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Long Time Behavior for Semiclassical NLS

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Abstract—In a recent paper, Jin, Levermore and McLaughlin analyze the semiclassical behavior of solutions to the defocusing, completely integrable nonlinear Schrödinger equation. We complete their analysis, by providing the long time behavior of the semiclassical solutions. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords—Solitons, Nonlinear Schrödinger equation, Semiclassical asymptotics.

In their well-known papers Lax and Levermore [1] analyze the solutions of the zero dispersion limit of KdV, under fairly general initial data, either belonging in the Schwartz class, or of shock type. In the third of the series, they also give the long-time asymptotics for such solutions.

Following Lax and Levermore, an analogous discussion of the semiclassical defocusing NLS equation is given in [2]. Whitham equations are introduced and weak limits of the squared density, the momentum, and the energy of solutions are expressed in terms of the Riemann invariants of the Whitham system. Although long-time formulae are not given in [2], they would be of some value[1]. The statement and proof of such formulae is the aim of this note.

THEOREM. *Let $u(x,t;h)$ solve*

$$ihu_t(x,t;h) + \frac{h^2}{2}u_{xx}(x,t;h) + \left(1 - |u(x,t;h)|^2\right)u(x,t;h) = 0,$$

with the far − field boundary condition

$$u(x,t) \sim exp\left(\frac{\pm iS_\infty}{h}\right), \; for\ some\ S_\infty \in \mathbb{R}, \tag{1}$$

and the initial condition

$$u(x,0;h) = A(x)exp\left(\frac{iS(x)}{h}\right),$$

[1]They could be used, for example, in evaluating the $k - \epsilon$ turbulence model for the Navier-Stokes equations [3].

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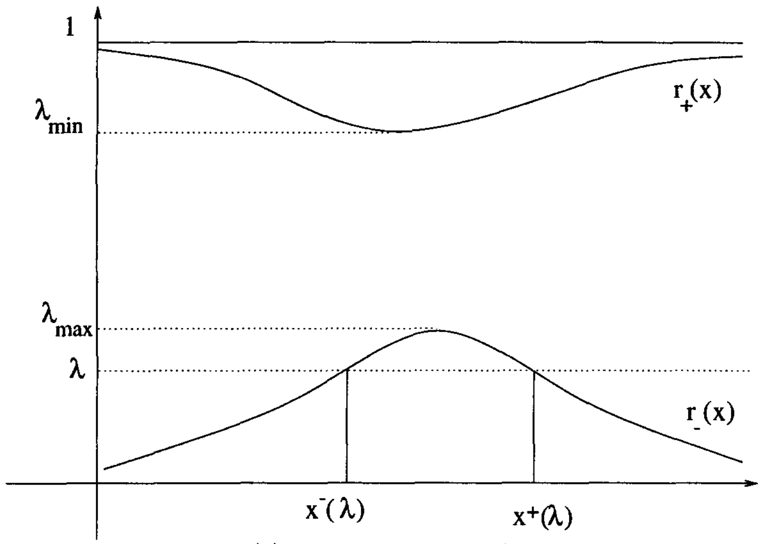

Figure 1. The initial data $r_{\pm}(x)$. Note their critical values $\lambda_{\min}$ and $\lambda_{\max}$, and the indicated defining relations for the turning points $x^{\pm}(\lambda)$.

where $A(x) - 1$ and the x-derivative S_x belong in the Schwartz class. Let us also assume, for simplicity, that the initial data are 'single well' in the sense of [2] (see Figure 1). In other words, the r_- defined below has only one maximum $= \lambda_{\max}$ (respectively, r_+ has only one minimum $= \lambda_{\min}$) and $-1 < \lambda_{\max} < \lambda_{\min} < 1$. Then, the weak limit $\bar{\rho}(x,t) = \lim_{h\to 0} |u(x,t;h)|^2$ exists and in the 'Whitham' region $x/t \in (-1, \lambda_{\max}) \cup (\lambda_{\min}, 1)$, we have, as $t \to \infty$,

$$\begin{gathered}\bar{\rho}(x,t) \sim 1 - \frac{4}{\pi t}\phi\left(\frac{x}{t}\right)\left(1 - \left(\frac{x}{t}\right)^2\right)^{1/2}, \quad \text{where} \\ \phi(\lambda) = \int_{x_-(\lambda)}^{x_+(\lambda)} \frac{|\lambda - 1/2(r_+(s) + r_-(s))|}{(\lambda - r_+(s))^{1/2}(\lambda - r_-(s))^{1/2}}\,ds, \\ r_{\pm}(x) = \frac{S_x}{2} \pm A(x), \\ \text{and } x_{\pm} \text{ are defined by } r_-(x_{\pm}(\lambda)) = \lambda, \qquad x_- < x_+.\end{gathered} \tag{2}$$

Outside the Whitham region, $\bar{\rho} \sim 1$.

PROOF. The existence of the weak limit is proved in [2]. The long-time behavior can be derived following [1] in two ways. One can use the semiclassical formulae to derive long-time asymptotics for the Riemann invariants of the Whitham equations. We prefer to follow an alternative way (also suggested in [1]) of beginning with the multisoliton formula for fixed h and then taking $h \to 0$.

For fixed h, the long-time behavior of $|u|^2$ is as follows [4, pp. 168–176]. In the solitonless regions $|x/t| > 1$ and $\lambda_{\max} < x/t < \lambda_{\min}$, we have $|u|^2 = 1 - O(t^{-1/2})$, as $t \to \infty$. In the Whitham region, the solution is a multisoliton solution:

$$\begin{gathered}|u(x,t;h)|^2 \sim 1 - \sum_{n=1}^{N(h)} s(x - \eta_n t - x_n, \eta_n), \qquad \text{where} \\ s(x,\eta) = \frac{1-\eta^2}{\cosh^2((1-\eta^2)^{1/2}(x/2h))},\end{gathered} \tag{3}$$

with exponentially small error. The eigenvalues of the associated Lax operator η_n accumulate in the set $(-1, \lambda_{\max}) \cup (\lambda_{\min}, 1)$. The x_ns are some phase constants of no importance.

The width of each soliton $s(x,\eta)$ is $O(h/(1-\eta^2)^{1/2})$. By Weyl's Law for the distribution of eigenvalues in $(-1,\lambda_{\max})\cup(\lambda_{\min},1)$ as $h\to 0$,

$$\eta_{n+1}-\eta_n=\frac{\pi h}{\phi(\bar\eta_n)}, \tag{4}$$

where $\bar\eta_n\in(\eta_n,\eta_{n+1})$.

Peaks of solitons are located at $\eta_n t$. As $t\to\infty$, they are separated by $\pi ht/\phi(\eta_n)$, so for large t, they are well separated. The wave number η of the soliton that peaks at x at time t is $\eta=x/t$, if t is large and either $-1<x/t<\lambda_{\max}$ or $\lambda_{\min}<x/t<1$. So the density of the solitons is

$$\frac{\phi(x/t)}{\pi ht}. \tag{5}$$

The area between a soliton and the line $u=1$ is

$$4h\left(1-\eta^2\right)^{1/2}\sim 4h\left(1-\left(\frac{x}{t}\right)^2\right)^{1/2}, \tag{6}$$

so the asymptotic area density is the product of (5) and (6):

$$\frac{4\phi(x/t)}{\pi}\left(1-\left(\frac{x}{t}\right)^2\right)^{1/2}.$$

The asymptotic area density is $1-\bar\rho$. Hence, the asymptotic formula for the weak limit $\bar\rho$ follows readily.

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