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On Two Theorems by Hartman and Wintner. An Application of the Ważewski Retract Method

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1. INTRODUCTION

In [3, 4] Hartman and Wintner proved the existence of monotone decreasing, nonnegative, and nontrivial solutions of certain ordinary differential systems in R^n,

$$\dot{x} = g(t, x), \qquad \text{for } t \geqslant 0.$$

In [3] they considered a linear system and in [4] a more general nonlinear case.

In this paper, we will show that these theorems of Hartman and Wintner are special cases of a much more general theorem (our Theorem 3), which we prove by an application of the Ważewski retract method.

For background material and applications, we refer to [3, 4]. The emphasis here is on the retract method. We are not looking for monotone solutions, but rather solutions which remain inside a given domain. The translation to the cases considered by Hartman and Wintner is, however, immediate.

2. RESULTS

LEMMA. *Let A and B be sets in a topological space X. Let $A \cup B$ be contractible, but not A. Then A is not a retract of $A \cup B$.*

Proof. This is clear from the definitions of contraction and retraction.

EXAMPLE. We give an example in R^n (see Fig. 1). Put

$$V = \left\{ x \mid x \geqslant 0, \sum_{i=1}^{n} x_i \leqslant 1, \prod_{i=1}^{n} x_i = 0, x \neq 0 \right\},$$

$$\Sigma = \left\{ x \mid x > 0, \sum_{i=1}^{n} x_i = 1 \right\},$$

$$\Omega = \left\{ x \mid x > 0, \sum_{i=1}^{n} x_i < 1 \right\}.$$

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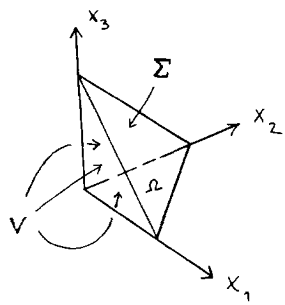

FIG. 1. $n = 3$.

Obviously, V is not contractible whereas $V \cup \Sigma$ is.

Hence V is not a retract of $V \cup \Sigma$.

In [3] Hartman and Wintner proved the following theorem (see also [2, p. 506]):

THEOREM 1. *Let $A(t)$ be an $n \times n$ continuous matrix for $t \geqslant 0$ such that $A(t) \geqslant 0$.*

Then the differential system

$$\dot{x} = -A(t)\,x$$

has at least one solution $x(t) \neq 0$ such that $x(t) \geqslant 0$ for $t \geqslant 0$ (and consequently $\dot{x}(t) \leqslant 0$).

Remark. Hartman and Wintner considered the interval $0 < t < \infty$, but the difference is unimportant.

Proof. We shall use the Ważewski retraction method; see [5], [1], or [2, p. 279]. Introduce the $n \times n$ matrix

$$M = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

and choose a sequence $\epsilon_k \searrow 0$. Consider the system

$$\dot{x} = -(A(t) + \epsilon_k M)\,x \qquad (*)$$

with the condition $x(0) = x_0$, where x_0 is a variable point on Σ. (V, Σ, Ω are as in the example.) Now, if $\bar{x} \in V$ and $\sum_{i=1}^{n} \bar{x}_i < 1$, it follows (since $\dot{x} < 0$ there) that $(t, \bar{x})$ is a strict egress point, for any $t > 0$, and relative to Ω. Further, there are no other egress points on $\partial\Omega$, for any $t \geqslant 0$, so all egress points are strict.

Now let $x_0 \in \Sigma$, and let $x_0(t)$ satisfy (*) and $x_0(0) = x_0$. If $x_0(t)$ does not stay in Ω for $t > 0$, let τ be the first instant, when $x_0(t) \in V$. Clearly, τ is well defined. Put $x_0(\tau) = \phi(x_0)$. Suppose now that $x_0(t)$ leaves Ω for any $x_0 \in \Sigma$. If we extend

ϕ as the identity on V, then ϕ is continuous on $V \cup \Sigma$ (see [2, p. 280]). Thus, ϕ is a retraction of $V \cup \Sigma$ onto V, which is impossible, according to the example. Hence $x_0(t) \in \Omega$ for $t > 0$, and for some $x_{0,k} \in \Sigma$.

Now the proof is completed by choosing a subsequence $x_{0,k_j}(t)$, which converges uniformly on every interval $0 \leqslant t \leqslant T < \infty$. The remaining arguments are obvious.

Remark 1. It is clear that if $A(t) > 0$ for $t \geqslant 0$, then we obtain a solution $x(t)$ such that $x(t) > 0$ for $t \geqslant 0$. But under the assumption $A(t) \geqslant 0$, there need not exist a strictly positive solution. A counterexample is easily given.

Remark 2. The retract argument could equally well be based on the obvious fact that the relative boundary of Σ, i.e., $\bar{\Sigma} \cap V$, is not a retract of $\bar{\Sigma}$, but is a retract of V.

We shall next consider a nonlinear system. We retain the notations Ω, Σ, V and we put $E = \bar{\Omega} \times [0, \infty) \subset R^{n+1}$.

THEOREM 2. *Let $U \subset R^{n+1}$ be an open set containing E and let $f(t, x)$: $U \to R^n$ be continuous on U together with the partials*

$$\frac{\partial f_i}{\partial x_j}(t, x), \qquad i, j = 1, 2,..., n.$$

Assume that $f(t, 0) = 0$ for $t \geqslant 0$. Assume that $f_i(t, x) \geqslant 0$ as soon as $x \in V$ and $x_i = 0$. Finally, assume that $\sum_{i=1}^{n} f_i(t, x) \geqslant 0$ if $x \in \Sigma$.

Consider the system

$$\dot{x} = -f(t, x). \tag{1}$$

Then there is a solution $x_0(t)$ of (1), *such that $x_0(t) \geqslant 0$ and $x_0(t) \neq 0$ for $t \geqslant 0$. Further, $\sum_{i=1}^{n} x_{0,i}(t) \leqslant 1$, with equality for $t = 0$. Observe that we have made no assumption concerning the sign of $f(t, x)$ for $x \in \Omega$.*

Proof of the Theorem. This is analogous to the proof of Theorem 1. The auxiliary systems are now

$$\dot{x} = -[f(t, x) + \epsilon_k Mx], \qquad x(0) = x_0 \in \Sigma.$$

The details are left to the reader.

THEOREM 3. Let *$U \subset R^{n+1}$ be an open set containing E and let the mapping $f(t, x)$: $U \to R^n$ be continuous. Assume that $f(t, 0) = 0$ for $t \geqslant 0$. Let $f_i(t, x) \geqslant 0$ if $x \in V$, $x_i = 0$, and $t \geqslant 0$. Finally, assume that $\sum_{i=1}^{n} f_i(t, x) \geqslant 0$ if $x \in \Sigma$ and $t \geqslant 0$.*

Then there is a solution $x_0(t)$ of the system $\dot{x} = -f(t, x)$ such that $x_0(t) \geqslant 0$ and $\sum_{i=1}^{n} (x_0(t))_i \leqslant 1$, with equality for $t = 0$.

Proof. (1) Suppose we can construct a solution of $\dot{x} = -f(t, x)$, with required properties on $0 \leqslant t \leqslant N$, for any natural number N. Then a standard application of Arzela's theorem will give us a solution on $0 \leqslant t < \infty$. Therefore, we may restrict our attention to $0 \leqslant t \leqslant N$.

(2) We thus consider $[0, N]$ and put $E_N = \bar{\Omega} \times [0, N]$. Introduce a function $\psi(r) \in C^\infty$ such that $\psi(r) = 0$ for $r \leqslant 1$, $0 \leqslant \psi(r) \leqslant 1$ for $1 \leqslant r \leqslant 2$, and $\psi(r) = 1$ for $r \geqslant 2$.

Take a sequence $r_k \searrow 0$ and consider

$$f_k(t, x) = \psi(\| x \|/r_k) f(t, x).$$

Since $f(t, 0) = 0$, it easily follows that $f_k(t, x) \to f(t, x)$ uniformly on E_N. Suppose that we can find a solution $x_k(t)$ of $\dot{x} = -f_k(t, x)$ for $0 \leqslant t \leqslant N$, with the required properties, for $k = 1, 2, 3,\ldots$. Then an easy application of Arzela's theorem gives a solution of $\dot{x} = -f(t, x)$ for $0 \leqslant t \leqslant N$. Therefore, we may assume that $f(t, x) = 0$ if $\| x \| \leqslant \Theta$, for some $\Theta > 0$.

(3) We shall now construct a solution of $\dot{x} = -f(t, x)$ over $0 \leqslant t \leqslant N$. Besides the assumptions in the theorem, we assume that $f(t, x) = 0$, if $\| x \| \leqslant \Theta$, for some $\Theta > 0$. Take a function $\phi(x) \in C_0^\infty(R^n)$ such that $\phi(x) \geqslant 0$, $\int \phi \, dx = 1$ and $\operatorname{supp} \phi \subset \{x \mid \| x \| \leqslant 1\}$. Form the convolution with respect to x,

$$g_\delta(t, x) = 1/\delta^n \int_{R^n} f(t, x - y)\, \phi(y/\delta)\, dy,$$

where $\delta > 0$ is a parameter. Clearly, it is well defined in a neighborhood of E_N if $\delta \leqslant \delta_0$, for some $\delta_0 > 0$. Suppose $0 < \delta \leqslant \delta_0 < \Theta$. Then $g_\delta(t, x) = 0$ if $\| x \| \leqslant \Theta - \delta_0$. Further, g_δ has continuous partial derivatives with respect to x. Finally,

$$\lim_{\delta \to +0} g_\delta(t, x) = f(t, x), \qquad \text{uniformly for } (t, x) \in E_N .$$

Take a sequence $\delta_k \searrow 0$, and put $h_{\delta_k, \epsilon}(t, x) = g_{\delta_k}(t, x) + \epsilon M x$. Clearly, we can choose a sequence $\epsilon_k \to 0$ such that $h_{\delta_k, \epsilon_k}(t, x)$ satisfies the conditions of Theorem 2, if we let h_{δ_k, ϵ_k} be independent of t for $t \geqslant N$. (Here we use the fact that $g_{\delta_k}(t, x) = 0$ for $\| x \| \leqslant \Theta - \delta_0$, in an obvious manner.)

Thus, by Theorem 2, there is a solution of $\dot{x} = -h_{\delta_k, \epsilon_k}(t, x)$ over $0 \leqslant t \leqslant N$, with appropriate properties. Further, since $\lim_{k \to \infty} h_{\delta_k, \epsilon_k}(t, x) = f(t, x)$ uniformly on E_N, we can apply Arzela's theorem a last time to obtain the desired solution to $\dot{x} = -f(t, x)$. In view of points (1) and (2) this completes the proof.

Remark 1. In contrast to Theorems 1 and 2, it does not follow that $x_0(t) \neq 0$ for $t \geqslant 0$. In fact, one can easily construct an example, where each nonnegative solution becomes zero in finite time. By imposing some extra condition on $f(t, x)$ near $x = 0$ one can make sure that $x_0(t) \neq 0$ for all $t \geqslant 0$.

Remark 2. This theorem is a considerable generalization of theorem (*) in [4, p. 861]. Whereas they assume $f(t, x) \geqslant 0$ on E, we have restrictions on the sign of $f_i(t, x)$ only for $x \in \partial\Omega$, and this is a typical feature for the Ważewski retract method.

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