ELSEVIER

Discrete Mathematics 140 (1995) 119–127

DISCRETE MATHEMATICS

A Helly theorem for geodesic convexity in strongly dismantlable graphs

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Received 27 March 1992; revised 4 April 1994

Abstract

A (finite or infinite) graph G is *strongly dismantlable* if its vertices can be linearly ordered $x_0, \ldots, x_\alpha$ so that, for each ordinal $\beta < \alpha$, there exists a strictly increasing finite sequence $(i_j)_{0 \leqslant j \leqslant n}$ of ordinals such that $i_0 = \beta$, $i_n = \alpha$ and $x_{i_{j+1}}$ is adjacent with x_{i_j} and with all neighbors of x_{i_j} in the subgraph of G induced by $\{x_\gamma\colon \beta \leqslant \gamma \leqslant \alpha\}$. We show that the Helly number for the geodesic convexity of such a graph equals its clique number. This generalizes a result of Bandelt and Mulder (1990) for dismantlable graphs. We also get an analogous equality dealing with infinite families of convex sets.

0. Introduction

A *convexity* on a connected graph G is an algebraic closure system $\mathscr{C}$ on $V(G)$, such that every element of $\mathscr{C}$, the *convex sets*, induces a connected subgraph of G.

Several kinds of graph convexities have already been investigated (see [2–6, 8]). Two of them seem the most natural: the *geodesic convexity* and the *minimal path convexity*. In the first (resp. second) a subset C of $V(G)$ is convex if it contains the set of vertices of any geodesic, i.e., shortest path (resp. chordless path) joining two vertices in C. For these convexities the standard parameters such as Carathéodory, Helly and Radon numbers, have been studied. In particular, the Helly parameter, the only one we will consider in this paper, has received the most attention. The *Helly number* $h(G)$ of a graph G is the smallest integer (if there is one) such that any finite family of $h(G)$-wise non-disjoint convex sets has a non-empty intersection. This integer is clearly not smaller than the cardinality of any simplex (i.e., complete subgraph) of G, thus than the supremum of these cardinalities, the *clique number* $\omega(G)$ of G. For the minimal path convexity, Duchet [5], and independently Jamison and Nowakowski [8], proved that the equality $h(G) = \omega(G)$ holds for any connected graph, finite or infinite. As for the geodesic convexity, except for distance-hereditary graphs (i.e., graphs for which all induced paths are geodesics, such as trees for example) where the

SSDI 0012-365X(93)E0178-7

equality $h(G)=\omega(G)$ is an obvious consequence of the Duchet–Jamison–Nowakowski theorem, and for pseudo-modular graphs (cf. [3]), it seems that, so far, this equality has been proved for some classes of finite graphs only: first by Cepoj [4] for chordal graphs, then, generalizing this result, by Bandelt and Mulder [3] for dismantlable graphs (a graph G is *dismantlable* if its vertices can be linearly ordered $x_0, \ldots, x_n$ so that, for each $i<n$, there is a vertex of the subgraph G_i induced by $\{x_i, \ldots, x_n\}$ which is adjacent with x_i and with all neighbors of x_i in G_i).

To study certain problems of invariant simplices in infinite graphs, we recently extended this concept of dismantlable graphs by introducing the *strongly dismantlable* (or *subretract collapsible*) graphs. Since the finite strongly dismantlable graphs are the dismantlable ones, it seems quite reasonable to check if Bandelt and Mulder's result [3, Theorem 1] can be extended to strongly dismantlable graphs. This is what we first do in this paper (Theorem 4.1). Then, in Section 5, we consider infinite families of convex sets.

1. Notation

The graphs we consider are undirected, without loops and multiple edges. If x and y are two vertices of a graph G we denote $x \equiv_G y$ if $x=y$ or $\{x,y\}\in E(G)$. If $x\in V(G)$, the set $V(x;G):=\{y\in V(G)\colon \{x,y\}\in E(G)\}$ is the *neighborhood* of x. For $A\subseteq V(G)$ we denote by $G|A$ the subgraph of G induced by A, and we set $G-A:=G|(V(G)-A)$. A *path* $W:=\langle x_0, \ldots, x_n\rangle$ is a graph with $V(W)=\{x_0, \ldots, x_n\}$, $x_i\neq x_j$ if $i\neq j$, and $E(W)=\{\{x_i, x_{i+1}\}\colon 0\leqslant i<n\}$. A *ray* or *one-way infinite path* $R:=\langle x_0, x_1, \ldots\rangle$ is defined similarly. A vertex x_i of a path $W:=\langle x_0, \ldots, x_n\rangle$ with $0<i<n$ will be called an *internal vertex* of W. A subset C of $V(G)$ is *geodesically convex*, for short *convex*, if it contains the set $I(x,y)$ of vertices of every xy-geodesic, for all $x,y\in C$. The *convex hull* $\mathrm{co}_G(C)$ *of* C *in* G is the smallest convex set of G containing C, thus $\mathrm{co}_G(C)=\bigcup_{n\geqslant 0}C_n$ where $C_0=C$ and $C_{n+1}=\bigcup_{x,y\in C_n}I(x,y)$. The usual *distance in* G *between two vertices* x *and* y, that is the length of an xy-geodesic in G, will be denoted by $\mathrm{dist}_G(x,y)$. The *diameter* of G is $\mathrm{diam}(G):=\sup\{\mathrm{dist}_G(x,y)\colon x,y\in V(G)\}$. A graph is *bounded* if its diameter is finite. A subgraph H of G is *isometric* if $\mathrm{dist}_H(x,y)=\mathrm{dist}_G(x,y)$ for all vertices x and y of H. If x is a vertex of G and r a non-negative integer, the set $B_G(x,r):=\{y\in V(G)\colon \mathrm{dist}_G(x,y)\leqslant r\}$ is the *ball of center* x *and radius* r *in* G. We will write $x\leqslant_G y$ (resp. $x<_G y$) if $B_G(x,1)\subseteq B_G(y,1)$ (resp. $B_G(x,1)\subset B_G(y,1)$), and we will say that x is *dominated* (resp. *strictly dominated*) *by* y *in* G.

If G and H are two graphs, a map $f\colon V(G)\to V(H)$ is a *contraction* if f preserves the relation $\equiv$, i.e., if $x\equiv_G y$ implies $f(x)\equiv_H f(y)$. Notice that a contraction $f\colon G\to H$ is a non-expansive map between the metric spaces $(V(G), \mathrm{dist}_G)$ and $(V(H), \mathrm{dist}_H)$, i.e., $\mathrm{dist}_H(f(x), f(y))\leqslant \mathrm{dist}_G(x,y)$ for all $x,y\in V(G)$. A contraction f from G onto an induced subgraph H of G is a *retraction*, and H is a *retract* of G, if its restriction $f|H$ to H is the identity.

2. Strongly dismantlable graphs and retract-collapsible graphs

The concept of dismantlability can be straightforwardly extended to infinite graphs as follows.

Definition 2.1. A graph G is said to be *dismantlable* if there is a well-ordering $\leqslant$ on $V(G)$ such that, any vertex x which is not the greatest element of $(V(G), \leqslant)$ if such a greatest element exists, is dominated by some vertex $y \neq x$ in the subgraph of G induced by the set $\{z \in V(G)\colon x \leqslant z\}$.

As this extension seems too much general to get interesting results, we introduced the following restricted concept.

Definition 2.2. A graph G is *strongly dismantlable* if there is a well-ordering $\leqslant$ on $V(G)$ with a greatest element m such that, for every vertex $x \neq m$, there is a strictly increasing finite sequence $x = x_0 < \cdots < x_n = m$ where, for $0 \leqslant i < n$, the vertex x_i is dominated by x_{i+1} in the subgraph of G induced by the set $\{z \in V(G)\colon x_i \leqslant z\}$.

Clearly *any strongly dismantlable graph is dismantlable.* Furthermore, by [10, Theorem 4.4], *any rayless connected dismantlable graph is strongly dismantlable.* Thus, in particular *the finite strongly dismantlable graphs are the dismantlable ones.*

In order to characterize those graphs, and thus to work more easily with them, we will recall other classes of graphs.

2.3. Let $D(G) := \{x \in V(G)\colon x <_G y \text{ for some } y \in V(G)\}$. For an ordinal α, we define $G^{(\alpha)}$ inductively as follows:

- $G^{(0)} := G$,
- $G^{(\alpha+1)} := G^{(\alpha)} - D(G^{(\alpha)})$,
- $G^{(\alpha)} := \bigcap_{\beta<\alpha} G^{(\beta)}$ if α is a limit ordinal.

The ordinal $d(G) := \min\{\alpha\colon G^{(\alpha)} = G^{(\alpha+1)}\}$ will be called the *depth* of G, and the subgraph $G^{(\infty)} := G^{(d(G))}$ the *base* of G. Finally, for $x \in V(G)$ the *depth* of x will be $d(x) := \max\{\alpha\colon \alpha \leqslant \mathrm{d}(G) \text{ and } x \in V(G^{(\alpha)})\}$.

Definition 2.4. A graph G is said to be *collapsible* if $G^{(\infty)}$ is empty or is a simplex.

Definition 2.5. A graph G is said to be *retractable* if, for any ordinals α and β with $\alpha \leqslant \beta \leqslant d(G)$, there exists a retraction $f_{\alpha\beta}\colon G^{(\alpha)} \to G^{(\beta)}$ satisfying the following properties:

(i) $f_{\alpha\beta} = f_{\gamma\beta} \circ f_{\alpha\gamma}$ for any ordinal γ with $\alpha \leqslant \gamma \leqslant \beta$;

(ii) $x <_{G^{(\alpha)}} f_{\alpha\alpha+1}(x)$ for any $x \in D(G^{(\alpha)})$;

(iii) if β is a limit ordinal $>\alpha$, then, for any $x \in V(G^{(\alpha)})$, there is an ordinal δ with $\alpha \leqslant \delta < \beta$, such that $\delta \leqslant \gamma \leqslant \beta$ implies $f_{\alpha\delta}(x) = f_{\alpha\gamma}(x)$.

In particular any graph of depth 0 is retractable. A graph is said to be *retract-collapsible* if it is retractable and collapsible.

Lemma 2.6 (Polat [10, Lemmas 3.6 and 3.7]). *If G contains no infinite simplices and has a finite depth, or if G is rayless, then G is retractable.*

2.7. Let G be a graph, and $(T_x)_{x\in V(G)}$ a family of pairwise-disjoint rayless trees such that $T_x \cap G = \langle x \rangle = T_x^{(\infty)}$ for every $x \in V(G)$. Note that any rayless tree is retract-collapsible by Lemma 2.6. Then $H := G \cup \bigcup_{x\in V(G)} T_x$ will be called a *tree-extension* of G. Observe that $H^{(\infty)}$ is a subgraph of G since the base of T_x is $\langle x \rangle$ for every $x \in V(G)$.

Definition 2.8. A graph G will be said *subretractable* (resp. *subretract-collapsible*) if there is a tree-extension of G which is retractable (resp. retract-collapsible). The base of a retractable tree-extension of G will be called a *subbase* of G. The *subdepth* of G will be the ordinal

$$\mathrm{sd}(G) := \min\{d(H)\colon H \text{ is a retractable tree-extension of } G\}.$$

Clearly, $\mathrm{sd}(G) = d(G)$ if G is retractable.

Proposition 2.9 (Polat [11, Theorem 3.11]). *A graph is strongly dismantlable if and only if it is subretract-collapsible.*

Therefore, the retract-collapsible graphs are particular strongly dismantlable graphs.

3. Examples of strongly dismantlable graphs

3.1 [11, Theorem 3.4]. *A ball-Helly graph* (i.e., a graph such that any family of pairwise-non-disjoint balls has a non-empty intersection) *is strongly dismantlable if and only if it contains no isometric rays.*

3.2 [11, *Theorem* 4.5]. *A ball-Helly graph G is retract-collapsible if it has one of the following properties:*

(i) *G is rayless;*

(ii) *G is bounded; and in this case $d(G) < \mathrm{diam}(G)$.*

In particular *any rayless tree is retract-collapsible* since any tree is a ball-Helly graph.

3.3 [11, Theorem 5.2]. *Let G be subretractable graph. Then G is cop-win* (the cop-win graphs are the graphs characterized by Nowakovski and Winkler [9] where, in some pursuit game, a cop can always catch a robber) *if and only if it is subretract-collapsible.*

Consequently *any strongly dismantlable graphs is cop-win*, but we do not yet know if the converse holds.

3.4. We will complete this section by proving a result that gives another example of strongly dismantlable graph. We recall that a graph is *chordal* if it contains no induced cycles of length greater than three.

Theorem 3.5. *Let G be a connected rayless graph. Then G is chordal if and only if every connected induced subgraph of G is retract-collapsible.*

Proof. (a) Suppose that G is not chordal, then G contains an induced cycle C of length greater than three. But such a cycle C is not collapsible since $C^{(\infty)}=C$ is not a simplex.

(b) Conversely, suppose that G is chordal, and let H be a non-empty connected subgraph of G. H is clearly chordal and also rayless since so is G. Hence, it is retractable by Lemma 2.6. We have then to show that $H^{(\infty)}$ is a simplex to complete the proof. As H is non-empty, connected and retractable, $H^{(\infty)}$ is a non-empty connected graph. Furthermore it is chordal since so is H. Suppose that $H^{(\infty)}$ is not a simplex. Define vertices $x_0, x_1, \ldots$ of $H^{(\infty)}$ such that $\langle x_0, \ldots, x_n \rangle$ is an induced path of H, for every n. Since $H^{(\infty)}$ is connected and not complete, there exists an induced path $\langle x_0, x_1, x_2 \rangle$ of length two. Suppose that $x_0, \ldots, x_p$ have already been defined, for some $p \geqslant 2$. Since $D(H^{(\infty)}) = \emptyset$, the vertex x_p is not strictly dominated by x_{p-1} in $H^{(\infty)}$. Hence, there is a vertex $x_{p+1} \in V(x_p; H^{(\infty)}) - B_{H^{(\infty)}}(x_{p-1}, 1)$. This vertex is distinct from $x_0, \ldots, x_p$ since $x_p \not\simeq_{H^{(\infty)}} x_i$ for $0 \leqslant i \leqslant p-2$, as $\langle x_0, \ldots, x_p \rangle$ is induced. Suppose that x_{p+1} is adjacent with a vertex x_i for some $i < p-1$. Let i be the greatest such integer. Then $\langle x_i, x_{i+1}, \ldots, x_{p+1}, x_i \rangle$ is a cycle of length greater than three. Since $H^{(\infty)}$ is chordal, this cycle must have a chord, and this chord must be incident with x_{p+1} as $\langle x_i, \ldots, x_p \rangle$ is induced by hypothesis; but this contradicts the maximality of i. Thus $\langle x_0, \ldots, x_p, x_{p+1} \rangle$ is an induced path.

Therefore $\langle x_0, x_1, \ldots \rangle$ is a (induced) ray of $H^{(\infty)}$, a contradiction with the property of G. Consequently $H^{(\infty)}$ is complete. □

Corollary 3.6. *Let G be a connected rayless graph. Then G is chordal if and only if every connected induced subgraph of G is cop-win.*

That immediate consequence of 3.3 and 3.5 generalizes a result of Anstee and Farber [1, Corollary 2.8] about finite chordal graphs. Note that an infinite chordal graph may not be strongly dismantlable, even if it is bounded. Indeed Hahn et al. [7, Theorem 3.3] constructed a chordal graph of diameter two which was not cop-win, hence not strongly dismantlable by 3.3.

4. A Helly theorem

Theorem 4.1. *Let G be a strongly dismantlable graph such that $\omega(G)$ is finite. Then $h(G)=\omega(G)$.*

Proof. We have to prove that $h(G)\leqslant\omega(G)$.

(a) Suppose that G is retract-collapsible. Assume first that G is a tree, then, since the geodesic convexity and the minimal path convexity coincide for trees, Duchet–Jamison–Nowakowski's theorem gives $h(G)=\omega(G)=1$ or 2 according as G has only one or more than one vertex. Assume now that G is not a tree. Then, since it is retract-collapsible, it must contain a simplex of cardinality 3, hence $n:=h(G)\geqslant 3$. Suppose that G has no simplex of cardinality n. Let I be a set of cardinality n, and $(A_i)_{i\in I}$ a family of $(n-1)$-wise non-disjoint convex sets in G such that $\bigcap_{i\in I}A_i=\emptyset$.

In the following, for every $J\subseteq I$ and $\alpha\leqslant d(G)$, we put $A_J=\bigcap_{i\in J}A_i$ and $A_J^{(\alpha)}=A_J\cap V(G^{(\alpha)})$, with $A_i^{(\alpha)}=A_{\{i\}}^{(\alpha)}$ for all $i\in I$. Besides we call *critical set* for the family $(A_i)_{i\in I}$ a set $C:=\{a_i\colon i\in I\}$ where $a_i\in A_{I-\{i\}}$ for every $i\in I$. Note that $a_i\notin A_i$ since $\bigcap_{i\in I}A_i=\emptyset$, and that the convex hull $\mathrm{co}_G(C-\{a_i\})\subseteq A_i$, for all $i\in I$. Finally, we denote by $(f_{\alpha\beta})_{\alpha\leqslant\beta\leqslant d(G)}$ the family of retractions as defined in 2.3.

(a1) We will prove by induction on $\alpha\leqslant d(G)$ that $A_{I-\{i\}}^{(\alpha)}\neq\emptyset$ for all $i\in I$. This is clear if $\alpha=0$. Let $\alpha\geqslant 0$. Suppose that this holds for any $\beta<\alpha$, and assume that $A_{I-\{p\}}^{(\alpha)}=\emptyset$ for some $p\in I$.

Let $C:=\{a_i\colon i\in I\}$ be a critical set for the family $(A_i)_{i\in I}$ such that the integer $\lambda(C):=\sum_{i\neq j}\mathrm{dist}_G(a_i,a_j)$ is minimum. And let

$$\beta:=\sup\{\gamma<\alpha\colon f_{0\gamma}(a_i)\in A_{I-\{i\}}\text{ for all } i\in I\}.$$

If γ_p is the least ordinal γ such that $f_{0\gamma_p}(a_p)=f_{0\alpha}(a_p)$, then $f_{0\gamma_p}(a_p)\notin A_{I-\{p\}}$ by the assumption, hence $\beta<\gamma_p\leqslant\alpha$, with the strict inequality by 2.3(iii) if α is a limit ordinal. Furthermore, by 2.3(iii), $f_{0\beta}(a_i)\in A_{I-\{i\}}$ for all $i\in I$, with $f_{0\beta+1}(a_j)\notin A_{I-\{j\}}$ for some $j\in I$. The set $f_{0\beta}(C)$ is then a critical set for the family $(A_i)_{i\in I}$ which is included in $V(G^{(\beta)})$. Since $f_{0\beta}$ is a retraction and by the minimality of $\lambda(C)$, $\mathrm{dist}_{G^{(\beta)}}(f_{0\beta}(a_i), f_{0\beta}(a_{i'}))=\mathrm{dist}_G(a_i,a_i')$ for every i, $i'\in I$: hence $\lambda(f_{0\beta}(C))=\lambda(C)$.

For all $i\in I-\{j\}$, let b_i be a neighbor of $f_{0\beta}(a_j)$ on an $(f_{0\beta}(a_j))$, $(f_{0\beta}(a_i))$-geodesic in $G^{(\beta)}$. Note that $b_i\in A_{I-\{i,j\}}$, hence in particular $f_{0\beta+1}(a_j)\neq b_i$. Since $f_{0\beta+1}(a_j)$ is adjacent to b_i for every $i\in I-\{j\}$, all b_i's must be adjacent; otherwise there would be $i\neq i'$ such that the path $\langle f_{0\beta}(a_i), f_{0\beta+1}(a_j), f_{0\beta}(a_i')\rangle$ is a geodesic, hence $f_{0\beta+1}(a_j)$ would belong to $A_{I-\{j\}}$ by convexity, contrary to the hypothesis. Then, since G has no simplex of cardinality n, the b_i's cannot be all distinct, hence $b_k=b_h$ for some $k\neq h$. Thus $b_k\in A_{I-\{j\}}$. Therefore, $C':=(f_{0\beta}(C)-\{f_{0\beta}(a_j)\})\cup\{b_k\}$ is a critical set for $(A_i)_{i\in I}$ such that

$$\mathrm{dist}_G(b_k,f_{0\beta}(a_i))\leqslant\mathrm{dist}_G(b_i,f_{0\beta}(a_i))+1=\mathrm{dist}_G(f_{0\beta}(a_j),f_{0\beta}(a_i))$$

for all $i \in I - \{p\}$, with at least strict inequalities for $i = k$ and $i = h$. Hence, $\lambda(C') < \lambda(C)$, a contradiction with the minimality of $\lambda(C)$. Consequently $A_{I-\{i\}}^{(\alpha)} \neq \emptyset$ for all $i \in I$.

(a2) For $\alpha = d(G)$, $(A_i^{(\alpha)})_{i \in I}$ is then a family of $(n-1)$-wise non-disjoint convex sets in $G^{(\alpha)}$ such that $\bigcap_{i \in I} A_i^{(\alpha)} = \emptyset$. Thus, there is a critical set C for this family. This set is of cardinality n by the definition of critical sets, thus $G|C$ is not a simplex by the assumption we made for G, but this contradicts the hypothesis that G is collapsible.

Therefore $h(G) = \omega(G)$.

(b) Suppose now that G is strongly dismantlable, thus subretract-collapsible, but not retract-collapsible. Let $H := G \cup \bigcup_{x \in V(G)} T_x$ be a retract-collapsible tree-extension of G such that $d(H) = \mathrm{sd}(G)$. Any subset of $V(G)$ which is convex in G is also convex in H, and conversely. Furthermore, for every $x \in V(G)$, $h(T_x) = \omega(T_x) \leqslant 2$ by (a); hence $h(G) = h(H)$. Besides $\omega(G) = \omega(H)$. Hence, $h(G) = \omega(G)$, and the proof is complete. □

Corollary 4.2. *Let G be either a ball-Helly graph without isometric rays, or a rayless chordal graph. If $\omega(G)$ is finite, then $h(G) = \omega(G)$.*

This is a consequence of 4.1 and of 3.1 or 3.5, respectively.

5. Infinite families of convex sets

The Helly number of a graph G is related to finite families of convex sets only, even if G is infinite; if one omits this condition of finiteness by considering any family of convex sets, finite or infinite, then the equality $h(G) = \omega(G)$ does not hold in general. Take, for example, a one-way infinite path $R := \langle x_0, x_1, \ldots \rangle$ and the family $(\{x_i : n \leqslant i\})_{n \geqslant 0}$ of convex sets; then these sets are pairwise non-disjoint, but have an empty intersection. We will show that this is not the case for the strongly dismantlable graphs.

Notation 5.1. If I is a set and n a cardinal, $[I]^{<n}$ will denote the set of subsets of I of cardinality less than n. For a graph G, we will denote by $\omega^+(G)$ the least cardinal n such that G has no simplex of cardinality n; if $\omega^+(G)$ is finite, then it is the successor of the usual clique-number $\omega(G)$ of G. A family of sets will be said *$<n$-wise non-disjoint* $(n > 0)$ if it is p-wise non-disjoint for every cardinal $p < n$, i.e., if every subfamily of cardinality less then n has a non-empty intersection. For a graph G we will denote by $h^+(G)$ the least cardinal n such that any family of $<n$-wise non-disjoint convex sets in G has a non-empty intersection. If the Helly-number $h(G)$ of G exists, then $h^+(G) = h(G) + 1$.

Definition 5.2. A set S of vertices of a graph G is *fragmented* if there is a finite subset F of $V(G)$ such that the elements of S are pairwise separated by F (i.e., two different elements of S do not belong to the same component of $G - F$).

Lemma 5.3. *Let $A=\{a_\beta: \beta<\alpha\}$ be an infinite set of vertices of a connected rayless graph G. Then there is a cofinal subset Γ of α such that $\{a_\gamma: \gamma\in\Gamma\}$ is fragmented.*

Proof. Assume that this does not hold. Then, for any finite subset F of $V(G)$, there is $\beta<\alpha$ and a component $G(F)$ of $G-F$ which contains $\{a_\gamma: \beta\leqslant\gamma<\alpha\}$.

Construct vertices $x_0, x_1, \ldots$ such that, for $n>0$, x_{n-1} and x_n are adjacent and $x_n\in V(G(\{x_0,\ldots,x_{n-1}\}))$, as follows. Let x_0 be any vertex of G. Suppose that $x_0,\ldots,x_n$ have already been constructed. By the assumption, $G(\{x_0,\ldots,x_n\})$ is non-empty, and by the induction hypothesis, it is a strict subgraph of $G(\{x_0,\ldots,x_{n-1}\})$ for $n\geqslant 1$ since $x_n\in V(G(\{x_0,\ldots,x_{n-1}\}))$. Then let x_{n+1} be a vertex of $G(\{x_0,\ldots,x_n\})$ adjacent with x_n. Therefore, $\langle x_0, x_1, \ldots\rangle$ is a ray, thus contradicting the assumption on G. □

Theorem 5.4. *Let G be a strongly dismantlable graph which is rayless if its subdepth is infinite. Let κ be an infinite cardinal, and let $(A_i)_{i\in I}$ be a family of $\langle\kappa$-wise non-disjoint convex sets in G. If $\bigcap_{i\in I}A_i=\emptyset$, then there exists a simplex K in G such that $|V(K)\cap A_i|\geqslant\kappa$ for every $i\in I$; in particular, $\mathrm{sd}(G)$ is finite. Furthermore $h^+(G)=\omega^+(G)$.*

Proof. (a) Assume that G is retract-collapsible and that there is no simplex K in G such that $|V(K)\cap A_i|\geqslant\kappa$ for every $i\in I$.

We will use the notation we introduced in part (a) of the proof of Theorem 4.1. Notice that, for any $J\in[I]^{<\kappa}$, the subgraph $G|A_J^{(\alpha)}$ is a simplex if $A_J^{(\alpha+1)}=\emptyset$, since, for any two non-adjacent vertices a and b of $A_J^{(\alpha)}$, if c is an internal vertex of an ab-geodesic in $G^{(\alpha)}$, then so is $f_{0\alpha+1}(c)$.

(a1) We will prove by induction on $\alpha\leqslant d(G)$ that $A_J^{(\alpha)}\neq\emptyset$ for all $J\in[I]^{<\kappa}$. This is clear if $\alpha=0$. Let $\alpha\geqslant 0$. Suppose that this holds for any $\beta<\alpha$.

Case 1: $\alpha=\beta+1$. $(A_i^{(\beta)})_{i\in I}$ is then a family of $<\kappa$-wise non-disjoint convex sets in $G^{(\beta)}$ such that $\bigcap_{i\in I}A_i^{(\beta)}=\emptyset$, thus $|A_J^{(\beta)}|\geqslant\kappa$ for any $J\in[I]^{<\kappa}$; hence $A_J^{(\beta)}$ is not a simplex by the assumption and the fact that $|A_i\cap A_J^{(\beta)}|\geqslant\kappa$ for every $i\in I$. Thus, $A_J^{(\alpha)}$ is non-empty by the preceding remark.

Case 2: α is a limit ordinal. Then G is rayless. Let $J\in[I]^{<\kappa}$. Suppose that $A_J^{(\alpha)}=\emptyset$. Then, for every $\beta<\alpha$, there is $a_\beta\in A_J^{(\beta)}\cap D(G^{(\beta)})$. The set $\{a_\beta: \beta<\alpha\}$ is then infinite. By Lemma 5.3, there exists a non-empty finite set F_J of vertices of G, and a cofinal subset Γ_J of α such that, for every $\beta,\gamma\in\Gamma_J$, any $a_\beta a_\gamma$-path of G meets F_J. W.l.o.g. we can suppose that every vertex of F_J belongs to an $a_\beta a_\gamma$-geodesic for some $\beta,\gamma\in\Gamma'$, and for every cofinal subset Γ' of Γ_J. Thus, $F_J\subseteq V(A_J^{(\beta)})$ for every $\beta<\alpha$. Hence $F_J\subseteq V(G^{(\alpha)})$. Therefore $A_J^{(\alpha)}\neq\emptyset$.

(a2) For $\alpha=d(G)$, $(A_i^{(\alpha)})_{i\in I}$ is then a family of $<\kappa$-wise non-disjoint convex sets in $G^{(\alpha)}$ such that $\bigcap_{i\in I}A_i^{(\alpha)}=\emptyset$. Thus, for any $i\in I$, $|A_i^{(\alpha)}|\geqslant\kappa$ and thus $A_i^{(\alpha)}$ is not a simplex by the assumption on G and the fact that $|A_j\cap A_i^{(\alpha)}|\geqslant\kappa$ for every $j\in I$. Hence, the base $G^{(\alpha)}$ of G is not a simplex, a contradiction with the hypothesis that G is collapsible.

(b) Suppose now that G is subretract-collapsible but not retract-collapsible. Let H be a retract-collapsible tree-extension of G such that $d(H)=\mathrm{sd}(G)$. Clearly, any subset of $V(G)$ which is convex in G is also convex in H, and conversely. Besides any infinite simplex of H is a simplex of G. The result is then a consequence of (a).

Furthermore, using Theorem 4.1, we get immediately $h^+(G)=\omega^+(G)$. □

Corollary 5.5. *If G is a rayless or a bounded ball-Helly graph, or if it is a rayless chordal graph, then* $h^+(G)=\omega^+(G)$.

This is a consequence of 5.4 and of 3.2 or 3.5, respectively.

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