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On the fractal structure of the rescaled evolution set of Carlitz sequences of polynomials

F. von Haeseler *, H.-O. Peitgen, G. Skordev

Cevis – Universität Bremen, Universitätsallee 29, 28359 Bremen, Germany

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Abstract

Self-similarity properties of the coefficient patterns of the so-called m-Carlitz sequences of polynomials are considered. These properties are coded in an associated fractal set – the rescaled evolution set. We extend previous results on linear cellular automata with states in a finite field. Applications are given for the sequence of Legendre polynomials and sequences associated with the zero Bessel function.

Keywords: Lucas property; Self-similarity; m-automaticity; Legendre polynomials; Zero Bessel function

1. Introduction

It has been observed that the patterns representing the time evolution of many cellular automata starting with a finite initial configuration exhibit a fractal (self-similar) structure [25]. Willson proposed a rescaling procedure of the time evolution pattern which associated with every linear cellular automata with states in the ring of integers modulo a prime power, a compact set [23]. It was called later on rescaled evolution set [9]. Fractal properties of the rescaled evolution set reflect properties of the evolutions patterns.

The problem of describing this properties was considered in [24,21] and the references. In [9], the authors addressing the same problem introduced the class of m-Fermat cellular automata and showed that for any m-Fermat cellular automata starting with a finite initial condition there exists a rescaled evolution set. Furthermore, the self-similarity structure of the rescaled evolution set of an m-Fermat automaton is

* Corresponding author.
E-mail address: skordev@mathemalik.uni-breman.de (F. von Haeseler)

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described by a hierarchical iterated function system, or generated by an appropriate matrix substitution.

Among others, Bondarenko [5] and Sved [20], observed that many classical number sequences modulo m also exhibit certain self-similarity features. Using an idea from Hasseler, Peitgen and Skordev [9] the fractal structure of Gaussian binomial coefficients and the Stirling numbers of the first and second kind modulo a prime power was described in [14] . Earlier this was done for the binomial coefficient modulo a prime power with a different method in [10]. The basic idea is to consider these classical number sequences as orbits of (time dependent) cellular automata. For all number tables under investigation it turns out that the underlying cellular automata are linear automata with the m-Fermat property. Therefore, a rescaled evolution set exists and thus explains the observed self-similarities in the number table itself.

In this paper, we study a special type of time dependent linear cellular automata, for which there exist a rescaled evolution set. To this end, we need a generalization of the m-Fermat property which is closely connected with the Lucas property, cf. [17].

It was observed by I. Schur that the sequence of Legendre polynomials modulo an odd prime number has a property similar to the Lucas property [22,2] and the references there. This congruence is called Schur congruence for Legendre polynomials.

Carlitz generalized this congruence introducing m-Carlitz sequences of polynomials [6] (the name was introduced in [2]). The notion of m-Carlitz sequences is the proper generalization of the m-Fermat property.

In this note, we associate with every m-Carlitz sequence of polynomials a rescaled evolution set. This set is usually a fractal. The self-similarity structure of these sets is deciphered with an appropriate hierarchical iterated function system which is generated by a matrix substitution.

Several examples illustrate the general result. In particular, rescaled evolution sets for the sequence of Legendre polynomials modulo an odd prime number and the Carlitz sequences of polynomials associated with the Bessel function $J_0(X)$.

2. m-Carlitz sequences of polynomials

Let m be a natural number $\geqslant 2$. With $\mathbb{Q}_{(m)}$ we shall denote the rationals

$$\mathbb{Q}_{(m)} = \left\{ \frac{a}{b} \colon a \in \mathbb{Z}, b \in \mathbb{N}, \gcd(m,b) = 1 \right\}.$$

Note that $\mathbb{Q}_{(m)}$ is a commutative ring.

Definition 1. A sequence of polynomials $\mathscr{R} = (R_n(X))_{n\geqslant 0}$, $R_n(X) \in \mathbb{Q}_{(m)}[X]$ is called *m-Carlitz sequence* (or a sequence with the m-Carlitz property) if

- $R_0(X) = 1$, and
- $R_{nm+u}(X) \equiv R_n(X^m)R_u(X) \pmod m, n \in \mathbb{N}, u \in \{0,\dots,m-1\}$.

for $m \in \mathbb{N}, m\geqslant 2$.

It follows from the definition that

$$R_n(X) \equiv R_{n_s}(X^{m^s}) \dots R_{n_1}(X^m) R_{n_0}(X) \, (\operatorname{mod} m)$$

holds for $n = n_s m^s + \cdots + n_1 m + n_0, n_j \in \{0, \dots, m-1\}, 0 \leqslant j \leqslant s$.

Remark 1. 1. In [2], the authors introduce a more general notion of m-Carlitz sequences. The sequences we shall consider here are called simple m-Carlitz sequences.

2. Let $m \geqslant 2$ be a natural number and let $R_0(X) = 1, R_j(X) \in \mathbb{Q}_{(m)}[X], 0 \leqslant j \leqslant m-1$ be polynomials. Then these polynomials induce a unique m-Carlitz sequence by setting

$$R_n(X) = R_{n_s}(X^{m^s}) \dots R_{n_1}(X^m) R_{n_0}(X)$$

for $n = n_s m^s + \cdots + n_1 m + n_0$, $n_j \in \{0, \dots, m-1\}, 0 \leqslant j \leqslant s$.

3. Linear cellular automata with the m-Fermat property generate particular examples of m-Carlitz sequences of polynomials. Let $R(X) \in \mathbb{Q}_{(p)}[X]$ and let p be a prime number. The polynomials $R_j(X) = R(X)^j$, $0 \leqslant j \leqslant p-1$ generate a unique p-Carlitz sequence. This sequence is the orbit of 1 w.r.t. linear cellular automaton corresponding to the polynomial $R(X)$, e.g. [1].

4. Let $p_1, \dots, p_s$ be prime numbers and assume that $\mathscr{R}^{(j)} = (R_n^{(j)}(X))_{n \geqslant 0}$ is a p_j-Carlitz sequence for $0 \leqslant j \leqslant s$. Then the sequence $(R_n(X))_{n \geqslant 0}$ defined by

$$R_n(X) = \sum_{i=1}^{s} p_1 \dots p_{i-1} p_{i+1} \dots p_s R_n^{(i)}(X) + p_1 \dots p_s Q(X)$$

is a p_j-Carlitz sequence for all $p_j, 0 \leqslant j \leqslant s$ and all $Q(X) \in \mathbb{Z}[X]$

5. The Legendre polynomials provide another example of a Carlitz-sequence. One possible definition of the Legendre polynomials $\mathscr{P} = (P_n(X))_{n \geqslant 0}$ is to be the unique solution of the recurrence relation

$$(n+1) P_{n+1}(X) = (2n+1) X P_n(X) - n P_{n-1}(X), \quad n \geqslant 1$$

with the initial condition $P_0(X) = 1, P_1(X) = X$, [15, p. 46].

For a given odd prime number we consider the Legendre polynomials as an element of $\mathbb{Q}_{(p)}[X]$. By [15, p. 44], we obtain

$$P_n(X) = \frac{1}{2^n} \sum_{v=0}^{\lfloor n/2 \rfloor} (-1)^v \binom{n}{v} \binom{2n-2v}{n-2v} x^{n-2v}$$

a p-Carlitz sequence. A property discovered by Schur, and called Schur congruence for Legendre polynomials [22,6]. For a simple proof of this congruence see [2].

6. In [17], McIntosh introduced and considered number (and polynomial) sequences satisfying a property which he called Lucas Property (LP) or Double Lucas Property (DLP).

Definition 2. A sequence $r : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}_{(p)}$ has the p-Lucas property if it satisfies the following conditions.

- $r(m,n) \equiv 0 \pmod p$ for $m > n$ and $0 \leqslant n \leqslant p-1$.
- $r(m,n) \equiv r(m_s,n_s)\dots r(m_0,n_0) \pmod p$ for $m = m_s p^s + \cdots + m_1 p + m_0$ and $n = n_s p^s + \cdots + n_1 p + n_0$, where $m_j, n_j \in [p]$, $0 \leqslant j \leqslant s$, and at least one of m_s or n_s is different from zero.

Because of $r(m,n) \equiv 0 \pmod p$ for $m > n$ it follows that the sequence of polynomials $\mathscr{R} = (R_n(X))_{n \geqslant 0}$ defined as $R_n(X) = \sum_m r(m,n)X^m$ is a p-Carlitz sequence.

The converse is not true. The coefficients $(q(m,n))_{m,n \geqslant 0}$ of a given p-Carlitz sequence of polynomials

$$\mathscr{Q} = (Q_n(X))_n,\ Q_n(X) = \sum_m q(m,n)X^m$$

do not necessarily form a p-Lucas sequence.

If $\deg Q_j \leqslant p-1$ holds for all $0 \leqslant j \leqslant p-1$, then the double sequence $(q(m,n))_{m,n \geqslant 0}$ forms a sequence with p-Lucas property.

7. More examples for and properties of m-Carlitz sequences of polynomials can be found in [2].

3. Rescaled evolution set

In this section, we develop a graphical representation of and a proper rescaling procedure for p-Carlitz sequences of polynomials. The approach is the same as the one developed for cellular automata in [9,11].

The basic idea is to associate a compact subset in $\mathbb{R}^2$ – rescaled evolution set – with a p-Carlitz sequence of polynomials. In general, this compact set is a fractal. The self-similarity structure of the rescaled evolution set is encoded by a hierarchical iterated function system.

Let $\Sigma_c([p])$ be the set of all sequences $\underline{a} : \mathbb{Z} \to [p]$, where $[p] = \{0,\dots,p-1\}$, with compact support, i.e., $\mathrm{Card}\{m\colon m \in \mathbb{N}, \underline{a}(m) \neq 0\} < \infty$. By $\underline{0}$, we denote the zero sequence and $\Sigma_c^*([p]) = \Sigma_c([p]) \setminus \{\underline{0}\}$.

With $(\mathscr{H}(\mathbb{R}^2), h)$ we denote the space of nonempty compact subsets of $\mathbb{R}^2$ equipped with the Hausdorff metric h which is induced by the maximum norm on $\mathbb{R}^2$, e.g., [8,7]. Let $I = \{(x,y)\colon x,y \in [0,1]\}$ denote the unit square, then $I(n,m) = \{(x,y)+(n,m)\colon x,y \in [0,1]\}$ with $n,m \in \mathbb{N}$ denotes the translated unit square.

Definition 3. The map $G : \Sigma_c^*([p]) \to \mathscr{H}(\mathbb{R}^2)$, defined by

$$G(\underline{a}) = \bigcup \{I(m,0))\colon \underline{a}(m) \neq 0\}$$

is called graphical representation of the sequence $\underline{a}$.

Remark 2. 1. The choice of the graphical interpretation is not important. Another natural graphical representation is

$$G^*(\underline{a}) = \{(m,0)\colon \underline{a}(m) \neq 0\},$$

where $\underline{a} \neq \underline{0}$. We could and shall use also a more general graphical representation. Let K_j, $j = 1,\dots,p-1$ be a nonempty compact subsets in $\mathbb{R}^2$. Then a graphical representation $G_{K_1,\dots,K_{p-1}}$ is defined by

$$G_{K_1,\dots,K_{p-1}}(\underline{a}) = \bigcup \{(m,0) + K_j\colon \underline{a}(m) = j\},$$

where $\underline{a} \neq \underline{0}$.

2. Let $R(X) = \sum_{n=0}^{\infty} r_n X^n$ be a polynomial with coefficients in $\mathbb{Q}_{(m)}$, then the sequence of coefficients of the polynomial $R(X) \bmod p$ can be considered as an element of $\Sigma_{\mathrm{c}}^*([p])$. The graphical representation or $R(X)$ is defined as the graphical representation of its sequence of coefficients modulo p:

$$G(R(X)) = G((r_n \bmod p)_{n \geqslant 0}).$$

We say that the sequence $\underline{r} = (r(m,n))_{m,n\geqslant 0}$ is generated by a p-Carlitz sequence of polynomials $\mathscr{R} = (R_n(X))_{n\geqslant 0}$ if

$$R_n(X) \equiv \sum_m r(m,n) X^m \pmod p$$

holds for all $n \in \mathbb{N}$.

Definition 4. Let $\mathscr{R} = (R_n(X))_{n\geqslant 0}$ be a p-Carlitz sequence of polynomials and $\underline{r} = (r(m,n) \bmod p)_{m,n\geqslant 0}$ the double sequence associated with it. The set

$$X_k(\mathscr{R}) = \bigcup_{n=0}^{p^k-1} (G(R_n(X)) + (0,n))$$

is called the kth graphical representation of $\mathscr{R}$ (mod p).

Proposition 1. *Let $\mathscr{R} = (R_n(X))_{n\geqslant 0}$ be p-Carlitz sequence of polynomials. Then the sequence*

$$\left(\frac{1}{p^k} X_k(\mathscr{R})\right)_{k\geqslant 0}$$

converges in the space $(\mathscr{H}(\mathbb{R}^2), h)$.

Proof. The proof follows the same lines as the proof of Theorem 4.5 in [9]. It is sufficient to prove that

$$h\left(X_k(\mathscr{R}), \frac{1}{p} X_{k+1}(\mathscr{R})\right) \leqslant C,$$

where $C = \max\{\deg R_j(X)\colon 0 \leqslant j \leqslant p-1\}$. This implies that the sequence is a Cauchy sequence and therefore convergent, since $(\mathscr{H}(\mathbb{R}^2), h)$ is a complete metric space [8,7].

We have to show that

$$X_k(\mathscr{R}) \subset \left(\frac{1}{p}X_{k+1}(\mathscr{R})\right)_C,$$
$$\frac{1}{p}X_{k+1}(\mathscr{R}) \subset (X_k(\mathscr{R}))_C,$$

where $(M)_C = \{(x,y)\colon (x,y) \in \mathbb{R}^2, \mathrm{dist}((x,y),M) < C\}$ for $M \subset \mathbb{R}^2$ (C-neighborhood of the set M in $\mathbb{R}^2$).

Let $(m,n) \in X_k(\mathscr{R})$, i.e., $r(m,n) \not\equiv 0 \pmod p$ and $0 \leqslant n \leqslant p^k - 1$. Then the p-Carlitz property of the sequence $\mathscr{R}$ implies $R_{pn}(X) \equiv R_n(X^p) \pmod p$ and therefore $r(pm, pn) \not\equiv 0 \pmod p$, and $0 \leqslant pn \leqslant p^{k+1} - p \leqslant p^{k+1} - 1$, i.e., $(pm, pn) \in X_{k+1}(\mathscr{R})$. In other words,

$$X_k(\mathscr{R}) \subset \left(\frac{1}{p}X_{k+1}(\mathscr{R})\right)_1 \subset \left(\frac{1}{p}X_{k+1}(\mathscr{R})\right)_C$$

which proves the first inclusion.

For the second inclusion, we consider $(m,n) \in X_{k+1}(R)$, i.e., $r(m,n) \not\equiv 0 \pmod p$ and $0 \leqslant n \leqslant p^{k+1} - 1$. Let $n = n'p + u, 0 \leqslant u \leqslant p - 1$. Then

$$R_n(X) \equiv R_{n'}(X^p)R_u(X) \pmod p$$

which implies

$$r(m,n) \equiv \sum_s r(sp, n'p)r(sp,u) \pmod p,$$

where the summation is over all s with $m - \deg R_u(X) \leqslant s \leqslant m$. Since $r(n,m) \not\equiv 0 \pmod p$ there is at least one s satisfying the above condition and such that $r(sp, n'p) \not\equiv 0 \pmod p$. Then $(s,n') \in X_k(\mathscr{R})$ and $\mathrm{dist}(1/p(m,n),(s,n')) \leqslant (1/p)\deg R_u(X)$. Which gives the second inclusion. □

We call the limit of the sequence

$$X_\infty(\mathscr{R}) = \lim_{k\to\infty} \frac{1}{p^k}X_k(\mathscr{R})$$

the rescaled evolution set of the p-Carlitz sequence $\mathscr{R}$.

Remark 3. 1. The proposition does not depend on the graphical representation.

2. Figs. 1–3 represent some rescaled evolution sets of p-Carlitz sequences. All figures exhibit the simplest self-similarity features, namely the rescaled evolution set is generated by a iterated function system. In general rescaled evolution sets have a more complex self-similarity structure. For other examples of rescaled evolution sets of linear cellular automata see [9,13].

3. Let $\mathscr{R}$ be a p-Carlitz sequence $\mathscr{R}$ and $X_\infty(\mathscr{R})$ its associated rescaled evolution set. Since in most cases the rescaled evolution set is a fractal set we are interested in the box-counting dimension (see e.g. [8]) of $X_\infty(\mathscr{R})$. Let $l \in \mathbb{N}, s \in \mathbb{Z}$. We call a l-block $\boldsymbol{b} \in [p]^l$ s-accessible, or accessible if $s=1$, w.r.t. p-Carlitz sequence if there exist $m,n \in$

$\mathbb{N}$ with $(r(m,n), r(m+s,n), \ldots, r(m+s(l-1),n)) = \boldsymbol{b}$, where $\underline{r} = (r(m,n) \bmod p)_{m,n \geqslant 0}$ denotes the associated double sequence of $\mathscr{R}$.

Let $k \in \mathbb{N}$ and let $s, l \in \mathbb{N}$ then the set

$$X_k(\mathscr{R}, l, s) = \bigcup_{m,n} \left\{ \begin{array}{l} I(m,n)\colon (r(m,n), r(m+s,n), \ldots, r(m+s(l-1),n)) \\ \neq (0,\ldots,0), 0 \leqslant t \leqslant p^k - 1 \end{array} \right\}$$

is called the (l,s) representation of level k (w.r.t. $\mathscr{R}$).

For an s-accessible l-block $\boldsymbol{b}$ the set

$$X_k(\mathscr{R}, \boldsymbol{b}, s) = \bigcup \left\{ \begin{array}{l} I(m,n)\colon (r(m,n), r(m+s,n), \ldots, r(m+s(l-1),n) = \boldsymbol{b}, \\ 0 \leqslant n \leqslant p^k - 1 \end{array} \right\}$$

is called the $(s, \boldsymbol{b})$ representation of level k (w.r.t. $\mathscr{R}$).

Again, we consider the rescaled sequences of the (l,s) representation of level k and $(s, \boldsymbol{b})$ representation of level k, respectively, and we obtain:

(a) The sequence

$$\left(\frac{1}{p^k} X_k(\mathscr{R}, l, s) \right)_{k \geqslant 0}$$

converges to the rescaled evolution set $X_\infty(\mathscr{R})$.

(b) The sequence

$$\left(\frac{1}{p^k} X_k(\mathscr{R}, \boldsymbol{b}, s) \right)_{k \geqslant 0}$$

has a limit $X_\infty(\mathscr{R}, \boldsymbol{b}, s)$.

The first assertion is an immediate consequence of Proposition 1 and an appropriate choice of the graphical representation, which is defined as

$$G_s(\underline{a}) = \bigcup_m \{ I(m,0)\colon \underline{a}(m+s) \neq 0 \}.$$

The second assertion is essentially Lemma 5.2 in [12]. In general, the limit $X_\infty(\mathscr{R}, \boldsymbol{b}, s)$ depends on the choice of the block $\boldsymbol{b}$. In [12], Theorem 4 shows that for p-state linear cellular automata with p a prime number the limit is independent of the choice of $\boldsymbol{b}$ and therefore equal to the rescaled evolution set.

There are several ways to compute the box-counting dimension of $X_\infty(\mathscr{R})$. We fix $l \geqslant 1$ and define

$$\begin{aligned} N_k(\mathscr{R}, l) = \mathrm{Card}\{ &(m,n)\colon (r(m,n), r(m+1,n), \ldots, r(m+l-1,n)) \\ &\neq (0,\ldots,0), 0 \leqslant t \leqslant p^k - 1 \} \end{aligned}$$

which is equal to the number of black squares in the $(l,1)$ representation of level k. Then the box-counting dimension of the rescaled evolution set is given by

$$\lim_{k \to \infty} \frac{\log N_k(\mathscr{R}, l)}{k \log p} = \dim_{\mathrm{B}} X_\infty(\mathscr{R}).$$

In Section 7 we shall show that the limit exists and, furthermore, we present a formula for computing the box-counting dimension.

The existence of the limit yields that the growth rate of $N_k(\mathscr{R}, l)$ is proportional to p^{Dk} where $D = \dim_{\mathrm{B}} X_\infty(\mathscr{R})$. Moreover, the growth rate of the number of accessible l-blocks of a given p-Carlitz sequence is independent of l, see Corollary 3 in [12].

4. p-automaton corresponding to a p-Carlitz sequence

In this section, we study the double sequence generated by a p-Carlitz sequence under the aspect of automaticity. A (two-dimensional) p-automaton provides a device to compute the value of $r(n,m)$ modulo p from a knowledge of the p-adic expansion of n and m, respectively, [3,19].

Let $\mathscr{R} = (R_n(X))_{n \geqslant 0}$ be a p-Carlitz sequence of polynomials and $\underline{r} = (r(m,n) \bmod p)_{m,n \geqslant 0}$ the double sequence associated with it.

We start with the definition of the two-dimensional p-automaton, or equivalently a $(p \times p)$-substitution, $\mathscr{A}(\mathscr{R})$ associated with $\mathscr{R}$.

Let $d = \max\{ R_j(X)\colon 0 \leqslant j \leqslant p-1\}$. The p-automaton $\mathscr{A}_p(\mathscr{R})$ has

- state alphabet Z_p^d, where $Z_p = \mathbb{Z}/p\mathbb{Z}$. The elements of Z_p^d are denoted as $(a_{-d+1}, a_{-d+2}, \dots, a_{-1}, a_0)$.
- initial state $\boldsymbol{e}_0 = (0, \dots, 0, 1)$,
- output alphabet Z_p.
- output map $\pi : Z_p^d \to Z_p$ which is defined as

$$\pi(a_{-d+1}, \dots, a_0) = a_0.$$

What is left is the definition of the input maps

$$(i,j) : Z_p^d \to [p]^d, \quad i,j \in Z_p.$$

For that purpose we introduce some notations. Let $Z_p[[X^{-1}]]$ denote the Laurent series with coefficients in Z_p. We define the d-block map $b_d : Z_p[[X^{-1}]] \to Z_p^d$ as

$$\sum_{j \in \mathbb{Z}} l_j X^j \;\mapsto\; (l_{-d+1}, \dots, l_0).$$

The map b_d is a linear map of Z_p-linear spaces or finitely generated free Z_p-modules. By $\boldsymbol{e}_i$, $i = 0, \dots, d-1$, we denote the ith basis vector of the linear space Z_p^d. We then have

$$\boldsymbol{e}_i = b_d(X^{-i})$$

for $i = 0, \dots, d-1$.

With the help of the d-block map we are in a position to define the input maps of our automaton $\mathscr{A}_p(\mathscr{R})$. The input maps $(i,j) : Z_p^d \to Z_p^d$ will be linear maps and therefore it is sufficient to define $(i,j) : Z_p^d \to Z_p^d$ only on the elements $\boldsymbol{e}_l$, $l = 0, \dots, d-1$. We set

$$(i,j).\boldsymbol{e}_l \equiv b_d(X^{-lp-i} R_j(X)) \pmod p$$

for $i,j \in Z_p$, $0 \leqslant l \leqslant d-1$.

The input maps are extended to the maps $(m,n): Z_p^d \to Z_p^d$, $m,n \in \mathbb{N}$ with

$$(m,n).\boldsymbol{a} \equiv (m_0,n_0)(m_1,n_1)\dots(m_s,n_s).\boldsymbol{a} = r(m,n)\,(\mathrm{mod}\ p), \quad \boldsymbol{a} \in Z_p$$

for

$$m = m_s p^s + \cdots + m_1 p + m_0, n = n_s p^s + \cdots + n_1 p + n_0, \quad m_j, n_j \in [p].$$

Observe that $\boldsymbol{e}_0$ is a fixed point of the map $(0,0)$.

Proposition 2. *Let $\mathscr{R} = (R_n(X))_{n \geqslant 0}$ be a p-Carlitz sequence of polynomials and $\underline{r} = (r(m,n) \bmod p)_{m,n \geqslant 0}$ the double sequence generated by it. The p-automaton $\mathscr{A}_p(\mathscr{R})$ generates the sequence $\underline{r}$, i.e.,*

$$\pi((m,n).\boldsymbol{e}_0) \equiv r(m,n)\,(\mathrm{mod}\ p), \quad m,n \in \mathbb{N}.$$

Proof. As in the proof of Theorem 3 given in [1] it is sufficient to show that

$$(m,n).\boldsymbol{e}_0 \equiv b_d(X^{-m} R_n(X))\,(\mathrm{mod}\ p)$$

holds for all $m,n \in \mathbb{N}$.

For $m = n = 0$ there is nothing to show.

Let $n = n_s p^s + \cdots + n_1 p + n_0$ and $m = m_s p^s + \cdots + m_1 p + m_0$ denote the p-adic expansion of n and m, respectively. Furthermore, let us assume that at least one of the digits m_s, n_s is different from zero. The proof proceeds by induction with respect to s. For $s = 0$ we obtain the definition of the input maps (i,j). Let us now assume that assertion is true for all numbers of the set $\{0,\dots,p^{s-1}-1\}$ and that m,n are given by their p-expansions above. Then

$$\begin{aligned}
(m,n).\boldsymbol{e}_0 &\equiv (m_0 + m'p, n_0 + n'p).\boldsymbol{e}_0 = (m_0,n_0).(m',n').\boldsymbol{e}_0\\
&\equiv (m_0,n_0).b_d(X^{-m'} R_{t'}(X)),\\
&\quad \text{by the induction hypothesis,}\\
&\equiv \sum_u r(m'-u,n')(m_0,n_0).\boldsymbol{e}_u\\
&\equiv \sum_u r(m'-u,n') b_d(X^{-up-m_0} R_{n_0}(X))\\
&\equiv \sum_u r(m'p-up,n'p) b_d(X^{-up-m_0} R_{n_0}(X)),\\
&\quad \text{from the } p\text{-Carlitz property,}\\
&\equiv \left(\sum_u (m'p-up,n'p) r(up+m_0+j,n_0)\right)_{j=-d+1,\dots,0}\\
&\equiv (r(m-d+1,n),\dots,r(m,n))\,(\mathrm{mod}\ p),\\
&\quad \text{as } d \geqslant deg\, R_j(X), \quad 0 \leqslant j \leqslant p-1. \qquad \square
\end{aligned}$$

Remark 4. 1. In other words, Proposition 2 states that the sequence $(r(m,n) \bmod p)_{m,n\geqslant 0}$ associated with a p-Carlitz sequence is p-automatic, in the sense of Allouche and Mendes France [3] and Salon [19].

2. A different proof of the above proposition is given in [2].

5. Self-similarity structure of the rescaled evolution set of a p-Carlitz sequence of polynomials

In this section, we shall describe the self-similarity structure of the rescaled evolution set corresponding to a p-Carlitz sequence $\mathscr{R}=(R_n(X))_{n\geqslant 0}$. In the previous section, we have seen that the double sequence associated to a p-Carlitz sequence is p-automatic. The two-dimensional p-automaton $\mathscr{A}_p(\mathscr{R})$ associated to a p-Carlitz sequence $\mathscr{R}$ yields a $(p\times p)$ substitution (see [19,9,11,13]) which is the appropriate tool to describe the rescaled evolution set.

We denote this substitution by

$$\sigma_p=\sigma_p(\mathscr{R}):Z_p^d\to(Z_p^d)^{[p]\times[p]}$$

$$\boldsymbol{a}\mapsto((i,j).\boldsymbol{a})_{0\leqslant i,j\leqslant p-1}.$$

The substitution σ_p induces a map

$$\Sigma_p:\Sigma(\mathbb{N}^2,Z_p^d)\to\Sigma(\mathbb{N}^2,Z_p^d),$$

where $\Sigma(\mathbb{N}^2,Z_p^d)=\{\underline{\omega}\colon \underline{\omega}:\mathbb{N}^2\to Z_p^d\}$, as follows:

$$\Sigma_p(\underline{\omega})(m,n)=(m_0,n_0).\underline{\omega}(m',n'),$$

where $m=m'p+m_0, n=n'p+n_0, m_0,n_0\in[p], m',n'\in\mathbb{N}$.

This map may be interpreted in the following way: the elements $\underline{\omega}$ of $\Sigma(\mathbb{N}^2,Z_p^d)$ are (infinite) matrices with entries in Z_p^d. The map Σ_p replaces each entry $\underline{\omega}(m,n)$ by the $p\times p$-matrix $((i,j).(\underline{\omega}(m,n)))_{i,j}$.

Now we shall introduce as in [9] a geometrical object connected a nonzero element $\underline{\omega}$ with compact support. Denote by $\Sigma_{\mathrm{c}}^*(\mathbb{N}^2,Z_p^d)$ the set of all nonzero elements of $\Sigma(\mathbb{N}^2,Z_p^d)$ with compact support.

Definition 5. The map

$$g:\Sigma_{\mathrm{c}}^*(\mathbb{N}^2,Z_p^d)\to\mathscr{H}(\mathbb{R}^2)$$

with

$$g(\underline{\omega})=\bigcup_{m,n}\{I(m,n)\colon \underline{\omega}(m,n)\neq 0\}.$$

is called the graphical representation of $\underline{\omega}$.

The remarks following Definition 3 apply also. We consider the sequence of graphical representations associated with the iterations of the substitution Σ_p.

Lemma 1. *Let* $\underline{\omega} \in \Sigma_{\rm c}^*(\mathbb{N}^2, Z_p^d)$ *such that* $\Sigma_p(\underline{\omega})(0,0) = \underline{\omega}(0,0)$, *then the sequence*

$$\left(\frac{1}{p^n} g(\Sigma_p^n(\underline{\omega}))\right)_{n \geqslant 0}$$

converges in the space $(\mathscr{H}(R^2), h)$.

Proof. The assertion follows from

$$\frac{1}{p^{n+1}} g(\Sigma_p^{n+1}(\underline{\omega})) \subset \frac{1}{p^n} g(\Sigma_p^n(\underline{\omega}))$$

for all $n \in \mathbb{N}$. $\square$

We denote this limit by $\mathscr{A}_{p,\infty}(\underline{\omega})$ and call it rescaled evolution set of $\underline{\omega}$ w.r.t. the substitution Σ_p.

Now we shall describe the rescaled evolution set $X_\infty(\mathscr{R})$ of a p-Carlitz sequence of polynomials.

Let $\boldsymbol{e}_i$ be the ith basis vector in Z_p^d and $\underline{\varepsilon}_i$ the element of $\Sigma_{\rm c}(\mathbb{N}^2, Z_p^d)$ by $\underline{\varepsilon_i}(0,0) = \boldsymbol{e}_i$ and $\underline{\varepsilon_i}(m,n) = 0$ otherwise for $0 \leqslant i \leqslant d, d = \max\{\deg R_j(X), 0 \leqslant j \leqslant p-1\}$.

Proposition 3. *Let* $\mathscr{R} = (R_n(X))_{n \geqslant}$ *be a p-Carlitz sequence of polynomials and* Σ_p *its associated matrix* $p \times p$*-substitution. Then the rescaled evolution set of* $\mathscr{R}$ *is given by*

$$X_\infty(\mathscr{R}) = \bigcup_{i=0}^{d-1} \{\mathscr{A}_{p,\infty}(\underline{\varepsilon_i}) + (i,0)\}.$$

Proof. Let $\underline{\varepsilon} \in \Sigma_{\rm c}^*(\mathbb{N}^2, Z_p^d)$ be defined by $\underline{\varepsilon}(m,n) = 0$ if $n = 0$ or for $n = 0$, $m \geqslant d$ and $\underline{\varepsilon}(m,0) = \boldsymbol{e}_m$ if $m \leqslant d-1$. Then from the definition it follows that

$$\mathscr{A}_{p,\infty}(\underline{\varepsilon}) = \bigcup_{m=0}^{d-1} (\mathscr{A}_{p,\infty}(\underline{\varepsilon}_m) + (m,0))$$

with

$$\mathscr{A}_{p,\infty}(\underline{\varepsilon}) = \lim_{n \to \infty} g(\Sigma_p^n(\underline{\varepsilon})).$$

Now we shall prove that

$$X_\infty(\mathscr{R}) = \mathscr{A}_{p,\infty}(\underline{\varepsilon}).$$

The last equality follows from

$$(m,n).\boldsymbol{e}_0 = \Sigma_p^k(\underline{\varepsilon})(m,n)$$

which we prove next. Let $m = m^s p^s + \cdots + m_1 p + m_0$ and $n = n^{k-1} p^s + \cdots + n_1 p + n_0$ be the p-adic expansion of n and m, respectively.

If $s \leqslant k-1$ then we have

$$(m,n).\boldsymbol{e}_0 = \Sigma_p^k(\underline{\varepsilon_0})(m,n).$$

If $s \geqslant k$, then for $t = s-k$ we have

$$(m,n).\boldsymbol{e}_0 = (m_0,n_0)\ldots(m_{k-1},n_{k-1})(m_k,0)\ldots(m_s,0).\boldsymbol{e}_0.$$

For $\hat{m} = m_s p^{s-k} + \cdots + m_{k+1}p + m_k$ we have $(\hat{m},0).\boldsymbol{e}_0 = \boldsymbol{e}_{m_k}$ which gives

$$(m,n).\boldsymbol{e}_0 = \Sigma_p^k(\boldsymbol{e}_{m_k})(\hat{m},n)$$

and proves our assertion.

Remark 5. As in [9,11] the rescaled evolution set $X_\infty(\mathscr{R})$ is described by a hierarchical iterated function system (HIFS) defined by the $(p\times p)$-substitution $\sigma_p(\mathscr{R})$. To explain the construction we begin with an introduction of HIFS, which is adapted to our purposes, for a more general introduction see [4,16,18].

Let $(Z_p^*)^d$ be the set of all nonzero elements of Z_p^d. Furthermore, we consider the contracting mappings $f_{\alpha,\beta}$ of the unit square I defined by

$$f_{\alpha,\beta}(X,Y) = \left(\frac{X+\alpha}{p}, \frac{Y+\beta}{p}\right),$$

where $\alpha,\ \beta \in [p]$.

With a pair $(\boldsymbol{a},\boldsymbol{b}) \in (Z_p^*)^d \times (Z_p^*)^d$ we associate a subset $J(\boldsymbol{a},\boldsymbol{b}) \subset [p]^2$, which may be the empty set.

Let $N = p^d - 1 = \mathrm{Card}(Z_p^*)^d$. The N-fold product $\mathscr{H}(I)^N$ equipped with maximum metric

$$h_\infty(\boldsymbol{C},\boldsymbol{D}) = \max\{h(C_{\boldsymbol{a}},D_{\boldsymbol{b}}):\ \boldsymbol{a},\boldsymbol{b} \in (Z_p^*)^d\},$$

where $\boldsymbol{C} = (C_{\boldsymbol{a}})_{\boldsymbol{a}}$, $\boldsymbol{D} = (D_{\boldsymbol{a}})_{\boldsymbol{a}}$, is a complete metric space [7].

A mapping $\mathscr{F} : \mathscr{H}(I)^N \to \mathscr{H}(X)^N$ is called p-adic HIFS if $\mathscr{F}$ satisfies:

- For all $\boldsymbol{a},\ \boldsymbol{b} \in (Z_p^*)^d$ there exists a set $J(\boldsymbol{a},\boldsymbol{b}) \subset [p]^2$ such that the $\boldsymbol{a}$-component $\mathscr{F}(\boldsymbol{C})_{\boldsymbol{a}}$ of $\mathscr{F}(\boldsymbol{C})$ is of the form

  $$\mathscr{F}(\boldsymbol{C})_{\boldsymbol{a}} = \bigcup_{\boldsymbol{b}} F_{\boldsymbol{ab}}(C_{\boldsymbol{b}})$$

  and nonempty.
- The maps $F_{\boldsymbol{ab}} : \mathscr{H}(I) \to \mathscr{H}(I) \cup \{\emptyset\}$ which are defined as

  $$F_{\boldsymbol{ab}}(A) = \bigcup_{\alpha,\beta \in J(v,w)} f_{\alpha,\beta}(A), \quad A \in \mathscr{H}(I).$$

A map $\mathscr{F}$ that is a p-adic HIFS is a contraction map and therefore it has a unique fixed point $\boldsymbol{A} = (A_{\boldsymbol{a}})_{\boldsymbol{a}} \in \mathscr{H}(I)^N$, [4,16,18].

Given a p-Carlitz sequence $\mathscr{R}$ we can associate an HIFS to it by setting

$$J(\boldsymbol{a},\boldsymbol{b}) = \{(i,j):\ (i,j).\boldsymbol{a} = \boldsymbol{b}, i,j \in [p]\},$$

where (i,j) are the input maps of the automaton $\mathscr{A}(\mathscr{R})$. We denote this HIFS by $\mathscr{F}_p(\mathscr{R})$ and its fixed point by $\boldsymbol{A}(\mathscr{R}) = (A(\mathscr{R})_{\boldsymbol{a}})$. Similar arguments as in [12] show that

$$A_{p,\infty}(\boldsymbol{a}) = A(\mathscr{R})_{\boldsymbol{a}}, \quad \boldsymbol{a} \in (Z_p^*)^d.$$

6. Rescaled evolution sets of selected Carlitz sequences

In this section, as an application of the previous sections, we study first the Legendre polynomials as a p-Carlitz sequence. We present one image of the rescaled evolution for $p = 3$. Moreover, we state the associated $(p \times p)$-substitutions and compute the box-counting dimension of the rescaled evolution set.

Next, we study the substitutions defined by a Carlitz sequence of polynomials that are closely related to the Bessel function $J_0(X)$.

Lemma 2. *Let $p = 2s + 1$ be a prime number and let $\mathscr{P} = (P_j(X))j \in \mathbb{N}$ denote the Legendre polynomials. Then we have*

(1)

$$\deg P_j(X) = j, \quad 0 \leqslant j \leqslant s,$$

$$\deg P_{s+j}(X) = s - j, \quad 1 \leqslant j \leqslant s.$$

(2) *For $0 \leqslant j \leqslant s$ and j an even number*

$$r_p(2m, j) \neq 0 \bmod p, \quad \textit{for } 0 \leqslant m \leqslant \frac{j}{2},$$

$$r_p(2m+1, j) = 0 \bmod p, \quad \textit{otherwise.}$$

For $0 \leqslant j \leqslant s$, j and j an odd number

$$r_p(2m+1, j) \neq 0 \bmod p, \quad \textit{for } 0 \leqslant m \leqslant \frac{j-1}{2},$$

$$r_p(2m, j) = 0 \bmod p, \quad \textit{otherwise.}$$

(3) *For $1 \leqslant j \leqslant s$ and $s + j$ an even number*

$$r_p(2m, s+j) \neq 0 \bmod p, \quad \textit{for } \leqslant m \leqslant \frac{s-j}{2},$$

$$r_p(2m+1, s+j) = 0 \bmod p, \quad \textit{otherwise.}$$

For $1 \leqslant j \leqslant s$ such that $s + j$ is an odd number

$$r_p(2m+1, s+j) \neq 0 \bmod p, \quad \textit{for } \leqslant m \leqslant \frac{s-j-1}{2},$$

$$r_p(2m, s+j) = 0 \bmod p, \quad \textit{otherwise.}$$

The proof is a consequence of Lucas' Lemma and some easy computations.

If we consider the sequence Legendre polynomials modulo p, we simply write $\mathscr{P}_p$.

Proposition 4. *Let $p = 2s + 1$ be a prime number. The rescaled evolution set $X_\infty(\mathscr{P}_p)$ of the sequence of Legendre polynomials modulo p is the attractor of the iterated function system given by*

$$\mathscr{F}_p(\mathscr{P}) = \{I : f_{i,j}, (i,j) \in M(\mathscr{P}_p)\},$$

where $M(\mathrm{cal}\,P_p)=\{(i,j)\colon j=i+2k,\ 0\leqslant i\leqslant s-k, 0\leqslant k\leqslant s\}$ *and*

$$f_{i,j}(X,Y)=\left(\frac{X+i}{p},\frac{Y+j}{p}\right),\quad (X,Y)\in I.$$

The box-counting dimension of the rescaled evolution set is

$$\dim_{\mathrm{B}} X_\infty(\mathscr{P}_p)=\frac{\log(s+1)(s+2)/2}{\log p}.$$

Proof. The HIFS corresponding to the matrix $p\times p$-substitution $\sigma_p(\mathscr{P})$ is in fact an iterated function system (IFS). It is described by the preceding Lemma. The formula for the box-counting dimension is then a consequence of the definition of the IFS. Moreover, we have that the Hausdorff dimension of the rescaled evolution set is equal to the box-counting dimension [7]. Since the IFS $\mathscr{F}_p(\mathscr{P})$ satisfies the *open set condition* (this holds for all p-adic HIFS [18]). Then the formula for the box-counting dimension follows from the Moran–Hutchinson formula about the Hausdorff dimension (in this case also the box-counting dimension) of the attractor of the IFS $\mathscr{F}_p(\mathscr{P})$ [8,7]. $\square$

Corollary 1. *The growth rate of the number of nontrivial coefficients* mod p *of the sequence of Legendre polynomials is*

$$\lim_{k\to\infty}\frac{\log N(\mathscr{P}_p,k)}{k\log p}=\frac{\log(s+1)(s+2)/2}{\log p}.$$

6.1. Legendre polynomials modulo 3

The sequence of Legendre polynomials $\mathscr{P}_3=(P_n(X)\bmod 3)_{n\geqslant 0}$ is generated by the polynomials $P_0(X)=1,\ P_1(X)=X,\ P_2(X)=1\in\mathbb{F}_3[X]$. The matrix substitution $\sigma_3=\sigma_3(\mathscr{P}):\mathbb{F}_3\to\mathbb{F}_3^{[3]\times[3]}$ (here $d=\max\{\deg P_j(X)\bmod 3\colon 0\leqslant j\leqslant 2\}=1$) is given by

$$\sigma_3(u)=u\begin{vmatrix}1&0&0\\0&1&0\\1&0&0\end{vmatrix}$$

and $u\in\{0,1,2\}$]. The rescaled evolution set $X_\infty(\mathscr{P}_3)$ is the attractor of the IFS (iteration function system $\{I,f_{i,j}\colon (i,j)\in M(\mathscr{P}_3)\}$) where (see Fig. 1)

$$f_{i,j}(X,Y)=\left(\frac{X+i}{3},\frac{Y+j}{3}\right),\quad (X,Y)\in I$$

and $M(\mathscr{P}_3)=\{(0,0),(1,1),(0,2)\}$.

6.2. Legendre polynomials modulo 5

The sequence of Legendre polynomials $\mathscr{P}_5=(P_n(X)\bmod 5)_{n\geqslant 0}$ is generated by the polynomials $P_0(X)=1,\ P_1(X)=X,\ P_2(X)=2+4X^2,\ P_3(X)=X,\ P_4=1\in\mathbb{F}_5[X]$.

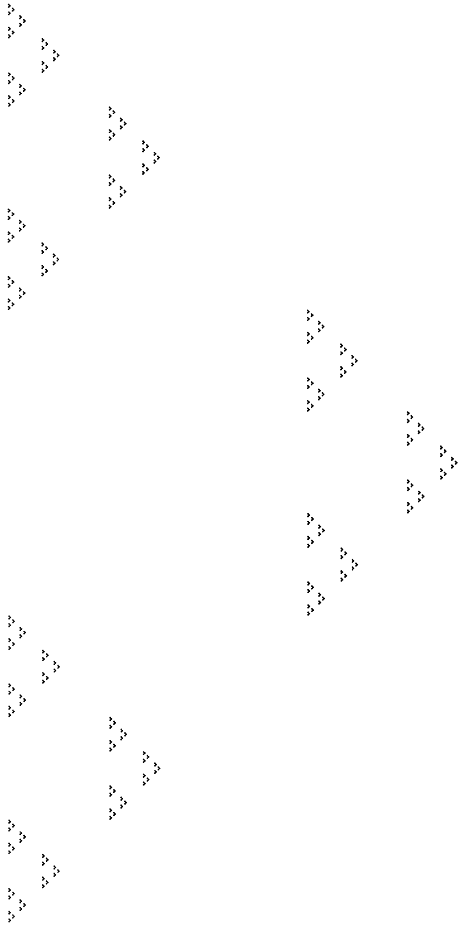

Fig. 1. Rescaled evolution set of the Legendre polynomials mod 3.

The matrix substitution $\sigma_5 = \sigma_5(\mathscr{P}) : \mathbb{F}_5^2 \to (\mathbb{F}_5^2)^{[5]\times[5]}$ (here $d = \max\{\deg P_j(X) \bmod 5 : 0 \leqslant j \leqslant 4\} = 2$) is given by

$$\sigma_5(01) = \begin{vmatrix} 01 & 10 & 00 & 00 & 00 \\ 00 & 01 & 10 & 00 & 00 \\ 02 & 20 & 04 & 40 & 00 \\ 00 & 01 & 10 & 00 & 00 \\ 01 & 10 & 00 & 00 & 00 \end{vmatrix}$$

and $\sigma_5(10) = \sigma_5(00) = 0$, $\sigma_5(\alpha, \beta) = \beta\sigma_5(01)$. The rescaled evolution set $X_\infty(\mathscr{P}_5)$ is the atractor of the IFS (iteration function system) $\{I, f_{i,j}\colon (i,j) \in M(\mathscr{P}_5)\}$ where

$$f_{i,j}(X,Y) = \left(\frac{X+i}{5}, \frac{Y+j}{5}\right), \quad (X,Y) \in I$$

and $M(\mathscr{P}_5) = \{(0,0),(1,1),(0,2),(2,2),(1,3),(0,4)\}$.

6.3. Legendre polynomials modulo 7

The sequence of Legendre polynomials $\mathscr{P}_7 = (P_n(X) \bmod 7)_{n \geqslant 0}$ is generated by the polynomials $P_0(X)=1$, $P_1(X)=X$, $P_2(X)=3+5X^2$, $P_3(X)=2X+6X^3$, $P_4(X)=3+5X^2$, $P_5(X)=X$, $P_6(X)=1 \in \mathbb{F}_7[X]$. The matrix substitution $\sigma_7 = \sigma_7(\mathscr{P}) : \mathbb{F}_7^3 \to (\mathbb{F}_7^3)^{[7]\times[7]}$ (here $d = \max\{\deg P_j(X) \bmod 7\colon 0 \leqslant j \leqslant 6\} = 3$) is given by

$$\sigma_7(001) = \left|\begin{array}{ccccccc} 001 & 010 & 100 & 000 & 000 & 000 & 000 \\ 000 & 001 & 010 & 100 & 000 & 000 & 000 \\ 003 & 030 & 305 & 050 & 500 & 000 & 000 \\ 000 & 002 & 020 & 206 & 060 & 600 & 000 \\ 003 & 030 & 305 & 050 & 500 & 000 & 000 \\ 000 & 001 & 010 & 100 & 000 & 000 & 000 \\ 001 & 010 & 100 & 000 & 000 & 000 & 000 \end{array}\right|$$

and $\sigma_7(010) = \sigma_7(100) = \sigma_7(000) = 0$, $\sigma_7(i,j,k) = k\sigma_7(001)$. The rescaled evolution set $X_\infty(\mathscr{P}_5)$ is the attractor of the IFS $\{I, f_{i,j}\colon (i,j) \in M(\mathscr{P}_7)\}$ where

$$f_{i,j}(X,Y) = \left(\frac{X+i}{7}, \frac{Y+j}{7}\right), \quad (X,Y) \in I$$

and $M(\mathscr{P}_7) = \{(0,0),(1,1),(0,2),(2,2),(1,3),(3,3),(0,4),(2,4),(1,5),(0,6)\}$.

6.4. Legendre polynomials modulo 11

The sequence of Legendre polynomials $\mathscr{P}_{11} = (P_n(X) \bmod 11)_{n \geqslant 0}$ is generated by the polynomials $P_0(X)=1$, $P_1(X)=X$, $P_2(X)=5+7X^2$, $P_3(X)=4X+8X^3$, $P_4(X)=10+10X^2+3X^4$, $P_5(X)=6+5X^3+X^5$, $P_6(X)=10+10X^2+3X^4$, $P_7(X)=4X+8X^3$, $P_8(X)=2+7X^2$, $P_9(X)=X$, $P_{10}(X)=1 \in \mathbb{F}_{11}[X]$. The matrix substitution is $\sigma_7(\mathscr{P}) : \mathbb{F}_{11}^5 \to (\mathbb{F}_{11}^5)^{[11]\times[11]}$ (here $d = \max\{\deg P_j(X) \bmod 7\colon 0 \leqslant j \leqslant 10\} = 5$). For simplicity we shall give a nontrivial restriction of the substitution $\sigma_{11}\colon \mathbb{F}_{11} \to \mathbb{F}_{11}^{[11]\times[11]}$ which generates the double sequence associated with Legendre polynomials mod 11:

$$\sigma_{11}(u) = u \left|\begin{array}{ccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 10 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 10 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right|,$$

where $u \in \{0,\dots,10\}$. The rescaled evolution set $X_\infty(\mathscr{P}_{11})$ is the attractor of the IFS $\{I, f_{i,j}\colon (i,j) \in M(\mathscr{P}_{11})\}$ where

$$f_{i,j}(X,Y) = \left(\frac{X+i}{11}, \frac{Y+j}{11}\right), \quad (X,Y) \in I$$

and

$$M(\mathscr{P}_{11}) = \{(0,0),(1,1),(0,2),(2,2),(1,3),(3,3),(0,4),(2,4),(4,4),(1,5),(3,5),(5,5),(0,6),(2,6),(4,6),(1,7),(3,7),(0,8),(2,8),(1,9),(0,10)\}.$$

6.5. The Bessel function $J_0(X)$

Using the Bessel function $J_0(X)$, Carlitz defined a sequence of polynomials $\omega_n(X)$ with p-Carlitz property, as follows [6]. Let

$$J_0(X) = \sum_0^\infty \frac{(-1)^n X^{2n}}{2^{2n}(n!)^2}$$

denote the power series of the Bessel function $J_0(X)$, then

$$\frac{J_0(2\sqrt{XZ})}{J_0(2\sqrt{Z})} = \sum_0^\infty \frac{\omega_n(X) Z^n}{(n!)^2}$$

defines a sequence of polynomials. The sequence of polynomials $\mathscr{J}_3 = (\omega_n(X) \bmod 3)_{n\geqslant 0}$ is a 3-Carlitz sequence of polynomials. Therefore, it is generated by the polynomials $\omega_0(X)=1, \omega_1(X)=1+2X, \omega_2(X)=2X+X^2 \in \mathbb{F}_3[X]$. The substitution associated with this sequence of polynomials is $\sigma_3(\mathscr{J}) : \mathbb{F}_3^2 \to (\mathbb{F}_3^2)^{[3]\times[3]}$. For simplicity, we shall give a nontrivial restriction of it $\sigma_3 : \mathbb{F}_3 \to (\mathbb{F}_3)^{[3]\times[3]}$ defined by

$$\sigma_3(u) = u \begin{vmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix},$$

where $u \in \{0,1,2\}$. The substitution σ_3 generates the double sequence $(\omega_n(X) \bmod 3)_{n\geqslant 0}$. The rescaled evolution set $X_\infty(\mathscr{J}_3)$, see Fig. 2, is the attractor of the IFS $\{I, f_{i,j}\colon (i,j) \in M_3(\mathscr{J}_3)\}$, where

$$f_{i,j}(X,Y) = \left(\frac{X+i}{3}, \frac{Y+j}{3}\right), \quad (X,Y) \in I, \quad (i,j) \in M(\mathscr{J}_3)$$

with $M(\mathscr{J}_3) = \{(0,0),(0,1),(1,1),(1,2),(2,2)\}$.

The sequence of polynomials $\mathscr{J}_5 = (\omega_n(X) \bmod 5)_{n\geqslant 0}$ is a 5-Carlitz sequence of polynomials. Therefore, it is generated by the polynomials $\omega_0(X) = 1, \omega_1(X) = 1 + 4X, \omega_2(X) = 3 + X + X^2, \omega_3(X) = 4 + 3X + 4X^2 + 4X^3, \omega_4 = 1 + X + 3X^2 + 4X^3 + X^4 \in \mathbb{F}_5[X]$.

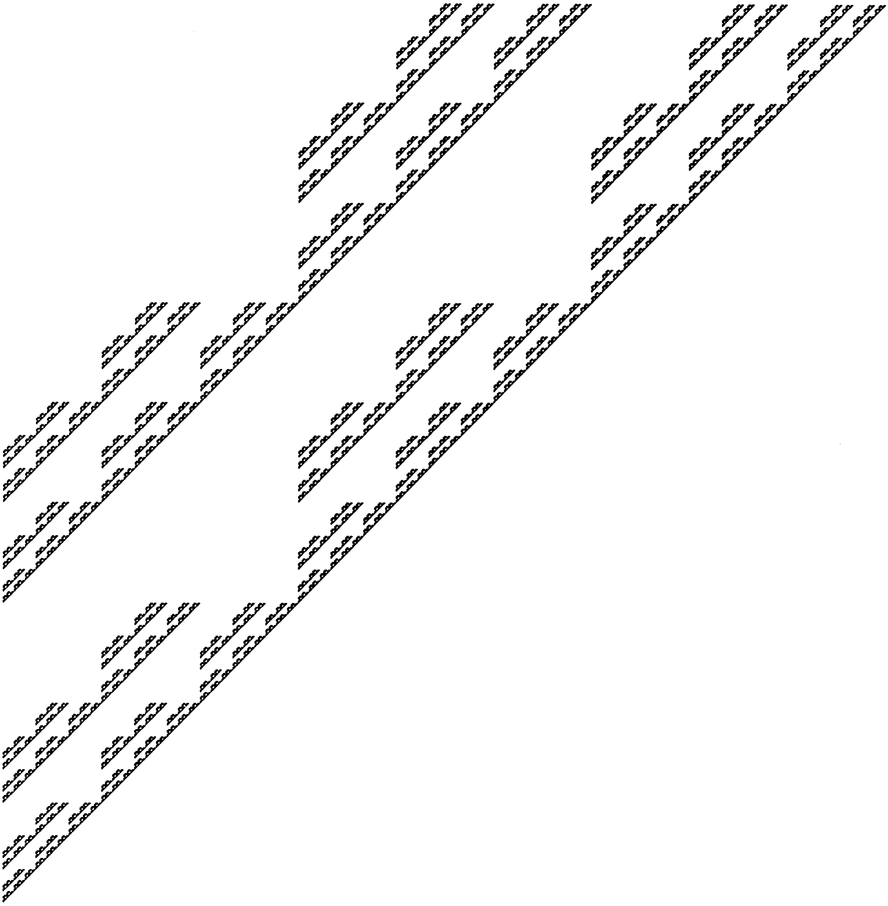

Fig. 2. Rescaled evolution set of $\mathscr{J}_3 = (\omega_n(X) \bmod 3)_{n \geqslant 0}$.

The substitution associated with this sequence of polynomials is $\sigma_5(\mathscr{J}) : \mathbb{F}_5^4 \to (\mathbb{F}_5^4)^{[5]\times[5]}$. For simplicity, we shall give a nontrivial restriction of it $\sigma_5 : \mathbb{F}_5 \to (\mathbb{F}_5)^{[5]\times[5]}$ defined by

$$\sigma_5(u) = u \begin{vmatrix} 1 & 1 & 3 & 4 & 1 \\ 4 & 3 & 4 & 4 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix},$$

where $u \in \{0, 1, \ldots, 4\}$, which generates the double sequence corresponding to $\mathscr{J}_5$.

The rescaled evolution set $X_\infty(\mathscr{J}_5)$ is the attractor of the IFS $\{I, f_{i,j}\colon (i,j) \in M_5(\mathscr{J}_5)\}$ where

$$f_{i,j}(X,Y) = \left(\frac{X+i}{5}, \frac{Y+j}{5}\right), \quad (X,Y) \in I, \quad (i,j) \in M(\mathscr{J}_5)$$

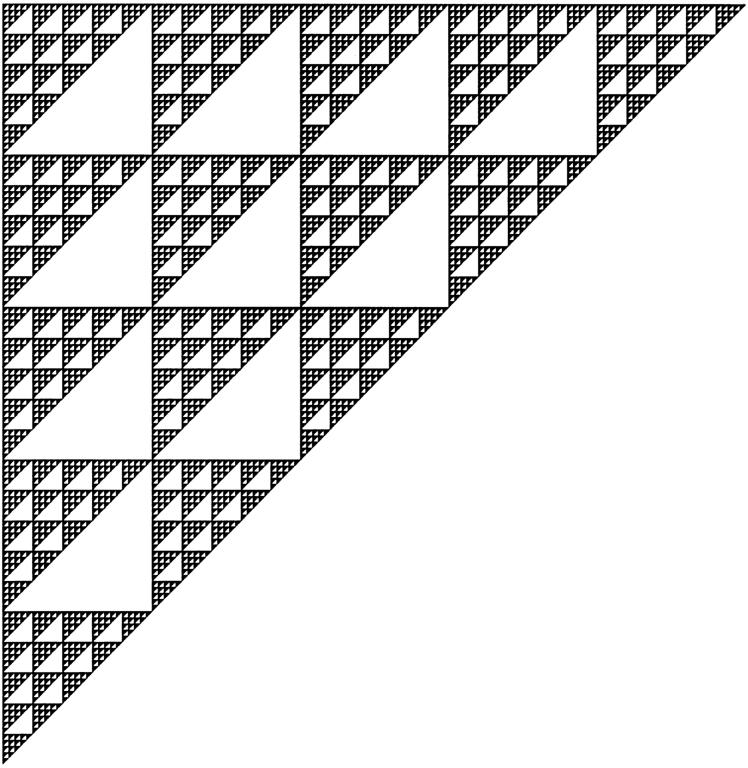

Fig. 3. Rescaled evolution set of $\mathscr{J}_5 = (\omega_n(X) \bmod 5)_{n\geqslant 0}$.

with

$$M(\mathscr{J}_5) = \{(0,0),(0,1),(0,2),(0,3),(0,4),(1,1),(1,2),\\ (1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}.$$

The rescaled evolution set $X_\infty(\mathscr{J}_5)$ shown in Fig. 3 coincides with the rescaled evolution set corresponding to the binomial coefficients modulo 5, [10].

7. Growth rate of blocks in p-Carlitz sequence of polynomials

In this section, we briefly discuss the growth rate of nontrivial blocks in the double sequence which is generated by a p-Carlitz sequence of polynomials.

Let $\mathscr{R} = (R_n(X))_{n\geqslant 0}$ be a p-Carlitz sequence of polynomials and $\underline{r} = (r(m,n) \bmod p)_{m,n\geqslant 0}$ the associated double sequence, and let $X_\infty(\mathscr{R})$ be the associated rescaled evolution set.

The geometrical representation of the substitution $\sigma_p = \sigma_p(\mathscr{R})$ corresponding to $\mathscr{R}$ is hierarchical iterated function system (HIFS [18]), which decodes not only the self-similarity properties of the double sequence $\underline{r}$. It gives also information about the growth rate of the number of nontrivial l-blocks in it. We remind the reader the definition of

$$N_k(\mathscr{R},l) = \mathrm{Card}\{(m,n)\colon (r(m,n), r(m-1,n)\ldots, r(m-l+1,n)) \neq (0,\ldots 0) \bmod p, 0 \leqslant n \leqslant p^k - 1\}.$$

We say that the growth rate of the nontrivial l-blocks is D_l if the limit

$$D_l = \lim_{k\to\infty} \frac{\log N_k(\mathscr{R},l)}{k \log p}$$

exists. Denote by $\underline{D_k}$ and $\overline{D_k}$ the liminf and limsup of the above sequence. Observe that $\underline{D_k}$ corresponding $\overline{D_k}$ are the low box-counting ($\underline{\dim}_{\mathrm{B}}$) and upper box-counting ($\overline{\dim}_{\mathrm{B}}$) dimensions of the rescaled evolution set $X_\infty(\mathscr{R})$ (for the box-counting dimensions see [9]). Since this set is constructed by hierarchical iterated function system the lower- and upper box-counting dimensions coincide and are also equal to the Hausdorff dimension of the rescaled evolution set [8]. Therefore we have in the notations adopted

Corollary 2. *The growth rate of nontrivial l-blocks in the double sequence associated with p-Carlitz sequence of polynomials does not depend on l and is equal to the box-counting* (*and Hausdorff*) *dimension of the rescaled evolution set*:

$$D_p(\mathscr{R}) = \dim_{\mathrm{B}} X_\infty(\mathscr{R}_p) = \dim_{\mathrm{H}} X_\infty(\mathscr{R}).$$

Remark 6. There are formulas for the box-counting (in this case equal to the Hausdorff) dimension of the components of the attractor vector of HIFS [16,4,7]. This general formulas are applicable since the p-adic HIFS corresponding to a p-Carlitz sequence of polynomials satisfy the *open set condition*. This formula generalizes the Moran–Hutchinson formula for the Hausdorff dimension of the attractor of IFS [8].

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