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Duality for finite multiple harmonic q-series

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Abstract

We define two finite q-analogs of certain multiple harmonic series with an arbitrary number of free parameters, and prove identities for these q-analogs, expressing them in terms of multiply nested sums involving the Gaussian binomial coefficients. Special cases of these identities—for example, with all parameters equal to 1—have occurred in the literature. The special case with only one parameter reduces to an identity for the divisor generating function, which has received some attention in connection with problems in sorting theory. The general case can be viewed as a duality result, reminiscent of the duality relation for the ordinary multiple zeta function.

Keywords: Multiple harmonic series; Finite q-analog; Gaussian binomial coefficients; q-series; Duality; Multiple zeta values

1. Introduction

One of the main results in [32] is the identity

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} q^{k(k+1)/2}}{(1-q)(1-q^2)\cdots(1-q^{k-1})(1-q^k)^2} = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}, \qquad |q|<1. \tag{1.1}$$

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As Uchimura showed, the series on the left-hand side of (1.1) is the limiting case ($n \to \infty$) of the polynomials

$$U_n(q) := \sum_{k=1}^{n} kq^k \prod_{j=k+1}^{n} (1 - q^j)$$

which arise in sorting: $U_n(1/2)$ is the average number of exchanges required to insert a new element into a heap-ordered binary tree with $2^n - 1$ elements [31]. Andrews and Uchimura [2] later proved the finite analog

$$\sum_{k=1}^{n} \frac{(-1)^{k+1} q^{k(k+1)/2}}{1 - q^k} \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{k=1}^{n} \frac{q^k}{1 - q^k}, \quad 0 < n \in \mathbf{Z}, \tag{1.2}$$

of (1.1) by differentiating the q-Chu–Vandermonde sum. Here,

$$\begin{bmatrix} n \\ k \end{bmatrix} := \prod_{j=1}^{k} \frac{1 - q^{n-k+j}}{1 - q^j} \tag{1.3}$$

is the (Gaussian) q-binomial coefficient, which vanishes by convention unless $0 \leqslant k \leqslant n$. Subsequently, Dilcher [16] proved the generalization

$$\sum_{k=1}^{n} \frac{(-1)^{k+1} q^{k(k+1)/2+(m-1)k}}{(1 - q^k)^m} \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{k_1=1}^{n} \frac{q^{k_1}}{1 - q^{k_1}} \sum_{k_2=1}^{k_1} \frac{q^{k_2}}{1 - q^{k_2}} \cdots \sum_{k_m=1}^{k_{m-1}} \frac{q^{k_m}}{1 - q^{k_m}} \tag{1.4}$$

of (1.2) by a double induction on n and m, and noted that (1.4) can be viewed as a q-analog of the identity

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k^m} \binom{n}{k} = \sum_{n \geqslant k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m \geqslant 1} \frac{1}{k_1 k_2 \cdots k_m}, \quad 0 < m \in \mathbf{Z}, \tag{1.5}$$

involving a finite multiple harmonic sum with unit exponents. In this paper, we provide a generalization of (1.4) that gives a q-analog of (1.5) with arbitrary positive integer exponents on the k_j. We also give a q-analog of (1.5) in which the inequalities on the indices are strict.

2. Main result

Henceforth, we assume q is real and $0 < q < 1$. The q-analog of a non-negative integer n is

$$[n]_q := \sum_{k=0}^{n-1} q^k = \frac{1 - q^n}{1 - q}.$$

Definition 1. Let n, m and $s_1, s_2, \ldots, s_m$ be non-negative integers. Define the multiply nested sums

$$Z_n[s_1,\ldots,s_m] := \sum_{n\geqslant k_1\geqslant\cdots\geqslant k_m\geqslant 1}\prod_{j=1}^m q^{k_j}[k_j]_q^{-s_j}, \tag{2.1}$$

$$A_n[s_1,\ldots,s_m] := \sum_{n\geqslant k_1\geqslant\cdots\geqslant k_m\geqslant 1}(-1)^{k_1+1}q^{k_1(k_1+1)/2}\begin{bmatrix} n \\ k_1\end{bmatrix} \times \prod_{j=1}^m q^{(s_j-1)k_j}[k_j]_q^{-s_j}, \tag{2.2}$$

with the understanding that $Z_0[s_1,\ldots,s_m]=A_0[s_1,\ldots,s_m]=0$, and with empty argument lists, $Z_n[\;]=A_n[\;]=1$ if $n>0$ and $m=0$. As in (1.4) and (1.5), the sums are over all integers k_j satisfying the indicated inequalities.

It will be convenient to make occasional use of the abbreviations $\mathbf{Cat}_{j=1}^m\{s_j\}$ for the concatenated argument sequence $s_1,\ldots,s_m$ and $\{s\}^m=\mathbf{Cat}_{j=1}^m\{s\}$ for $m\geqslant 0$ consecutive copies of the argument s. We can now state our main result.

Theorem 1. *Let n, r and $a_1, b_1, \ldots, a_r, b_r$ be positive integers. Then*

$$Z_n\left[\mathop{\mathbf{Cat}}_{j=1}^{r-1}\{\{1\}^{a_j-1}, b_j+1\}, \{1\}^{a_r-1}, b_r\right] = A_n\left[a_1, \{1\}^{b_1-1}, \mathop{\mathbf{Cat}}_{j=2}^{r}\{a_j+1, \{1\}^{b_j-1}\}\right].$$

Example 1. Putting $r=2$, $a_1=3$, $b_1=2$, $a_2=1$, $b_2=1$ in Theorem 1 gives the identity $Z_n[1,1,3,1]=A_n[3,1,2]$, or equivalently,

$$\sum_{n\geqslant j\geqslant k\geqslant m\geqslant p\geqslant 1}\frac{q^{j+k+m+p}}{[j]_q[k]_q[m]_q^3[p]_q} = \sum_{n\geqslant k\geqslant m\geqslant p\geqslant 1}(-1)^{k+1}q^{k(k+1)/2} \times \begin{bmatrix} n \\ k\end{bmatrix}\frac{q^{2k+p}}{[k]_q^3[m]_q[p]_q^2}.$$

Example 2. Putting $r=2$, $a_1=1$, $b_1=1$, $a_2=1$, $b_2=2$ in Theorem 1 gives the identity $Z_n[2,2]=A_n[1,2,1]$, or equivalently,

$$\sum_{n\geqslant k\geqslant m\geqslant 1}\frac{q^{k+m}}{[k]_q^2[m]_q^2} = \sum_{n\geqslant k\geqslant m\geqslant p\geqslant 1}(-1)^{k+1}q^{k(k+1)/2}\begin{bmatrix} n \\ k\end{bmatrix}\frac{q^m}{[k]_q[m]_q^2[p]_q}.$$

Theorem 1 has a concise reformulation in terms of involutions on sequences, or equivalently, dual words in the non-commutative polynomial algebra $\mathbf{Q}\langle x, y\rangle$: see Section 4. Additional consequences of Theorem 1 will be explored in the next section.

3. Special cases

For real x and y, we depart from convention and borrow the suggestive notation of [23] for the q-analog of $(x+y)^n$:

$$(x+y)_q^n := \prod_{k=0}^{n-1}(x+yq^k), \quad 0\leqslant n \in \mathbf{Z}.$$

It is easily seen that the q-binomial coefficient (1.3) has the alternative representation

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q)_q^n}{(1-q)_q^k(1-q)_q^{n-k}}, \quad 0\leqslant k\leqslant n,$$

from which follow the well-known limiting results

$$\lim_{n\to\infty}\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{(1-q)_q^k}, \quad \lim_{q\to 1}\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Notation for limiting cases of the sums in Definition 1 are as follows.

Definition 2.

$$Z[s_1,\ldots,s_m] := \lim_{n\to\infty} Z_n\left[\operatorname*{\mathbf{Cat}}_{j=1}^{m} s_j\right] = \sum_{k_1\geqslant\cdots\geqslant k_m\geqslant 1}\prod_{j=1}^{m} q^{k_j}[k_j]_q^{-s_j},$$

$$A[s_1,\ldots,s_m] := \lim_{n\to\infty} A_n\left[\operatorname*{\mathbf{Cat}}_{j=1}^{m} s_j\right]$$
$$= \sum_{k_1\geqslant\cdots\geqslant k_m\geqslant 1}\frac{(-1)^{k_1+1}q^{k_1(k_1+1)/2}}{(1-q)_q^{k_1}}\prod_{j=1}^{m}\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}.$$

Definition 3.

$$Z_n(s_1,\ldots,s_m) := \lim_{q\to 1} Z_n\left[\operatorname*{\mathbf{Cat}}_{j=1}^{m} s_j\right] = \sum_{n\geqslant k_1\geqslant\cdots\geqslant k_m\geqslant 1}\prod_{j=1}^{m} k_j^{-s_j},$$

$$A_n(s_1,\ldots,s_m) := \lim_{q\to 1} A_n\left[\operatorname*{\mathbf{Cat}}_{j=1}^{m} s_j\right] = \sum_{n\geqslant k_1\geqslant\cdots\geqslant k_m\geqslant 1}(-1)^{k_1+1}\binom{n}{k_1}\prod_{j=1}^{m} k_j^{-s_j}.$$

With this notation, the following consequences of Theorem 1 are immediate.

Corollary 1. *Let n, r and $a_1, b_1, \ldots, a_r, b_r$ be positive integers. Then*

$$Z_n\left(\operatorname*{\mathbf{Cat}}_{j=1}^{r-1}\{\{1\}^{a_j-1}, b_j+1\}, \{1\}^{a_r-1}, b_r\right)$$
$$= A_n\left(a_1, \{1\}^{b_1-1}, \operatorname*{\mathbf{Cat}}_{j=2}^{r}\{a_j+1, \{1\}^{b_j-1}\}\right),$$

$$Z\left[\mathop{\mathbf{Cat}}_{j=1}^{r-1}\{\{1\}^{a_j-1}, b_j+1\}, \{1\}^{a_r-1}, b_r\right] = A\left[a_1, \{1\}^{b_1-1}, \mathop{\mathbf{Cat}}_{j=2}^{r}\{a_j+1, \{1\}^{b_j-1}\}\right].$$

Corollary 2. *Let n, a and b be positive integers. Then*

$$Z_n[\{1\}^{a-1}, b] = A_n[a, \{1\}^{b-1}]. \tag{3.1}$$

Proof. Put $r=1$ in Theorem 1. $\square$

Remark 1. If we put $b=1$ and $a=m$ in (3.1), we find that

$$Z_n[\{1\}^m] = A_n[m], \tag{3.2}$$

which is (1.4). As we shall see, $A_n[0]=1$ is an easy consequence of the q-binomial theorem. Since we have defined $Z_n[\ ]=1$, it follows that (3.2) also holds when $m=0$. Concerning the equivalent equation (1.4), this point was also noted by Dilcher [16].

Remark 2. If we put $a=1$ and $b=m$ in (3.1), there follows $Z_n[m]=A_n[\{1\}^m]$, an identity dual to (1.4) and (3.2):

$$\sum_{k=1}^{n} \frac{q^k}{[k]_q^m} = \sum_{n \geqslant k_1 \geqslant \cdots \geqslant k_m \geqslant 1} (-1)^{k_1+1} q^{k_1(k_1+1)/2} \begin{bmatrix} n \\ k_1 \end{bmatrix} \prod_{j=1}^{m} \frac{1}{[k_j]_q}, \tag{3.3}$$

with respective limiting cases

$$\sum_{k=1}^{n} \frac{1}{k^m} = \sum_{n \geqslant k_1 \geqslant \cdots \geqslant k_m \geqslant 1} (-1)^{k_1+1} \binom{n}{k_1} \prod_{j=1}^{m} \frac{1}{k_j}$$

and

$$\sum_{k=1}^{\infty} \frac{q^k}{[k]_q^m} = \sum_{k_1 \geqslant \cdots \geqslant k_m \geqslant 1} \frac{(-1)^{k_1+1} q^{k_1(k_1+1)/2}}{(1-q)_q^{k_1}} \prod_{j=1}^{m} \frac{1}{[k_j]_q}.$$

Prodinger's main result [28, Theorem 1] is easily obtained from (3.3) by replacing q with $1/q$. The fact that Prodinger obtained his result from (1.4) as a consequence of an inverse-pair equivalence suggests that additional instances of our Theorem 1 may likewise be so related. That this is indeed the case is one of the insights of the next section.

4. Duality

Here, we recast Theorem 1 in the language of *duality*, a concept first formulated for multiple harmonic series in [19], and subsequently generalized [26,12]. Following a suggestion of Hoffman [21], define an involution on the set $\mathscr{S}$ of finite sequences of positive integers as follows. Let α be the map that sends a sequence in $\mathscr{S}$ to its sequence of partial sums. The image of α thus consists of the strictly increasing finite sequences of positive integers, on which we define an involution β by

$$\beta\left(\mathop{\mathbf{Cat}}_{j=1}^{m} t_j\right) = \{k \in \mathbf{Z} : 1 \leqslant k \leqslant t_m\} \backslash \{t_j : 1 \leqslant j \leqslant m-1\}, \quad 0 < m \in \mathbf{Z},$$

arranged in increasing order. In other words, β maps a strictly increasing sequence of positive integers $t_1, \ldots, t_m$ to its set-theoretic complement in the positive integers up to t_m, and then tacks t_m onto the end of the result. Clearly, the composition of maps $\alpha^{-1}\beta\alpha$ is an involution of $\mathscr{S}$, and it is easy to see that Theorem 1 can be restated as

$$Z_n[\vec{s}\,] = A_n[\alpha^{-1}\beta\alpha\vec{s}] \quad \forall \vec{s} \in \mathscr{S}, \quad 0 < n \in \mathbf{Z}.$$

For an alternative duality reformulation, let $\mathfrak{h} = \mathbf{Q}\langle x, y\rangle$ denote the non-commutative polynomial algebra over the field of rational numbers in two indeterminates x and y. Let $\mathfrak{h}' = \mathfrak{h}y$ and fix a positive integer n. Define $\mathbf{Q}$-linear maps $\widehat{A}_n$ and $\widehat{Z}_n$ on $\mathfrak{h}'$ by

$$\widehat{A}_n\left[\prod_{j=1}^{m} x^{s_j-1}y\right] := A_n\left[\mathop{\mathbf{Cat}}_{j=1}^{m} s_j\right], \quad \widehat{Z}_n\left[\prod_{j=1}^{m} x^{s_j-1}y\right] := Z_n\left[\mathop{\mathbf{Cat}}_{j=1}^{m} s_j\right],$$

for any positive integers $s_1, s_2, \ldots, s_m$. Let J be the automorphism of $\mathfrak{h}$ that switches x and y. Define an involution of $\mathfrak{h}'$ by

$$w^* = (Jw)x^{-1}y \quad \forall w \in \mathfrak{h}'. \tag{4.1}$$

It is now a routine matter to show that Theorem 1 can be restated as

$$\widehat{A}_n[w] = \widehat{Z}_n[w^*] \quad \forall w \in \mathfrak{h}'. \tag{4.2}$$

Now Prodinger [28, Lemma 1] proved that for positive integer n, the inverse pairs

$$\sum_{k=0}^{n} \beta_k = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \alpha_k \tag{4.3}$$

and

$$\sum_{k=0}^{n} q^{-k}\alpha_k = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2-kn} \begin{bmatrix} n \\ k \end{bmatrix} \beta_k \tag{4.4}$$

are equivalent. In other words, (4.3) holds for a pair of sequences $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\beta_0, \beta_1, \ldots, \beta_n$ if and only if (4.4) does. But our Theorem 1 states that (4.3) holds for the sequences

$$\alpha_k = -\sum_{k \geqslant k_2 \geqslant \cdots \geqslant k_m \geqslant 1} \frac{q^{s_1 k}}{[k]_q^{s_1}} \prod_{j=2}^{m} \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}},$$

$$\beta_k = \sum_{k \geqslant k_2 \geqslant \cdots \geqslant k_m \geqslant 1} \frac{q^{k}}{[k]_q^{s_1}} \prod_{j=2}^{m} \frac{q^{k_j}}{[k_j]_q^{s_j}},$$

where $s_1, \ldots, s_m$ are positive integers, $1 \leqslant k \leqslant n$, and $\alpha_0 = \beta_0 = 0$. Equivalently, (4.2) holds with $w = x^{s_1-1}y \cdots x^{s_m-1}y$. Since duality is an involution on words, $w^{**} = w$, and Theorem 1 also gives the dual statement $A_n[w^*] = Z_n[w]$. But this latter statement is easily seen to be equivalent to (4.4) if we replace q by $1/q$ throughout. Thus Prodinger's result implies that if $A_n[w] = Z_n[w^*]$ is known for a particular word w, then we also know $A_n[w^*] = Z_n[w]$, and vice versa. Of course, knowing only that the two statements are equivalent does not establish their truth in any particular instances—for this we need our Theorem 1. What it shows is that the notion of duality in the case of Theorem 1 coincides with the existence of a certain class of inverse pair identities.

For additional duality results concerning multiple harmonic q-series, see [12]. It is interesting to contrast (4.1) and (4.2) with the corresponding duality statement for multiple zeta values [20,22], for which the relevant involution is the *anti*-automorphism of $x\mathfrak{h}y$ that switches x and y.

5. Proof of Theorem 1

By induction, it suffices to establish the two recurrence relations for A_n stated in Propositions 1 and 2, along with the base cases $A_n[\;] = A_n[0] = 1$ for $0 < n \in \mathbf{Z}$.

Proposition 1. *Let n, m, and $s_1, s_2, \ldots, s_m$ be positive integers. Then*

$$A_n[s_1, s_2, \ldots, s_m] = \sum_{r=1}^{n} q^r [r]_q^{-1} A_r[s_1 - 1, s_2, \ldots, s_m].$$

Proposition 2. *Let n, m, and $s_2, s_3, \ldots, s_m$ be positive integers. Then*

$$A_n[0, s_2, s_3, \ldots, s_m] = [n]_q^{-1} A_n[s_2 - 1, s_3, \ldots, s_m].$$

The base case $A_n[\;] = 1$ for $n > 0$ is true by definition. As alluded to previously, the other base case is an easy consequence of the q-binomial theorem [1,18,23]

$$(x+y)_q^n = \sum_{m=0}^{n} q^{m(m-1)/2} \begin{bmatrix} n \\ m \end{bmatrix} x^{n-m} y^m. \tag{5.1}$$

Putting $x = 1$ and $y = -1$ in (5.1), we see that if $n > 0$, then

$$A_n[0] = \sum_{m=1}^{n} (-1)^{m+1} q^{m(m-1)/2} \begin{bmatrix} n \\ m \end{bmatrix} = 1 - (1-1)_q^n = 1.$$

Thus, it remains only to prove Propositions 1 and 2.

We shall make use of the (equivalent by symmetry) q-Pascal recurrences [23]

$$\begin{bmatrix} r \\ k \end{bmatrix} = \begin{bmatrix} r-1 \\ k \end{bmatrix} + q^{r-k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}, \quad \begin{bmatrix} r \\ k \end{bmatrix} = q^k \begin{bmatrix} r-1 \\ k \end{bmatrix} + \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} \tag{5.2}$$

and the following elementary summation formula.

Lemma 1. *Let k and n be positive integers with $k \leqslant n$. Then*

$$\sum_{r=k}^{n} q^r \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} = q^k \begin{bmatrix} n \\ k \end{bmatrix}.$$

Proof. Write the first q-Pascal recurrence (5.2) in the form

$$q^{r-k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} r \\ k \end{bmatrix} - \begin{bmatrix} r-1 \\ k \end{bmatrix},$$

multiply through by q^k, and sum on r. □

It will be helpful to introduce some further notation.

Definition 4. Let n, m, and $s_1, \ldots, s_m$ be non-negative integers. Define

$$W_n[s_1, \ldots, s_m] := \sum_{n \geqslant k_1 \geqslant \cdots \geqslant k_m \geqslant 1} \prod_{j=1}^{m} q^{(s_j-1)k_j} [k_j]_q^{-s_j},$$

with the understanding that $W_0[s_1, \ldots, s_m] = 0$ and $W_n[\ ] = 1$ if $n > 0$ and $m = 0$.

Proof of Proposition 1. By Lemma 1, we have

$$\begin{aligned}
&A_n[s_1, \ldots, s_m] \\
&\quad = \sum_{k=1}^{n} (-1)^{k+1} q^{k(k-1)/2+(s_1-1)k} [k]_q^{-s_1} W_k[s_2, \ldots, s_m] q^k \begin{bmatrix} n \\ k \end{bmatrix} \\
&\quad = \sum_{k=1}^{n} (-1)^{k+1} q^{k(k-1)/2+(s_1-1)k} [k]_q^{-s_1} W_k[s_2, \ldots, s_m] \sum_{r=k}^{n} q^r \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} \\
&\quad = \sum_{r=1}^{n} q^r \sum_{k=1}^{r} (-1)^{k+1} q^{k(k-1)/2+(s_1-1)k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} [k]_q^{-s_1} W_k[s_2, \ldots, s_m] \\
&\quad = \sum_{r=1}^{n} q^r [r]_q^{-1} \sum_{k=1}^{r} (-1)^{k+1} q^{k(k+1)/2+(s_1-2)k} \begin{bmatrix} r \\ k \end{bmatrix} [k]_q^{1-s_1} W_k[s_2, \ldots, s_m] \\
&\quad = \sum_{r=1}^{n} q^r [r]_q^{-1} A_r[s_1-1, s_2, \ldots, s_m]. \qquad \square
\end{aligned}$$

Proof of Proposition 2. By the second q-Pascal recurrence (5.2), we have

$$\begin{aligned}
&A_n[0, s_2, \ldots, s_m]\\
&= \sum_{k=1}^{n} (-1)^{k+1} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} W_k[s_2, \ldots, s_m]\\
&= \sum_{k=1}^{n} (-1)^{k+1} q^{k(k-1)/2} \left\{ q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \right\} W_k[s_2, \ldots, s_m]\\
&= \sum_{k=1}^{n} (-1)^{k+1} q^{k(k+1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix} W_k + \sum_{k=1}^{n} (-1)^{k+1} q^{k(k-1)/2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} W_k\\
&= \sum_{k=1}^{n-1} (-1)^{k+1} q^{k(k+1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix} W_k + \sum_{k=0}^{n-1} (-1)^{k} q^{k(k+1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix} W_{k+1}\\
&= W_1 + \sum_{k=1}^{n-1} (-1)^{k} q^{k(k+1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix} (W_{k+1} - W_k)\\
&= \sum_{k=0}^{n-1} (-1)^{k} q^{k(k+1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{(s_2-1)(k+1)}\, [k+1]_q^{-s_2}\, W_{k+1}[s_3, \ldots, s_m]\\
&= \sum_{k=1}^{n} (-1)^{k+1} q^{k(k-1)/2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} q^{(s_2-1)k}\, [k]_q^{-s_2}\, W_k[s_3, \ldots, s_m]\\
&= [n]_q^{-1} \sum_{k=1}^{n} (-1)^{k+1} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix} q^{(s_2-2)k}\, [k]_q^{1-s_2}\, W_k[s_3, \ldots, s_m]\\
&= [n]_q^{-1} A_n[s_2 - 1, s_3, \ldots, s_m]. \qquad \square
\end{aligned}$$

The following consequence of Proposition 2 may be worth noting.

Corollary 3. *Let n be a positive integer, and let m be a non-negative integer. Then $A_n[0, \{1\}^m] = [n]_q^{-m}$, or equivalently,*

$$\sum_{n \geqslant k \geqslant k_1 \geqslant \cdots \geqslant k_m \geqslant 1} (-1)^{k+1} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \prod_{j=1}^{m} [k_j]_q^{-1} = [n]_q^{-m}.$$

6. Multiple harmonic q-series with strict inequalities

Much of the recent literature concerning multiple harmonic series has focused on sums of the form [3–6,9–11,13,19,20,22,26]

$$\zeta(s_1, \ldots, s_m) := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} k_j^{-s_j} \tag{6.1}$$

and various multiple polylogarithmic extensions [7,8]. Thus, it may be of interest to consider finite q-analogs of (6.1) akin to (2.1), but in which the inequalities are strict as opposed to weak.

Definition 5. Let n, m and $s_1, s_2, \ldots, s_m$ be non-negative integers. Define

$$Z_n^{>}[s_1, \ldots, s_m] := \sum_{n \geqslant k_1 > \cdots > k_m \geqslant 1} \prod_{j=1}^{m} q^{k_j} [k_j]_q^{-s_j},$$

$$A_n^{>}[s_1, \ldots, s_m] := \sum_{n \geqslant k_1 > \cdots > k_m \geqslant 1} (-1)^{k_1} q^{k_1(k_1+1)/2} \begin{bmatrix} n \\ k_1 \end{bmatrix} \prod_{j=1}^{m} q^{(s_j-1)k_j} [k_j]_q^{-s_j},$$

with the understanding that if $m>0$, then $Z_0^{>}[s_1, \ldots, s_m] = A_0^{>}[s_1, \ldots, s_m] = 0$ and if $m=0$, then $Z_n^{>}[\,] = A_n^{>}[\,] = 1$ for all $n \geqslant 0$.

In light of (6.1), it is clear that $\lim_{n\to\infty} \lim_{q\to 1} Z_n^{>}[\vec{s}\,] = \zeta(\vec{s}\,)$ if $\vec{s} = s_1, \ldots, s_m$ is any vector of positive integers with $s_1 > 1$. Of course, there is an obvious relationship between Z_n and $Z_n^{>}$ and between A_n and $A_n^{>}$. For example,

$$Z_n[s] = Z_n^{>}[s], \qquad Z_n[s_1, s_2] = Z_n^{>}[s_1, s_2] + Z_n^{>}[s_1 + s_2],$$

and

$$\begin{aligned} Z_n[s_1, s_2, s_3] = {} & Z_n^{>}[s_1, s_2, s_3] + Z_n^{>}[s_1 + s_2, s_3] + Z_n^{>}[s_1, s_2 + s_3] \\ & + Z_n^{>}[s_1 + s_2 + s_3]. \end{aligned}$$

More generally, $Z_n[\vec{s}\,]$ is the sum of those $Z_n^{>}[\vec{t}\,]$, where $\vec{t}$ is obtained from $\vec{s}$ by replacing any number of commas by plus signs. Despite this relationship, the presence of strict inequalities in Definition 5 does appear to complicate matters insofar as there is no simple analog of Theorem 1 relating $Z_n^{>}$ to $A_n^{>}$. Nevertheless, there are recurrences for $A_n^{>}$ analogous to the recurrences satisfied by A_n. Arguing as in Section 5, we find that

$$A_n^{>}[s_1, \ldots, s_m] = \sum_{r=1}^{n} q^r [r]_q^{-1} A_r^{>}[s_1 - 1, s_2, \ldots, s_m],$$

$$A_n^{>}[0, s_2, \ldots, s_m] = -A_{n-1}^{>}[s_2, \ldots, s_m].$$

As a consequence, one can establish (using induction as in the proof of Corollary 3) the following result.

Theorem 2. *Let m and n be non-negative integers. Then* $(-1)^m Z_n^{>}[\{1\}^m] = A_n^{>}[\{1\}^m]$.

Corollary 4. *Let m be a positive integer. Then*

$$(-1)^m \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q} = \sum_{k_1 > \cdots > k_m > 0} (-1)^{k_1} \frac{q^{k_1(k_1+1)/2}}{(1-q)_q^{k_1}} \prod_{j=1}^{m} \frac{1}{[k_j]_q}.$$

Proof. Let $n \to \infty$ in Theorem 2. $\square$

Corollary 5. *Let m and n be positive integers. Then*

$$(-1)^m \sum_{n\geqslant k_1>\cdots>k_m\geqslant 1} \prod_{j=1}^m k_j^{-1} = \sum_{n\geqslant k_1>\cdots>k_m\geqslant 1} (-1)^{k_1}\binom{n}{k_1}\prod_{j=1}^m k_j^{-1}.$$

Proof. Let $q\to 1$ in Theorem 2. □

One can also prove Theorem 2 by differentiating the q-Chu–Vandermonde summation as in the Andrews–Uchimura [2] proof of (1.2):

Alternative Proof of Theorem 2. Let x be real, $x\neq 1$. Write the q-Chu–Vandermonde sum [18,30] in the form

$$\frac{(1-xq)_q^n}{(1-q)_q^n} = 1+\sum_{k=1}^n (-1)^k q^{k(k+1)/2}\begin{bmatrix} n\\ k\end{bmatrix}\frac{(x-1)_q^k}{(1-q)_q^k}$$

and differentiate both sides $m>0$ times with respect to x, obtaining

$$\begin{aligned}(-1)^m &\frac{(1-xq)_q^n}{(1-q)_q^n}\sum_{n\geqslant k_1>\cdots>k_m\geqslant 1}\prod_{j=1}^m \frac{q^{k_j}}{1-xq^{k_j}}\\ &=\sum_{k=1}^n (-1)^k q^{k(k+1)/2}\begin{bmatrix} n\\ k\end{bmatrix}\frac{(x-1)_q^k}{(1-q)_q^k}\sum_{k>k_1>\cdots>k_m\geqslant 0}\prod_{j=1}^m\frac{1}{x-q^{k_j}}.\end{aligned} \tag{6.2}$$

Now let $x\to 1$ and note that the sum on the right-hand side of (6.2) vanishes if $k_m>0$. Thus,

$$\begin{aligned}(-1)^m &\sum_{n\geqslant k_1>\cdots>k_m\geqslant 1}\prod_{j=1}^m \frac{q^{k_j}}{1-q^{k_j}}\\ &=\sum_{k=1}^n (-1)^k \begin{bmatrix} n\\ k\end{bmatrix}\frac{q^{k(k+1)/2}}{1-q^k}\sum_{k>k_1\cdots>k_{m-1}>0}\prod_{j=1}^{m-1}\frac{1}{1-q^{k_j}}.\end{aligned}$$

Finally, multiply both sides by $(1-q)^m$ to complete the proof. □

7. Final remarks

In [12,14,15,27,33], the multiple q-zeta function

$$\zeta[\vec{s};q]=\zeta[s_1,\ldots,s_m]:=\sum_{k_1>\cdots>k_m>0}\prod_{j=1}^m\frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}$$

is the central object of study. With the abbreviation

$$|\vec{s}|:=\sum_{j=1}^m s_j,$$

we note the relationship

$$\zeta[\vec{s};q]=q^{|\vec{s}|}Z_{\infty}^{>}[\vec{s};1/q],$$

where $Z_{\infty}^{>}[\vec{s};1/q]$ denotes the limit as $n\to\infty$ of the $Z_n^{>}$-function of Definition 5 with q replaced by $1/q$. As noted by one of the referees, often more than one q-analog is possible. Zudilin [34] has considered the sums

$$\sum_{k_1>\cdots>k_m>0}\prod_{j=1}^{m}\frac{q^{k_js_j}}{(1-q^{k_j})^{s_j}}.$$

See also [29,24,25]. The reason why the particular q-analog used in this paper was chosen is that, in addition to giving an intriguing extension of previously published results [2,16,28], it seemed a fairly natural choice based on the ideas in the related works [8,12,14,15].

Fu and Lascoux [17] have generalized (1.4) in a different direction. They proved that if n and m are positive integers, then

$$\begin{aligned}\sum_{n\geqslant k_1\geqslant\cdots\geqslant k_m\geqslant 1}\prod_{j=1}^{m}\frac{a-bq^{k_j}}{c-zq^{k_j}}&=\frac{c^n(1-zq/c)_q^n}{(1-q)_q^n(az-bc)^{n-1}}\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}\\&\quad\times\frac{(-1)^{k-1}q^{k(k+1)/2-nk}(1-q^k)(a-bq^k)^{m+n-1}}{(c-zq^k)^{m+1}}.\end{aligned}\tag{7.1}$$

Letting $c=z=1$, $a=0$, $b=-1$ in (7.1) gives (1.4).

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