Some congruences modulo 2 and 5 for bipartition with 5-core

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Abstract. We find some congruences modulo 2 and 5 for the number of bipartitions with 5-core for a positive integer \( n \) in the spirit of Ramanujan.

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1. Introduction

A bipartition of a positive integer \( n \) is a pair of partitions \((\lambda, \mu)\) such that the sum of all of the parts is \( n \). A bipartition with \( t \)-core is a pair of partitions \((\lambda, \mu)\) such that \( \lambda \) and \( \mu \) are both \( t \)-cores. If \( A_t(n) \) denotes the number of bipartitions with \( t \)-core of \( n \), then \( A_t(n) \) is defined by

\[
\sum_{n=0}^{\infty} A_t(n)q^n = \frac{(q^t; q^t)_\infty^{2t}}{(q; q)_\infty^{2t}}.
\]

(1.1)

where \((a; q)_\infty = \prod_{n=1}^{\infty} (1 - aq^n)\). We note the following well known congruence property which can be proved by using binomial theorem: For any prime \( p \) and positive integer \( k \),

\[
(q^k; q^k)_\infty \equiv (q^{pk}; q^{pk})_\infty \pmod{p}.
\]

(1.2)

The function \( A_t(n) \) defined in (1.1) have been studied by many mathematicians. Lin [8] discovered some interesting congruences modulo 4, 5, 7, and 8 for \( A_3(n) \). Yao [10] established several infinite families of congruences modulo 3 and 9 for \( A_9(n) \). Xia [9]

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established several infinite families of congruences modulo 4, 8 and \(\frac{4k-1}{3} \) \((k \geq 2)\) for \(A_3(n)\) and also generalized some results due to Lin and Yao. Baruah and Nath [1] also proved some results on \(A_3(n)\).

In this paper, we are concerned with the function \(A_5(n)\) which denotes the number of bipartition with 5-core of \(n\) and is given by

\[
\sum_{n=0}^{\infty} A_5(n)q^n = \frac{(q^5; q^5)_\infty^{10}}{(q; q)_\infty^2}. \tag{1.3}
\]

In Section 3, we find some congruences modulo 2 and 5 for \(A_5(n)\) in the spirit of Ramanujan. Section 2 is devoted to record some preliminary results.

2. Preliminaries

Ramanujan’s general theta-function \(f(a, b)\) [3, p. 35, Entry 19] is defined by

\[
f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty, \quad |ab| < 1. \tag{2.1}
\]

Lemma 2.1 ([4, Theorem 2.2]). For any prime \(p \geq 5\), we have

\[
(q; q)_\infty = \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{(3k^2+k)/2} f \left( -q^{3p^2+(6k+1)p}, -q^{3p^2-(6k+1)p} \right)
+ (-1)^{\frac{\pm p-1}{6}} q^{n/24} (q^{n^2}; q^{p^2})_\infty, \tag{2.2}
\]

where \(\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \text{ (mod } 6), \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \text{ (mod } 6). \end{cases}\)

Furthermore, if \(-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}\) and \(k \neq \frac{\pm p-1}{2}\), then \(3k^2+k \equiv 2p^2 \pm 24 \text{ (mod } p)\).

Lemma 2.2 ([7, Theorem 1]). We have

\[
\frac{(q^5; q^5)_\infty}{(q; q)_\infty} = \frac{(q^8; q^8)_\infty(q^{20}; q^{20})_\infty^{2}}{(q^2; q^2)_\infty^2(q^{40}; q^{40})_\infty} + q \frac{(q^4; q^4)_\infty^3(q^{10}; q^{10})_\infty(q^{40}; q^{40})_\infty}{(q^2; q^2)_\infty^3(q^{8}; q^{8})_\infty(q^{20}; q^{20})_\infty}.
\]

Lemma 2.3 ([6]). We have

\[
\frac{1}{(q; q)_\infty} = \frac{(q^{25}; q^{25})_\infty^5}{(q^5; q^5)_\infty^6} \left( F^{5-4}(q^5) + q F^{5-3}(q^5) + 2q^2 F^{2-2}(q^5) + 3q^3 F^{1-1}(q^5) + 5q^4 - 3q^5 F(q^5) + 2q^6 F^2(q^5) - q^7 F^3(q^5) + q^8 F^4(q^5) \right),
\]

where \(F(q) := q^{-1/5} R(q)\) and \(R(q)\) is Rogers-Ramanujan continued fraction defined by

\[
R(q) := \frac{1}{q^{1/5}} \frac{1}{1 + q \frac{q^2}{1 + q \frac{q^3}{1 + q \frac{q^4}{1 + q \frac{q^5}{1 + q \frac{q^6}{1 + q \cdots}}}}}; \quad |q| < 1.
\]
Lemma 2.4 ([3, p. 39, Entry 24(ii)]). We have
\[
(q; q)^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/2}.
\]

Lemma 2.5 ([2, p. 648, Theorem 2.1; Eqns. (2.1), (2.5) & (2.13)]). If
\[
\sum_{n=0}^{\infty} p_3(n) q^n = (q; q)^3,
\]
then for any positive integer \(k\),
\[
p_3 \left( \frac{3^{2k} n + 3^{2k} - 1}{8} \right) = (-3)^k p_3(n),
\]
\[
p_3 \left( \frac{5^{2k} n + 5^{2k} - 1}{8} \right) = 5^k p_3(n)
\]
and
\[
p_3 \left( \frac{7^{2k} n + 7^{2k} - 1}{8} \right) = (-7)^k p_3(n).
\]

3. CONGRUENCES MODULO 2 AND 5 FOR \(A_5(n)\)

Theorem 3.1. We have
(i) \(A_5(2n + 1) \equiv 0 \pmod{2}\).
(ii) \(A_5(8n + 4) \equiv 0 \pmod{2}\).

Proof. Using (1.2) with \(p = 2\) in (1.3), we find that
\[
\sum_{n=0}^{\infty} A_5(n) q^n = \frac{(q^5; q^5)^{10}}{(q; q)^2} \equiv \frac{(q^{10}; q^{10})^5}{(q^2; q^2)^5} \pmod{2}.
\]
The right hand side of (3.1) contains no term involving odd power of \(q\), so extracting the terms involving \(q^{2n+1}\) from (3.1), we arrive at (i).

Extracting the terms involving \(q^{2n}\) from (3.1) and replacing \(q^2\) by \(q\) and simplifying using (1.2), we obtain
\[
\sum_{n=0}^{\infty} A_5(2n) q^n \equiv \frac{(q^5; q^5)^5}{(q; q)^5} \equiv \frac{(q^{20}; q^{20}) (q^5; q^5)^5}{(q; q)^5} \pmod{2}.
\]
Employing Lemma 2.2 in (3.2) and simplifying using (1.2), we deduce that
\[
\sum_{n=0}^{\infty} A_5(2n) q^n \equiv (q^4; q^4) (q^{20}; q^{20}) + q \frac{(q^{10}; q^{10}) (q^{40}; q^{40})}{(q^2; q^2)} \pmod{2}.
\]
The right hand side of (3.3) contains no term involving \( q^{4n+2} \), so extracting the terms involving \( q^{4n+2} \) from (3.3), we complete the proof of (ii).

**Theorem 3.2.** Let \( p \geq 5 \) be a prime with \( \left( \frac{-5}{p} \right) = -1 \). Then for non-negative integers \( \alpha \) and \( n \), we have

\[
\sum_{n=0}^{\infty} A_5(8p^{2\alpha} n + 2p^{2\alpha} - 2) q^n \equiv (q; q)_{\infty}(q^5; q^5)_{\infty} \pmod{2},
\]

(3.4)

where, here and throughout the paper \( \langle : \rangle \) denotes the Legendre symbol.

**Proof.** Extracting the terms involving \( q^{4n} \) from (3.3) and replacing \( q^4 \) by \( q \), we obtain

\[
\sum_{n=0}^{\infty} A_5(8n)q^n \equiv (q; q)_{\infty}(q^5; q^5)_{\infty} \pmod{2},
\]

(3.5)

which is the case \( \alpha = 0 \).

Assume (3.4) holds for \( \alpha \). Employing Lemma 2.1 in (3.4), we obtain

\[
\sum_{n=0}^{\infty} A_5(8p^{2\alpha} n + 2p^{2\alpha} - 2) q^n \\
\equiv \left[ \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{(3k^2+k)/2} f \left( -q^{\frac{3p^2-6k+1}{2}}, -q^{\frac{3p^2-(6k+1)}{2}} \right) \right] \\
+ (\pm \frac{p-1}{6}) q^{\frac{p^2-1}{24}} (q^p; q^p)_{\infty} \\
\times \left[ \sum_{m=-\frac{p-1}{6}}^{\frac{p-1}{6}} (-1)^m q^{5(3m^2+m)/2} f \left( -q^{\frac{5p^2+6m+1}{2}}, -q^{\frac{5p^2-(6m+1)}{2}} \right) \right] \\
+ (\pm \frac{p-1}{6}) q^{\frac{5p^2-1}{24}} (q^{5p}; q^{5p})_{\infty} \pmod{2},
\]

(3.6)

Consider the congruence

\[
\frac{3k^2+k}{2} + 5 \left( \frac{3m^2+m}{2} \right) \equiv 6 \left( \frac{p^2-1}{24} \right) \pmod{p}.
\]

(3.7)

The congruence (3.7) is equivalent to

\[
(6k+1)^2 + 5(6m+1)^2 \equiv 0 \pmod{p}.
\]

(3.8)

For \( \left( \frac{-5}{p} \right) = -1 \) the congruence (3.8) has a unique solution \( k = m = \pm \frac{p-1}{6} \). So extracting the terms involving \( q^{p^{4n+(p^2-1)/4}} \) from (3.6), dividing by \( q^{(p^2-1)/4} \) and replacing \( q^p \) by \( q \), we
deduce that
\[ \sum_{n=0}^{\infty} A_5 \left( 8p^{2\alpha+1}n + 2p^{2\alpha+2} - 2 \right) q^n \equiv (q^p; q^p)_{\infty} (q^5p; q^5p)_{\infty} \pmod{2}. \quad (3.9) \]

Extracting the terms involving \( q^{\alpha n} \) from (3.9) and replacing \( q^p \) by \( q \), we obtain
\[ \sum_{n=0}^{\infty} A_5 \left( 8p^{2\alpha+2}n + 2p^{2\alpha+2} - 2 \right) q^n \equiv (q; q)_{\infty} (q^5; q^5)_{\infty} \pmod{2}, \quad (3.10) \]
which is the case \( \alpha + 1 \) of (3.4). Hence, the proof is complete. \( \square \)

**Corollary 3.3.** Let \( p \geq 5 \) be a prime with \( \left( \frac{-5}{p} \right) = -1 \). Then for non-negative integers \( \alpha \) and \( n \), we have
\[ A_5 \left( 8p^{2\alpha+2}n + 2p^{2\alpha+1}(4j + p) - 2 \right) \equiv 0 \pmod{2}, \quad (3.11) \]
where \( j = 1, 2, 3, \ldots, p-1 \).

**Proof.** Extracting the terms involving \( q^{\alpha n+j} \) for \( j = 1, 2, 3, \ldots, p-1 \) from (3.9), we arrive at the desired result. \( \square \)

**Theorem 3.4.** For any positive integer \( k \), we have
\[ A_5 \left( 2^k n + 2^k - 2 \right) \equiv A_5(2n) \pmod{2}. \]

**Proof.** Extracting the terms involving \( q^{2n+1} \) from (3.3), dividing by \( q \) and replacing \( q^2 \) by \( q \), we obtain
\[ \sum_{n=0}^{\infty} A_5(2^n + 2)q^n \equiv \frac{(q^{20}; q^{20})_{\infty} (q^5; q^5)_{\infty}}{(q; q)_{\infty}} \pmod{2}. \quad (3.12) \]
Combining (3.2) and (3.12), we deduce that
\[ A_5(2^n + 2) \equiv A_5(2n) \pmod{2}. \quad (3.13) \]
Iterating (3.13) by replacing \( n \) by \( 2n + 1 \) and for any positive integer \( k \), we obtain
\[ A_5 \left( 2^k n + 2^{k-1} + 2^{k-2} + \cdots + 2 \right) \equiv A_5(2n) \pmod{2}. \quad (3.14) \]
Simplifying (3.14), we arrive at the desired result. \( \square \)

**Theorem 3.5.** We have
(i) \( \sum_{n=0}^{\infty} A_5 \left( 16n \right) q^n \equiv (q; q)_{\infty}^3 \pmod{2} \),
(ii) \( A_5 \left( 80n + 16i + 8 \right) \equiv 0 \pmod{2} \), where \( i = 1, 2, 3 \) and 4.
Proof. Simplifying (3.5) using (1.2), we obtain
\[
\sum_{n=0}^{\infty} A_5(8n)q^n \equiv \left(\frac{q^{5}q^5}{q^{2}q^2}\right)_{\infty} (\mod 2). \tag{3.15}
\]
Employing Lemma 2.2 in (3.15) and simplifying using (1.2), we deduce that
\[
\sum_{n=0}^{\infty} A_5(8n)q^n \equiv (q^{2}q^2)_{\infty} (\mod 2). \tag{3.16}
\]
Extracting the terms involving \(q^{2n}\) from (3.16) and replacing \(q^{2}\) by \(q\), we arrive at (i). Again, extracting the terms involving \(q^{2n+1}\) in (3.16), dividing by \(q\) and replacing \(q^{2}\) by \(q\), we obtain
\[
\sum_{n=0}^{\infty} A_5(16n+8)q^n \equiv (q^{5}q^{5})_{\infty} (\mod 2). \tag{3.17}
\]
Extracting the terms involving \(q^{5n+i}\) for \(i = 1, 2, 3,\) and 4 from (3.17), we arrive at (ii). □

**Theorem 3.6.** For any positive integer \(k\), then

(i) \(A_5(16 \cdot 3^{2k}n + 2 \cdot 3^{2k} - 2) \equiv A_5(16n) \pmod{2},\)

(ii) \(A_5(16 \cdot 5^{2k}n + 2 \cdot 5^{2k} - 2) \equiv A_5(16n) \pmod{2},\)

(iii) \(A_5(16 \cdot 7^{2k}n + 2 \cdot 7^{2k} - 2) \equiv A_5(16n) \pmod{2}.\)

**Proof.** Employing (2.3) in Theorem 3.5(i), we deduce that
\[
A_5(16n) \equiv p_3(n) \pmod{2}. \tag{3.18}
\]
Employing (3.18) in (2.4), (2.5), and (2.6), we arrive at (i), (ii), and (iii), respectively. □

**Corollary 3.7.** If \(n\) is not a triangular number, then
\[
A_5(16n) \equiv 0 \pmod{2}.
\]

**Proof.** Employing Lemma 2.4 in Theorem 3.5(i), we obtain
\[
\sum_{n=0}^{\infty} A_5(16n)q^n \equiv \sum_{n=0}^{\infty} (-1)^n(2n+1)q^{n(n+1)/2} \pmod{2}. \tag{3.19}
\]
The desired result now follows easily from (3.19). □

**Corollary 3.8.** If \(n\) is not a triangular number, we have
\[
A_5(16 \cdot 3^{2k}n + 2 \cdot 3^{2k} - 2) \equiv 0 \pmod{2},
\]
\[
A_5(16 \cdot 5^{2k}n + 2 \cdot 5^{2k} - 2) \equiv 0 \pmod{2},
\]
\[
A_5(16 \cdot 7^{2k}n + 2 \cdot 7^{2k} - 2) \equiv 0 \pmod{2}.
\]
Proof. We employ Corollary 3.7 in Theorem 3.6 to complete the proof. □

Theorem 3.9. We have
(i) \( A_5(5n + 2) \equiv 0 \pmod{5} \),
(ii) \( A_5(5n + 3) \equiv 0 \pmod{5} \),
(iii) \( A_5(5n + 4) \equiv 0 \pmod{5} \).

Proof. Squaring the identity in Lemma 2.3, we find that

\[
\frac{1}{(q; q^2)_\infty^2} = \frac{(q^{25}; q^{25})_{\infty}^{10}}{(q^5; q^5)_{\infty}^{12}} \{ F^{-8}(q^5) + 2qF^{-7}(q^5) \\
+ 5q^2F^{-6}(q^5) + 10q^3F^{-5}(q^5) + 20q^4F^{-4}(q^5) \\
+ 16q^5F^{-3}(q^5) + 27q^6F^{-2}(q^5) + 20q^7F^{-1}(q^5) + 15q^8 - 20q^9F(q^5) \\
+ 27q^{10}F^2(q^5) - 16q^{11}F^3(q^5) + 20q^{12}F^4(q^5) - 10q^{13}F^5(q^5) \\
+ 5q^{14}F^6(q^5) - 2q^{15}F^7(q^5) + q^{16}F^8(q^5) \}.
\] (3.20)

Employing (3.20) in (1.3), we find that

\[
\sum_{n=0}^{\infty} A_5(n)q^n = \frac{(q^{25}; q^{25})_{\infty}^{10}}{(q^5; q^5)_{\infty}^{12}} \{ F^{-8}(q^5) + 2qF^{-7}(q^5) \\
+ 5q^2F^{-6}(q^5) + 10q^3F^{-5}(q^5) + 20q^4F^{-4}(q^5) \\
+ 16q^5F^{-3}(q^5) + 27q^6F^{-2}(q^5) + 20q^7F^{-1}(q^5) + 15q^8 - 20q^9F(q^5) \\
+ 27q^{10}F^2(q^5) - 16q^{11}F^3(q^5) + 20q^{12}F^4(q^5) - 10q^{13}F^5(q^5) \\
+ 5q^{14}F^6(q^5) - 2q^{15}F^7(q^5) + q^{16}F^8(q^5) \}.
\] (3.21)

Extracting the terms involving \( q^{5n+2} \) from (3.21), then dividing by \( q^2 \) and replacing \( q^5 \) by \( q \), we find that

\[
\sum_{n=0}^{\infty} A_5(5n + 2)q^n = 5 \frac{(q^5; q^5)_{\infty}^{10}}{(q; q^2)_{\infty}^{12}} \{ F^{-6}(q) + 4qF^{-1}(q) + 4q^2F^4(q) \}.
\] (3.22)

Now (i) follows easily from (3.22).

Extracting the terms involving \( q^{5n+3} \) from (3.21), then dividing by \( q^3 \) and replacing \( q^5 \) by \( q \), we find that

\[
\sum_{n=0}^{\infty} A_5(5n + 3)q^n = 5 \frac{(q^5; q^5)_{\infty}^{10}}{(q; q^2)_{\infty}^{12}} \{ 2F^{-5}(q) + 3q - 2q^2F^5(q) \}.
\] (3.23)

Now (ii) follows easily from (3.23).

Extracting the terms involving \( q^{5n+4} \) from (3.21), then dividing by \( q^4 \) and replacing \( q^5 \) by \( q \), we find that

\[
\sum_{n=0}^{\infty} A_5(5n + 4)q^n = 5 \frac{(q^5; q^5)_{\infty}^{10}}{(q; q^2)_{\infty}^{12}} \{ 4F^{-4}(q) - 4qF(q) + q^2F^6(q) \}.
\] (3.24)

Now (iii) follows easily from (3.24). □
Remark 3.10. Theorem 3.9(ii) also follows as a particular case of a general result in [5, p. 4, Theorem 8].

Theorem 3.11. For any positive integer \( k \), we have
\[
A_5(5^k n + 2 \cdot 5^k - 2) \equiv 5^k A_5(n) \quad \text{(mod 10)}.
\]

Proof. From (3.23), we note that
\[
\sum_{n=0}^{\infty} A_5(5n + 3)q^n = \frac{(q^5; q^5)_{10}^{\infty}}{(q; q^2)_{\infty}^{\infty}} \left\{ 10 \left( F_5^{-5}(q) + q - q^2 F_5^5(q) \right) + 5q \right\}
\equiv 5q \frac{(q^5; q^5)_{10}^{\infty}}{(q; q^2)_{\infty}^{\infty}} \quad \text{(mod 10)}. \quad (3.25)
\]
Employing (1.3) in (3.25), we obtain
\[
\sum_{n=0}^{\infty} A_5(5n + 3)q^n \equiv 5 \sum_{n=0}^{\infty} A_5(n)q^{n+1} \quad \text{(mod 10)}. \quad (3.26)
\]
Extracting the term involving \( q^{n+1} \) on both sides of (3.26), we obtain
\[
A_5(5n + 8) \equiv 5A_5(n) \quad \text{(mod 10)}. \quad (3.27)
\]
Iterating (3.27) by replacing \( n \) by \( 5n + 8 \) \( k \) times, we deduce that
\[
A_5 \left( 5^k n + \left( 5^{k-1} + 5^{k-2} + \cdots + 5 + 1 \right) 8 \right) \equiv 5^k A_5(n) \quad \text{(mod 10)}. \quad (3.28)
\]
Simplifying (3.28), we arrive at the desired result. \( \square \)

Corollary 3.12. For any positive integer \( k \), we have
(i) \( A_5 \left( 5^k n + 2 \cdot 5^k - 2 \right) \equiv 0 \quad \text{(mod 5)}, \)
(ii) \( A_5 \left( 5^k n + 2 \cdot 5^k - 2 \right) \equiv A_5(n) \quad \text{(mod 2)}. \)

Proof. Proof follows from Theorem 3.11. \( \square \)

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