Journal of Multivariate Analysis 101 (2010) 2234–2249

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Nonparametric rank-based tests of bivariate extreme-value dependence

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ARTICLE INFO

Article history:
Received 16 February 2010
Available online 1 June 2010

AMS subject classifications:
62H15
62G32

Keywords:
Contiguity
Extreme-value copulas
Local power comparisons
Multiplier central limit theorem
Pseudo-observations
Ranks

ABSTRACT

A new class of tests of extreme-value dependence for bivariate copulas is proposed. It is based on the process comparing the empirical copula with a natural nonparametric rank-based estimator of the unknown copula under extreme-value dependence. A multiplier technique is used to compute approximate p-values for several candidate test statistics. Extensive Monte Carlo experiments were carried out to compare the resulting procedures with the tests of extreme-value dependence recently studied in Ben Ghorbal et al. (2009) [1] and Kojadinovic and Yan (2010) [19]. The finite-sample performance study of the tests is complemented by local power calculations.

1. Introduction

Extreme-value copulas appear in extreme-value theory as the limits of copulas of componentwise maxima in random samples [8,15,16]. This makes them natural tools for modeling the dependence between extreme observations in fields such as finance [23], insurance [7] or hydrology [27]. Their use, however, is not restricted to the statistical modeling of extremes as such copulas may prove to be appropriate dependence models for any data set exhibiting positive dependence.

Any extreme-value copula can be represented in terms of its *Pickands dependence function* [25,5,17,16]. For a bivariate copula C, this representation becomes a characterization and takes the form

$$C(u,v) = \exp\left[\log(uv) A\left\{\frac{\log(v)}{\log(uv)}\right\}\right], \quad u, v \in (0,1), \tag{1}$$

where $A : [0,1] \to [1/2,1]$, the Pickands dependence function, is convex and satisfies $\max(t, 1-t) \le A(t) \le 1$ for all $t \in [0,1]$.

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be a random sample from an unknown bivariate cumulative distribution function (c.d.f.) H with unknown continuous margins F and G, and unknown copula C. In order to reduce the number of candidate copula families that could be used as models for C, one natural step is to test whether C belongs to the class of extreme-value copulas. Ghoudi, Khoudraji and Rivest were the first to propose a test of bivariate extreme-value dependence [15]. Their test, based on the bivariate probability integral transformation, was thoroughly revisited in [1]. A second test, based on the characterization of extreme-value copulas as max-stable copulas, was recently proposed in [19].

The aim of this work is to derive a third class of tests of extreme-value dependence for bivariate copulas, and to compare its finite-sample performance and local limiting power with those of its two competitors mentioned above.

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doi:10.1016/j.jmva.2010.05.004

The proposed class of tests is based on the process comparing the empirical copula with another natural nonparametric estimator of the unknown copula derived under the hypothesis of extreme-value dependence. The latter estimator is constructed from a rank-based version of the Capéraà–Fougères–Genest estimator [3] of the Pickands dependence function recently studied in [14]. As the empirical process on which the proposed class of tests is based has an unwieldy limiting distribution, a multiplier approach inspired by that suggested in [26] is used to compute approximate p-values for several candidate test statistics.

The paper is organized as follows. Section 2 is devoted to an in-depth description of the proposed tests: the empirical process on which the tests are based is thoroughly studied and the computation of asymptotically valid approximate p-values for the test statistics is explained in detail. Section 3 partially reports the results of a large scale Monte Carlo study comparing the finite-sample performance of various versions of the tests with those of the tests of extreme-value dependence proposed in [1,19]. These experiments are complemented by asymptotic local power calculations in Section 4. The last section contains methodological recommendations and concluding remarks. All the proofs are relegated to the Appendices.

The following notational conventions are used in the paper. For any $x, y \in \mathbb{R}$, $\min(x, y)$ and $\max(x, y)$ are denoted by $x \wedge y$ and $x \vee y$, respectively. Furthermore, $\ell^\infty(\mathcal{S})$ represents the space of bounded real-valued functions on the set $\mathcal{S}$, while $\mathcal{C}([a, b])$ represents the space of continuous real-valued functions on the real closed interval $[a, b]$; both are equipped with the uniform metric. The arrow $\rightsquigarrow$ denotes weak convergence while the set of bivariate extreme-value copulas, i.e., copulas characterized by (1), is denoted by $\mathcal{EV}$.

Note finally that all the tests studied in this work are implemented in the R package `copula` [20] available on the Comprehensive R Archive Network.

2. Description of the test

The empirical process at the root of the proposed new class of tests of extreme-value dependence involves the comparison of the empirical copula with a natural nonparametric estimator of the unknown copula derived under the hypothesis $\mathcal{H}_0$: $C \in \mathcal{EV}$. The latter estimator is obtained by replacing the unknown Pickands dependence function in characterization (1) by a consistent rank-based estimator of it recently studied in [14].

2.1. Nonparametric estimation of C

A natural nonparametric estimator of the underlying copula $C(u, v) = H\{F^{-1}(u), G^{-1}(v)\}$ is the empirical copula [4]. It is usually defined as

$$C_n(u, v) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(U_{i,n} \le u, V_{i,n} \le v), \quad u, v \in [0, 1],$$

where $(U_{1,n}, V_{1,n}), \dots, (U_{n,n}, V_{n,n})$ are *pseudo-observations* from C computed from the data by $(U_{i,n}, V_{i,n}) = (F_n(X_i), G_n(Y_i))$ for all $i \in \{1, \dots, n\}$ with F_n and G_n being the rescaled empirical counterparts of F and G respectively defined by

$$F_n(x) = \frac{1}{n+1}\sum_{i=1}^{n} \mathbf{1}(X_i \le x) \quad \text{and} \quad G_n(y) = \frac{1}{n+1}\sum_{i=1}^{n} \mathbf{1}(Y_i \le y), \quad x, y \in \mathbb{R}.$$

Under the assumption that C has continuous partial derivatives on $(0, 1)^2$, it is well known [9,6,31] that the weak limit of the empirical copula process $\sqrt{n}(C_n - C)$ is

$$\mathbb{C}(u, v) = \alpha(u, v) - C^{[1]}(u, v)\alpha(u, 1) - C^{[2]}(u, v)\alpha(1, v), \quad u, v \in [0, 1], \tag{2}$$

where $C^{[j]}$ denotes the partial derivative of C with respect to the jth argument and α is a C-Brownian bridge, i.e., a tight centered Gaussian process on $[0, 1]^2$ with covariance function $E[\alpha(u, v)\alpha(u', v')] = C(u \wedge u', v \wedge v') - C(u, v)C(u', v')$, $u, v, u', v' \in [0, 1]$.

2.2. Nonparametric estimation of A

Genest and Segers [14] have recently studied two rank-based estimators of the Pickands dependence function A appearing in representation (1). These two estimators are the rank-based versions of the two best-known nonparametric estimators of A, namely the Pickands estimator [25] and the Capéraà–Fougères–Genest estimator [3]. The latter estimator was found to behave better in finite samples in several studies [14,12]. The results of the Monte Carlo experiments carried out in this work and partially reported in Section 3 concur with this conclusion. For this reason, we present the derivation of the proposed tests only when based on the Capéraà–Fougères–Genest estimator. The analogue expressions based on the Pickands estimator can be recovered *mutatis mutandis*.

Assume that C is an extreme-value copula and, as in the previous subsection, let $(U_{1,n}, V_{1,n}), \dots, (U_{n,n}, V_{n,n})$ be the pseudo-observations from C computed from the original data. Furthermore, let

$$S_{i,n} = -\log(U_{i,n}) \quad \text{and} \quad T_{i,n} = -\log(V_{i,n}),$$

for every $i \in \{1, \ldots, n\}$, and let

$$\xi_{i,n}(0) = S_{i,n}, \qquad \xi_{i,n}(1) = T_{i,n}, \quad \text{and} \quad \xi_{i,n}(t) = \left(\frac{S_{i,n}}{1-t}\right) \wedge \left(\frac{T_{i,n}}{t}\right),$$

for every $i \in \{1, \ldots, n\}$ and any $t \in (0, 1)$. The rank-based version of the Capéraà–Fougères–Genest estimator is then defined by

$$A_n(t) = \exp\left\{-\gamma - \frac{1}{n}\sum_{i=1}^{n} \log \xi_{i,n}(t)\right\}, \quad t \in [0, 1],$$

where $\gamma = -\int_0^\infty \log(x)\mathrm{e}^{-x}\mathrm{d}x \approx 0.577$ is Euler's constant. The previous estimator can be expressed in terms of the empirical copula as

$$A_n(t) = \exp\left[-\gamma + \int_0^1 \left\{C_n(x^{1-t}, x^t) - \mathbf{1}(x > e^{-1})\right\} \frac{\mathrm{d}x}{x \log x}\right], \quad t \in [0, 1].$$

The limiting behavior of A_n follows from [14, Theorem 3.2]. Provided that the true Pickands dependence function A is twice continuously differentiable on (0, 1) (which we will assume in the rest of the paper), we have that

$$\sqrt{n}\{A_n(t) - A(t)\} \rightsquigarrow \mathbb{A}(t) = A(t) \int_0^1 \mathbb{C}(x^{1-t}, x^t) \frac{\mathrm{d}x}{x \log x}, \tag{3}$$

in $\mathcal{C}([0, 1])$, where $\mathbb{C}$ is defined in (2).

To ensure that the endpoint constraints $A_n(0) = A_n(1) = 1$ are satisfied, the previous estimator can be corrected as suggested in [3]. This yields the corrected version

$$A_{n,c}(t) = \exp\{\log A_n(t) - (1-t)\log A_n(0) - t \log A_n(1)\}, \quad t \in [0, 1],$$

which generally behaves better in small samples than the uncorrected one. For this reason, in the rest of the paper, we shall always work with the above corrected version. Note however that A_n and $A_{n,c}$ become indistinguishable as n tends to infinity [14, Section 2.4].

2.3. Test process and test statistics

In view of the previous subsection and of representation (1), it seems sensible to define a nonparametric estimator of the unknown copula under extreme-value dependence as

$$C_{A_{n,c}}(u, v) = \exp\left[\log(uv) A_{n,c}\left\{\frac{\log(v)}{\log(uv)}\right\}\right], \quad u, v \in (0, 1).$$

A natural way of testing extreme-value dependence then consists of comparing the empirical copula C_n, which is a nonparametric estimator of C whether $\mathcal{H}_0 : C \in \mathcal{EV}$ is true or not, with $C_{A_{n,c}}$. More formally, this amounts to basing tests of extreme-value dependence on the empirical process

$$\mathbb{D}_n = \sqrt{n}(C_n - C_{A_{n,c}}),$$

i.e.,

$$\mathbb{D}_n(u, v) = \sqrt{n}\left(C_n(u, v) - \exp\left[\log(uv) A_{n,c}\left\{\frac{\log(v)}{\log(uv)}\right\}\right]\right), \quad u, v \in (0, 1). \tag{4}$$

The following result, proved in Appendix A, describes the asymptotic behavior of the test process (4) under $\mathcal{H}_0$.

Proposition 1. *Let $a, b \in (0, 1)$, $a < b$, and suppose that A is twice continuously differentiable on* (0, 1)*. Then, under $\mathcal{H}_0$, $\mathbb{D}_n \rightsquigarrow \mathbb{D}$ in $\ell^\infty([a, b]^2)$, where*

$$\mathbb{D}(u, v) = \mathbb{C}(u, v) - \exp\left[\log(uv) A\left\{\frac{\log(v)}{\log(uv)}\right\}\right] \log(uv) \mathbb{A}\left\{\frac{\log(v)}{\log(uv)}\right\}. \tag{5}$$

The reals a and b in the previous proposition can be chosen arbitrarily close to 0 and 1, respectively. We will explain in Section 3 how they were chosen in practice.

As candidate test statistics, we restricted our attention to the two Cramér–von Mises functionals

$$S_n = \int_{[a,b]^2} \mathbb{D}_n(u, v)^2 \mathrm{d}u\mathrm{d}v \quad \text{and} \quad T_n = \int_{[a,b]^2} \mathbb{D}_n(u, v)^2 \mathrm{d}C_n(u, v).$$

Kolmogorov–Smirnov statistics were not considered because, from our experience, Cramér–von Mises statistics generally lead to more powerful tests.

2.4. Multiplier central limit theorems

The use of the weak limit of the test process $\mathbb{D}_n$ established in Proposition 1 to compute asymptotic p-values for the statistics S_n and T_n appears unwieldy. We therefore resort to a multiplier approach to obtain approximate p-value for S_n and T_n in the spirit of that used in [26,21]. The idea is to use multipliers to generate a large number of approximate independent realizations of the weak limit $\mathbb{D}$ of the test process, derive the corresponding approximate independent realizations of S_n and T_n, and finally compute approximate p-values using the resulting empirical c.d.f.s.

Before stating the key result that provides an asymptotic justification to the adopted approach, let us first introduce additional notation. Let N be a large integer and let $Z_i^{(k)}$, $i = 1, \ldots, n$, $k = 1, \ldots, N$, be i.i.d. random variables with mean 0 and variance 1 independent of the data $(X_1, Y_1), \ldots, (X_n, Y_n)$. For any $k \in \{1, \ldots, N\}$, let

$$\begin{aligned}\alpha_n^{(k)}(u, v) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i^{(k)} \left\{ \mathbf{1}(U_{i,n} \le u, V_{i,n} \le v) - C_n(u, v) \right\} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Z_i^{(k)} - \bar{Z}^{(k)}) \mathbf{1}(U_{i,n} \le u, V_{i,n} \le v), \quad u, v \in [0, 1], \end{aligned} \tag{6}$$

where $\bar{Z}^{(k)} = n^{-1} \sum_{i=1}^{n} Z_i^{(k)}$.

In order to obtain approximate independent copies of the process $\mathbb{D}$ defined in (5), it is necessary to estimate the unknown partial derivatives of C that appear in the expression of $\mathbb{C}$, and therefore in that of $\mathbb{A}$; see (2) and (3), respectively. A first way to proceed consists of using the generic estimators proposed in [26, page 380] (see also [19, Proposition 2]). For $(u, v) \in (0, 1)^2$, these are respectively defined by

$$C_n^{[1]}(u, v) = \frac{C_n(u + n^{-1/2}, v) - C_n(u - n^{-1/2}, v)}{2n^{-1/2}},$$

and

$$C_n^{[2]}(u, v) = \frac{C_n(u, v + n^{-1/2}) - C_n(u, v - n^{-1/2})}{2n^{-1/2}}.$$

Under extreme-value dependence, starting from characterization (1), alternative natural nonparametric estimators were proposed in [21]. For any $(u, v) \in (0, 1)^2$, let $t_{uv} = \log(v)/\log(uv)$. The partial derivatives $C^{[1]}$ and $C^{[2]}$ can then be estimated, for $(u, v) \in (0, 1)^2$, by

$$C_{A_{n,c}}^{[1]}(u, v) = \{\hat{A}_{n,c}(t_{uv}) - t_{uv} A'_{n,c}(t_{uv})\} (uv)^{A_{n,c}(t_{uv}) - (1 - t_{uv})},$$

and

$$C_{A_{n,c}}^{[2]}(u, v) = \{\hat{A}_{n,c}(t_{uv}) + (1 - t_{uv}) A'_{n,c}(t_{uv})\} (uv)^{A_{n,c}(t_{uv}) - t_{uv}},$$

where $\hat{A}_{n,c} = (A_{n,c} \wedge 1) \vee I \vee (1 - I)$, I is the identity function, and

$$A'_{n,c}(t) = \frac{A_{n,c}\{(t + n^{-1/2}) \wedge 1\} - A_{n,c}\{(t - n^{-1/2}) \vee 0\}}{2n^{-1/2}}, \quad t \in (0, 1).$$

Now, for any $k \in \{1, \ldots, N\}$ and $(u, v) \in (0, 1)^2$, let

$$\mathbb{C}_n^{(k)}(u, v) = \alpha_n^{(k)}(u, v) - C_n^{[1]}(u, v) \alpha_n^{(k)}(u, 1) - C_n^{[2]}(u, v) \alpha_n^{(k)}(1, v), \tag{7}$$

let

$$\mathbb{C}_{A_{n,c}}^{(k)}(u, v) = \alpha_n^{(k)}(u, v) - C_{A_{n,c}}^{[1]}(u, v) \alpha_n^{(k)}(u, 1) - C_{A_{n,c}}^{[2]}(u, v) \alpha_n^{(k)}(1, v), \tag{8}$$

let

$$\mathbb{A}_n^{(k)}(t) = A_{n,c}(t) \int_0^1 \mathbb{C}_{A_{n,c}}^{(k)}(x^{1-t}, x^t) \frac{\mathrm{d}x}{x \log x}, \tag{9}$$

and let

$$\mathbb{D}_n^{(k)}(u, v) = \mathbb{C}_n^{(k)}(u, v) - \exp\left[\log(uv) A_{n,c}\left\{\frac{\log(v)}{\log(uv)}\right\}\right] \log(uv) \mathbb{A}_n^{(k)}\left\{\frac{\log(v)}{\log(uv)}\right\}. \tag{10}$$

The following result, proved in Appendix B, is at the root of the proposed new class of tests of extreme-value dependence.

Proposition 2. *Let $a, b \in (0,1)$, $a < b$, and suppose that A is twice continuously differentiable on $(0,1)$. Then, under $\mathscr{H}_0$,*

$$\left(\mathbb{D}_n, \mathbb{D}_n^{(1)}, \ldots, \mathbb{D}_n^{(N)}\right) \rightsquigarrow \left(\mathbb{D}, \mathbb{D}^{(1)}, \ldots, \mathbb{D}^{(N)}\right)$$

in $\ell^\infty([a,b]^2)^{\otimes(N+1)}$, where $\mathbb{D}^{(1)}, \ldots, \mathbb{D}^{(N)}$ are independent copies of the process $\mathbb{D}$ defined in (5).

As shall be discussed in Section 2.6, alternative definitions of the process $\mathbb{D}_n^{(k)}$ can be considered depending on whether $\mathbb{C}_n^{(k)}$ or $\mathbb{C}_{A_{n,c}}^{(k)}$ are used in (9) and (10). The definitions adopted above led to the tests with the best finite-sample behavior. More details will be given in Section 3.

Next, for any $k \in \{1, \ldots, N\}$, let

$$S_n^{(k)} = \int_{[a,b]^2} \mathbb{D}_n^{(k)}(u,v)^2 \mathrm{d}u\mathrm{d}v.$$

From the previous proposition and the continuous mapping theorem, we immediately have that, under $\mathscr{H}_0$,

$$\left(S_n, S_n^{(1)}, \ldots, S_n^{(N)}\right) \rightsquigarrow \left(S, S^{(1)}, \ldots, S^{(N)}\right)$$

in $[0,\infty)^{\otimes(N+1)}$, where S is the weak limit of S_n, and $S^{(1)}, \ldots, S^{(N)}$ are independent copies of S. This suggests computing an approximate p-value for S_n as

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{1}\left(S_n^{(k)} \geq S_n\right).$$

Similarly, for any $k \in \{1, \ldots, N\}$, let

$$T_n^{(k)} = \int_{[a,b]^2} \mathbb{D}_n^{(k)}(u,v)^2 \mathrm{d}C_n(u,v).$$

An approximate p-value for T_n is then computed by $N^{-1}\sum_{k=1}^{N} \mathbf{1}\left(T_n^{(k)} \geq T_n\right)$.

In order to carry out the tests, it is necessary to compute the integral appearing in the expression of $\mathbb{A}_n^{(k)}$ given in (9). As shown in [21], for any $k \in \{1, \ldots, N\}$ and any $t \in (0,1)$, we have

$$\begin{aligned}\int_0^1 \mathbb{C}_{A_{n,c}}^{(k)}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x} &= -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Z_i^{(k)} - \bar{Z}^{(k)})\log\left(\frac{S_{i,n}}{1-t} \wedge \frac{T_{i,n}}{t}\right) \\ &\quad - \frac{1}{\sqrt{n}}\{\hat{A}_{n,c}(t) - tA'_{n,c}(t)\}\sum_{i=1}^{n} Z_i^{(k)} \int_0^1 x^{\hat{A}_{n,c}(t)-(1-t)}\left\{\mathbf{1}(U_{i,n} \leq x^{1-t}) - \frac{\lfloor x^{1-t}(n+1)\rfloor}{n}\right\}\frac{\mathrm{d}x}{x\log x} \\ &\quad - \frac{1}{\sqrt{n}}\{\hat{A}_{n,c}(t) + (1-t)A'_{n,c}(t)\}\sum_{i=1}^{n} Z_i^{(k)} \int_0^1 x^{\hat{A}_{n,c}(t)-t}\left\{\mathbf{1}(V_{i,n} \leq x^{t}) - \frac{\lfloor x^{t}(n+1)\rfloor}{n}\right\}\frac{\mathrm{d}x}{x\log x},\end{aligned}$$

where, for any $y \geq 0$, $\lfloor y \rfloor$ denotes the integer part of y. Note that the two integrals appearing in the right-hand side of the previous expression are not indefinite as the integrands are zero when x gets close to 0 or 1. They are computed numerically in our implementation.

2.5. Consistency of the tests

The work of Garralda-Guillem [10] implies that extreme-value copulas are left-tail decreasing (LTD) in both arguments; see e.g. [24, Section 5.2.2]. These dependence conditions are actually satisfied by the most popular bivariate copulas with positive dependence such as the Clayton, Frank, normal, t and Plackett. If C has a continuous density and is LTD in both arguments but is not necessarily an extreme-value copula, it was shown in [12, Proposition 2] that $\sqrt{n}(A_{n,c} - A_C) \rightsquigarrow \mathbb{A}_C$ in $\mathcal{C}([0,1])$, where

$$A_C(t) = \exp\left[-\gamma + \int_0^1 \left\{C(x^{1-t}, x^t) - \mathbf{1}(x > e^{-1})\right\}\frac{\mathrm{d}x}{x\log x}\right], \quad t \in [0,1], \tag{11}$$

and

$$\mathbb{A}_C(t) = A_C(t)\int_0^1 \mathbb{C}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x}, \quad t \in [0,1].$$

The function A_C actually turns out to be well defined for any copula C and reduces to the Pickands dependence function A when C is an extreme-value copula.

To study the consistency of the proposed tests, assume that C has a continuous density and is LTD in both arguments without being an extreme-value copula. Also, let

$$C_{A_C}(u,v) = \exp\left[\log(uv) A_C\left\{\frac{\log(v)}{\log(uv)}\right\}\right], \quad u,v \in (0,1).$$

Then, the test process $\mathbb{D}_n$ can be decomposed as

$$\sqrt{n}(C_n - C_{A_{n,c}}) = \sqrt{n}(C_n - C) - \sqrt{n}(C_{A_{n,c}} - C_{A_C}) + \sqrt{n}(C - C_{A_C}).$$

Proceeding as in the proof of Proposition 1, it can be verified that $\sqrt{n}(C_n - C) - \sqrt{n}(C_{A_{n,c}} - C_{A_C})$ converges weakly to

$$\mathbb{C}(u,v) - \exp\left[\log(uv) A_C\left\{\frac{\log(v)}{\log(uv)}\right\}\right] \log(uv) \mathbb{A}_C\left\{\frac{\log(v)}{\log(uv)}\right\}$$

in $\ell^\infty([a,b]^2)$. If $C \neq C_{A_C}$, then $\sup_{(u,v)\in(0,1)^2} \sqrt{n}|C(u,v) - C_{A_C}(u,v)|$ tends to infinity, which implies that any sensible statistic derived from the process $\sqrt{n}(C_n - C_{A_{n,c}})$ will tend to infinity.

Interestingly enough, conclusions about the consistency of the studied tests can then be drawn when the function A_C is convex, as stated in the following result proved in Appendix C.

Proposition 3. *Assume that C has a continuous density and is LTD in both arguments without being an extreme-value copula. If the function A_C is convex, then $C \neq C_{A_C}$.*

As can be seen from [12, Figure 3], the function A_C appears convex for the most frequently used bivariate copulas with positive dependence such as the Clayton, Frank, normal and Plackett. This suggests that the proposed class of tests will be consistent under a wide range of alternatives. An analytical proof of the convexity of A_C for non-extreme-value copulas that have a continuous density and that are LTD in both arguments is however still missing.

2.6. Alternative versions of the tests

Alternative versions of the tests can be obtained by replacing $\mathbb{C}^{(k)}_{A_{n,c}}$ by $\mathbb{C}^{(k)}_n$, or vice versa, in (9) and (10), respectively. Among the four possible definitions for $\mathbb{D}^{(k)}_n$, only two led to tests that were not too liberal for small sample size. The best rejection rates were obtained using the definition adopted in Section 2.4. Slightly less powerful but faster tests were obtained by defining $\mathbb{A}^{(k)}_n$ as

$$\mathbb{A}^{(k)}_n(t) = A_{n,c}(t) \int_0^1 \mathbb{C}^{(k)}_n(x^{1-t}, x^t) \frac{dx}{x \log x} \tag{12}$$

instead of (9). Although all four possible versions are expected to be asymptotically equivalent, we were not able to prove an analogue of Proposition 2 in this last case.

Unlike the version of the test described in Section 2.4, the version based on the above definition of $\mathbb{A}^{(k)}_n$ does not require the use of numerical integration to compute the integral appearing in its expression, and is therefore substantially faster. To see this, let

$$S^+_{i,n} = -\log\left\{(U_{i,n} + n^{-1/2}) \wedge \frac{n}{n+1}\right\}, \qquad S^-_{i,n} = -\log\left\{(U_{i,n} - n^{-1/2}) \vee \frac{1}{n+1}\right\},$$

and

$$T^+_{i,n} = -\log\left\{(V_{i,n} + n^{-1/2}) \wedge \frac{n}{n+1}\right\}, \qquad T^-_{i,n} = -\log\left\{(V_{i,n} - n^{-1/2}) \vee \frac{1}{n+1}\right\},$$

for every $i \in \{1,\dots,n\}$. Then, as shown in [21], for any $k \in \{1,\dots,N\}$ and any $t \in (0,1)$,

$$\begin{aligned}\int_0^1 \mathbb{C}^{(k)}_n(x^{1-t}, x^t)\frac{dx}{x\log x} = \frac{1}{\sqrt{n}}\sum_{i=1}^n (Z^{(k)}_i - \bar{Z}^{(k)}) \Bigg[&- \log\left(\frac{S_{i,n}}{1-t} \wedge \frac{T_{i,n}}{t}\right) \\ &- \frac{1}{2\sqrt{n}}\sum_{j=1}^n \Bigg\{ - \log\left(\frac{S^-_{j,n}}{1-t} \wedge \frac{T_{j,n}}{t} \wedge \frac{S_{i,n}}{1-t}\right) + \log\left(\frac{S^+_{j,n}}{1-t} \wedge \frac{T_{j,n}}{t} \wedge \frac{S_{i,n}}{1-t}\right) \\ &- \log\left(\frac{S_{j,n}}{1-t} \wedge \frac{T^-_{j,n}}{t} \wedge \frac{T_{i,n}}{t}\right) + \log\left(\frac{S_{j,n}}{1-t} \wedge \frac{T^+_{j,n}}{t} \wedge \frac{T_{i,n}}{t}\right)\Bigg\}\Bigg].\end{aligned}$$

3. Finite-sample performance

The finite-sample performance of the tests described in the previous section was investigated in a large scale Monte Carlo experiment. As explained in Section 2.6, four versions of the test based on S_n and four versions of the test based on T_n were considered. The reals a and b appearing in the expressions of the statistics S_n and $S_n^{(k)}$ were set to $1/m$ and $1 - 1/m$, with $m = 30$. The resulting integrals were computed numerically using a grid of m^2 uniformly spaced points on $[1/m, 1-1/m]^2$. Larger values of m were considered but this did not seem to improve the results. The statistics T_n and $T_n^{(k)}$ were computed by setting a and b to 0 and 1, respectively, that is, as

$$T_n = \frac{1}{n}\sum_{i=1}^{n} \mathbb{D}_n(U_{i,n}, V_{i,n})^2 \quad \text{and} \quad T_n^{(k)} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{D}_n^{(k)}(U_{i,n}, V_{i,n})^2,$$

respectively.

The rejection rates of the eight tests mentioned above were compared with those of the test of extreme-value dependence described in [1] based on a variance estimator denoted $\hat{\sigma}_n^2$, and with those of the test proposed in [19] based on a statistic denoted $T_{3,4,5,n}$.

To investigate the level of the tests, only the Gumbel–Hougaard (GH) copula and an asymmetric version of it were used. Indeed, one of the most surprising findings of the recent study of bivariate extreme-value copulas carried out in [12] is that the most frequently used such copulas, *viz.* the Gumbel–Hougaard, Galambos, Hüsler–Reiss and Student extreme-value, show striking similarities for a given degree of dependence. It therefore does not seem necessary to work with different extreme-value families. More variety in this class is obtained by using asymmetric extreme-value copulas constructed using Khoudraji's device [18,11,22]. The asymmetric version of the Gumbel–Hougaard copula used in our study is denoted by aGH and is defined by

$$\mathrm{aGH}_{\theta,\lambda,\kappa}(u, v) = u^{1-\lambda} v^{1-\kappa} \mathrm{GH}_\theta(u^\lambda, v^\kappa), \quad u, v \in [0, 1],\ \lambda, \kappa \in (0, 1],\ \lambda \neq \kappa,$$

where GH_θ is the c.d.f. of the Gumbel–Hougaard copula with parameter θ. Note that $\mathrm{aGH}_{\theta,\lambda,\kappa}$ is nothing else but the asymmetric logistic model introduced in [29,30]. In the experiments, θ was set to 4, while λ and κ were set to 0.4 and 0.95, respectively, so that data generated from this copula display strong asymmetries. For the Gumbel–Hougaard (GH) copula, three values of θ were considered, corresponding respectively to a Kendall's τ of 0.25, 0.5 and 0.75.

To study the power of the tests, five non-extreme-value copulas were used: the Clayton (Cl), Frank (F), normal (N), t with 4 degrees of freedom (t-4) and Plackett (P). As previously, three levels of dependence, i.e., $\tau \in \{0.25, 0.5, 0.75\}$, were considered.

Sample of sizes $n = 100, 200, 400$ and 800 were generated. All the tests were carried out at the 5% significance level and empirical rejection rates were computed from 1000 random samples per scenario.

A first finding of our extensive Monte Carlo study is that, among the four possible ways of defining the processes $\mathbb{D}_n^{(k)}$, only those described in Sections 2.4 and 2.6 gave tests that were not too liberal for $n = 100$ and 200.

A second rather accidental finding is that the use of $\hat{C}_n$ defined by

$$\hat{C}_n(u, v) = \frac{1}{n+1}\left\{\sum_{i=1}^{n} \mathbf{1}(U_{i,n} \le u, V_{i,n} \le v) + \frac{1}{2}\right\}, \quad u, v \in [0, 1],$$

instead of C_n in the expressions of the statistics S_n and T_n (the expressions of $S_n^{(k)}$ and $T_n^{(k)}$ remaining unchanged) gave consistently less conservative and more powerful tests. Clearly, $\hat{C}_n$ and C_n are asymptotically equivalent since $\sup_{(u,v)\in[0,1]^2} |\hat{C}_n(u, v) - C_n(u, v)| \le 1/n$. The results to be presented in the forthcoming tables are therefore those obtained when C_n is replaced by $\hat{C}_n$ in the expressions of S_n and T_n, i.e., in the expression of $\mathbb{D}_n$ given in (4). The statistics resulting from this asymptotically negligible modification will be denoted by $\hat{S}_n$ and $\hat{T}_n$, respectively.

A third finding is that the tests based on the statistics $\hat{T}_n$ and T_n were more powerful than those based on $\hat{S}_n$ and S_n in all the scenarios under consideration.

The rejection rates of the two best versions of the test based on $\hat{T}_n$ are given in Tables 1–3. They differ according to whether approximate p-values were computed using the alternative expressions given in Section 2.6 or those of Section 2.4. We will refer to these two versions as $\hat{T}_n^C$ and $\hat{T}_n^A$, respectively.

As can be seen from Table 1, the test $\hat{T}_n^C$ appears to be too conservative for small sample size. Its empirical levels seem to improve overall as n increases, though the improvement seems to be slow when $\tau = 0.75$. The empirical levels of $\hat{T}_n^A$, although not perfect, are globally more satisfactory for small sample size.

In terms of power, the test $\hat{T}_n^A$ outperforms that based on $\hat{T}_n^C$. As can be seen from Table 2 and Table 3, the difference between their rejection rates is substantial for $n = 100$ but decreases rather rapidly as n increases. For $n = 800$, the two tests are virtually equivalent as could have been expected.

When compared to the tests based on $T_{3,4,5,n}$ and $\hat{\sigma}_n^2$, the test $\hat{T}_n^A$ is the most powerful when the data arise from the Frank or the Plackett copula and $\tau \in \{0.5, 0.75\}$. For $\tau = 0.25$, it is outperformed by the test based on $T_{3,4,5,n}$. For data sets

Table 1
Rejection rate (in %) of the null hypothesis as observed in 1000 random samples of size $n = 100$, 200, 400 and 800 from the Gumbel–Hougaard copula (GH) and its asymmetric version (aGH) with $\theta = 4$ and $(\lambda, \kappa) = (0.4, 0.95)$.

Copula	τ	$T_{3,4,5,n}$	$\hat{T}_n^C$	$\hat{T}_n^A$	$\hat{\sigma}_n^2$	$T_{3,4,5,n}$	$\hat{T}_n^C$	$\hat{T}_n^A$	$\hat{\sigma}_n^2$
		$n = 100$				$n = 200$			
GH	0.25	3.8	2.3	3.3	4.5	4.7	3.4	3.8	5.3
	0.50	6.0	3.8	4.7	4.5	4.0	3.6	3.9	4.3
	0.75	2.3	1.9	3.7	4.7	3.1	2.8	3.2	5.2
aGH		5.3	4.8	5.4	7.3	6.1	5.1	5.3	5.7
		$n = 400$				$n = 800$			
GH	0.25	4.9	4.9	4.3	5.1	4.8	5.0	4.5	6.0
	0.50	3.9	3.5	3.4	4.5	4.4	4.7	4.2	5.8
	0.75	2.5	2.3	2.4	5.5	3.3	4.1	3.7	5.2
aGH		6.3	6.3	5.5	5.8	5.1	6.0	5.1	4.1

Table 2
Rejection rate (in %) of the null hypothesis as observed in 1000 random samples of size $n = 100$ and 200 from the Clayton (C), Frank (F), normal (N), t with 4 degrees of freedom (t-4), and Plackett copula (P).

Copula	τ	$T_{3,4,5,n}$	$\hat{T}_n^C$	$\hat{T}_n^A$	$\hat{\sigma}_n^2$	$T_{3,4,5,n}$	$\hat{T}_n^C$	$\hat{T}_n^A$	$\hat{\sigma}_n^2$
		$n = 100$				$n = 200$			
C	0.25	70.3	70.8	76.8	81.1	93.2	96.1	97.0	98.4
	0.50	98.3	99.1	99.5	99.8	100.0	100.0	100.0	100.0
	0.75	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
F	0.25	40.3	25.7	32.5	22.5	64.6	51.4	58.3	36.3
	0.50	67.1	57.4	67.5	32.0	95.2	93.9	95.7	58.7
	0.75	83.0	84.9	91.6	33.9	99.3	99.8	99.9	58.9
N	0.25	20.5	15.6	19.4	20.7	37.4	31.7	36.5	35.8
	0.50	28.7	29.4	36.7	36.3	49.9	57.2	61.8	60.8
	0.75	24.1	27.5	39.8	45.2	44.6	57.1	66.5	76.0
P	0.25	33.4	18.6	26.5	21.5	59.5	44.6	51.5	36.1
	0.50	49.9	44.4	54.3	32.7	82.2	81.0	84.7	57.8
	0.75	53.3	51.7	64.5	34.3	78.9	86.8	90.9	59.7
t-4	0.25	12.7	15.0	17.9	16.9	17.9	23.0	23.9	26.8
	0.50	19.7	25.1	29.0	31.2	35.2	47.3	50.1	57.5
	0.75	19.5	22.9	33.3	38.7	32.9	44.4	50.6	65.6

Table 3
Rejection rate (in %) of the null hypothesis as observed in 1000 random samples of size $n = 400$ and 800 from the Clayton (C), Frank (F), normal (N), t with 4 degrees of freedom (t-4), and Plackett copula (P).

Copula	τ	$T_{3,4,5,n}$	$\hat{T}_n^C$	$\hat{T}_n^A$	$\hat{\sigma}_n^2$	$T_{3,4,5,n}$	$\hat{T}_n^C$	$\hat{T}_n^A$	$\hat{\sigma}_n^2$
		$n = 400$				$n = 800$			
C	0.25	99.8	99.8	99.8	100.0	100.0	100.0	100.0	100.0
	0.50	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
	0.75	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
F	0.25	91.4	88.1	90.5	67.1	99.7	99.7	99.7	92.0
	0.50	100.0	100.0	100.0	86.0	100.0	100.0	100.0	99.3
	0.75	100.0	100.0	100.0	87.1	100.0	100.0	100.0	98.8
N	0.25	62.3	61.9	64.8	66.0	90.2	91.6	92.5	90.5
	0.50	80.2	86.6	88.3	89.5	99.0	99.8	99.8	99.3
	0.75	76.8	88.2	90.8	95.7	98.4	99.6	99.6	99.9
P	0.25	86.5	80.2	83.1	62.3	99.3	99.1	99.2	88.1
	0.50	97.4	98.5	98.9	85.5	100.0	100.0	100.0	99.3
	0.75	98.5	99.8	99.9	88.0	100.0	100.0	100.0	99.5
t-4	0.25	27.9	38.6	39.2	44.5	52.8	67.2	67.7	74.3
	0.50	58.9	74.1	74.3	85.6	85.8	96.2	96.1	98.6
	0.75	61.3	79.6	81.8	94.2	91.4	99.0	99.1	99.9

generated from the normal copula, $\hat{T}_n^A$ is overall slightly less powerful than the test based on $\hat{\sigma}_n^2$, both tests outperforming the test based on $T_{3,4,5,n}$. The test based on $\hat{\sigma}_n^2$ has an edge over its competitors when data arise from the t copula with 4 degrees of freedom. As could have been expected, the difference between the four tests becomes very small when n reaches 800.

4. Local power comparisons

The tests whose finite-sample performance was investigated in the previous section can also be compared in terms of their ability to detect small departures from extreme-value dependence. To that effect, we consider in this section sequences of distributions defined by

$$H_{\delta_n}(x,y) = Q_{\delta_n}\{F(x), G(y)\}, \quad x, y \in \mathbb{R},$$

where

$$Q_{\delta_n}(u,v) = (1-\delta_n)C(u,v) + \delta_n D(u,v), \tag{13}$$

$\delta_n = \delta/\sqrt{n}$ for some $\delta \geq 0$, and C and D are absolutely continuous copulas such that $C \in \mathcal{EV}$ and $D \notin \mathcal{EV}$. Furthermore, let q_δ be the density associated with Q_δ and let $\dot{q}_\delta = \partial q_\delta/\partial\delta$. Also, notice that $Q_0 = C$ and $H_0 = H$. To ensure that the processes $\sqrt{n}(C_n - C)$ and $\sqrt{n}(A_{n,c} - A)$ have a non-degenerate joint limiting distribution under the sequence $(H_{\delta_n})_{n\geq 1}$, it is further assumed that the condition given in [32, Equation (3.10.10)] with $h = \delta\dot{q}_0/q_0$ holds, i.e., that

$$\lim_{n\to\infty} \int_{(0,1)^2} \left[\sqrt{n}\left\{\sqrt{q_{\delta_n}(u,v)} - \sqrt{q_0(u,v)}\right\} - \frac{\delta\dot{q}_0(u,v)}{2\sqrt{q_0(u,v)}}\right]^2 \mathrm{d}u\mathrm{d}v = 0. \tag{14}$$

The above criterion entails that the sequence $(H_{\delta_n})_{n\geq 1}$ is contiguous with respect to H. A similar setting was for instance considered in [13] and [2] for studying the local power of independence and goodness-of-fit tests, respectively.

The following result, proved in Appendix D, will enable us to identify the asymptotic distribution of the test process $\mathbb{D}_n$ defined in (4) under the sequence $(H_{\delta_n})_{n\geq 1}$.

Proposition 4. *Let C be an extreme-value copula whose Pickands dependence function A is twice continuously differentiable on $(0,1)$, and let D be an absolutely continuous copula. Then, under $(H_{\delta_n})_{n\geq 1}$,*

$$\begin{pmatrix} \sqrt{n}\{C_n(u,v) - C(u,v)\} \\ \sqrt{n}\{A_{n,c}(t) - A(t)\} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathbb{C}(u,v) + \delta\{D(u,v) - C(u,v)\} \\ \mathbb{A}(t) + \delta A(t)\{\log A_D(t) - \log A(t)\} \end{pmatrix}$$

in $\ell^\infty([0,1]^2) \times \mathcal{C}([0,1])$, where $\mathbb{C}$ (resp. $\mathbb{A}$) is the weak limit of $\sqrt{n}(C_n - C)$ (resp. $\sqrt{n}(A_{n,c} - A)$) under H, and A_D is defined as in (11).

Note that by proceeding as in [12, Appendix C], the previous result can be extended to the situation where C and D are absolutely continuous copulas such that C has a continuous density and is LTD in both arguments.

Let $a, b \in (0,1)$, $a < b$. Under the conditions of the previous proposition and by proceeding as in the proof of Proposition 1, we immediately obtain that, under $(H_{\delta_n})_{n\geq 1}$, the test process $\mathbb{D}_n$ converges weakly to

$$\mathbb{D}_\delta(u,v) = \mathbb{C}(u,v) + \delta\{D(u,v) - C(u,v)\} - \exp\left[\log(uv)A\left\{\frac{\log(v)}{\log(uv)}\right\}\right]\log(uv)$$
$$\times\left(\mathbb{A}\left\{\frac{\log(v)}{\log(uv)}\right\} + \delta A\left\{\frac{\log(v)}{\log(uv)}\right\}\left[\log A_D\left\{\frac{\log(v)}{\log(uv)}\right\} - \log A\left\{\frac{\log(v)}{\log(uv)}\right\}\right]\right)$$

in $\ell^\infty([a,b]^2)$.

Let $q_T(\alpha)$ be the asymptotic critical value of level α of the test statistic T_n. The limiting local power function of the test based on T_n is then defined as

$$\beta_T(\alpha,\delta) = \lim_{n\to\infty} \Pr\left\{T_n \geq q_T(\alpha) \mid H_{\delta_n}\right\}.$$

It appears unfortunately impossible to obtain an analytical expression for β_T in the setting under consideration. We therefore resort again to a multiplier approach similar to that presented in Section 2.6. Let m be a large integer and let $(X_1,Y_1),\ldots,(X_m,Y_m)$ be a random sample from c.d.f. $C \in \mathcal{EV}$. Let N be a large integer and let $Z_i^{(k)}$, $i = 1,\ldots,m$, $k = 1,\ldots,N$, be i.i.d. random variables with mean 0 and variance 1 independent of $(X_1,Y_1),\ldots,(X_m,Y_m)$. For any $k \in \{1,\ldots,N\}$ and any $(u,v) \in [a,b]^2$, let

$$\mathbb{D}_{\delta,m}^{(k)}(u,v) = \mathbb{C}_m^{(k)}(u,v) + \delta\{D(u,v) - C(u,v)\} - \exp\left[\log(uv)A\left\{\frac{\log(v)}{\log(uv)}\right\}\right]\log(uv)$$
$$\times\left(\mathbb{A}_m^{(k)}\left\{\frac{\log(v)}{\log(uv)}\right\} + \delta A\left\{\frac{\log(v)}{\log(uv)}\right\}\left[\log A_D\left\{\frac{\log(v)}{\log(uv)}\right\} - \log A\left\{\frac{\log(v)}{\log(uv)}\right\}\right]\right),$$

where $\mathbb{C}_m^{(k)}$ and $\mathbb{A}_m^{(k)}$ are defined as in (7) and (12), respectively. Furthermore, for any $k \in \{1,\ldots,N\}$, let

$$T_{\delta,m}^{(k)} = \frac{1}{m}\sum_{\mathrm{i}=1}^{m} \mathbb{D}_{\delta,m}^{(k)}(U_{\mathrm{i},m}, V_{\mathrm{i},m})^2.$$

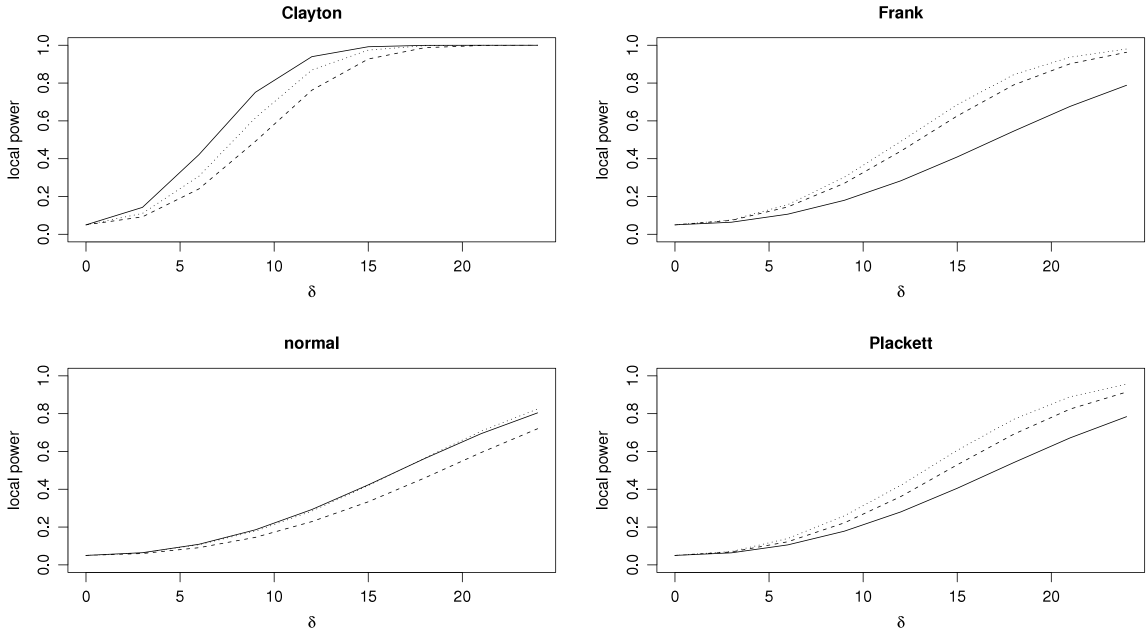

Fig. 1. Asymptotic local power functions of the tests based on $\hat{\sigma}_n^2$ (solid line), $T_{3,4,5,n}$ (dashed line) and T_n (dotted line) when C is the Gumbel–Hougaard copula with $\tau = 0.25$ and D is either the Clayton, Frank, normal or Plackett copula with $\tau = 0.25$.

As in Section 2, the statistics $T_{\delta,m}^{(1)}, \ldots, T_{\delta,m}^{(N)}$ can be thought of as approximate independent realizations of T, where T is the weak limit of T_n under $(H_{\delta_n})_{n\geq 1}$. It is thus natural to estimate the asymptotic critical value of T_n of level α as the empirical quantile of $T_{0,m}^{(1)}, \ldots, T_{0,m}^{(N)}$ of order $1-\alpha$. We will denote it by $\hat{q}_T(\alpha)$ as we continue. The limiting local power function can then be estimated as

$$\hat{\beta}_T(\alpha, \delta) = \frac{1}{N} \sum_{k=1}^{N} \mathbf{1}\left\{T_{\delta,m}^{(k)} \geq \hat{q}_T(\alpha)\right\}.$$

The limiting local powers to be represented in the forthcoming graphs were estimated using $m = 2500$ and $N = 10\,000$.

A similar approach was used to compute the asymptotic local power function of the test of extreme-value dependence proposed in [19] and based on the statistic $T_{3,4,5,n}$. For the test based on $\hat{\sigma}_n^2$, the limiting local power function was computed using the expression given in [1, Proposition 3] in which $\mu'(0)$ was estimated by Monte Carlo integration from samples of size 500 000, and $\sigma(0)$ was estimated using the large-sample variance estimator defined in [1, Section 4] from 2500 observations from C. The last step was performed using code generously provided by Johanna Nešlehová and now available in the R package `copula`.

Asymptotic local power calculations were performed in the following settings: C was taken to be the Gumbel–Hougaard copula with $\tau = 0.25$, 0.5 or 0.75, and D was either the Clayton, Frank, normal or Plackett copula with $\tau = 0.25$, 0.5 or 0.75. For the sake of clarity, we only report the results when C and D have the same degree of dependence.

As can be seen from Figs. 1–3, the results are consistent with those obtained in the simulations. Local powers increase in all settings as τ increases and, for a given τ, the greatest local powers are obtained when D is the Clayton copula. In the latter case, the test based on $\hat{\sigma}_n^2$ slightly outperforms its competitors when $\tau = 0.25$ and 0.5. When $\tau = 0.75$, all three tests are very close. The test based on $\hat{\sigma}_n^2$ is also more powerful then its competitors, overall, when D is the normal copula, the difference in local power being more pronounced for higher dependence. When D is the Frank or the Plackett copula, the test based on T_n is the most powerful and the test based on $T_{3,4,5,n}$ is second best.

5. Concluding remarks

Among the various tests of bivariate extreme-value dependence considered in this work, no single test was found to be consistently better than the others. The test proposed in [1] based on $\hat{\sigma}_n^2$ appears to be better suited for elliptical alternatives, while for the other alternatives considered in the experiments, the tests $\hat{T}_n^A$ and $T_{3,4,5,n}$ have the highest rejection rates. Overall, the test $\hat{T}_n^A$ displays the best behavior. From a computational perspective, the test based on $\hat{\sigma}_n^2$ is the fastest; it is followed by the tests based on $T_{3,4,5,n}$, $\hat{T}_n^C$ and finally $\hat{T}_n^A$.

Based on the Monte Carlo experiments and local power comparisons presented in this work, our recommendations are as follows. We suggest the use of the test $\hat{T}_n^A$ in the case of small samples. When n reaches 400, a faster yet almost as powerful

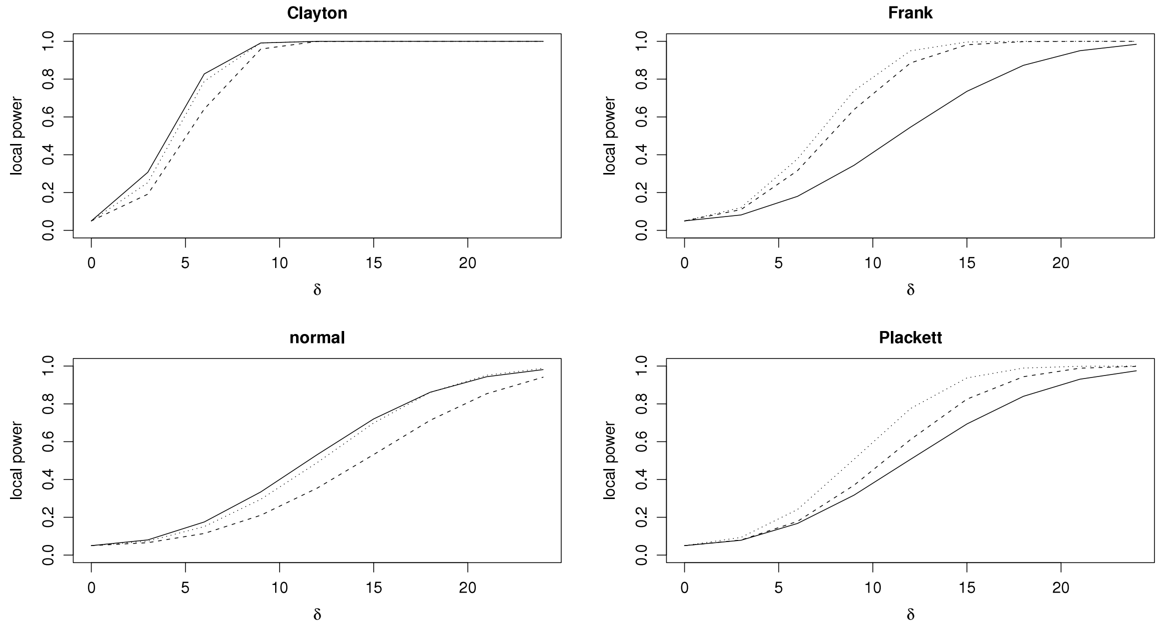

Fig. 2. Asymptotic local power functions of the tests based on $\hat{\sigma}_n^2$ (solid line), $T_{3,4,5,n}$ (dashed line) and T_n (dotted line) when C is the Gumbel–Hougaard copula with $\tau = 0.5$ and D is either the Clayton, Frank, normal or Plackett copula with $\tau = 0.5$.

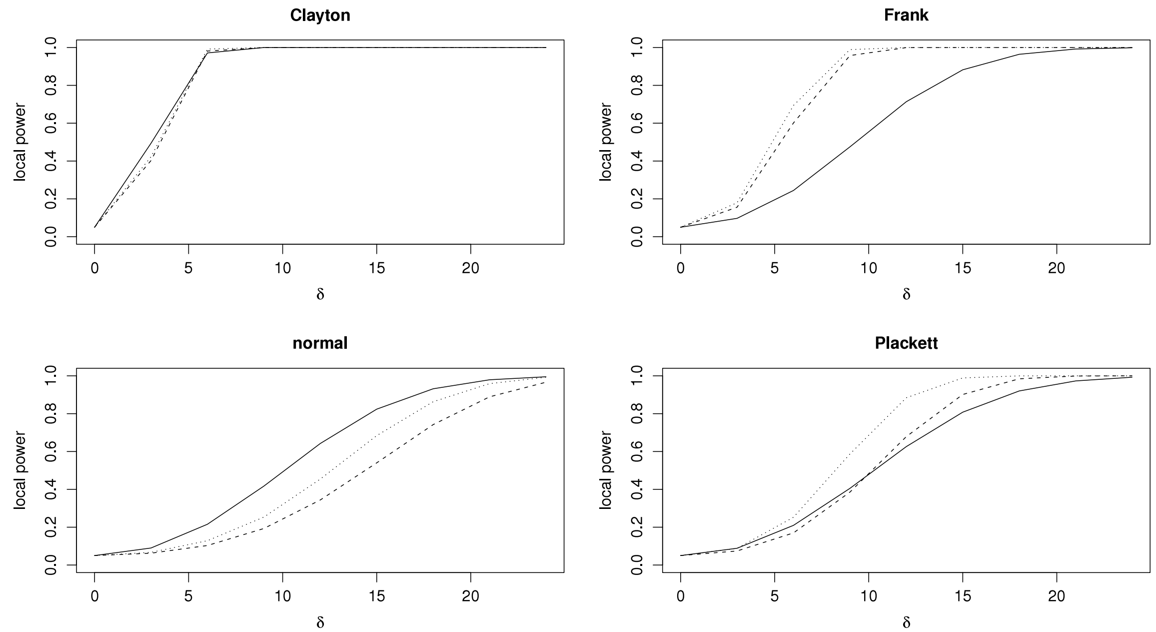

Fig. 3. Asymptotic local power functions of the tests based on $\hat{\sigma}_n^2$ (solid line), $T_{3,4,5,n}$ (dashed line) and T_n (dotted line) when C is the Gumbel–Hougaard copula with $\tau = 0.75$ and D is either the Clayton, Frank, normal or Plackett copula with $\tau = 0.75$.

alternative is the test $\hat{T}_n^C$ based on the expressions given in Section 2.6. If one suspects that the dependence might be elliptical or in the case of very large samples, the test based on $\hat{\sigma}_n^2$ should be preferred.

Acknowledgments

The authors are very grateful to Johanna Nešlehová for providing R routines implementing the test based on $\hat{\sigma}_n^2$, and to Mark Holmes for fruitful discussions, as always. The study of the finite-sample performance of the tests was carried out on a Beowulf cluster at the Department of Statistics, University of Connecticut, which was partially supported by NSF grant 0723557 ''Scientific Computing Research Environments for the Mathematical Sciences'' (SCREMS).

Appendix A. Proof of Proposition 1

Proof. Setting $\delta = 0$ in (13), we have from Proposition 4 that

$$\begin{pmatrix} \sqrt{n}(C_n - C) \\ \sqrt{n}(A_{n,c} - A) \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathbb{C} \\ \mathbb{A} \end{pmatrix}$$

in $\ell^\infty([0,1]^2) \otimes \mathcal{C}([0,1])$. Let ϑ be the map from $\mathcal{C}([0,1])$ to $\ell^\infty([a,b]^2)$ defined by

$$\vartheta(B)(u,v) = \exp\left[\log(uv) B\left\{\frac{\log(v)}{\log(uv)}\right\}\right], \quad B \in \mathcal{C}([0,1]),\ (u,v) \in [a,b]^2.$$

Now, let $B \in \mathcal{C}([0,1])$, let $(t_n)_{n\geq 1}$ be a sequence of reals converging to 0 and let $(h_n)_{n\geq 1}$ be a sequence of functions in $\mathcal{C}([0,1])$ converging to $h \in \mathcal{C}([0,1])$. Then, as n tends to infinity and uniformly in $(u,v) \in [a,b]^2$,

$$\begin{aligned} &\frac{1}{t_n}\{\vartheta(B + t_n h_n)(u,v) - \vartheta(B)(u,v)\} \\ &\quad = \exp\left[\log(uv) B\left\{\frac{\log(v)}{\log(uv)}\right\}\right] \frac{1}{t_n}\left(\exp\left[\log(uv) t_n h_n\left\{\frac{\log(v)}{\log(uv)}\right\}\right] - 1\right) \\ &\quad \to \exp\left[\log(uv) B\left\{\frac{\log(v)}{\log(uv)}\right\}\right] \log(uv) h\left\{\frac{\log(v)}{\log(uv)}\right\} = \vartheta'_B(h)(u,v). \end{aligned}$$

It is easy to verify that the map ϑ'_B is continuous with respect to the topologies of uniform convergence on $\mathcal{C}([0,1])$ and $\ell^\infty([a,b]^2)$, and linear. It follows that ϑ is Hadamard-differentiable tangentially to $\mathcal{C}([0,1])$; see e.g. [32, Chapter 3.9]. From the functional version of Slutsky's theorem, we then have that

$$\begin{pmatrix} \sqrt{n}\{C_n(u,v) - C(u,v)\} \\ \sqrt{n}\{\vartheta(A_{n,c}) - \vartheta(A)\} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathbb{C}(u,v) \\ \vartheta'_A(\mathbb{A}) \end{pmatrix}$$

in $\ell^\infty([a,b]^2)^{\otimes 2}$. The continuous mapping theorem then implies that

$$\sqrt{n}\{C_n(u,v) - C(u,v)\} - \sqrt{n}\left(\exp\left[\log(uv) A_{n,c}\left\{\frac{\log(v)}{\log(uv)}\right\}\right] - \exp\left[\log(uv) A\left\{\frac{\log(v)}{\log(uv)}\right\}\right]\right)$$

converges in $\ell^\infty([a,b]^2)$ to (5). Under $\mathcal{H}_0$, representation (1) immediately implies that this is also the weak limit of the test process $\mathbb{D}_n$ defined in (4). □

Appendix B. Proof of Proposition 2

Proof. Let $(U_i, V_i) = (F(X_i), G(Y_i))$ for all $i \in \{1, \ldots, n\}$, let $\bar{C}_n$ be the empirical c.d.f. computed from the unobservable random sample $(U_1, V_1), \ldots, (U_n, V_n)$, and let $\alpha_n = \sqrt{n}(\bar{C}_n - C)$. Furthermore, following [14], let

$$\mathbb{E} = \{(u,v) \in [0,1]^2 : 0 < u \wedge v < 1\} = (0,1]^2 \setminus \{(1,1)\},$$

let $\omega \in (0, 1/2)$, let $q_\omega(t) = t^\omega(1-t)^\omega$, $t \in [0,1]$, and let

$$\mathbb{G}_{n,\omega}(u,v) = \begin{cases} \dfrac{\alpha_n(u,v)}{q_\omega(u \wedge v)} & \text{if } (u,v) \in \mathbb{E}, \\ 0 & \text{if } u = 0 \text{ or } v = 0 \text{ or } (u,v) = (1,1). \end{cases} \tag{B.1}$$

From [14, Theorem G.1], we known that the process $\mathbb{G}_{n,\omega}$ converges weakly in $\ell^\infty([0,1]^2)$ to a centered Gaussian process $\mathbb{G}_\omega$ with continuous sample paths.

Now, for any $k \in \{1, \ldots, N\}$ and any $u, v \in [0,1]$, define

$$\mathbb{G}_{n,\omega}^{(k)}(u,v) = \begin{cases} \dfrac{1}{\sqrt{n}} \displaystyle\sum_{i=1}^n (Z_i^{(k)} - \bar{Z}^{(k)}) \dfrac{\mathbf{1}(U_{i,n} \leq u, V_{i,n} \leq v)}{q_\omega(u \wedge v)} = \dfrac{\alpha_n^{(k)}(u,v)}{q_\omega(u \wedge v)} & \text{if } (u,v) \in \mathbb{E}, \\ 0 & \text{if } (u,v) \in [0,1]^2 \setminus \mathbb{E}, \end{cases}$$

where $\alpha_n^{(k)}$ is defined in (6). From Lemma 2 of [21], we have that

$$\left(\mathbb{G}_{n,\omega}, \mathbb{G}_{n,\omega}^{(1)}, \ldots, \mathbb{G}_{n,\omega}^{(N)}\right) \rightsquigarrow \left(\mathbb{G}_\omega, \mathbb{G}_\omega^{(1)}, \ldots, \mathbb{G}_\omega^{(N)}\right)$$

in $\ell^\infty([0,1]^2)^{\otimes(N+1)}$, where $\mathbb{G}_\omega^{(1)}, \ldots, \mathbb{G}_n^{(N)}$ are independent copies of $\mathbb{G}_\omega$.

Next, let ψ_ω be the map from $\ell^\infty([0,1]^2)$ to $\ell^\infty([0,1]^2)$ defined by

$$\psi_\omega(B)(u,v) = q_\omega(u \wedge v)B(u,v), \quad B \in \ell^\infty([0,1]^2),\ (u,v) \in [0,1]^2. \tag{B.2}$$

It is easy to verify that ψ_ω is continuous with respect to the topology of uniform convergence on $\ell^\infty([0,1]^2)$, and that ψ_ω transforms $\mathbb{G}_{n,\omega}$ into α_n, and $\mathbb{G}_{n,\omega}^{(k)}$ into $\alpha_n^{(k)}$ for all $k \in \{1,\ldots,N\}$. From the continuous mapping theorem, we then obtain that

$$\left(\alpha_n(u,v), \mathbb{G}_{n,\omega}(u,v), \alpha_n^{(1)}(u,v), \mathbb{G}_{n,\omega}^{(1)}(u,v), \ldots, \alpha_n^{(N)}(u,v), \mathbb{G}_{n,\omega}^{(N)}(u,v)\right)$$

converges weakly to

$$\left(q_\omega(u \wedge v)\mathbb{G}_\omega(u,v), \mathbb{G}_\omega(u,v), q_\omega(u \wedge v)\mathbb{G}_\omega^{(1)}(u,v), \mathbb{G}_\omega^{(1)}(u,v), \ldots, q_\omega(u \wedge v)\mathbb{G}_\omega^{(N)}(u,v), \mathbb{G}_\omega^{(N)}(u,v)\right)$$

in $\ell^\infty([0,1]^2)^{\otimes 2(N+1)}$. From [26, Lemma A.1], we also know that

$$\left(\alpha_n, \alpha_n^{(1)}, \ldots, \alpha_n^{(N)}\right) \rightsquigarrow \left(\alpha, \alpha^{(1)}, \ldots, \alpha^{(N)}\right)$$

in $\ell^\infty([0,1]^2)^{\otimes(N+1)}$, where $\alpha^{(1)}, \ldots, \alpha^{(N)}$ are independent copies of the C-Brownian bridge α. As we continue, we shall therefore denote $q_\omega(u \wedge v)\mathbb{G}_\omega(u,v)$ by $\alpha(u,v)$, and $q_\omega(u \wedge v)\mathbb{G}_\omega^{(k)}(u,v)$ by $\alpha^{(k)}(u,v)$.

Now, for any $(u,v) \in [0,1]^2$, let

$$\mathbb{C}_n(u,v) = \alpha_n(u,v) - C^{[1]}(u,v)\alpha_n(u,1) - C^{[2]}(u,v)\alpha_n(1,v).$$

From the continuous mapping theorem and Proposition 2 in [19] stating that

$$\sup_{(u,v)\in[a,b]^2} \left|C_n^{[1]}(u,v) - C^{[1]}(u,v)\right| \xrightarrow{\Pr} 0 \quad \text{and} \quad \sup_{(u,v)\in[a,b]^2} \left|C_n^{[2]}(u,v) - C^{[2]}(u,v)\right| \xrightarrow{\Pr} 0,$$

we then obtain that

$$\left(\mathbb{C}_n, \mathbb{G}_{n,\omega}, \mathbb{C}_n^{(1)}, \mathbb{G}_{n,\omega}^{(1)}, \ldots, \mathbb{C}_n^{(N)}, \mathbb{G}_{n,\omega}^{(N)}\right) \rightsquigarrow \left(\mathbb{C}, \mathbb{G}_\omega, \mathbb{C}^{(1)}, \mathbb{G}_\omega^{(1)}, \ldots, \mathbb{C}^{(N)}, \mathbb{G}_\omega^{(N)}\right)$$

in $\ell^\infty([a,b]^2)^{\otimes 2(N+1)}$, where $\mathbb{C}_n^{(k)}$ is defined in (7), $\mathbb{C}$ is defined by (2), and $\mathbb{C}^{(1)}, \ldots, \mathbb{C}^{(N)}$ are independent copies of $\mathbb{C}$.

From [28, page 371] (see also [31, Proposition 1]), for any $(u,v) \in [0,1]^2$, we have that

$$\sqrt{n}\{C_n(u,v) - C(u,v)\} = \mathbb{C}_n(u,v) + R_n(u,v), \tag{B.3}$$

where $\sup_{(u,v)\in[0,1]^2} |R_n(u,v)| \xrightarrow{\Pr} 0$. Hence,

$$\left(\sqrt{n}(C_n - C), \mathbb{G}_{n,\omega}, \mathbb{C}_n^{(1)}, \mathbb{G}_{n,\omega}^{(1)}, \ldots, \mathbb{C}_n^{(N)}, \mathbb{G}_{n,\omega}^{(N)}\right) \rightsquigarrow \left(\mathbb{C}, \mathbb{G}_\omega, \mathbb{C}^{(1)}, \mathbb{G}_\omega^{(1)}, \ldots, \mathbb{C}^{(N)}, \mathbb{G}_\omega^{(N)}\right)$$

in $\ell^\infty([a,b]^2)^{\otimes 2(N+1)}$.

Next, let $(\xi_n)_{n\ge1}$ be the sequence of deterministic maps from $\ell^\infty([0,1]^2)$ to $\mathcal{C}([0,1])$ defined, for any $B \in \ell^\infty([0,1]^2)$, by

$$\xi_n(B)(t) = -\int_{l_n}^{k_n} B(\mathrm{e}^{-s(1-t)}, \mathrm{e}^{-st})K_1(s,t)\frac{\mathrm{d}s}{s} + \int_{l_n}^{k_n} B(\mathrm{e}^{-s(1-t)}, 1)K_2(s,t)\frac{\mathrm{d}s}{s} + \int_{l_n}^{k_n} B(1, \mathrm{e}^{-st})K_3(s,t)\frac{\mathrm{d}s}{s}, \tag{B.4}$$

where $l_n = 1/(n+1)$, $k_n = 2\log(n+1)$, $K_1(s,t) = q_\omega(\mathrm{e}^{-s(1-t)} \wedge \mathrm{e}^{-st})$, $K_2(s,t) = q_\omega(\mathrm{e}^{-s(1-t)})C^{[1]}(\mathrm{e}^{-s(1-t)}, \mathrm{e}^{-st})$ and $K_3(s,t) = q_\omega(\mathrm{e}^{-st})C^{[2]}(\mathrm{e}^{-s(1-t)}, \mathrm{e}^{-st})$ for all $s \in (0,\infty)$ and $t \in [0,1]$. As shown in [14, proof of Theorem 3.2], the sequence of maps $(\xi_n)_{n\ge1}$ can be used in the framework of the extended continuous mapping theorem; see e.g. [33, Theorem 18.11].

From [14, proof of Theorem 3.2], we also have that the process

$$\sqrt{n}\{\log A_n(t) - \log A(t)\} = \int_0^1 \sqrt{n}\{C_n(x^{1-t}, x^t) - C(x^{1-t}, x^t)\}\frac{\mathrm{d}x}{x\log x}, \quad t \in [0,1],$$

is asymptotically equivalent to the process $\xi_n(\mathbb{G}_{n,\omega})(t)$, $t \in [0,1]$. Now, let $c = \log(b)/\log(ab)$ and $d = \log(a)/\log(ab)$. Similarly, from the proof of Theorem 2 of [21], we have that, for any $k \in \{1,\ldots,N\}$, the process

$$\int_0^1 \mathbb{C}_{A_{n,c}}^{(k)}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x}, \quad t \in [c,d],$$

where $\mathbb{C}^{(k)}_{A_{n,c}}$ is defined in (8), is asymptotically equivalent to the process $\zeta_\infty(\mathbb{G}^{(k)}_{n,\omega})(t)$, $t \in [c, d]$, where ζ_∞ is defined as in (B.4) with $l_n = 0$ and $k_n = \infty$. Furthermore, using (2) and after a change of variable, it can be verified that

$$\xi_\infty(\mathbb{G}_\omega)(t) = \int_0^1 \mathbb{C}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x} \quad \text{and} \quad \zeta_\infty(\mathbb{G}^{(k)}_\omega)(t) = \int_0^1 \mathbb{C}^{(k)}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x}.$$

From the extended continuous mapping theorem, we then obtain that

$$\Big(\sqrt{n}\{C_n(u, v) - C(u, v)\}, \sqrt{n}\{\log A_n(t) - \log A(t)\},$$
$$\mathbb{C}^{(1)}_n(u, v), \int_0^1 \mathbb{C}^{(1)}_{A_{n,c}}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x}, \ldots, \mathbb{C}^{(N)}_n, \int_0^1 \mathbb{C}^{(N)}_{A_{n,c}}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x}\Big)$$

converges weakly to

$$\left(\mathbb{C}(u, v), \int_0^1 \mathbb{C}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x}, \mathbb{C}^{(1)}(u, v), \int_0^1 \mathbb{C}^{(1)}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x}, \ldots, \mathbb{C}^{(N)}, \int_0^1 \mathbb{C}^{(N)}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x}\right)$$

in $\{\ell^\infty([a, b]^2) \otimes \mathcal{C}([c, d])\}^{\otimes(N+1)}$.

Using the functional version of Slutsky's theorem, the fact that A_n converges uniformly in probability to A, and the fact that $A_{n,c}$ and A_n are asymptotically indistinguishable, it follows that

$$\left(\sqrt{n}(C_n - C), \sqrt{n}(A_{n,c} - A), \mathbb{C}^{(1)}_n, \mathbb{A}^{(1)}_n, \ldots, \mathbb{C}^{(N)}_n, \mathbb{A}^{(N)}_n\right) \rightsquigarrow \left(\mathbb{C}, \mathbb{A}, \mathbb{C}^{(1)}, \mathbb{A}^{(1)}, \ldots, \mathbb{A}^{(N)}, \mathbb{C}^{(N)}\right)$$

in $\{\ell^\infty([a, b]^2) \otimes \mathcal{C}([c, d])\}^{\otimes(N+1)}$, where $\mathbb{A}^{(k)}_n$ is defined in (9), $\mathbb{A}$ is defined in (3) and, for any $k \in \{1, \ldots, N\}$,

$$\mathbb{A}^{(k)}(t) = A(t)\int_0^1 \mathbb{C}^{(k)}(x^{1-t}, x^t)\frac{\mathrm{d}x}{x\log x}, \quad t \in [c, \mathrm{d}].$$

The desired result finally follows from the functional version of Slutsky's theorem based on the map ϑ used in the proof of Proposition 1, the continuous mapping theorem and the fact that $A_{n,c}$ converges uniformly in probability to A. □

Appendix C. Proof of Proposition 3

Proof. From [12, Proposition 3], we know that $A_C(t) \geq \max(t, 1 - t)$ for all $t \in [0, 1]$. Furthermore, C being LTD in both arguments, it is positive quadrant dependent, which, according again to [12, Proposition 3], implies that $A_C(t) \leq 1$ for all $t \in [0, 1]$. If A_C is additionally convex, then C_{A_C} is an extreme-value copula, which implies that $C \neq C_{A_C}$ since C is not an extreme-value copula. □

Appendix D. Proof of Proposition 4

Proof. As previously, let $(U_\mathrm{i}, V_\mathrm{i}) = (F(X_\mathrm{i}), G(Y_\mathrm{i}))$ for all $\mathrm{i} \in \{1, \ldots, n\}$, and let

$$\Lambda_n = \sum_{\mathrm{i}=1}^n \log\frac{q_{\delta_n}(U_\mathrm{i}, V_\mathrm{i})}{q_0(U_\mathrm{i}, V_\mathrm{i})}.$$

Let c (resp. d) be the p.d.f. associated with C (resp. D). According to [32, Lemma 3.10.11], under condition (14),

$$\int_{[0,1]^2}\frac{\mathrm{d}(u, v) - c(u, v)}{c(u, v)}\mathrm{d}C(u, v) = 0, \qquad \int_{[0,1]^2}\frac{\{\mathrm{d}(u, v) - c(u, v)\}^2}{\{c(u, v)\}^2}\mathrm{d}C(u, v) < \infty$$

and the log-likelihood ratio Λ_n can be expressed as

$$\Lambda_n = \frac{1}{\sqrt{n}}\sum_{\mathrm{i}=1}^n \delta\,\frac{\mathrm{d}(U_\mathrm{i}, V_\mathrm{i}) - c(U_\mathrm{i}, V_\mathrm{i})}{c(U_\mathrm{i}, V_\mathrm{i})} - \frac{1}{2}\int_{[0,1]^2}\delta^2\frac{\{\mathrm{d}(u, v) - c(u, v)\}^2}{c(u, v)}\mathrm{d}u\mathrm{d}v + R_n,$$

where R_n converges to 0 in probability both under C and under $(Q_{\delta_n})_{n\geq 1}$.

Now, using the notation defined in Appendix B, rewrite (B.1) as

$$\mathbb{G}_{n,\omega}(u, v) = \begin{cases} \dfrac{1}{\sqrt{n}}\displaystyle\sum_{\mathrm{i}=1}^n \frac{\mathbf{1}(U_\mathrm{i} \leq u, V_\mathrm{i} \leq v) - C(u, v)}{q_\omega(u \wedge v)} & \text{if } (u, v) \in \mathbb{E}, \\ 0 & \text{if } u = 0 \text{ or } v = 0 \text{ or } (u, v) = (1, 1). \end{cases}$$

Then, for any finite collection of points $(u_1, v_1), \ldots, (u_k, v_k)$ in $[0, 1]^2$, we have, from the multivariate central limit theorem, that, under C,

$$\left(\mathbb{G}_{n,\omega}(u_1, v_1), \ldots, \mathbb{G}_{n,\omega}(u_k, v_k), \Lambda_n\right) \rightsquigarrow \left(\mathbb{G}_{\omega}(u_1, v_1), \ldots, \mathbb{G}_{\omega}(u_k, v_k), \Lambda\right),$$

in $\mathbb{R}^{\otimes(k+1)}$, where Λ is the weak limit of Λ_n. Tightness follows from the weak convergence of $\mathbb{G}_{n,\omega}$ in $\ell^\infty([0, 1]^2)$. Hence, $(\mathbb{G}_{n,\omega}, \Lambda_n) \rightsquigarrow (\mathbb{G}_\omega, \Lambda)$ in $\ell^\infty([0, 1]^2) \otimes \mathbb{R}$.

The continuous mapping theorem with the map ψ_ω defined in (B.2) then implies that

$$\left(\alpha_n(u, v), \mathbb{G}_{n,\omega}(u, v), \Lambda_n\right) \rightsquigarrow \left(q_\omega(u \wedge v)\mathbb{G}_\omega(u, v), \mathbb{G}_\omega(u, v), \Lambda\right)$$

in $\ell^\infty([0, 1]^2)^{\otimes 2} \otimes \mathbb{R}$. As previously, the C-Brownian bridge $q_\omega(u \wedge v)\mathbb{G}_\omega(u, v)$ will be denoted by $\alpha(u, v)$ in what follows. Proceeding as in Appendix B, from the continuous mapping theorem and Stute's representation given in (B.3), we obtain that

$$\left(\sqrt{n}(C_n - C), \mathbb{G}_{n,\omega}, \Lambda_n\right) \rightsquigarrow \left(\mathbb{C}, \mathbb{G}_\omega, \Lambda\right)$$

in $\ell^\infty([0, 1]^2)^{\otimes 2} \otimes \mathbb{R}$, where $\mathbb{C}$ is defined by (2).

Next, by proceeding as in [14, proof of Theorem 3.2], and as already explained in Appendix B, under H, the process

$$\sqrt{n}\{\log A_n(t) - \log A(t)\} = \int_0^1 \sqrt{n}\{C_n(x^{1-t}, x^t) - C(x^{1-t}, x^t)\}\frac{dx}{x \log x}$$

is asymptotically equivalent to the process $\xi_n(\mathbb{G}_{n,\omega})(t)$, where the sequence of maps $(\xi_n)_{n\geq 1}$ is defined in (B.4). The extended continuous mapping theorem then implies that, under H,

$$\left(\sqrt{n}\{C_n(u, v) - C(u, v)\}, \sqrt{n}\{\log An(t) - \log A(t)\}, \Lambda_n\right) \rightsquigarrow \left(\mathbb{C}(u, v), \int_0^1 \mathbb{C}(x^{1-t}, x^t)\frac{dx}{x \log x}, \Lambda\right)$$

in $\ell^\infty([0, 1]^2) \otimes \mathcal{C}([0, 1]) \otimes \mathbb{R}$.

All is now set for an application of Le Cam's third lemma [32, Theorem 3.10.7, Example 3.10.8 and page 407]. We therefore obtain that, under $(H_{\delta_n})_{n\geq 1}$,

$$\begin{pmatrix} \sqrt{n}\{C_n(u, v) - C(u, v)\} \\ \sqrt{n}\{\log A_n(t) - \log A(t)\} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathbb{C}(u, v) + \mathrm{cov}\{\mathbb{C}(u, v), \Lambda\} \\ \int_0^1 \mathbb{C}(x^{1-t}, x^t)\frac{dx}{x \log x} + \mathrm{cov}\left\{\int_0^1 \mathbb{C}(x^{1-t}, x^t)\frac{dx}{x \log x}, \Lambda\right\} \end{pmatrix}$$

in $\ell^\infty([0, 1]^2) \otimes \mathcal{C}([0, 1])$.

From Example 3.10.8 and Theorem 3.10.12 in [32] applied to the process α_n, we know that

$$\begin{aligned} \mathrm{cov}\{\alpha(u, v), \Lambda\} &= \int_{[0,1]^2} \{\mathbf{1}(s \leq u, t \leq v) - C(u, v)\}\delta\frac{d(s, t) - c(s, t)}{c(s, t)}dC(s, t) \\ &= \delta\{D(u, v) - C(u, v)\}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{cov}\{\mathbb{C}(u, v), \Lambda\} &= \mathrm{cov}\{\alpha(u, v), \Lambda\} - C^{[1]}(u, v)\mathrm{cov}\{\alpha(u, 1), \Lambda\} - C^{[2]}(u, v)\mathrm{cov}\{\alpha(1, v), \Lambda\} \\ &= \mathrm{cov}\{\alpha(u, v), \Lambda\} = \delta\{D(u, v) - C(u, v)\}. \end{aligned}$$

We then obtain that

$$\begin{aligned} \mathrm{cov}\left\{\int_0^1 \mathbb{C}(x^{1-t}, x^t)\frac{dx}{x \log x}, \Lambda\right\} &= \int_0^1 \mathrm{cov}\left\{\mathbb{C}(x^{1-t}, x^t), \Lambda\right\}\frac{dx}{x \log x} \\ &= \delta\int_0^1 \{D(x^{1-t}, x^t) - C(x^{1-t}, x^t)\}\frac{dx}{x \log x} \\ &= \delta\{\log A_D(t) - \log A(t)\}, \end{aligned}$$

where the last equality comes from (11). The desired result finally follows from the functional version of Slutsky's theorem and the fact that $A_{n,c}$ and A_n are asymptotically indistinguishable. □

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