On the maps preserving the equality of distance✩

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Abstract

In this paper, in connection with the known result of Baker and Vogt, we get if $f : X \rightarrow Y$ preserves equality of distance with dimension $X \geq 2$ and $Y$ is strictly convex, the range of $f$ contains a segment, then $f$ is affine.

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1. Introduction

A theorem due to Mazur and Ulam [1] states that every isometry of a real vector space is linear up to translation. Charzyński [2] and Rolewicz [3] have shown, respectively, that surjective isometries of finite-dimensional $F$-space and of locally bounded spaces with concave norm are also linear.

Vogt [4] extends the result of Mazur and Ulam in a different direction. The space remain real vector space, but he replace isometries by the more general notion of equality of distance preserving maps, maps with the property that the distance between image points depends functionally on the distance between domain points.

Definition 1.1. A map $f : X \rightarrow Y$ preserves equality of distance iff there exists a function $p : \mathbb{R}^+_0 \rightarrow \mathbb{R}^+_0$ such that for each $x$ and $y$ in $X$, $\|f(x) - f(y)\| = p(\|x - y\|)$. The function $p$ is called the gauge function for $f$.

Vogt proved that every continuous equality of distance preserving map from a normed real vector space onto a normed real vector space is affine. In connection with the known result of Baker [5], a question is raised:

Question 1.2. Let $X$ and $Y$ be two real normed spaces. If $f : X \rightarrow Y$ preserves equality of distance, $Y$ is strictly convex, then is $f$ affine?

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2. Properties of maps preserving equality of distance

Lemma 2.1. Let \( X \) and \( Y \) be two real normed spaces, \( f : X \to Y \) preserves equality of distance, \( p \) is the gauge function of \( f \). If \( f \) is continuous, then \( p \) is continuous.

Lemma 2.2. (See [41].) Given \( \varepsilon > 0 \) and a real normed space \( X \) with dimension \( X \geq 2 \), then for each \( x \) in \( X \) with \( \|x\| \leq 2\varepsilon \), there exist \( x_1 \) and \( x_2 \) in \( X \) such that \( \|x_1\| = \|x_2\| = \varepsilon \) and \( x = x_1 + x_2 \).

Proof. If \( x = 0 \), choose \( u \) in \( X \) with \( \|u\| = 1 \). Letting \( x_1 = \varepsilon u = -x_2 \), we have our conclusion. Assume then that \( x \neq 0 \).

Consider \( S_\varepsilon = \{ y : y \in X, \|y\| = \varepsilon \} \). If \( v \) and \( w \) are independent vectors in \( S_\varepsilon \), then

\[ \alpha_{v,w}(t) = \frac{\varepsilon[(\cos t)v + (\sin t)w]}{\|\varepsilon[(\cos t)v + (\sin t)w]\|} \]

is a path in \( S_\varepsilon \) from \( v \) to \( w \). If \( v \) and \( w \) are dependent vectors in \( S_\varepsilon \), then \( w = \pm v \). Using dimension \( X \geq 2 \), choose \( z \) in \( S_\varepsilon \) such that \( z \) is independent of \( v \) (and hence of \( w \) also). Then

\[ \beta_{v,w}(t) = \begin{cases} \alpha_{v,z}(t), & t \leq \pi/2, \\ \alpha_{z,w}(t - \pi/2), & t \geq \pi/2, \end{cases} \]

is a path in \( S_\varepsilon \) from \( v \) to \( w \). We conclude that \( S_\varepsilon \) is path-connected.

Now define \( \mu : S_\varepsilon \to \mathbb{R}_{0}^+ \) by \( \mu(y) = \|x - y\| \) for \( y \in S_\varepsilon \). Then \( \mu \) is a continuous function on \( S_\varepsilon \). Since \( \pm \varepsilon(x/\|x\|) \) is in \( S_\varepsilon \), we have:

\[ \mu(\varepsilon(x/\|x\|)) = \|x - \varepsilon(x/\|x\|)\| = 2\varepsilon - \varepsilon(x/\|x\|) \leq \varepsilon < \varepsilon(x/\|x\|) = \|x - (\varepsilon(x/\|x\|))\| = \mu(\varepsilon(x/\|x\|)). \]

By the path-connectedness of \( S_\varepsilon \) there exists an element \( x_1 \) in \( S_\varepsilon \) such that the intermediate value \( \varepsilon = \mu(x_1) \).

Let \( x_2 = x - x_1 \). Then \( x = x_1 + x_2 \), \( \|x_1\| = \varepsilon \) since \( x_1 \) is in \( S_\varepsilon \), and \( \|x_2\| = \|x - x_1\| = \mu(x_1) = \varepsilon \). \( \Box \)

Lemma 2.3. (See [41].) Let \( X \) and \( Y \) be two real normed spaces with dimension \( X \geq 2 \), \( f : X \to Y \) preserves equality of distance. If for every \( \varepsilon > 0 \) there exist \( x \) and \( y \) in \( X \) such that \( x \neq y \) and \( \|f(x) - f(y)\| < \varepsilon \), then \( f \) is uniformly continuous.

Proof. Given \( \varepsilon > 0 \), choose \( x \) and \( y \) in \( X \) such that \( x \neq y \) and \( \|f(x) - f(y)\| < \varepsilon/3 \). Let \( \delta = 2\|x - y\| > 0 \).

If \( v \) and \( w \) are in \( X \) and \( \|v - w\| \leq \delta \), by Lemma 2.2 there exist \( x_1 \) and \( x_2 \) in \( X \) with \( \|x_1\| = \|x_2\| = \delta/2 \) and \( v - w = x_1 + x_2 \). Then

\[ \|f(v) - f(w)\| = \|p(\|v - w\|)\| = \|p(\|x_1 - (-x_2)\|)\| = \|f(x_1) - f(-x_2)\| \leq \|f(x_1) - f(0)\| + \|f(0) - f(x_2)\| = 2p(\|x_1 - 0\|) + 2p(\|0 - (-x_2)\|) = 2p(\|x - y\|) = 2\|f(x) - f(y)\| < 2(\varepsilon/3) < \varepsilon. \] \( \Box \)

Lemma 2.4. Let \( X \) and \( Y \) be two real normed spaces with dimension \( X \geq 2 \), \( f : X \to Y \) preserves equality of distance. If \( f \) is continuous, \( f(0) = 0 \), and \( f \neq 0 \), then \( f \) is injective and \( f^{-1} \) is continuous.

Proof. (1) Suppose \( f \) is not injective, then there exist \( u \) and \( v \) in \( X \) such that \( u \neq v \) and \( f(u) = f(v) \). We have

\[ p(\|u - v\|) = \|f(u) - f(v)\| = 0. \]

Denote \( \|u - v\| \) by \( \varepsilon \). By Lemma 2.2 we know for each \( x_0 \) in \( X \) with \( \|x_0\| \leq 2\varepsilon \) there exist \( x_1 \) and \( x_2 \) such that \( \|x_1\| = \|x_2\| = \varepsilon \) and \( x_0 = x_1 - x_2 \). Then

\[ \|f(x_0)\| = \|f(x_0) - f(0)\| = \|p(\|x_0\|)\| = \|p(\|x_1 - x_2\|)\| = \|f(x_1) - f(x_2)\| \leq \|f(x_1)\| + \|f(x_2)\| \]

\[ = \|f(x_1) - f(0)\| + \|f(x_2) - f(0)\| = 2p(\varepsilon) = 0. \]

So \( f(x_0) = 0 \) for any \( x_0 \) in \( X \) with \( \|x_0\| \leq 2\varepsilon \).

By induction we know \( f(x) = 0 \) for any \( x \) in \( X \). It is a contradiction.

(2) Next we will prove \( f^{-1} \) is continuous. Let \( \{y_n = f(x_n)\}_{n=0}^{\infty} \) is a subset of the range of \( f \), \( y_n \to y_0 \) \( (n \to \infty) \), where \( \{x_n\} \subseteq X \). We only need to prove \( x_n \to x_0 \) \( (n \to \infty) \).

By the assumption,

\[ p(\|x_n - x_0\|) = \|f(x_n) - f(x_0)\| = \|y_n - y_0\| \to 0 \quad (n \to \infty). \] (2.1)
Denote $\|x_n - x_0\|$ by $\varepsilon_n$. If $\varepsilon_n$ is unbounded, by passing to a subsequence we may assume that $\varepsilon_n$ converges to infinity. For arbitrary $\delta > 0$, there exists $N \in \mathbb{N}$ such that $p(\varepsilon_n) < \delta$ for any $n > N$. By Lemma 2.2 we know for each $x$ in $X$ with $\|x\| \leq 2\varepsilon_n$ there exist $x_1$ and $x_2$ such that $\|x_1\| = \|x_2\| = \varepsilon_n$ and $x = x_1 - x_2$. Then
\[
\|f(x)\| = p(\|x\|) = p(\|x_1 - x_2\|) = \|f(x_1) - f(x_2)\| \leq \|f(x_1)\| + \|f(x_2)\| = \|f(x_1) - f(0)\| + \|f(x_2) - f(0)\| = 2p(\varepsilon_n) = 2\delta.
\]

Since $\varepsilon_n$ converges to infinity, $\|f(x)\| \leq 2\delta$ for any $x$ in $X$. By the arbitrariness of $\delta$ we have $f(x) = 0$ for any $x$ in $X$, which is impossible. Then $\varepsilon_n$ is bounded. Assume $\varepsilon'$ is an accumulation point of $\{\varepsilon_n\}$ and $\{\varepsilon_n\} \subseteq \{\varepsilon_n\}$ converges to $\varepsilon'$. Choose $x$ in $X$ with $\|x\| = 1$, then
\[
p(\varepsilon_n x) = p(f(\varepsilon_n x)) = p(f(\varepsilon' x)) = p(\varepsilon').
\]
By Eq. (2.1), $p(\varepsilon') = 0$, $f(\varepsilon' x) = 0$. But $f$ is injective, so $\varepsilon' = 0$. From this we know 0 is the only accumulation point of $\{\varepsilon_n\}$, then $\|x_n - x_0\| = \varepsilon_n \to 0$ ($n \to \infty$). That is $x_n \to x_0$ ($n \to \infty$). □

3. Main theorem

**Theorem 3.1.** Let $X$ be a real normed space with dimension $X \geq 2$, $Y$ be a real normed strictly convex space, $f : X \to Y$ preserves equality of distance with $f(0) = 0$. If there exists $y \neq 0$ in $Y$ such that $[-y, y]$ is a subset of the range of $f$, then $f$ is linear and $f = \lambda g$ where $\lambda$ is a non-zero real number and $g$ is an isometry of $X$ onto $Y$.

**Proof.** By Lemmas 2.3 and 2.4 we know $f$ and $f^{-1}$ are continuous, then the gauge function $p$ is also continuous. Define $\alpha_0 = \inf\{\varepsilon: p(\varepsilon) = \|y\|\}$. By the continuity of $p$ we have $p(\alpha_0) = \|y\|$ and $p(\alpha) < \|y\|$ for all $0 < \alpha < \alpha_0$.

(1) First we prove that $p$ is monotone increasing on $[0, \alpha_0]$. Indeed, we only need to prove $p$ is monotone on $[0, \alpha_0]$. If not, there exist $\alpha_1$ and $\alpha_2$ in $[0, \alpha_0]$ such that $p(\alpha_1) = p(\alpha_2)$. Without loss of generality, assume $\alpha_1 < \alpha_2$. By the continuity of $p$ we know there exists $\alpha_3$ such that $\alpha_1 < \alpha_3 < \alpha_2$ and $p$ has a local extremum at $\alpha_3$. Since $f^{-1}$ is continuous, $\|f^{-1}(\varepsilon y)\|$ is continuous on $[-1, 1]$. Then there exists $\varepsilon_0$ in $[0, 1]$ such that $\|f^{-1}(\varepsilon_0 y)\| = \alpha_3$. Take two sequences $\{\varepsilon_n\}$ and $\{\varepsilon'_n\}$ such that $0 < \varepsilon_n < \varepsilon_0 < \varepsilon'_n < 1$, $\varepsilon_n \not< \varepsilon_0, \varepsilon'_n \not< \varepsilon_0$, then $\{\varepsilon_n y, \varepsilon'_n y\}$ is contained in the range of $f$ and
\[
f^{-1}(\varepsilon_n y) \rightarrow f^{-1}(\varepsilon_0 y), \quad f^{-1}(\varepsilon'_n y) \rightarrow f^{-1}(\varepsilon_0 y),
\]
\[
\|f^{-1}(\varepsilon_n y)\| \rightarrow \alpha_3, \quad \|f^{-1}(\varepsilon'_n y)\| \rightarrow \alpha_3.
\]
Denote $\|f^{-1}(\varepsilon_n y)\|$, $\|f^{-1}(\varepsilon'_n y)\|$ by $\beta_n, \beta'_n$. Then
\[
\beta_n \to \alpha_3, \quad \beta'_n \to \alpha_3,
\]
\[
p(\beta_n) = \|f(f^{-1}(\varepsilon_n y))\| = \|\varepsilon_n p(\alpha_3) \not< p(\alpha_3),
\]
\[
p(\beta'_n) = \|f(f^{-1}(\varepsilon'_n y))\| = \|\varepsilon'_n p(\alpha_3) \not< p(\alpha_3).
\]
This is contradict to that $p$ has a local extremum at $\alpha_3$.

(2) Second we prove that $p$ is linear on $[0, \beta]$, where $0 < \beta \leq \alpha_0/2$. Similarly to the proof of Lemma 2.4 we can prove that $\eta = \inf\{\varepsilon: p(\varepsilon) > \|y\|\} < \eta$ we know $\|f^{-1}(\varepsilon y')\| < \alpha_0$. Let $\beta = \min(\alpha_0/2, \alpha')$, where $0 < \alpha' \leq \alpha_0$ and $p(\alpha') = \eta$. From (1) for each $\gamma_1, \gamma_2$ in $[0, \beta]$ there exist $x_1$ and $x_2$ in $X$ such that $\|x_1\| = \|x_2\| = 1$ and $f(\gamma_1 x_1) = \frac{p(\gamma_1)}{p(\alpha_0)} y$, $f(\gamma_2 x_2) = -\frac{p(\gamma_2)}{p(\alpha_0)} y$. Then
\[
p(\gamma_1 + \gamma_2) = \|f(\gamma_1 x_1) - f(\gamma_2 x_2)\| \leq \|f(\gamma_1 x_1)\| + \|f(\gamma_2 x_2)\| = p(\gamma_1) + p(\gamma_2)
\]
\[
= \|f(\gamma_1 x_1)\| + \|f(\gamma_2 x_2)\| = \|f(\gamma_1 x_1) - f(\gamma_2 x_2)\| = p(\|\gamma_1 x_1 - \gamma_2 x_2\|)
\]
\[
\leq p(\gamma_1 + \gamma_2).
\]
So $p(\gamma_1 + \gamma_2) = p(\gamma_1) + p(\gamma_2)$ for all $\gamma_1, \gamma_2$ in $[0, \beta]$.

By the continuity of $p$ it is easy to see $p$ is linear on $[0, \beta]$. 
Third we prove \( p \) is linear on \( \mathbb{R}^1_+ \). Let \( p(\alpha) = \lambda \alpha \) on \([0, \beta]\). Obviously \( \lambda > 0 \). Then \( \| f(x') - f(x'') \| = \lambda \| x' - x'' \| \) for any \( x' \) and \( x'' \) in \( X \) and \( \| x' - x'' \| \leq \beta \). Let \( \beta < \| x' - x'' \| \), firstly suppose that \( \| x' - x'' \| \leq 2 \beta \). Set \( z = \frac{x' + x''}{2} \), so

\[
\| x' - z \| = \frac{\| x' - x'' \|}{2} \leq \beta, \quad \| x'' - z \| = \frac{\| x' - x'' \|}{2} \leq \beta.
\]

We define

\[
u = \beta \frac{x' - z}{\| x' - x'' \|}, \quad v = \beta \frac{x'' - z}{\| x' - x'' \|}.
\]

Then \( \| u - v \| = \beta \), thus implies that

\[
\| f(u) - f(v) \| = \lambda \beta, \quad \| u - z \| = \frac{\beta}{2} < \beta,
\]

so

\[
\| f(u) - f(z) \| = \frac{\lambda \beta}{2}.
\]

We also have

\[
\| u - x' \| = \frac{\| x' - x'' \| - \beta}{2} < \beta,
\]

which implies

\[
\| x' - z \| = \| u - x' \| + \| u - z \|.
\]

Combining these facts we claim that

\[
\| f(x') - f(z) \| = \| f(u) - f(x') \| + \| f(u) - f(z) \|.
\]

Because \( Y \) is strictly convex, then there exists \( \delta \) such that \( f(x') - f(u) = \delta (f(u) - f(z)) \), and we denote that \( \delta = \frac{\| x' - x'' \|}{\| p \|} - 1 \). Thus we obtain the fact as follows

\[
f(x') = (1 + \delta) f(u) - \delta f(z).
\]

Similarly

\[
f(x'') = (1 + \delta) f(v) - \delta f(z).
\]

It follows that

\[
\| f(x') - f(x'') \| = (1 + \delta) \| f(u) - f(v) \| = (1 + \delta) \lambda \beta = \lambda \| x' - x'' \|.
\]

Using inductive method, we have

\[
\| f(x') - f(x'') \| = \lambda \| x' - x'' \| \quad \forall x', x'' \in X.
\]

(4) Let \( g = f/\lambda \), then \( g \) is an isometry. By the result of Baker [5], \( g \) is linear. So \( f \) is also linear. Since \( B_r \) is a subset of the range of \( f \), then \( B_{r/\lambda} \) is a subset of the range of \( g \), we know \( g \) is surjective. \( \square \)

**Corollary 3.2.** Let \( X \) be a real normed space with dimension \( X \geq 2 \), \( Y \) be a real normed strictly convex space, \( f : X \to Y \) preserves equality of distance. If the range of \( f \) contains a segment, then \( f \) is affine.

**Proof.** Suppose \([f(x_0), f(y_0)]\) is contained in the range of \( f \), let \( f(z_0) = (f(x_0) + f(y_0))/2 \) and \( g(x) = f(x + z_0) - f(z_0) \) for all \( x \) in \( X \). By Theorem 3.1 we know \( g \) is linear. Then \( f(x) = g(x - z_0) + f(z_0) = g(x) - g(z_0) + f(z_0) \) is affine. \( \square \)

If we define \( f : R \to R^2 \) as \( f(x) = (\sin x, \cos x) \), it is easy to see that \( f \) preserves the equality of distance but is not affine. We still do not know whether \( f \) is affine if \( \dim X = 1 \) and the range of \( f \) contains a segment or \( \dim X \geq 2 \) and the range of \( f \) does not contain a segment.
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