]$, $\varepsilon : \boldsymbol{k}[G] \longrightarrow \boldsymbol{k}$ and an anti-algebra map $S : \boldsymbol{k}[G] \longrightarrow \boldsymbol{k}[G]$ as follows:*

$$\Delta(h) = h \otimes h, \qquad \Delta(t) = h^N wt \otimes e_1 t + t \otimes e_0 t, \qquad \Delta(w) = w \otimes e_0 w + w^{-1} \otimes e_1 w,$$
$$\varepsilon(h) = 1, \qquad \varepsilon(t) = 1, \qquad \varepsilon(w) = 1,$$
$$S(h) = h^{-1}, \qquad S(t) = (e_0 - e_1 w)t, \qquad S(w) = e_0 w^{-1} + e_1 w,$$

where $e_0 = \frac{1+h^N}{2}$, $e_1 = \frac{1-h^N}{2}$. Then the algebra isomorphism $\varphi : \boldsymbol{k}[G] \longrightarrow A_{Nn}^{+\lambda}$ is a Hopf algebra isomorphism.

A.2. The representation ring of $A_{Nn}^{+\lambda}$

Via the algebra isomorphism φ given in Appendix A.1, one can determine the structure of the representation ring of $A_{Nn}^{+\lambda}$.

Let us recall the definition of the representation ring of a semisimple Hopf algebra, which is a natural extension of that of a finite group [30,31]. Let A be a semisimple Hopf algebra of finite dimension over a field $\boldsymbol{k}$. By $\mathfrak{R}(A)$ we denote the set of isomorphism classes of finite-dimensional left A-modules, and for a finite-dimensional left A-module V we denote by $[V]$ the isomorphism class of V. Then $\mathfrak{R}(A)$ has a semiring structure induced by $[V] + [W] = [V \oplus W]$ and $[V][W] = [V \otimes W]$. Also, $\mathfrak{R}(A)$ has the unit element, which is given by $[\boldsymbol{k}]$, where the left A-module action of $\boldsymbol{k}$ is due to the counit ε. Let $\mathrm{Rep}(A)$ denote the Grothendieck group constructed from the enveloping group of $\mathfrak{R}(A)$ as an abelian semigroup. Then the semiring structure of $\mathfrak{R}(A)$ uniquely determines a ring structure of $\mathrm{Rep}(A)$. Furthermore, the ring $\mathrm{Rep}(A)$ has an anti-homomorphism of rings $* : \mathrm{Rep}(A) \longrightarrow \mathrm{Rep}(A)$, which induced from the antipode S. Explicitly, this anti-homomorphism $*$ is defined by $[V] \longmapsto [V^*]$ for a finite-dimensional left A-module V. We call the ring $\mathrm{Rep}(A)$ with $*$ the *representation ring* of A. In general, $* : \mathrm{Rep}(A) \longrightarrow \mathrm{Rep}(A)$ is not an involution, and the representation ring $\mathrm{Rep}(A)$ is not commutative. However, if the Hopf algebra A possesses a universal R-matrix, then $*$ is an involution, and for two left A-modules V and W an A-linear isomorphism $c_{V,W} : V \otimes W \longrightarrow W \otimes V$ is defined by use of R, and hence we see that $\mathrm{Rep}(A)$ is commutative. We note that $\mathrm{Rep}(A)$ is a free $\mathbb{Z}$-module with finite rank, and a $\mathbb{Z}$-basis of $\mathrm{Rep}(A)$ is given by the isomorphism classes of simple left A-modules.

Lemma A.2. *Let $\boldsymbol{k}$ be a field whose characteristic does not divide $2nN$, and suppose that there is a primitive $4nN$-th root of unity in $\boldsymbol{k}$. For integers i, j and an even integer k let χ_{ijk} be the one-dimensional representation of the algebra $A_{Nn} = \boldsymbol{k}[G_{Nn}]$ given by (5.20), and for integers j, k with $j \equiv k \pmod 2$ let ρ_{jk} be the two-dimensional representation of the algebra $A_{Nn} = \boldsymbol{k}[G_{Nn}]$ given by (5.21). We set $\epsilon(n) = 0$ if n is even, and $\epsilon(n) = 1$ if n is odd. Then as representations of the Hopf algebra A_{Nn} the following hold for $i, j, k, i', j', k' \in \mathbb{Z}$.*

- (i) $[\rho_{2n+j,k}] = [\rho_{jk}] = [\rho_{-j,k}]$ *for* $j \equiv k \pmod 2$, $[\rho_{n+j,k}] = [\rho_{n-j,k}]$ *for* $n + j \equiv k \pmod 2$,
  (ii) $[\rho_{0k}] = [\chi_{00k} \oplus \chi_{10k}]$ *for* $k \equiv 0 \pmod 2$, $[\rho_{nk}] = [\chi_{01,k-\epsilon(n)} \oplus \chi_{11,k-\epsilon(n)}]$ *for* $k \equiv n \pmod 2$.
- *On representations of tensor products*
  (iii) $[\chi_{ijk} \otimes \chi_{i'j'k'}] = [\chi_{i+i',j+j',k+k'}]$, *where* k, k' *are even,*
  (iv) $[\chi_{ijk} \otimes \rho_{j'k'}] = [\rho_{j'k'} \otimes \chi_{ijk}] = [\rho_{nj+j',k+k'+\epsilon(n)j}]$ *for* $k \equiv 0, j' \equiv k' \pmod 2$,

(v) $[\rho_{jk} \otimes \rho_{j'k'}] = [\rho_{j+j',k+k'} \oplus \rho_{j-j',k+k'}]$ *for* $j \equiv k, j' \equiv k' \pmod 2$.

- *On contragredient representations* χ^*_{ijk} *and* ρ^*_{jk}

(vi) $[\chi^*_{ijk}] = [\chi_{i,-j,-k}]$ *for* $k \equiv 0 \pmod 2$,

(vii) $[\rho^*_{jk}] = [\rho_{j,-k}]$ *for* $j \equiv k \pmod 2$.

Proof. (i) By definition, $[\rho_{2n+j,k}] = [\rho_{j,k}]$ and $[\rho_{-j,k}] = [\rho_{jk}]$ are obtained immediately. By using these equations we have $[\rho_{n+j,k}] = [\rho_{-(n+j),k}] = [\rho_{2n-(n+j),k}] = [\rho_{n-j,k}]$.

(ii) Let $\{e_1, e_2\}$ be the standard basis of $V_{0k} = \boldsymbol{k} \oplus \boldsymbol{k}$. Then the subspaces $\boldsymbol{k}(e_1 + e_2)$ and $\boldsymbol{k}(e_1 - e_2)$ are ρ_{0k}-invariant, and $\boldsymbol{k}(e_1 + e_2) = V_{00k}$ and $\boldsymbol{k}(e_1 - e_2) = V_{10k}$ as submodules of V_{jk}. This implies that $[\rho_{0k}] = [\chi_{00k} \oplus \chi_{10k}]$. Similarly, the subspaces $\boldsymbol{k}(e_1 + e_2)$ and $\boldsymbol{k}(e_1 - e_2)$ of V_{nk} are also ρ_{nk}-invariant, and

$$\boldsymbol{k}(e_1 + e_2) = \begin{cases} V_{01k} & (n \text{ is even}), \\ V_{01,k-1} & (n \text{ is odd}), \end{cases} \quad \text{and} \quad \boldsymbol{k}(e_1 - e_2) = \begin{cases} V_{11k} & (n \text{ is even}), \\ V_{11,k-1} & (n \text{ is odd}), \end{cases}$$

as submodules of V_{nk}. This proves that $[\rho_{nk}] = [\chi_{01,k-\epsilon(n)} \oplus \chi_{11,k-\epsilon(n)}]$.

(iii) This follows from $\chi_{ijk} \otimes \chi_{i'j'k'} = \chi_{i+i',j+j',k+k'}$.

(iv) By using the coproduct Δ given in Proposition A.1, we see that $(\chi_{ijk} \otimes \rho_{j'k'})(t)$, $(\chi_{ijk} \otimes \rho_{j'k'})(w)$, $(\chi_{ijk} \otimes \rho_{j'k'})(h)$ are represented by the matrices

$$(-1)^{i+k'j}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \omega^{2(nj+j')N} & 0 \\ 0 & \omega^{-2(nj+j')N} \end{pmatrix}, \qquad \omega^{2(k+k'+\epsilon(n)j)n}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

respectively. Let $\{e_1, e_2\}$ be the standard basis of $\boldsymbol{k}^2$. Then by considering the matrix presentation of $\chi_{ijk} \otimes \rho_{j'k'}$ with respect to the basis $\{e_1, (-1)^{i+k'j}e_2\}$, we see that $[\chi_{ijk} \otimes \rho_{j'k'}] = [\rho_{nj+j',k+k'+\epsilon(n)j}]$.

Similarly, we see that, if n or j is even, then $(\rho_{j'k'} \otimes \chi_{ijk})(t)$, $(\rho_{j'k'} \otimes \chi_{ijk})(w)$, $(\rho_{j'k'} \otimes \chi_{ijk})(h)$ are represented by the matrices

$$(-1)^{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \omega^{2(nj+j')N} & 0 \\ 0 & \omega^{-2(nj+j')N} \end{pmatrix}, \qquad \omega^{2(k+k'+\epsilon(n)j)n}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

respectively, and, if n and j are odd, then they are represented by the matrices

$$(-1)^{i}\begin{pmatrix} 0 & \omega^{2(nj'+j')N} \\ \omega^{-2(nj'+j')N} & 0 \end{pmatrix}, \qquad \begin{pmatrix} \omega^{-2(nj+j')N} & 0 \\ 0 & \omega^{2(nj+j')N} \end{pmatrix}, \qquad \omega^{2(k+k'+j)n}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

respectively. So, in the case when n or j is even, by changing basis from $\{e_1, e_2\}$ to $\{e_1, (-1)^i e_2\}$ we see that $[\rho_{j'k'} \otimes \chi_{ijk}] = [\rho_{nj+j',k+k'+\epsilon(n)j}]$, and in the case when n and j are odd, by changing basis from $\{e_1, e_2\}$ to $\{e_2, (-1)^i \omega^{2(nj'+j')N} e_1\}$ we have the same equation, $[\rho_{j'k'} \otimes \chi_{ijk}] = [\rho_{nj+j',k+k'+j}] = [\rho_{nj+j',k+k'+\epsilon(n)j}]$.

(v) Let $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ be the bases of V_{jk} and $V_{j'k'}$, such that the matrix representations of ρ_{jk} and $\rho_{j'k'}$ with respect to the bases are given by (5.21), respectively. Then the action of $\rho_{jk} \otimes \rho_{j'k'}$ on $V_{jk} \otimes V_{j'k'}$ is given by

$$t \cdot e_a \otimes e'_b = \begin{cases} e_{3-a} \otimes e'_{3-b} & (\text{if } k' \text{ is even}), \\ \omega^{2(-1)^b(nj+j)N} e_{3-a} \otimes e'_{3-b} & (\text{if } k' \text{ is odd}), \end{cases}$$

$$w \cdot e_a \otimes e'_b = \begin{cases} \omega^{2((-1)^{1-a}j+(-1)^{1-b}j')N} e_a \otimes e'_b & (\text{if } k' \text{ is even}), \\ \omega^{2((-1)^{a}j+(-1)^{1-b}j')N} e_a \otimes e'_b & (\text{if } k' \text{ is odd}), \end{cases}$$

$$h \cdot e_a \otimes e'_b = \omega^{2(k+k')n} e_a \otimes e'_b$$

for $a, b = 1, 2$. Therefore, we see that $[\rho_{jk} \otimes \rho_{j'k'}] = [\rho_{j+j',k+k'} \oplus \rho_{j-j',k+k'}]$ by considering the matrix presentation of $\rho_{jk} \otimes \rho_{j'k'}$ with respect to the basis $\{e_1 \otimes e'_1, e_2 \otimes e'_2, e_1 \otimes e'_2, e_2 \otimes e'_1\}$ or $\{e_2 \otimes e'_1, \omega^{2(jn-j)N} e_1 \otimes e'_2, e_2 \otimes e'_2, \omega^{2(jn+j)N} e_1 \otimes e'_1\}$ according to the case whether k' is even or odd.

(vi) Since $\chi^*_{ijk}(t) = (-1)^i$, $\chi^*_{ijk}(w) = (-1)^{-j}$, $\chi^*_{ijk}(h) = \omega^{-2(k+\epsilon(n)j)n}$, we have $\chi^*_{ijk} = \chi_{i,-j,-k}$.

(vii) Let $\{e_1, e_2\}$ be the standard basis of $V_{jk} = \boldsymbol{k} \oplus \boldsymbol{k}$. Then with respect to the dual basis $\{e^*_1, e^*_2\}$ of $\{e_1, e_2\}$, the contragredient representation ρ^*_{jk} is represented as follows: $\rho^*_{jk}(h) = \begin{pmatrix} \omega^{-2kn} & 0 \\ 0 & \omega^{-2kn} \end{pmatrix}$, and, if k is even, then $\rho^*_{jk}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\rho^*_{jk}(w) = \begin{pmatrix} \omega^{-2jN} & 0 \\ 0 & \omega^{2jN} \end{pmatrix}$, and, if k is odd, then $\rho^*_{jk}(t) = \begin{pmatrix} 0 & -\omega^{-2jN} \\ -\omega^{2jN} & 0 \end{pmatrix}$, $\rho^*_{jk}(w) = \begin{pmatrix} \omega^{2jN} & 0 \\ 0 & \omega^{-2jN} \end{pmatrix}$. Thus in the case when k is even, by considering the matrix presentation of ρ^*_{jk} with respect to the basis $\{e^*_2, e^*_1\}$, we see that $[\rho^*_{jk}] = [\rho_{j,-k}]$. In the case when k is odd, by considering the matrix presentation of ρ^*_{jk} with respect to the basis $\{e^*_1, -\omega^{2jN} e^*_2\}$, we have the same result: $[\rho^*_{jk}] = [\rho_{j,-k}]$. □

From the above lemma, we see that the representation ring of A_{Nn} is described as in the following proposition. In the case when $N = 1$, this result has already proved by [7, Proposition 3.9].

Proposition A.3. *Let $\boldsymbol{k}$ be a field whose characteristic does not divide $2nN$, and suppose that there is a primitive $4nN$-th root of unity in $\boldsymbol{k}$.*

(1) *If n is even, then the representation rings of A_{Nn} and $\boldsymbol{k}[G_{Nn}]$ are isomorphic as rings with $*$-structure, and both of them are isomorphic to the commutative ring generated by $a, b, c, x_1, \ldots, x_{n-1}$ subject to the commutativity relations and the following relations:*

$$a^2 = b^2 = c^N = 1, \tag{A.5}$$
$$ax_i = x_i \quad (i = 1, 2, \ldots, n-1), \tag{A.6}$$
$$bx_i = x_{n-i} \quad (i = 1, 2, \ldots, n-1), \tag{A.7}$$
$$x_i x_j = c^{\frac{1-(-1)^{ij}}{2}}(x_{|i-j|} + x_{i+j}) \quad (i, j = 1, 2, \ldots, n-1), \tag{A.8}$$

where the indices of x in the right-hand side of (A.8) are treated under the rules

$$x_0 = 1 + a, \qquad x_n = b(1 + a), \qquad x_{n+i} = x_{n-i} \quad (i = 1, 2, \ldots, n-1).$$

The $*$-structure is given by $a^* = a$, $b^* = b$, $c^* = c^{-1}$, $x_i^* = c^{\frac{(-1)^i-1}{2}} x_i$ $(i = 1, \ldots, n-1)$.

(2) *If n is odd, then the representation rings of A_{Nn} and $\boldsymbol{k}[G_{Nn}]$ are isomorphic as rings with $*$-structure, and both of them are isomorphic to the commutative ring generated by $a, b, x_1, \ldots, x_{n-1}$ subject to the commutativity relations and the following relations:*

$$a^2 = b^{2N} = 1, \tag{A.9}$$
$$ax_i = x_i \quad (i = 1, \ldots, n-1), \tag{A.10}$$
$$bx_i = x_{n-i} \quad (i = 2, 4, \ldots, n-1), \tag{A.11}$$
$$x_i x_j = b^{1-(-1)^{ij}}(x_{|i-j|} + x_{i+j}) \quad (i, j = 1, \ldots, n-1) \tag{A.12}$$

where the indices of x in the right-hand side of (A.12) are treated under the rules

$$x_0 = 1 + a, \qquad x_n = b(1 + a), \qquad x_{n+i} = x_{n-i} \quad (i = 1, 2, \ldots, n-1).$$

The $*$-structure is given by $a^* = a$, $b^* = b^{-1}$, $x_i^* = b^{(-1)^i-1} x_i$ $(i = 1, \ldots, n-1)$.

Proof. Since the same results as in Lemma A.2 hold for the group Hopf algebra $\boldsymbol{k}[G_{Nn}]$, it is sufficient to prove that the representation ring of A_{Nn} is isomorphic to the commutative ring $\mathcal{R}$ which is presented by the given generators and relations described in the proposition. By Proposition A.1, we may assume that $A_{Nn} = \boldsymbol{k}[G_{Nn}]$ as algebras.

(1) By using Lemma A.2, we see that the representation ring $\mathrm{Rep}(A_{Nn})$ is the commutative ring generated by $a = [\chi_{100}]$, $b = [\chi_{010}]$, $c = [\chi_{002}]$, $x_i = [\rho_{i\epsilon(i)}]$ $(i = 1, \ldots, n-1)$ with relations (A.5)–(A.8), where χ_{ijk} and ρ_{jk} are the one-dimensional and two-dimensional representations of the algebra $A_{Nn} = \boldsymbol{k}[G_{Nn}]$ given by (5.20) and (5.21), respectively, and $\epsilon(i)$ is equal to 0 or 1 according to whether i is even or odd, respectively. Furthermore, we have $x_0 = [\chi_{000}] + [\chi_{100}] = 1 + a$, $x_n = [\chi_{010}] + [\chi_{110}] = [\chi_{010}] + [\chi_{010}][\chi_{100}] = b(1+a)$, and $x_{n+i} = [\rho_{n+i,\epsilon(n+i)}] = [\rho_{n-i,\epsilon(n+i)}] = [\rho_{n-i,\epsilon(n-i)}] = x_{n-i}$. From the results argued so far, we see that there is a ring homomorphism $f : \mathcal{R} \longrightarrow \mathrm{Rep}(A_{Nn})$ such that

$$f(a) = [\chi_{100}], \qquad f(b) = [\chi_{010}], \qquad f(c) = [\chi_{002}], \qquad f(x_i) = [\rho_{i\epsilon(i)}] \quad (i = 1, 2, \ldots, n-1).$$

The map f is bijective. The inverse map $g : \mathrm{Rep}(A_{Nn}) \longrightarrow \mathcal{R}$ is the $\mathbb{Z}$-linear map defined by

$$g([\chi_{ijk}]) = a^i b^j c^{\frac{k}{2}} \quad (i, j = 0, 1, k = 0, 2, \ldots, 2N-2),$$
$$g([\rho_{jk}]) = c^{\frac{k-\epsilon(j)}{2}} x_j \quad (k = 0, 1, \ldots, 2N-1, j = 1, 2, \ldots, n-1, j \equiv k \ (\mathrm{mod}\ 2)).$$

We conclude that $f : \mathcal{R} \longrightarrow \mathrm{Rep}(A_{Nn})$ is a ring isomorphism.

The $*$-structure in $\mathrm{Rep}(A_{Nn})$ is also determined by Parts (vi) and (vii) of Lemma A.2.

(2) In the same manner as above, it can be verified that there is a ring isomorphism $f : \mathcal{R} \longrightarrow \mathrm{Rep}(A_{Nn})$ such that $f(a) = [\chi_{100}], f(b) = [\chi_{010}], f(x_i) = [\rho_{i\epsilon(i)}]$ $(i = 1, \ldots, n-1)$ preserving $*$-structures. Here, what we should take account of is the following fact. If we set $b = [\chi_{010}]$, then $[\chi_{002}] = b^2$, and since n is odd, it follows from Parts (i) and (iv) of Lemma A.2 that

$$bx_i = [\chi_{010}][\rho_{i\epsilon(i)}] = [\rho_{n+i,1+\epsilon(i)}] = [\rho_{n-i,1+\epsilon(i)}] = \begin{cases} [\chi_{002} \otimes \rho_{n-i,0}] = b^2 x_{n-i} & (i \text{ is odd}), \\ [\chi_{000} \otimes \rho_{n-i,1}] = x_{n-i} & (i \text{ is even}). \end{cases}$$

This is equivalent to $bx_i = x_{n-i}$ for all even integers i. □

Remark A.4. In the case when $(\lambda, n) = (+, \text{even})$ or $(\lambda, n) = (-, \text{odd})$, as an algebra $A_{Nn}^{+\lambda}$ is isomorphic to the group algebra $\boldsymbol{k}[G'_{Nn}]$, where

$$G'_{Nn} = \langle h, t, w \mid t^2 = h^{2N} = 1, w^n = 1, tw = w^{-1}t, ht = th, hw = wh \rangle.$$

In a similar manner as in the proofs of Lemma A.2 and Proposition A.3, one can determine the structure of $\mathrm{Rep}(A_{Nn}^{++})$ in the case when n is even, and the structure of $\mathrm{Rep}(A_{Nn}^{+-})$ in the case when n is odd. As a result, we see that in the case when n is even the representation ring $\mathrm{Rep}(A_{Nn}^{++})$ is not commutative (see Lemma A.7 in Appendix A.3), whereas the representation ring $\mathrm{Rep}(\boldsymbol{k}[G'_{Nn}])$ is commutative since $\boldsymbol{k}[G'_{Nn}]$ is cocommutative. Therefore, two representation rings of A_{Nn}^{++} and $\boldsymbol{k}[G'_{Nn}]$ are not isomorphic. On the contrary, in the case when n is odd, we see that $\mathrm{Rep}(A_{Nn}^{+-}) \otimes_{\mathbb{Z}} \boldsymbol{k}$ and $\mathrm{Rep}(\boldsymbol{k}[G'_{Nn}]) \otimes_{\mathbb{Z}} \boldsymbol{k}$ are isomorphic as algebras with $*$-structure over $\boldsymbol{k}$.

A.3. Self-duality of $A_{Nn}^{+\lambda}$

Based on Proposition A.1, we identify $A_{Nn}^{+\lambda}$ with $\boldsymbol{k}[G]$ such as

$$h = x_{11}^2 - x_{12}^2,$$
$$t = x_{12}^N + x_{22}^N = x_{11}^{N-1}x_{22} + x_{12}^N,$$
$$w = x_{11}^{2N-1}x_{22} - x_{21}^{2N-1}x_{12} = x_{11}^{2N-1}\chi_{22} - x_{12}^{2N-2}\chi_{21}^2.$$

Then $w^{-1} = x_{22}x_{11}^{2N-1} - x_{12}x_{21}^{2N-1} = x_{11}^{2N-2}\chi_{22}^2 - x_{12}^{2N-1}\chi_{21}$.

Suppose that $\alpha, \beta \in \boldsymbol{k}$ satisfy $(\alpha\beta)^N = 1$ and $(\alpha\beta^{-1})^n = \lambda$, and consider the braiding $\sigma_{\alpha\beta}$ of $A_{Nn}^{+\lambda}$ defined in Theorem 5.7. We set $\xi := \alpha\beta$ and $\eta := \alpha\beta^{-1}$. By induction one can determine the values of $\sigma_{\alpha\beta}$ on the elements of the basis $\{h^i w^k t^p \mid 0 \le i \le 2N-1, 0 \le k \le n-1, p = 0, 1\}$ of $A_{Nn}^{+\lambda}$ as follows. For $i, j, k, l \ge 0$,

$$\sigma_{\alpha\beta}(h^i w^k, h^j w^l) = \xi^{2ij}\eta^{-2kl}, \tag{A.13}$$

$$\sigma_{\alpha\beta}(h^i w^k, h^j w^l t) = (-1)^{i+k}\xi^{2ij}\eta^{-k(2l-1)}, \tag{A.14}$$

$$\sigma_{\alpha\beta}(h^i w^k t, h^j w^l) = (-1)^{j+l}\xi^{2ij}\eta^{(2k-1)l}, \tag{A.15}$$

$$\sigma_{\alpha\beta}(h^i w^k t, h^j w^l t) = (-1)^{i+j+k+l}\xi^{2ij+\frac{N^2-1}{2}}\eta^{2kl-k-l}\alpha. \tag{A.16}$$

From here to the end of the paper, we suppose that $\boldsymbol{k}$ is a field whose characteristic does not divide $2nN$, and that it contains a primitive $4nN$-th root of unity.

Theorem A.5. *Let α, β be elements in $\boldsymbol{k}$ satisfying $(\alpha\beta)^N = 1$ and $(\alpha\beta^{-1})^n = \lambda$. Suppose that $\lambda = -1$ or $(\lambda, n) = (1, \text{odd})$. Then the braiding $\sigma_{\alpha\beta}$ of $A_{Nn}^{+\lambda}$ is non-degenerate if and only if*

(i) *$\alpha\beta$ is a primitive N-th root of unity, and*
(ii) *$\alpha\beta^{-1}$ is a primitive n-th root of λ.*

Proof. To show the "if" part, we show the contraposition.

Suppose that $\alpha\beta$ is not a primitive N-th root of unity. Then $N \ge 3$ is required since, if $N = 1$, then $\alpha\beta = (\alpha\beta)^N = 1$. Let ξ be a primitive N-th root of unity. Then $\alpha\beta$ is represented by $\alpha\beta = \xi^m$ for some divisor $m\ (\neq 1)$ of N. Hence, by setting $m' := N/m\ (< N)$, we have $\sigma_{\alpha\beta}(h^{2m'}, h^j w^l) = (\alpha\beta)^{4m'j} = 1 = \sigma_{\alpha\beta}(1,\ h^j w^l)$, $\sigma_{\alpha\beta}(h^{2m'}, h^j w^l t) = (-1)^{2m'}(\alpha\beta)^{4m'j} = 1 = \sigma_{\alpha\beta}(1,\ h^j w^l t)$. Thus $\sigma_{\alpha\beta}(1 - h^{2m'}, a) = 0$ for all $a \in A_{Nn}^{+\lambda}$. Since $1 < 2m' < 2N$, we see that $1 - h^{2m'} \neq 0$. Therefore, $\sigma_{\alpha\beta}$ degenerates as a bilinear form on $A_{Nn}^{+\lambda}$.

Next, suppose that $\alpha\beta^{-1}$ is not a primitive n-th root of λ. Then there is an $r \in \mathbb{N}$ such that $1 \le r < n$ and $(\alpha\beta^{-1})^r = \lambda$. So, $(\alpha\beta^{-1})^{n-r} = 1$. By (A.13) and (A.14), we have $\sigma_{\alpha\beta}(w^{n-r}, h^j w^l) = (\alpha\beta^{-1})^{-2(n-r)l} = 1$, $\sigma_{\alpha\beta}(w^{n-r}, h^j w^l t) = (-1)^{n-r}(\alpha\beta^{-1})^{-(n-r)(2l-1)} = (-1)^{n-r}$. Thus, if $n - r$ is even, then $\sigma_{\alpha\beta}(1 - w^{n-r}, a) = 0$ for all $a \in A_{Nn}^{+\lambda}$. It follows from $0 < n-r \le n-1$ that $1 - w^{n-r} \neq 0$, and hence $\sigma_{\alpha\beta}$ degenerates. If $n-r$ is odd, then $\sigma_{\alpha\beta}(h^N - w^{n-r}, a) = 0$ for all $a \in A_{Nn}^{+\lambda}$. Since $h^N - w^{n-r} \neq 0$, the braiding $\sigma_{\alpha\beta}$ also degenerates as a bilinear form.

We will show the "only if" part. Let us consider the linear map $F : A_{Nn}^{+\lambda} \longrightarrow (A_{Nn}^{+\lambda})^*$ defined by

$$F(a) = \sum_{j=0}^{2N-1}\sum_{l=0}^{n-1} \sigma_{\alpha\beta}(a, h^j w^l)(h^j w^l)^* + \sum_{j=0}^{2N-1}\sum_{l=0}^{n-1} \sigma_{\alpha\beta}(a, h^j w^l t)(h^j w^l t)^* \qquad (a \in A_{Nn}^{+\lambda}).$$

Here, $\{(h^j w^l t^p)^* \mid 0 \le j \le 2N-1, 0 \le l \le n, p = 0, 1\}$ stands for the dual basis of the basis $\{h^j w^l t^p \mid 0 \le j \le 2N-1, 0 \le l \le n, p = 0, 1\}$ of $A_{Nn}^{+\lambda}$. Setting $\xi = \alpha\beta$, $\eta = \alpha\beta^{-1}$, we have

$$F(h^iw^k)=\sum_{j=0}^{2N-1}\sum_{l=0}^{n-1}\xi^{2ij}\eta^{-2kl}\big((h^jw^l)^*+(-1)^{i+k}\eta^k(h^jw^lt)^*\big),$$
$$F(h^iw^kt)=\sum_{j=0}^{2N-1}\sum_{l=0}^{n-1}(-1)^{j+l}\xi^{2ij}\eta^{2kl-l}\big((h^jw^l)^*+(-1)^{i+k}\xi^{\frac{N^2-1}{2}}\eta^{-k}\alpha(h^jw^lt)^*\big).$$

In what follows, let ξ be a primitive N-th root of unity, and η be a primitive n-th root of λ.
Since N is odd, for $j,j'\in\mathbb{Z}$ we have

$$\sum_{i=0}^{2N-1}\xi^{2i(j-j')}=\begin{cases}2N & \text{if } j\equiv j' \pmod N,\\ 0 & \text{otherwise,}\end{cases}\qquad \sum_{i=0}^{2N-1}(-1)^i\xi^{2i(j-j')}=0.$$

Furthermore, since $\eta^{2(l-l')}=1$ if and only if $l-l'\equiv 0 \pmod n$ under the condition $\lambda=-1$ or $(\lambda,n)=(1,\text{odd})$, it follows that, for $l'\in\mathbb{Z}$,

$$\sum_{k=0}^{n-1}\eta^{2k(l-l')}=\begin{cases}0 & \text{otherwise,}\\ n & \text{if } l\equiv l' \pmod n.\end{cases}\tag{A.17}$$

Therefore,

$$\sum_{i=0}^{2N-1}\sum_{k=0}^{n-1}\eta^{2kl'}\xi^{-2ij'}F(h^iw^k)=2nN\big((h^{j'}w^{l'})^*+(h^{j'+N}w^{l'})^*\big),$$
$$\sum_{i=0}^{2N-1}\sum_{k=0}^{n-1}\eta^{-2kl'}\xi^{-2ij'}F(h^iw^kt)=2nN(-1)^{j'+l'}\eta^{-l'}\big((h^{j'}w^{l'})^*-(h^{j'+N}w^{l'})^*\big).$$

From these equations, we have

$$(h^{j'}w^{l'})^*=\frac{1}{4nN}\left(\sum_{i=0}^{2N-1}\sum_{k=0}^{n-1}\eta^{2kl'}\xi^{-2ij'}F(h^iw^k)+(-1)^{j'+l'}\eta^{l'}\sum_{i=0}^{2N-1}\sum_{k=0}^{n-1}\eta^{-2kl'}\xi^{-2ij'}F(h^iw^kt)\right),$$
$$(h^{j'+N}w^{l'})^*=\frac{1}{4nN}\left(\sum_{i=0}^{2N-1}\sum_{k=0}^{n-1}\eta^{2kl'}\xi^{-2ij'}F(h^iw^k)-(-1)^{j'+l'}\eta^{l'}\sum_{i=0}^{2N-1}\sum_{k=0}^{n-1}\eta^{-2kl'}\xi^{-2ij'}F(h^iw^kt)\right).$$

In a similar manner, it can be proved that $(h^{j'}w^{l'}t)^*$ and $(h^{j'+N}w^{l'}t)^*$ are linear combinations of $\{F(h^iw^k),F(h^iw^kt)\mid 0\le i\le 2N-1,\ 0\le k\le n-1\}$. Thus F is surjective, and hence F is an isomorphism. This implies that $\sigma_{\alpha\beta}$ is non-degenerate. □

Corollary A.6. *Suppose that $\lambda=-1$, or $(\lambda,n)=(1,\text{odd})$. Then the Hopf algebra $A_{Nn}^{+\lambda}$ is self-dual.*

Proof. In general, for a finite-dimensional Hopf algebra A, any braiding $\sigma: A\otimes A\longrightarrow \boldsymbol{k}$ gives rise to the Hopf pairing $\langle\ ,\ \rangle: A^{\mathrm{cop}}\otimes A\longrightarrow \boldsymbol{k}$ defined by $\langle x,y\rangle=\sigma(x,y)$ for $x,y\in A$, and this pairing induces a Hopf algebra map $F:A\longrightarrow (A^{\mathrm{cop}})^*$ defined by $(F(a))(b)=\sigma(a,b)$ for $a,b\in A$. Applying this fact to the Hopf algebra $A_{Nn}^{+\lambda}$ and the braiding $\sigma_{\alpha\beta}$, we have a Hopf algebra map $F:A_{Nn}^{+\lambda}\longrightarrow ((A_{Nn}^{+\lambda})^{\mathrm{cop}})^*$. Furthermore, an algebra isomorphism $\phi: A_{Nn}^{+\lambda}\longrightarrow A_{Nn}^{+\lambda}$ can be defined by $\phi(x_{ij})=x_{ji}$ $(i,j=1,2)$, and we see that it becomes a Hopf algebra isomorphism form $A_{Nn}^{+\lambda}$ to $(A_{Nn}^{+\lambda})^{\mathrm{cop}}$ [12]. So, if $\sigma_{\alpha\beta}$ is non-degenerate, then the composition ${}^t\phi\circ F: A_{Nn}^{+\lambda}\longrightarrow (A_{Nn}^{+\lambda})^*$ gives a Hopf algebra isomorphism.

To complete the proof, by Theorem A.5, it suffices to show that there are α,β such that $\alpha\beta$ is a primitive N-th root of unity, and $\alpha\beta^{-1}$ is a primitive n-th root of λ. Let $\omega\in\boldsymbol{k}$ be a primitive $4nN$-th root of unity. In the case when $\lambda=-1$, we take $\alpha=\omega^{N+2n}$, $\beta=\omega^{2n-N}$. Then $\alpha\beta^{-1}=\omega^{2N}$ is a primitive n-th root of -1, and $\alpha\beta=\omega^{4n}$ is a primitive N-th root of unity. In the case when $\lambda=1$ and n is odd, we take $\alpha=\omega^{2N+2n}$, $\beta=\omega^{2n-2N}$. Then $\alpha\beta^{-1}=\omega^{4N}$ and $\alpha\beta=\omega^{4n}$ are primitive n-th and primitive N-th roots of unity, respectively. □

To show that A_{Nn}^{++} is not self-dual for any even integer n, we compare the groups of group-like elements of A_{Nn}^{++} and $(A_{Nn}^{++})^*$. The structure of $G(A_{Nn}^{\nu\lambda})$ for all $N\ge 1$, $n\ge 2$ and $\lambda,\nu=\pm 1$ has already been determined by Suzuki [12]. In the case when N is odd, and $\nu=+$, the group $G(A_{Nn}^{\nu\lambda})$ is given as follows.

$$G(A_{Nn}^{+\lambda})\cong\begin{cases}C_2\times C_{2N} & (n \text{ is even, or } (n,\lambda)=(\text{odd},1)),\\ C_{4N} & (n \text{ is odd, and } \lambda=-1).\end{cases}\tag{A.18}$$

On the contrary, we have:

Lemma A.7. *The structure of $G((A_{Nn}^{+\lambda})^*)$ is given as follows.*

$$G((A_{Nn}^{++})^*) \cong \begin{cases} SA_{8N} & (n \text{ is even}), \\ C_2 \times C_{2N} & (n \text{ is odd}), \end{cases} \qquad G((A_{Nn}^{+-})^*) \cong \begin{cases} C_2 \times C_2 \times C_N & (n \text{ is even}), \\ C_{4N} & (n \text{ is odd}), \end{cases}$$

where SA_{8N} is the finite group of order $8N$ defined by $SA_{8N} = \langle\, b, c \mid b^2 = c^{4N} = 1, cb = bc^{2N+1} \,\rangle$.

Proof. Let ω be a primitive $4nN$-th root of unity.

(1) If n is even, then $G((A_{Nn}^{++})^*) = \{\chi_{ijk} \mid i, j = 0, 1, k = 0, 1, \ldots, 2N-1\}$, where $\chi_{ijk} : A_{Nn}^{++} \longrightarrow \boldsymbol{k}$ is the algebra map defined by $\chi_{ijk}(t) = (-1)^i$, $\chi_{ijk}(w) = (-1)^j$, $\chi_{ijk}(h) = \omega^{2nk}$. Since the product of $G((A_{Nn}^{++})^*)$ is given by $\chi_{ijk}\chi_{i'j'k'} = \chi_{i+i'+(j+k)k', j+j', k+k'}$ for all integers i, i', j, j', k, k', and $a := \chi_{100}$, $b := \chi_{010}$, $c := \chi_{001}$ satisfy the equations $a^2 = b^2 = 1$, $c^{2N} = a$, $cb = bc^{2N+1}$, we have $G((A_{Nn}^{++})^*) = \langle\, b, c \mid b^2 = c^{4N} = 1, cb = bc^{2N+1} \,\rangle = SA_{8N}$.
If n is odd, then $G((A_{Nn}^{++})^*) = \{\, \chi_{ik} \mid i = 0, 1, k = 0, 1, \ldots, 2N-1 \,\}$, where $\chi_{ik} : A_{Nn}^{++} \longrightarrow \boldsymbol{k}$ is the algebra map defined by $\chi_{ik}(t) = (-1)^i$, $\chi_{ik}(w) = (-1)^k$, $\chi_{ik}(h) = \omega^{2nk}$. Since the product of $G((A_{Nn}^{++})^*)$ is given by $\chi_{ik}\chi_{i'k'} = \chi_{i+i', k+k'}$ for all integers i, i', k, k', we see that $G((A_{Nn}^{++})^*) \cong C_2 \times C_{2N}$.

(2) If n is even, then we see that $G((A_{Nn}^{+-})^*) = \{\, \chi_{ijk} \mid i, j = 0, 1, k = 0, 2, \ldots, 2N-2 \,\}$, where χ_{ijk} is the algebra map defined in the same way as in the proof of Part (1). Since the product of $G((A_{Nn}^{+-})^*)$ is given by $\chi_{ijk}\chi_{i'j'k'} = \chi_{i+i', j+j', k+k'}$, and $a := \chi_{100}$, $b := \chi_{010}$, $c := \chi_{002}$ satisfy the equations $a^2 = b^2 = 1$, $c^N = 1$, we see that $G((A_{Nn}^{+-})^*) = C_2 \times C_2 \times C_N$.
If n is odd, then $G((A_{Nn}^{+-})^*) = \{\chi_{ik} \mid i = 0, 1, k = 0, 2, \ldots, 2N-2\}$, where $\chi_{ik} : A_{Nn}^{+-} \longrightarrow \boldsymbol{k}$ is the algebra map such that $\chi_{ik}(t) = (-1)^i$, $\chi_{ik}(w) = 1$, $\chi_{ik}(h) = \omega^{2nk}$. The product of $G((A_{Nn}^{+-})^*)$ is given by $\chi_{ik}\chi_{i'k'} = \chi_{i+i'+kk', k+k'}$, and $a := \chi_{10}$ and $b := \chi_{01}$ satisfy the equations $a^2 = 1$, $b^{2N} = a$. Hence $G((A_{Nn}^{+-})^*) = C_{4N}$. □

If a semisimple Hopf algebra A possesses a quasitriangular structure, the representation ring needs to be commutative. In the case when $N \geq 1$ is odd, and n is even, by Lemma A.7 the representation ring of the dual Hopf algebra $(A_{Nn}^{++})^*$ is not commutative, and therefore there is no quasitriangular structure of $(A_{Nn}^{++})^*$. By (A.18) and Lemma A.7 we have:

Proposition A.8. *If n is even, then the Hopf algebra A_{Nn}^{++} is not self-dual.*

Proof. The group $G(A_{Nn}^{++}) \cong C_2 \times C_{2N}$ is commutative by (A.18); meanwhile, $G((A_{Nn}^{++})^*) \cong SA_{8N}$ is not by Lemma A.7. This implies that $G(A_{Nn}^{++}) \not\cong G((A_{Nn}^{++})^*)$, and $A_{Nn}^{++} \not\cong (A_{Nn}^{++})^*$. □

Corollary A.9. *If n is even, then all braidings of A_{Nn}^{++} degenerate.*

Proof. Assume that there is a non-degenerate braiding of A_{Nn}^{++}. Then there is a Hopf algebra isomorphism $F : A_{Nn}^{++} \longrightarrow ((A_{Nn}^{++})^{\text{cop}})^*$. Let us consider the Hopf algebra isomorphism $\phi : A_{Nn}^{++} \longrightarrow (A_{Nn}^{++})^{\text{cop}}$ defined by $\phi(x_{ij}) = x_{ji}$ $(i, j = 1, 2)$. Then the composition ${}^t\phi \circ F : A_{Nn}^{++} \longrightarrow (A_{Nn}^{++})^*$ is also a Hopf algebra isomorphism. This contradicts Proposition A.8. □

References

[1] P. Schauenburg, Hopf bigalois extensions, Comm. Algebra 24 (1996) 3797–3825.
[2] P. Schauenburg, Hopf-Galois and Bi-Galois extensions, in: G. Janelidze, B. Pareigis, W. Tholen (Eds.), Galois Theory, Hopf Algebras, and Semiabelian Categories, in: Fields Inst. Comm., vol. 43, AMS, Providence, RI, 2004, pp. 469–515.
[3] Y. Doi, Braided bialgebras and quadratic bialgebras, Comm. Algebra 21 (1993) 1731–1749.
[4] A. Masuoka, Cleft extensions for a Hopf algebra generated by a nearly primitive element, Comm. Algebra 22 (1994) 4537–4559.
[5] Y. Doi, M. Takeuchi, Quaternion algebras and Hopf crossed products, Comm. Algebra 23 (1995) 3291–3325.
[6] P. Schauenburg, Bi-Galois objects over the Taft algebras, Israel J. Math. 115 (2000) 101–123.
[7] A. Masuoka, Cocycle deformations and Galois objects for some cosemisimple Hopf algebras of finite dimension, Contemp. Math. 267 (2000) 195–214.
[8] S. Majid, Representation-theoretic rank and double Hopf algebras, Comm. Algebra 18 (1990) 3705–3712.
[9] D. Tambara, S. Yamagami, Tensor categories with fusion rules of self-duality for finite abelian groups, J. Algebra 209 (1998) 692–707.
[10] G.I. Kac, V.G. Paljutkin, Finite ring groups, in: Transactions of the Moscow Mathematical Society for the year 1966, AMS, 1967, pp. 251–294 (original Russian paper: Trudy Moscov. Math. Ob. 15 (1966) 224–261).
[11] A. Masuoka, Semisimple Hopf algebras of dimension 6, 8, Israel J. Math. 92 (1995) 361–373.
[12] S. Suzuki, A family of braided cosemisimple Hopf algebras of finite dimension, Tsukuba J. Math. 22 (1998) 1–29.
[13] E. Abe, Hopf Algebras, Cambridge Univ. Press, Cambridge, 1980 (original Japanese version published by Iwanami Shoten, Tokyo, 1977).
[14] S. Montgomery, Hopf Algebras and their Action on Rings, in: C.B.M.S., vol. 82, American Mathematical Society, 1993.
[15] M.E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[16] S. MacLane, Categories for the Working Mathematician, in: G.T.M., vol. 5, Springer-Verlag, New York, 1971.
[17] A. Joyal, R. Street, Braided tensor categories, Adv. Math. 102 (1993) 20–78.
[18] V.G. Drinfel'd, Quantum groups, in: Proceedings of the International Congress of Mathematics, Berkeley, CA., 1987, pp. 798–820.
[19] P. Etingof, S. Gelaki, On the exponent of finite-dimensional Hopf algebras, Math. Res. Lett. 6 (1999) 131–140.
[20] D.E. Radford, On the quasitriangular structures of a semisimple Hopf algebra, J. Algebra 141 (1991) 354–358.
[21] R.G. Larson, Characters of Hopf algebras, J. Algebra 17 (1971) 352–368.
[22] D.E. Radford, On the antipode of a quasitriangular Hopf algebra, J. Algebra 151 (1992) 1–11.
[23] D.E. Radford, Minimal quasitriangular Hopf algebras, J. Algebra 157 (1993) 285–315.
[24] R.G. Larson, D.E. Radford, Semisimple Hopf algebras, J. Algebra 171 (1995) 5–35.
[25] P. Etingof, S. Gelaki, On finite-dimensional semisimple and cosemisimple Hopf algebras in positive characteristic, Int. Math. Res. Not. (16) (1998) 851–864.
[26] R.G. Larson, D.E. Radford, Finite dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple, J. Algebra 117 (1988) 267–289.

[27] P. Schauenburg, On Coquasitriangular Hopf Algebras and the Quantum Yang–Baxter Equation, in: Algebra Berichte, vol. 67, Verlag Reinhard Fischer, 1992.
[28] M. Wakui, On the universal R-matrices of the dihedral groups, Sūrikaisekikenkyūsho Kōkyūroku 1057 (1998) 41–53 (in Japanese).
[29] C. Călinescu, S. Dăscălescu, A. Masuoka, C. Menini, Quantum lines over non-cocommutative cosemisimple Hopf algebras, J. Algebra 273 (2004) 753–779.
[30] W.D. Nichols, M.B. Richmond, The Grothendieck group of a Hopf algebra, J. Pure Appl. Algebra 106 (1996) 297–306.
[31] D. Nikshych, K_0-rings and twisting of finite dimensional semisimple Hopf algebras, Comm. Algebra 26 (1998) 321–342 (Erratum: Comm. Algebra 26 (1998) 1347).