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Approximately vanishing of topological cohomology groups

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Abstract

In this paper, we establish the pexiderized stability of coboundaries and cocycles and use them to investigate the Hyers–Ulam stability of some functional equations. We prove that for each Banach algebra A, Banach A-bimodule X and positive integer n, $H^n(A, X) = 0$ if and only if the nth cohomology group approximately vanishes.

Keywords: Hyers–Ulam stability; Approximate cocycle; Approximate coboundary; Topological cohomology groups; Approximately vanishing; Derivation; Approximate amenability; Approximate contractibility

1. Introduction

Topological cohomology arose from the problems concerning extensions by H. Kamowitz who introduced the Banach version of Hochschild cohomology groups in 1962 [12], derivations by Kadison and Ringrose [10,11] and amenability by Johnson [9] and has been extensively developed by A.Ya. Helemskii and his school [4]. In addition, this area includes a lot of problems concerning automorphism groups of operator algebras, fixed point theorems, stability, perturbations, invariant means [4] and their applications to quantum physics [24].

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Consider the functional equation $\mathcal{E}_1(f) = \mathcal{E}_2(f)$ $(\mathcal{E})$ in a certain framework. We say a function f_0 is an approximate solution of $(\mathcal{E})$ if $\mathcal{E}_1(f_0)$ and $\mathcal{E}_2(f_0)$ are close in some sense. The stability problem is whether or not there is a true solution of $(\mathcal{E})$ near f_0.

The stability of functional equations started with the following question concerning stability of group homomorphisms proposed by S.M. Ulam [25] during a talk before a Mathematical Colloquium at the University of Wisconsin, Madison, in 1940:

> Let G_1 be a group and let (G_2, d) be a metric group. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$.

In 1941, Hyers [6] provide the first (partial) answer to Ulam's problem as follows:

> If E_1, E_2 are Banach spaces and $f : E_1 \to E_2$ is a mapping for which there is $\epsilon > 0$ such that $\|f(x+y) - f(x) - f(y)\| < \epsilon$ for all $x, y \in E_1$, then there is a unique additive mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| < \epsilon$ for all $x \in E_1$.

In 1978, Th.M. Rassias [17] established a generalization of the Hyers' result as the first theorem in the subject of stability of functional equations which allows the Cauchy difference $f(x+y) - f(x) - f(y)$ to be unbounded. This phenomenon has extensively influenced the development of what is called Hyers–Ulam–Rassias stability; cf. [18–22].

During the last decades the problem of Hyers–Ulam–Rassias stability for various functional equations has been widely investigated by many mathematicians. Four methods are used to establish the stability: the Hyers–Ulam sequences, fixed points, invariant means, and sandwich theorems. For a comprehensive account on the stability, the reader is referred to [2,3,7].

In this paper, using Hyers sequence [6] and some ideas of [15] and [17], we study the pexiderized stability of n-cocycles and n-coboundaries and investigate approximately vanishing of topological cohomology groups as well. In particular, for $n = 1$, our results can be regarded as generalizations of C.-G. Park's results on derivations [16] and multilinear mappings [15].

Throughout this paper, all spaces are assumed to be over the complex field $\mathbb{C}$.

2. Stability of cocycles and coboundaries

Throughout this section, A denotes a normed algebra and X is a Banach A-bimodule. Suppose that $f_1, f_2, f_3 : \prod_{j=1}^n A \to X$ are mappings. Fix $n \geqslant 1$ and scalars $\lambda_1, \ldots, \lambda_n$. Set

$$
\begin{aligned}
&D^n_{\lambda_1,\ldots,\lambda_n}[f_1, f_2, f_3](a_1, b_1, \ldots, a_n, b_n) \\
&\quad := \sum_{j=1}^{n} \bigl(f_1(a_1, \ldots, a_{j-1}, \lambda_j a_j + \lambda_j b_j, a_{j+1}, \ldots, a_n) \\
&\qquad - \lambda_j f_2(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)
\end{aligned}
$$

$$- \lambda_j f_3(a_1, \dots, a_{j-1}, b_j, a_{j+1}, \dots, a_n)\big),$$

and

$$\delta^0 x(a) := ax - xa,$$
$$\delta^n[f_1, f_2, f_3](a_1, a_2, \dots, a_{n+1})$$
$$:= a_1 f_1(a_2, \dots, a_{n+1}) + \sum_{j=1}^{n} (-1)^j f_2(a_1, \dots, a_{j-1}, a_j a_{j+1}, a_{j+2}, \dots, a_{n+1})$$
$$+ (-1)^{n+1} f_3(a_1, \dots, a_n) a_{n+1},$$

where $x \in X$ and $a_1, \dots, a_n, a_{n+1}, b_1, \dots, b_n \in A$.

If $f_1 = f_2 = f_3 = f$ we denote $D^n_{\lambda_1,\dots,\lambda_n}[f_1, f_2, f_3]$ and $\delta^n[f_1, f_2, f_3]$ simply by $D^n_{\lambda_1,\dots,\lambda_n} f$ and $\delta^n f$, respectively. A mapping $f : \prod_{j=1}^n A \to X$ is called multi-linear (multi-additive) if $D^n_{\lambda_1,\dots,\lambda_n} f = 0$ for all $\lambda_1, \dots, \lambda_n$ ($D^n_{1,\dots,1} f = 0$). A multi-linear mapping f is said to be n-cocycle if $\delta^n f = 0$. By an n-coboundary we mean a multi-linear mapping of the form $\delta^0(x)$ or $\delta^{n-1} g$ in which g is multi-linear.

Theorem 2.1. *Let α, β be positive numbers, $n \geqslant 1$, $f_1, f_2, f_3 : \prod_{j=1}^n A \to X$ be mappings such that*

$$\big\| D^n_{\lambda_1,\dots,\lambda_n}[f_1, f_2, f_3](a_1, b_1, \dots, a_n, b_n) \big\| \leqslant \alpha, \tag{1}$$

$$\big\| \delta^n[f_1, f_2, f_3](a_1, a_2, \dots, a_{n+1}) \big\| \leqslant \beta \tag{2}$$

for all $a_1, \dots, a_n, a_{n+1}, b_1, \dots, b_n \in A$ and all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$.

Suppose that for each $1 \leqslant k \leqslant 3$, $f_k(a_1, \dots, a_n)$ vanishes if $a_i = 0$ for any i. Then there exists a unique n-cocycle F such that

$$\big\| f_1(a_1, \dots, a_n) - F(a_1, \dots, a_n) \big\| \leqslant 3.2^n \alpha,$$
$$\big\| f_2(a_1, \dots, a_n) - F(a_1, \dots, a_n) \big\| \leqslant 3\left(1 + \frac{1}{n}\right).2^n \alpha,$$
$$\big\| f_3(a_1, \dots, a_n) - F(a_1, \dots, a_n) \big\| \leqslant 6.2^n \alpha.$$

Furthermore, if f_1 is continuous at a point $(e_1, \dots, e_n)$ of $\prod_{j=1}^n A$, then F is continuous on whole $\prod_{j=1}^n A$.

Proof. We shall establish the theorem in three steps:

Step (I). *Existence of the multi-linear mapping F.* Let $1 \leqslant i \leqslant n$ be fixed. Putting $\lambda_1 = \dots = \lambda_n = 1$, $b_1 = \dots = b_n = 0$ in (1), we get

$$\left\| \sum_{j=1}^n f_1(a_1, \dots, a_j, \dots, a_n) - f_2(a_1, \dots, a_j, \dots, a_n) \right\| \leqslant \alpha,$$

whence

$$\left\| f_1(a_1,\dots,a_n) - f_2(a_1,\dots,a_n)\right\| \leqslant \frac{\alpha}{n} \tag{3}$$

for all $a_1,\dots,a_n \in A$. Putting $a_j = (1-\delta_{ij})b_j$ in (1), we obtain

$$\left\| f_1(b_1,\dots,b_n) - f_3(b_1,\dots,b_n)\right\| \leqslant \alpha \tag{4}$$

for all $b_1,\dots,b_n \in A$. Putting $\lambda_1 = \cdots = \lambda_n = 1,\ b_j = \delta_{ij}a_i$ in (1), we get

$$\begin{aligned}&\Bigg\| \sum_{j\in\{1,\dots,n\}-\{i\}} \big(f_1(a_1,\dots,a_j,\dots,a_n) - f_2(a_1,\dots,a_j,\dots,a_n)\big)\\ &\quad + f_1(a_1,\dots,a_{i-1},2a_i,a_{i+1},\dots,a_n) - f_2(a_1,\dots,a_n) - f_3(a_1,\dots,a_n)\Bigg\| \leqslant \alpha\end{aligned}$$

so that

$$\begin{aligned}&\left\| f_1(a_1,\dots,a_{i-1},2a_i,a_{i+1},\dots,a_n) - 2f_1(a_1,\dots,a_n)\right\|\\ &\quad \leqslant \Bigg\| \sum_{j\in\{1,\dots,n\}-\{i\}} f_1(a_1,\dots,a_j,\dots,a_n) - f_2(a_1,\dots,a_j,\dots,a_n)\\ &\qquad + f_1(a_1,\dots,a_{i-1},2a_i,a_{i+1},\dots,a_n) - f_2(a_1,\dots,a_n) - f_3(a_1,\dots,a_n)\Bigg\|\\ &\qquad + \Bigg\| \sum_{j=1}^{n} f_2(a_1,\dots,a_j,\dots,a_n) - f_1(a_1,\dots,a_j,\dots,a_n)\Bigg\|\\ &\qquad + \left\| f_3(a_1,\dots,a_n) - f_1(a_1,\dots,a_n)\right\| \leqslant \alpha+\alpha+\alpha = 3\alpha.\end{aligned}$$

Hence

$$\left\| f_1(a_1,\dots,a_{i-1},2a_i,a_{i+1},\dots,a_n) - 2f_1(a_1,\dots,a_n)\right\| \leqslant 3\alpha. \tag{5}$$

Replacing $a_1,\dots,a_{i-1}$ by $2a_1,\dots,2a_{i-1}$, respectively, in (5), we get

$$\begin{aligned}&\Bigg\| \frac{1}{2^{i-1}} f_1(2a_1,\dots,2a_{i-1},a_i,a_{i+1},\dots,a_n)\\ &\quad - \frac{1}{2^i} f_1(2a_1,\dots,2a_{i-1},2a_i,a_{i+1},\dots,a_n)\Bigg\| \leqslant \frac{3}{2^i}\alpha\end{aligned}$$

so that

$$\begin{aligned}&\left\| f_1(a_1,\dots,a_n) - \frac{1}{2^n} f_1(2a_1,\dots,2a_n)\right\|\\ &\quad \leqslant \sum_{i=1}^{n} \Bigg\| \frac{1}{2^{i-1}} f_1(2a_1,\dots,2a_{i-1},a_i,a_{i+1},\dots,a_n)\\ &\qquad - \frac{1}{2^i} f_1(2a_1,\dots,2a_{i-1},2a_i,a_{i+1},\dots,a_n)\Bigg\|\\ &\quad \leqslant \frac{2^n-1}{2} 3\alpha.\end{aligned} \tag{6}$$

Replacing $a_1, \dots, a_n$ by $2^j a_1, \dots, 2^j a_n$ in (6), we get

$$\left\| f_1\big(2^j a_1, \dots, 2^j a_n\big) - \frac{1}{2^n} f_1\big(2^{j+1} a_1, \dots, 2^{j+1} a_n\big)\right\| \leqslant \frac{2^n - 1}{2} 3\alpha,$$

whence

$$\begin{aligned}
&\left\| f_1(a_1, \dots, a_n) - \frac{1}{2^{mn}} f_1\big(2^m a_1, \dots, 2^m a_n\big)\right\| \\
&\quad \leqslant \sum_{j=0}^{m-1} \left\| \frac{1}{2^{nj}} f_1\big(2^j a_1, \dots, 2^j a_n\big) - \frac{1}{2^{n+nj}} f_1\big(2^{j+1} a_1, \dots, 2^{j+1} a_n\big)\right\| \\
&\quad \leqslant \frac{2^n - 1}{2} 3\alpha \sum_{j=0}^{m-1} \frac{1}{2^{nj}}.
\end{aligned}$$

Hence

$$\left\| f_1(a_1, \dots, a_n) - \frac{1}{2^{mn}} f_1\big(2^m a_1, \dots, 2^m a_n\big)\right\| \leqslant 3\left(1 - \frac{1}{2^{mn}}\right) 2^n \alpha \tag{7}$$

for all m and all $a_1, \dots, a_n \in A$. Furthermore,

$$\begin{aligned}
&\left\| \frac{1}{2^{m_1 n}} f_1\big(2^{m_1} a_1, \dots, 2^{m_1} a_n\big) - \frac{1}{2^{m_2 n}} f_1\big(2^{m_2} a_1, \dots, 2^{m_2} a_n\big)\right\| \\
&\quad \leqslant \frac{2^n - 1}{2} 3\alpha \sum_{j=m_1}^{m_2-1} \left(\frac{1}{2^n}\right)^j
\end{aligned} \tag{8}$$

for all $m_2 > m_1$.

Inequality (8) shows that the sequence $\{\frac{1}{2^{mn}} f_1(2^m a_1, \dots, 2^m a_n)\}_{m\in\mathbb{N}}$ is Cauchy in the Banach module X and so is convergent. Set

$$F(a_1, \dots, a_n) := \lim_{m\to\infty} \frac{1}{2^{mn}} f_1\big(2^m a_1, \dots, 2^m a_n\big). \tag{9}$$

Inequality (7) yields

$$\big\| f_1(a_1, \dots, a_n) - F(a_1, \dots, a_n)\big\| \leqslant 3.2^n \alpha.$$

By (3),

$$\big\| 2^{-mn} f_1\big(2^m a_1, \dots, 2^m a_n\big) - 2^{-mn} f_2\big(2^m a_1, \dots, 2^m a_n\big)\big\| \leqslant \frac{\alpha}{2^{mn} n}.$$

Using (9), we have

$$F(a_1, \dots, a_n) = \lim_{m\to\infty} \frac{1}{2^{mn}} f_2\big(2^m a_1, \dots, 2^m a_n\big). \tag{10}$$

By (5) and (7), we get

$$\begin{aligned}
&\big\| f_2(a_1, \ldots, a_{i-1}, 2a_i, a_{i+1}, \ldots, a_n) - 2f_2(a_1, \ldots, a_n)\big\| \\
&\quad \leqslant \big\| f_2(a_1, \ldots, a_{i-1}, 2a_i, a_{i+1}, \ldots, a_n) - f_1(a_1, \ldots, a_{i-1}, 2a_i, a_{i+1}, \ldots, a_n)\big\| \\
&\qquad + \big\| f_1(a_1, \ldots, a_{i-1}, 2a_i, a_{i+1}, \ldots, a_n) - 2f_1(a_1, \ldots, a_n)\big\| \\
&\qquad + 2\big\| f_1(a_1, \ldots, a_n) - f_2(a_1, \ldots, a_n)\big\| \\
&\quad \leqslant \frac{\alpha}{n} + 3\alpha + 2\frac{\alpha}{n} = 3\left(1 + \frac{1}{n}\right)\alpha
\end{aligned}$$

so that

$$\big\| f_2(a_1, \ldots, a_{i-1}, 2a_i, a_{i+1}, \ldots, a_n) - 2f_2(a_1, \ldots, a_n)\big\| \leqslant 3\left(1 + \frac{1}{n}\right)\alpha. \tag{11}$$

As the same way as we obtained inequality (7), one can deduce from (11) that

$$\left\| f_2(a_1, \ldots, a_n) - \frac{1}{2^{mn}} f_2\big(2^m a_1, \ldots, 2^m a_n\big)\right\| \leqslant 3\left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{2^{mn}}\right)2^n\alpha.$$

Letting m tend to ∞, we obtain

$$\big\| f_2(a_1, \ldots, a_n) - F(a_1, \ldots, a_n)\big\| \leqslant 3\left(1 + \frac{1}{n}\right).2^n\alpha.$$

Similarly, by applying (4), we obtain

$$F(a_1, \ldots, a_n) = \lim_{m\to\infty} \frac{1}{2^{mn}} f_3\big(2^m a_1, \ldots, 2^m a_n\big) \tag{12}$$

and

$$\big\| f_3(a_1, \ldots, a_n) - F(a_1, \ldots, a_n)\big\| \leqslant 6.2^n\alpha.$$

Replacing $a_1, \ldots, a_n, b_1, \ldots, b_n, \lambda_1, \ldots, \lambda_i, \ldots, \lambda_n$ by $2^m a_1, \ldots, 2^m a_n, 0, \ldots, 0, 1, \ldots, \lambda, \ldots, 1$, respectively, in (1), we get

$$\begin{aligned}
&\bigg\| \frac{1}{2^{mn}} D^n_{1,\ldots,\lambda_i,\ldots,1}[f_1, f_2, f_3]\big(2^m a_1, 0, \ldots, 2^m a_{i-1}, 0, 2^m a_i, 2^m b_i, 2^m a_{i+1}, 0, \\
&\quad \ldots, 2^m a_n, 0\big)\bigg\| \leqslant \frac{\alpha}{2^{mn}}.
\end{aligned}$$

Passing to the limit as $m \to \infty$, we conclude that

$$\begin{aligned}
&F(a_1, \ldots, a_{i-1}, \lambda_i a_i + \lambda_i b_i, a_{i+1}, \ldots, a_n) \\
&\quad = \lambda_i F(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) + \lambda_i F(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n).
\end{aligned}$$

Therefore F is linear in the ith variable for each $i = 1, \ldots, n$.

If $F' : \prod_{j=1}^n A \to X$ is a multi-linear mapping with $\| f(a_1, \ldots, a_n) - F'(a_1, \ldots, a_n)\| \leqslant 3.2^n\alpha$ for all $a_1, \ldots, a_n \in A$, then

$$\begin{aligned}
&\big\| F(a_1, \ldots, a_n) - F'(a_1, \ldots, a_n)\big\| \\
&\quad = \lim_{m\to\infty} 2^{-mn}\big\| f\big(2^m a_1, \ldots, 2^m a_n\big) - F'\big(2^m a_1, \ldots, 2^m a_n\big)\big\| \leqslant \lim_{m\to\infty} \frac{3.2^n\alpha}{2^{mn}} = 0,
\end{aligned}$$

whence $F = F'$.

Step (II). *Proving F to be cocycle.* For each fixed $1 \leqslant i \leqslant n$ and $a_1, \dots, a_n \in A$ one can apply (11) and induction on m to prove

$$\begin{aligned}&\bigl\|2^{-m} f_2\bigl(a_1, \dots, a_{i-1}, 2^m a_i, a_{i+1}, \dots, a_n\bigr) - f_2(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n)\bigr\| \\ &\quad \leqslant 3\bigl(1 - 2^{-m}\bigr)\left(1 + \frac{1}{n}\right)\alpha. \qquad (13)\end{aligned}$$

Now we can replace a_i by $2^m a_i$ in (13) to get

$$\begin{aligned}&\bigl\|2^{-(n+1)m} f_2\bigl(2^m a_1, \dots, 2^m a_{i-1}, 2^{2m} a_i, 2^m a_{i+1}, \dots, 2^m a_n\bigr) \\ &\qquad - 2^{-mn} f_2\bigl(2^m a_1, \dots, 2^m a_{i-1}, 2^m a_i, 2^m a_{i+1}, \dots, 2^m a_n\bigr)\bigr\| \\ &\quad \leqslant 3\left(\frac{1}{2^{mn}} - \frac{1}{2^{m(n+1)}}\right)\alpha. \qquad (14)\end{aligned}$$

Then (10) and (14) yield

$$F(a_1, \dots, a_n) = \lim_{m\to\infty} \frac{1}{2^{m(n+1)}} f_2\bigl(2^m a_1, \dots, 2^m a_{i-1}, 2^{2m} a_i, 2^m a_{i+1}, \dots, 2^m a_n\bigr). \qquad (15)$$

By (2), we have

$$\begin{aligned}&\bigl\|2^{-(n+1)m}\delta^n[f_1, f_2, f_3](a_1, \dots, a_{n+1})\bigr\| \\ &= \Bigg\| 2^{-mn} a_1 f_1\bigl(2^m a_2, \dots, 2^m a_{n+1}\bigr) \\ &\quad + 2^{-(n+1)m} \sum_{j=1}^{n} (-1)^j f_2\bigl(2^m a_1, \dots, 2^m a_{j-1}, 2^{2m} a_j a_{j+1}, a_{j+2}, \dots, 2^m a_{n+1}\bigr) \\ &\quad + (-1)^{n+1} 2^{-mn} f_3\bigl(2^m a_1, \dots, 2^m a_n\bigr) a_{n+1} \Bigg\| \leqslant 2^{-(n+1)m}\beta\end{aligned}$$

for all m and all $a_1, \dots, a_{n+1} \in A$.

Next by passing to the limit as $m \to \infty$ and noting to (9), (12) and (15), we get

$$\begin{aligned}\delta^n F(a_1, \dots, a_{n+1}) &= a_1 F(a_2, \dots, a_{n+1}) \\ &\quad + \sum_{j=1}^{n} (-1)^j F(a_1, \dots, a_{j-1}, a_j a_{j+1}, a_{j+2}, \dots, a_{n+1}) \\ &\quad + (-1)^{n+1} F(a_1, \dots, a_n) a_{n+1} = 0\end{aligned}$$

for all $a_1, \dots, a_{n+1} \in A$. Hence F is a cocycle.

Step (III). *Continuity of F.* We use the strategy of Hyers [6]. If F were not continuous at the point $(e_1, \dots, e_n)$, then there would be an integer P and a sequence $\{(a_1^m, \dots, a_n^m)\}$ of $\prod_{j=1}^n A$ converging to zero such that $\|F(a_1^m, \dots, a_n^m)\| > \frac{1}{P}$. Let K be an integer greater than $7P2^n\alpha$. Since $\lim_{m\to\infty} f_1(K(a_1^m, \dots, a_n^m) + (e_1, \dots, e_n)) = f_1(e_1, \dots, e_n)$, there is an integer N such that $\|f_1(K(a_1^m, \dots, a_n^m) + (e_1, \dots, e_n)) - f_1(e_1, \dots, e_n)\| < 2^n\alpha$ for all $n \geqslant N$. Hence

$$\begin{aligned} 7.2^n\alpha < \frac{K}{P} &< \big\|F\big(K\big(a_1^m, \dots, a_n^m\big)\big)\big\| \\ &= \big\|F\big(K\big(a_1^m, \dots, a_n^m\big) + (e_1, \dots, e_n)\big) - F(e_1, \dots, e_n)\big\| \\ &\leqslant \big\|F\big(K\big(a_1^m, \dots, a_n^m\big) + (e_1, \dots, e_n)\big) - f_1\big(K\big(a_1^m, \dots, a_n^m\big) - (e_1, \dots, e_n)\big)\big\| \\ &\quad + \big\|f_1\big(K\big(a_1^m, \dots, a_n^m\big) - (e_1, \dots, e_n)\big) - f_1(e_1, \dots, e_n)\big\| \\ &\quad + \big\|f_1(e_1, \dots, e_n) - F(e_1, \dots, e_n)\big\| \leqslant 3.2^n\alpha + 2^n\alpha + 3.2^n\alpha = 7.2^n\alpha \end{aligned}$$

for all $n > N$, a contradiction. Now the multi-linearity of F guarantees continuity of F on whole $\prod_{j=1}^n A$. $\square$

Theorem 2.2. *Let α, β, γ be positive numbers, $x \in X$ and $f_1, f_2, f_3 : A \to X$ be mappings such that*

$$\begin{aligned} &\big\|D_\lambda^1[f_1, f_2, f_3](a, b)\big\| \leqslant \alpha, \\ &\big\|\delta^1[f_1, f_2, f_3](a, b)\big\| \leqslant \beta, \\ &\big\|ax - xa - f(a)\big\| \leqslant \gamma \end{aligned} \tag{16}$$

for all $a, b \in A$ and all $\lambda \in \mathbb{C}$.

Suppose that for each $1 \leqslant k \leqslant 3$, $f_k(0) = 0$. Then there exists a unique 1-cocycle F such that

$$\begin{aligned} &\big\|f_1(a) - F(a)\big\| \leqslant 6\alpha, \\ &\big\|f_2(a) - F(a)\big\| \leqslant 12\alpha, \\ &\big\|f_3(a) - F(a)\big\| \leqslant 12\alpha, \\ &F(a) = ax - xa \end{aligned}$$

for all $a \in A$.

Proof. By Theorem 2.1, there is a unique 1-cocycle F defined by $F(a) := \lim_{m\to\infty} 2^{-m} \times f_1(2^m a)$ satisfying the required inequalities. It follows from (16) that $\|ax - xa - 2^{-m} f_1(2^m a)\| \leqslant 2^{-m}\gamma$. Passing to the limit we conclude that $F(a) = ax - xa$. $\square$

Remark 2.3. Theorem 2.2 gives us the Hyers–Ulam stability of any one of the following function equations:

(i) $f(ab) = af(b) + f(a)b$; cf. [16];
(ii) $af(b) = f(a)b$;
(iii) $f(ab) = af(b)$;
(iv) $f(ab) = f(a)b$,

together with the Cauchy equation $f(a + b) = f(a) + f(b)$.

To see this, put in Theorem 2.2 $f_1 = f_2 = f_3 = f$ to get (i); $f_1 = f_3 = f$, $f_2 = 0$ to obtain (ii); $f_1 = f_2 = f$, $f_3 = 0$ to get (iii); and $f_1 = 0$, $f_2 = f_3 = f$ to obtain (iv).

Proposition 2.4. *Let A be linearly spanned by a set $S \subseteq A$, α, β be positive numbers, $n \geqslant 1$, $f_1, f_2, f_3 : \prod_{j=1}^{n} A \to X$ be mappings such that*

$$\left\| D^n_{\lambda_1,\dots,\lambda_n}[f_1, f_2, f_3](a_1, b_1, \dots, a_n, b_n) \right\| \leqslant \alpha$$

for all $a_1, \dots, a_n, b_1, \dots, b_n \in A$ and all $\lambda_1, \dots, \lambda_n \in \mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$; and

$$\left\| \delta^n [f_1, f_2, f_3](a_1, a_2, \dots, a_{n+1}) \right\| \leqslant \beta$$

for all $a_1, \dots, a_{n+1} \in S$.

Suppose that for each $1 \leqslant k \leqslant 3$, $f_k(a_1, \dots, a_n)$ vanishes if $a_i = 0$ for any i. Then there exists a unique n-cocycle F such that

$$\left\| f_1(a_1, \dots, a_n) - F(a_1, \dots, a_n) \right\| \leqslant 3.2^n \alpha,$$

$$\left\| f_2(a_1, \dots, a_n) - F(a_1, \dots, a_n) \right\| \leqslant 3\left(1 + \frac{1}{n}\right).2^n \alpha,$$

$$\left\| f_3(a_1, \dots, a_n) - F(a_1, \dots, a_n) \right\| \leqslant 6.2^n \alpha.$$

Proof. By the same argument as in the proof of Theorem 2.1 there exists a multi-additive mapping F satisfying the required inequalities such that $\delta^n F(a_1, a_2, \dots, a_{n+1}) = 0$ holds for all $a_1, \dots, a_{n+1} \in S$.

Fix $1 \leqslant i \leqslant n$. Assume that $\lambda_i \in \mathbb{C}$ and $\lambda \neq 0$. If N is a positive integer greater than $4\|a_i\|$ then $\|\frac{a_i}{N}\| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [13, Theorem 1] there are three numbers $z_1, z_2, z_3 \in \mathbb{T}$ such that $3\frac{a_i}{N} = z_1 + z_2 + z_3$. By virtue of the multi-additivity of F we easily conclude that F is multi-linear. Since each element of A is a linear combination of elements of S, we infer that F is a cocycle. □

Proposition 2.5. *Let A be linearly spanned by a set $S \subseteq A$, α, β be positive numbers, $n \geqslant 1$, $f_1, f_2, f_3 : \prod_{j=1}^{n} A \to X$ be mappings such that*

$$\left\| D^n_{\lambda_1,\dots,\lambda_n}[f_1, f_2, f_3](a_1, b_1, \dots, a_n, b_n) \right\| \leqslant \alpha$$

for all $a_1, \dots, a_n, b_1, \dots, b_n \in A$ and all $\lambda_1, \dots, \lambda_n \in \{1, i\}$ and

$$\left\| \delta^n [f_1, f_2, f_3](a_1, a_2, \dots, a_{n+1}) \right\| \leqslant \beta$$

for all $a_1, \dots, a_{n+1} \in S$.

Suppose that for each $1 \leqslant k \leqslant 3$, $f_k(a_1, \dots, a_n)$ vanishes if $a_i = 0$ for any i. Assume that for each $1 \leqslant i \leqslant n$ and each fixed $(a_1, \dots, a_n)$ the function $t \mapsto f(a_1, \dots, a_{i-1}, ta_i, a_{i+1}, \dots, a_n)$ is continuous on $\mathbb{R}$. Then there exists a unique n-cocycle F such that

$$\left\| f_1(a_1, \dots, a_n) - F(a_1, \dots, a_n) \right\| \leqslant 3.2^n \alpha,$$

$$\left\| f_2(a_1, \dots, a_n) - F(a_1, \dots, a_n) \right\| \leqslant 3\left(1 + \frac{1}{n}\right).2^n \alpha,$$

$$\left\| f_3(a_1, \dots, a_n) - F(a_1, \dots, a_n) \right\| \leqslant 6.2^n \alpha.$$

Proof. By the same argument as in the proof of Theorem 2.1 there exists a unique multi-additive mapping F satisfying the required inequalities such that $\delta^n F(a_1, a_2, \ldots, a_{n+1}) = 0$ holds for all $a_1, \ldots, a_{n+1} \in S$.

Fix $1 \leqslant i \leqslant n$. By the same reasoning as in the proof of theorem of [17], the mapping F is multi-$\mathbb{R}$-linear. Since $\mathbb{C}$ as a vector space over $\mathbb{R}$ is generated by $\{1, i\}$, we conclude that F is multi-$\mathbb{C}$-linear. Since each element of A is a linear combination of elements of S, we infer that F is a cocycle. □

Theorem 2.6. *Let $\alpha, \beta, \gamma, \delta$ be positive numbers, $n \geqslant 2$, $f_1, f_2, f_3 : \prod_{j=1}^{n} A \to X$ and $g_1, g_2, g_3 : \prod_{j=1}^{n-1} A \to X$ be mappings such that*

$$\begin{aligned}
&\left\| D^n_{\lambda_1,\ldots,\lambda_n}[f_1, f_2, f_3](a_1, b_1, \ldots, a_n, b_n) \right\| \leqslant \alpha, \\
&\left\| \delta^n[f_1, f_2, f_3](a_1, a_2, \ldots, a_{n+1}) \right\| \leqslant \beta, \\
&\left\| D^{n-1}_{\lambda_1,\ldots,\lambda_{n-1}}[g_1, g_2, g_3](a_1, b_1, \ldots, a_{n-1}, b_{n-1}) \right\| \leqslant \gamma, \\
&\left\| \delta^{n-1}[g_1, g_2, g_3](a_1, a_2, \ldots, a_n) - f_1(a_1, \ldots, a_{n+1}) \right\| \leqslant \delta
\end{aligned} \tag{17}$$

for all $a_1, \ldots, a_{n-1}, a_n, a_{n+1}, b_1, \ldots, b_{n-1}, b_n \in A$ and all $\lambda_1, \ldots, \lambda_{n-1}, \lambda_n \in \mathbb{C}$.

Suppose that for each $1 \leqslant k \leqslant 3$, $f_k(a_1, \ldots, a_n)$ and $g_k(a_1, \ldots, a_{n-1})$ vanish if $a_i = 0$ for any i and g_1 is continuous at a point of $\prod_{j=1}^{n-1} A$. Then there exist a unique n-cocycle F and a unique continuous multi-linear mapping $G : \prod_{j=1}^{n-1} A \to X$ such that

$$\begin{aligned}
&\left\| f_1(a_1, \ldots, a_n) - F(a_1, \ldots, a_n) \right\| \leqslant 3.2^n \alpha, \\
&\left\| f_2(a_1, \ldots, a_n) - F(a_1, \ldots, a_n) \right\| \leqslant 3\left(1 + \frac{1}{n}\right).2^n \alpha, \\
&\left\| f_3(a_1, \ldots, a_n) - F(a_1, \ldots, a_n) \right\| \leqslant 6.2^n \alpha, \\
&\left\| g_1(a_1, \ldots, a_{n-1}) - G(a_1, \ldots, a_{n-1}) \right\| \leqslant 3.2^n \gamma, \\
&\left\| g_2(a_1, \ldots, a_{n-1}) - G(a_1, \ldots, a_{n-1}) \right\| \leqslant 3\left(1 + \frac{1}{n}\right).2^n \gamma, \\
&\left\| g_3(a_1, \ldots, a_{n-1}) - G(a_1, \ldots, a_{n-1}) \right\| \leqslant 6.2^n \gamma
\end{aligned}$$

and

$$F = \delta^n G.$$

Proof. Theorem 2.1 gives rise the existence of a unique n-cocycle F with requested properties. Using the same reasoning as in the proof of Theorem 2.1 one can show that there exists a unique continuous multi-linear mapping G defined by

$$G(a_1, \ldots, a_{n-1}) := \lim_{m \to \infty} \frac{1}{2^{m(n-1)}} g_1\left(2^m a_1, \ldots, 2^m a_{n-1}\right) \tag{18}$$

satisfying the required inequalities and

$$G(a_1,\dots,a_{n-1}) := \lim_{m\to\infty}\frac{1}{2^{m(n-1)}}g_3\big(2^m a_1,\dots,2^m a_{n-1}\big), \tag{19}$$

$$\begin{aligned} &G(a_1,\dots,a_{n-1})\\ &\quad=\lim_{m\to\infty}\frac{1}{2^{mn}}g_2\big(2^m a_1,\dots,2^m a_{i-1},2^{2m}a_i,2^m a_{i+1},\dots,2^m a_{n-1}\big). \end{aligned}\tag{20}$$

Clearly,

$$\delta^{n-1}G(a_1,\dots,a_n)=\lim_{m\to\infty}2^{-mn}\delta^{n-1}[g_1,g_2,g_3]\big(2^m a_1,\dots,2^m a_n\big).$$

Inequality (17) yields

$$\begin{aligned} &\big\|2^{-mn}\delta^{n-1}[g_1,g_2,g_3]\big(2^m a_1,\dots,2^m a_n\big)-2^{-mn}f_1\big(2^m a_1,\dots,2^m a_n\big)\big\|\\ &\quad=\Bigg\|2^{-m(n-1)}a_1g_1\big(2^m a_2,\dots,2^m a_{n+1}\big)\\ &\qquad+2^{-nm}\sum_{j=1}^{n-1}(-1)^j g_2\big(2^m a_1,\dots,2^m a_{j-1},2^{2m}a_ja_{j+1},a_{j+2},\dots,2^m a_n\big)\\ &\qquad+(-1)^n2^{-m(n-1)}g_3\big(2^m a_1,\dots,2^m a_{n-1}\big)a_n-2^{-mn}f_1\big(2^m a_1,\dots,2^m a_n\big)\Bigg\|\\ &\quad\leqslant 2^{-mn}\delta. \end{aligned}$$

Letting $m\to\infty$ and using (9), (18)–(20), we conclude that

$$\begin{aligned} &\Bigg\|a_1G(a_2,\dots,a_n)+\sum_{j=1}^{n-1}(-1)^jG(a_1,\dots,a_ja_{j+1},a_{j+2},\dots,a_n)\\ &\quad+(-1)^nG(a_1,\dots,a_{n-1})a_n-F(a_1,\dots,a_n)\Bigg\|=0. \end{aligned}$$

Thus $\delta^{n-1}(G)=F$. □

Remark 2.7. There are statements similar to Propositions 2.4, 2.5 for coboundaries.

3. Vanishing of cohomology groups

Throughout this section, A denotes a Banach algebra and X is a Banach A-bimodule. For $n=0,1,2,\dots,$ let $C^n(A,X)$ be the Banach space of all bounded n-linear mappings from $A\times\cdots\times A$ into X equipped with multi-linear operator norm $\|f\|=\sup\{\|f(a_1,\dots,a_n)\|\colon\ a_i\in A,\ \|a_i\|\leqslant 1,\ 1\leqslant i\leqslant n\}$, and $C^0(A,X)=X$. The elements of $C^n(A,X)$ are called n-dimensional cochains. Consider the sequence

$$0\to C^0(A,X)\xrightarrow{\delta^0}C^1(A,X)\xrightarrow{\delta^1}\cdots\to\big(\tilde{C}(A,X)\big).$$

It is not hard to show that the above sequence is a complex, i.e., for each n, $\delta^{n+1} \circ \delta^n = 0$; cf. [23].

$\tilde{C}(A, X)$ is called the standard cohomology complex or Hochschild–Kamowitz complex for A and X. The nth cohomology group of $\tilde{C}(A, X)$ is said to be n-dimensional (ordinary or Hochschild) cohomology group of A with coefficients in X and denoted by $H^n(A, X)$. The spaces $\operatorname{Ker} \delta^n$ and $\operatorname{Im} \delta^{n-1}$ are denoted by $Z^n(A, X)$ and $B^n(A, X)$, respectively. Hence $H^n(A, X) = Z^n(A, X)/B^n(A, X)$. The cohomology groups of small dimensions $n = 0, 1, 2, 3$ are very important and applicable.

$H^0(A, X) = Z^0(A, X)$ is the so-called center of X.

Any element of

$$Z^1(A, X) = \{\mathrm{d} : A \to X;\ \mathrm{d} \text{ is bounded and linear, and } \mathrm{d}(ab) = a\mathrm{d}(b) + \mathrm{d}(a)b\}$$

is called a derivation of A in X and any element of $B^1(A, X) = \{\mathrm{d}_x : A \to X;\ \mathrm{d}_x(a) = ax - xa,\ a \in A,\ x \in X\}$ is called an inner derivation. The Banach algebra A is said to be contractible if $H^1(A, X) = Z^1(A, X)/B^1(A, X) = 0$ for all X and to be amenable (according to Johnson) if $H^1(A, X^*) = 0$ for all X; cf. [9].

$H^2(A, X)$ is the equivalence classes of singular extensions of A by X; cf. [1].

$H^3(A, X)$ can be used in the study of stable properties of Banach algebras; cf. [8].

For $n \geqslant 4$ there is no known interesting interpretation of $H^n(A, X)$. But their vanishing is what homological dimension is about [5]. Given $n \geqslant 1$, by an approximate n-cocycle we mean a mapping $f : \prod_{j=1}^n A \to X$ which is continuous at a point and $f(a_1, \ldots, a_n) = 0$ whenever $a_i = 0$ for any i, and such that

$$\begin{aligned} &\left\| D^n_{\lambda_1,\ldots,\lambda_n} f(a_1, b_1, \ldots, a_n, b_n) \right\| \leqslant \alpha, \\ &\left\| \delta^n f(a_1, a_2, \ldots, a_{n+1}) \right\| \leqslant \beta \end{aligned}$$

for some positive numbers α and β and for all $a_1, \ldots, a_n, a_{n+1}, b_1, \ldots, b_n \in A$ and all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$.

Given $n \geqslant 2$, by an approximate n-coboundary we mean a mapping of the form $\delta^{n-1} g$ in which $g : \prod_{j=1}^{n-1} A \to X$ that is continuous at a point and $g(a_1, \ldots, a_{n-1}) = 0$ whenever $a_i = 0$ for any i, and such that

$$\left\| D^{n-1}_{\lambda_1,\ldots,\lambda_{n-1}} g(a_1, b_1, \ldots, a_{n-1}, b_{n-1}) \right\| \leqslant \gamma$$

for some positive number γ and for all $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1} \in A$ and all $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{C}$.

By an approximate 1-coboundary we mean a mapping of the form $\delta^0(a) = ax - xa$ for some $x \in X$, i.e., a usual 1-coboundary.

If every approximate n-cocycle f is near an approximate n-coboundary, in the sense that there exist $\eta > 0$ and an approximate n-coboundary h such that $\|h(a_1, \ldots, a_n) - f(a_1, \ldots, a_n)\| \leqslant \eta$ for all $a_1, \ldots, a_n \in A$, we say the nth cohomology group of A with coefficients in X approximately vanishes.

Theorem 3.1. *For a positive integer n, $H^n(A, X) = 0$ if and only if the nth cohomology groups of A in X approximately vanishes.*

Proof. Suppose that $H^n(A, X) = 0$ and f is an approximate n-cocycle. By Theorem 2.1 there is an n-cocycle $F \in Z^n(A, X)$ such that $\|F(a_1, \dots, a_n) - f(a_1, \dots, a_n)\| \leqslant 3.2^n\alpha$ where α is given by (1). Since $H^n(A, X) = 0$, there exists $G \in C^{n-1}(A, X)$ such that $\delta^{n-1}G = F$. Hence $\|\delta^{n-1}G(a_1, \dots, a_n) - f(a_1, \dots, a_n)\| \leqslant 3.2^n\alpha$. Hence f is approximated by an approximate coboundary.

For the converse, let $F \in Z^n(A, X)$. Then F is trivially an approximate n-cocycle. Since nth cohomology group of A in X approximately vanishes, there exist $\eta > 0$ and an approximate n-coboundary h such that $\|h(a_1, \dots, a_n) - F(a_1, \dots, a_n)\| \leqslant \eta$. By Theorems 2.2 and 2.6 there exist $G \in C^{n-1}(A, X)$ such that $F = \delta^{n-1}G$. hence $F \in B^n(A, X)$. Therefore, $H^n(A, X) = 0$. □

Corollary 3.2. *The Banach algebra A is contractible if and only if every continuous approximate derivation is near an inner derivation* (*see* [14] *for another approach*).

Corollary 3.3. *The Banach algebra A is amenable if and only if every continuous approximate derivation into a dual Banach bimodule is near an inner derivation.*

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