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Moser–Trudinger type inequalities for the Hessian equation ☆

Gu-Ji Tian [a], Xu-Jia Wang [b,*]

[a] *Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, China*
[b] *Centre for Mathematics and Its Applications, The Australian National University, Canberra, ACT 0200, Australia*

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Abstract

The k-Hessian equation for $k \geqslant 2$ is a class of fully nonlinear partial differential equation of divergence form. A Sobolev type inequality for the k-Hessian equation was proved by the second author in 1994. In this paper, we prove the Moser–Trudinger type inequality for the k-Hessian equation.

Keywords: Hessian equation; Moser–Trudinger inequality; A priori estimates

1. Introduction

Let Ω be a bounded smooth domain in the Euclidean space $\mathbb{R}^n$. For a function $u \in C^2(\Omega)$, the k-Hessian operator $S_k[u]$ is defined by

$$S_k[u] = \left[D^2 u\right]_k, \tag{1.1}$$

where $1 \leqslant k \leqslant n$, $[A]_k$ denotes the sum of all $k \times k$ principal minors of the matrix A. The k-Hessian operator $S_k[u]$ is also equal to the k^{th}-elementary symmetric polynomial of the eigenvalues of the Hessian matrix D^2u. When $k = 1$, it is the Laplace operator, when $k = n$, it is the Monge–Ampère operator.

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* Corresponding author.
E-mail addresses: tianguji@wipm.ac.cn (G.-J. Tian), Xu-Jia.Wang@anu.edu.au (X.-J. Wang).

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The existence and a priori estimates of smooth solutions to the k-Hessian equation were first proved in [1] and also in [10] for some cases, and were extended to more general equations in [12,21], see also [7,8,14,24,25] for related results. The k-Hessian equation for $k = 2, \dots, n-1$ can be regarded as a series of fully nonlinear partial differential equations linking the Laplace equation to the Monge–Ampère equation, and in particular they are also of divergence form. Therefore one may expect various variational and potential theoretic properties for these equations. There has been a lot of research in this direction indeed [2,5,13,18,19,23,27]. Due to its variational structure, Sobolev and Moser–Trudinger type inequalities for the k-Hessian equation are fundamental and of particular interest.

A Sobolev type inequality for the k-Hessian equation has been obtained by the second author in [27] (and also in [3] for convex functions), which was used in [5] to study the associated variational problems. In this paper we prove a Moser–Trudinger type inequality for the k-Hessian equation, which occurs in the case when $k = n/2$. First we recall the divergence structure of the k-Hessian operator,

$$S_k[u] = \frac{1}{k} \sum u_{ij} S_k^{ij}[u] = \frac{1}{k} \sum \partial_i \bigl(u_j S_k^{ij}[u]\bigr), \tag{1.2}$$

where $u_i = u_{x_i}$, $u_{ij} = u_{x_i x_j}$, and $S_k^{ij}[u] = \frac{\partial}{\partial u_{ij}} S_k[u]$. The second equality is due to the fact that the coefficients S_k^{ij} are divergence free,

$$\sum_i \partial_i S_k^{ij}[u] = 0 \quad \forall j. \tag{1.3}$$

Following [1], we say a function $u \in C^2(\Omega)$ is *k-admissible* if $S_j[u] \geqslant 0$ for all $j = 1, \dots, k$. A function is k-admissible if and only if it is subharmonic when $k = 1$, or convex when $k = n$. If u is k-admissible, $\{S_k^{ij}[u]\}$ is positive semi-definite [1].

Denote by $\Phi^k(\Omega)$ the set of all k-admissible functions in Ω, and by $\Phi_0^k(\Omega)$ the set of all k-admissible functions vanishing on $\partial\Omega$. The set $\Phi_0^k(\Omega)$ is non-empty (containing nonzero functions) if and only if $\partial\Omega$ is $(k-1)$-convex, namely for any point $x \in \partial\Omega$, $\sigma_{k-1}(\kappa(x)) > 0$, where $\kappa(x) = (\kappa_1(x), \dots, \kappa_{n-1}(x))$ are the principal curvatures of $\partial\Omega$ at x [1]. In this paper we will always assume that $\partial\Omega$ is $(k-1)$-convex.

Denote

$$I_k(u) = \int_\Omega (-u) S_k[u]\, dx = \frac{1}{k} \int_\Omega u_i u_j S_k^{ij}[u], \tag{1.4}$$

the associated functional, and

$$\|u\|_{\Phi_0^k} = \bigl[I_k(u)\bigr]^{\frac{1}{k+1}}, \quad u \in \Phi_0^k. \tag{1.5}$$

One easily verifies that $\|\cdot\|_{\Phi_0^k}$ is a norm in Φ_0^k [27]. The following Sobolev type inequalities were proved in [27]. For convex functions, they were first established in [3,2].

Theorem 1.1. *Let $u \in \Phi_0^k(\Omega)$.*

(i) *If $1 \leqslant k < \frac{n}{2}$, we have*

$$\|u\|_{L^{p+1}(\Omega)} \leqslant C\|u\|_{\Phi_0^k} \quad \forall\, p+1 \in \left[1, k^*\right], \tag{1.6}$$

where $k^* = \frac{n(k+1)}{n-2k}$, C depends only on n, k, p, and $|\Omega|$.

(ii) *If $k = \frac{n}{2}$,*

$$\|u\|_{L^p(\Omega)} \leqslant C\|u\|_{\Phi_0^k} \tag{1.7}$$

for any $p < \infty$, where C depends only on n, p, and $\operatorname{diam}(\Omega)$.

(iii) *If $\frac{n}{2} < k \leqslant n$,*

$$\|u\|_{L^\infty(\Omega)} \leqslant C\|u\|_{\Phi_0^k}, \tag{1.8}$$

where C depends on n, k, and $\operatorname{diam}(\Omega)$.

The exponent k^* in (1.6) is optimal. In [27] it was proved that the best constant in (1.6) is achieved by radially symmetric functions. For radial function $u \in \Phi_0^k(B_R(0))$, we have

$$\|u\|_{\Phi_0^k}^{k+1} = \int\limits_{B_R} (-u)S_k[u]\,dx = \frac{\omega_{n-1}}{k}\binom{n-1}{k-1}\int\limits_0^R r^{n-k}\left(u'\right)^{k+1}dr. \tag{1.9}$$

Therefore when $p+1 = k^*$ and $k < \frac{n}{2}$, by the classical Sobolev embedding, the best constant C is attained when $\Omega = \mathbb{R}^n$ by the function

$$u(x) = \left[1+|x|^2\right]^{(2k-n)/2k}. \tag{1.10}$$

Moreover, when $p+1 < k^*$, by Hölder's inequality we see that the constant C depends on the volume $|\Omega|$. When $k > \frac{n}{2}$, it was shown that any k-admissible function is Hölder continuous with the optimal exponent $\alpha = 2 - \frac{n}{k}$ [23]. Theorem 1.1 was used in [5] to study the associated variational problems. In [22] it was also shown that for any k-admissible function $u \in \Phi_0^k(\Omega)$,

$$\|u\|_{\Phi_0^l(\Omega)} \leqslant C\|u\|_{\Phi_0^k(\Omega)} \tag{1.11}$$

where $1 \leqslant l < k \leqslant n$ and C is a constant depending on n, k, l, and Ω.

Note that by the compactness of the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $p < \frac{2n}{n-2}$, we also have the compactness of the embedding $\Phi_0^k(\Omega) \hookrightarrow L^p(\Omega)$ for $p < k^*$ if $1 \leqslant k < \frac{n}{2}$, or $p < \infty$ for $p \geqslant \frac{n}{2}$. See [28] for details.

The purpose of this paper is to prove the following Moser–Trudinger type inequality for the k-Hessian equation with $k = \frac{n}{2}$. When $n = 2$, $k = 1$, it coincides with the special Moser–Trudinger inequality $W_0^{1,2}(\Omega) \hookrightarrow L_{\varphi^*}(\Omega)$ with $\varphi(t) = e^{t^2}$.

Theorem 1.2. *Let $k = \frac{n}{2}$. Then for any $u \in \Phi_0^k(\Omega)$,*

$$\sup\left\{ \int_\Omega \exp\left(\alpha\left(\frac{u}{\|u\|_{\Phi_0^k}}\right)^\beta\right):\ u \in \Phi_0^k(\Omega)\right\} < C, \tag{1.12}$$

where $0 < \alpha \leqslant \alpha_0$, $1 \leqslant \beta \leqslant \beta_0$,

$$\alpha_0 = n\left[\frac{\omega_{n-1}}{k}\binom{n-1}{k-1}\right]^{2/n}, \tag{1.13}$$

$$\beta_0 = \frac{n+2}{n}, \tag{1.14}$$

ω_n is the surface area of the unit sphere in $\mathbb{R}^{n+1}$, and C is a positive constant depending only on n and $\operatorname{diam}(\Omega)$.

By Theorem 1.2, the set $\Phi_0^k(\Omega)$ can be embedded in the Orlicz space associated with the function $\exp(t^{(n+2)/n})$. Recall that for an even, convex function φ in $\mathbb{R}^1$ satisfying $\lim_{t\to\infty}\varphi(t)/t = \infty$, the Orlicz class $L_\varphi(\Omega)$ is the set of functions satisfying $\int_\Omega \varphi(u(x))\,dx < \infty$, and the Orlicz space associated with φ, $L_{\varphi^*}(\Omega)$, is the linear hull of $L_\varphi(\Omega)$ with the norm $\|u\|_{L_{\varphi^*}(\Omega)} = \inf\{k;\ \int_\Omega \varphi(\frac{u}{k}) \leqslant 1\}$. In [20,16], Trudinger and Moser proved the embedding $W_0^{1,n}(\Omega) \hookrightarrow L_{\varphi^*}(\Omega)$ with $\varphi(t) = \exp(t^{n/(n-1)}) - 1$. Trudinger proved by the Taylor expansion that there exists a small $\lambda > 0$ such that for all $u \in W_0^{1,n}(\Omega)$ with $\|Du\|_{L^n(\Omega)} = 1$,

$$\int_\Omega e^{\lambda|u|^{\frac{n}{n-1}}}\,dx \leqslant C. \tag{1.15}$$

Moser improved the exponent λ to the optimal one

$$\lambda = n(\omega_{n-1})^{1/(n-1)}. \tag{1.16}$$

About the proof of Theorem 1.2, since the norm $\|\cdot\|_{\Phi_0^k}$ involves both the first and second derivatives, the proofs for the classical Moser–Trudinger inequalities do not apply to the k-Hessian equations. A weak version of (1.12) (namely when $\alpha > 0$ is small) can be obtained by using the Sobolev type inequality (1.7) and the Taylor expansion. But to obtain the optimal exponent α_0, we cannot use the symmetrization techniques as in Moser's proof. The associated symmetrization for the k-Hessian equation is not available. One of the main ingredients of the paper is to prove a monotonicity formula (Lemma 3.1) so that the proof of (1.12) can be reduced to radial functions.

Another difficulty in proving (1.12) is that we cannot use the variational principle directly, as the k-Hessian equation is fully nonlinear. More precisely, we cannot prove directly that a maximizer of (1.12) satisfies the associated Euler equation, even in a very weak sense of measure [23]. We have to employ a gradient flow and establish the global existence and convergence of smooth solutions, and show that the limit of the gradient flow at $t \to \infty$ converges to a maximizer of (1.12) (Theorems 4.1 and 4.2).

The proof in this paper is inspired by that in [27] but technically the argument in this paper is much difficult. For example, a similar gradient flow was used in [27], but the gradient flow in this paper contains a constraint involving second derivatives, new techniques (see Remarks 4.1–4.4) are needed to obtain the global existence of smooth solutions, which make the proof very involved. A simpler and more direct proof is desirable. In a recent paper [26], Verbitsky found a different proof of the inequalities (1.6) and (1.11). He also proved some new inequalities related to the k-Hessian equalities. His proof uses the Wolff potential estimate for the k-Hessian equation [13], which was based on the Hessian measures developed in [23]. He also obtained the inequality (1.12) for small $\alpha > 0$ (Theorem 2.1). In [9] we also proved a new class of Sobolev type inequalities.

This paper is arranged as follows. In Section 2 we prove (1.12) for $\beta = \beta_0$ and a small $\alpha > 0$, using the Sobolev type inequality (1.7) and the Taylor expansion. In Section 3 we prove a monotonicity formula, which reduces the inequality (1.12) to radial symmetric functions in a ball, and obtain (1.12) from the sharp Moser–Trudinger inequality in [16]. In Section 4 we prove the global regularity of solutions to a gradient flow of an approximation problem of the functional in (1.12). Finally in Section 5 we use the gradient flow to prove the existence of a smooth maximizer of (1.12).

2. Taylor expansion

In this section we prove (1.12) for $\beta = \beta_0$ and a small α by the Taylor expansion, making use of the Sobolev type inequality (1.7). Set

$$T_p(\Omega) = \inf_{u \in \Phi_0^k(\Omega)} \|u\|^{k+1}_{\Phi_0^k(\Omega)} / \|u\|^{k+1}_{L^{p+1}(\Omega)}. \tag{2.1}$$

It was proved in [27] that $T_p(\Omega_1) \geqslant T_p(\Omega_2)$ if $\Omega_1 \subset \Omega_2$, and

$$T_p(B_1) = T_p^*, \tag{2.2}$$

where B_1 is the unit ball, and

$$\begin{aligned} T_p^* &= \inf_{u \in \Phi_0^k(B_1)} \left\{ \frac{\|u\|^{k+1}_{\Phi_0^k(B_1)}}{\|u\|^{k+1}_{L^{p+1}(B_1)}} \colon u \text{ is radial} \right\} \\ &= \inf_{u \in \Phi_0^k(B_1)} \left\{ \frac{\int_0^1 \frac{1}{k} \omega_{n-1} \binom{n-1}{k-1} r^{n-k} (u')^{k+1}\, dr}{[\int_0^1 \omega_{n-1} r^{n-1} |u|^{p+1}\, dr]^{(k+1)/(p+1)}} \right\}. \end{aligned} \tag{2.3}$$

Without loss of generality, let us assume that $\Omega \subset B_1(0)$. Then $T_p(\Omega) \geqslant T_p^*$. We estimate T_p^* as follows.

Lemma 2.1. *For any $p > 1$, we have*

$$\frac{\int_0^1 |u|^{p+1} r^{n-1}\, dr}{[\int_0^1 |u'|^{\frac{n+2}{2}} r^{n/2}\, dr]^{\frac{2(p+1)}{n+2}}} \leqslant \left(\frac{1}{n} + \frac{p+1}{n+2} \right)^{1 + \frac{n(p+1)}{n+2}} e^{-\frac{n(p+1)}{n+2}}. \tag{2.4}$$

Proof. First by Hardy's inequality (see Theorem 1.14, [17]), we have

$$\int_0^1 |u|^{p+1} r^{n-1}\,dr \leqslant D\left(1+\frac{n(p+1)}{n+2}\right)\left(1+\frac{n+2}{n(p+1)}\right)^{\frac{n(p+1)}{n+2}}\left[\int_0^1 |u'|^{\frac{n+2}{2}} r^{\frac{n}{2}}\,dr\right]^{\frac{2(p+1)}{n+2}},$$

where

$$D=\sup_{x\in(0,1)}\int_x^1 (1-r)^{n-1}\,dr\left[\int_0^x (1-r)^{-1}\,dr\right]^{\frac{n(p+1)}{n+2}}.$$

By direct computation,

$$D=\frac{1}{n}\left(\frac{p+1}{n+2}\right)^{\frac{n(p+1)}{n+2}} e^{-\frac{n(p+1)}{n+2}}.$$

Hence we obtain (2.4). □

Now we compute, by (1.9),

$$\begin{aligned}\int_{B_1(0)}\left(\frac{|u|}{\|u\|_{\Phi_0^k}}\right)^{\frac{n+2}{n}j} dx &= \int_0^1 \omega_{n-1} r^{n-1}|u|^{\frac{n+2}{n}j}\,dr/\|u\|_{\Phi_0^k}^{\frac{n+2}{n}j}\\ &= \frac{\int_0^1 \omega_{n-1} r^{n-1}|u|^{\frac{n+2}{n}j}}{[\int_0^1 \frac{\omega_{n-1}}{k}\binom{n-1}{k-1} r^{n-k}(u')^{k+1}\,dr]^{\frac{2j}{n}}}.\end{aligned}$$

By (2.4) with $p+1=\frac{n+2}{n}j$, we obtain

$$\int_{B_1(0)}\left(\frac{|u|}{\|u\|_{\Phi_0^k}}\right)^{\frac{n+2}{n}j} dx \leqslant \omega_{n-1}\left(\frac{\omega_{n-1}}{k}\binom{n-1}{k-1}\right)^{-\frac{2j}{n}}\left(\frac{j+1}{n}\right)^{j+1} e^{-j}$$

Hence

$$\begin{aligned}\int_{B_1(0)} \exp\left(\alpha\left(\frac{u}{\|u\|_{\Phi_0^k}}\right)^{\frac{n+2}{n}}\right) dx &= \omega_{n-1}\int_0^1 r^{n-1}\sum_{j=1}^{\infty}\frac{\alpha^j}{j\,!}\left(\frac{|u|}{\|u\|_{\Phi_0^k}}\right)^{\frac{n+2}{n}j}\\ &\leqslant \omega_{n-1}\sum_{j=1}^{\infty}\alpha^j\left(\frac{\omega_{n-1}}{k}\binom{n-1}{k-1}\right)^{-\frac{2}{n}j}\left(\frac{j+1}{n}\right)^{1+j}\frac{e^{-j}}{j\,!}\\ &\leqslant C\sum_{j=1}^{\infty}\alpha^j\left(\frac{\omega_{n-1}}{k}\binom{n-1}{k-1}\right)^{-\frac{2}{n}j}.\end{aligned}$$

When the constant $\alpha < (\frac{\omega_{n-1}}{k}\binom{n-1}{k-1})^{\frac{2}{n}}$, the right-hand side is bounded. Recall that $T_p(\Omega) \geqslant T_p^*$. We have thus proved the following weak form of Theorem 1.2.

Theorem 2.1. *Let $k = \frac{n}{2}$. Then for any $u \in \Phi_0^k(\Omega)$,*

$$\sup\left\{ \int_\Omega \exp\left(\alpha\left(\frac{u}{\|u\|_{\Phi_0^k}}\right)^{\frac{n+2}{n}}\right):\ u \in \Phi_0^k(\Omega)\right\} < C \tag{2.5}$$

if $\alpha < (\frac{\omega_{n-1}}{k}\binom{n-1}{k-1})^{\frac{2}{n}}$, where C depends only on n and the diameter of Ω.

3. A monotonicity formula

Denote

$$F(t) = F_{\alpha,\beta}(t) = e^{\alpha|t|^\beta} - \sum_{j=1}^{k-1}\frac{\alpha^j}{j\,!}|t|^{j\beta} = \sum_{j=k}^{\infty}\frac{\alpha^j}{j\,!}|t|^{j\beta},$$

$$f(t) = f_{\alpha,\beta}(t) = \sum_{j=k}^{\infty}\frac{j\beta\alpha^j}{j\,!}|t|^{j\beta-1},$$

where $\alpha > 0$, $\beta \geqslant 1$ are constants. To prove Theorem 1.2, it suffices to prove

$$Y(\Omega) := \sup\left\{ \int_\Omega F\left(\frac{u}{\|u\|_{\Phi_0^k}}\right):\ u \in \Phi_0^k(\Omega)\right\} < C \tag{3.1}$$

for $\alpha = \alpha_0$, $\beta = \beta_0$. We wish to reduce (3.1) to radial functions. Our purpose is to prove the monotonicity formula

$$Y(\Omega_1) \leqslant Y(\Omega_2) \tag{3.2}$$

for any bounded k-convex domains $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ with $\Omega_1 \subset \Omega_2$.

For any integer $m \geqslant k$, denote

$$F_m(t) = F_{\alpha,\beta,m}(t) = \sum_{j=k}^{m}\frac{\alpha^j}{j\,!}|t|^{j\beta},$$

$$f_m(t) = f_{\alpha,\beta,m}(t) = \sum_{j=k}^{m}\frac{j\beta\alpha^j}{j\,!}|t|^{j\beta-1},$$

and

$$Y_m(\Omega) = Y_{\alpha,\beta,m}(\Omega) = \sup\left\{ \int_\Omega F_m\left(\frac{u}{\|u\|_{\Phi_0^k}}\right):\ u \in \Phi_0^k(\Omega)\right\} < C. \tag{3.3}$$

By (1.7), $Y_m(\Omega)$ is well defined and finite. The following monotonicity is crucial for the proof of Theorem 1.2. Our proof uses the existence of maximizers of (3.3), which will be established in the next section.

Lemma 3.1. *For any $m \geqslant k$, $\alpha > 0$, and $\beta \geqslant 1$, there holds*

$$Y_m(\Omega_1) \leqslant Y_m(\Omega_2) \tag{3.4}$$

provided $\Omega_1 \subset \Omega_2$. Sending $m \to \infty$ we obtain (3.2).

Proof. Assume to the contrary that (3.4) does not hold. Then there exist $\Omega_2 \supset \Omega_1$, $\Omega_2 \neq \Omega_1$, such that $Y_m(\Omega_2) < Y_m(\Omega_1)$. Let u be a smooth maximizer of (3.3) with $\Omega = \Omega_1$ (the existence of u is proved in Theorem 4.1). By multiplying a constant we assume that

$$\int_{\Omega_1} (-u) S_k[u]\, dx = 1.$$

Then u satisfies the equation

$$S_k[u] = \lambda f_m(u)$$

with

$$\lambda \int (-u) f_m(u) = 1.$$

Let φ be a k-admissible solution to

$$\begin{aligned} S_k[\varphi] &= S_k[u] \quad \text{in } \Omega_2,\\ \varphi &= 0 \quad \text{on } \partial\Omega_2, \end{aligned} \tag{3.5}$$

where we let the right-hand side $S_k[u] = 0$ in $\Omega_2 - \Omega_1$ and use the notion of weak solutions in [23]. Noting that $\varphi \leqslant 0$ on $\partial\Omega_1$, by the comparison principle, we have $\varphi \leqslant u$ in Ω_1. We also have

$$\int_{\Omega_2} F_m\left(\frac{\varphi}{\|\varphi\|_{\Phi_0^k(\Omega_2)}}\right) \leqslant Y_m(\Omega_2) < Y_m(\Omega_1), \tag{3.6}$$

where the first inequality is by definition and the second one is by assumption. Denote

$$\Phi(\psi) = \sum_{j=k}^{m} \frac{\alpha^j}{j\,!} \frac{\int_{\Omega_2} |\psi|^{j\beta}}{[\int_{\Omega_2} (-\psi) S_k[u]]^{\frac{j\beta}{k+1}}}, \tag{3.7}$$

then (as $u = 0$ in $\Omega_2 - \Omega_1$)

$$\Phi(u) = \sum_{j=1}^{m} \frac{\alpha^j}{j\,!} \frac{\int_{\Omega_2} |u|^{j\beta}}{[\int_{\Omega_2} (-u) S_k[u]]^{\frac{j\beta}{k+1}}} = Y_m(\Omega_1).$$

We will prove that

$$\frac{d}{dt}\Phi\big(u+t(\varphi-u)\big)\big|_{t=0}\geqslant 0, \tag{3.8}$$

$$\frac{d^2}{dt^2}\Phi\big(u+t(\varphi-u)\big)\geqslant 0 \quad \forall t\in(0,1). \tag{3.9}$$

Suppose (3.8) and (3.9) hold, then we have $\Phi(u)\leqslant\Phi(\varphi)$, which, together with (3.6), leads to

$$Y_m(\Omega_1)=\Phi(u)\leqslant\Phi(\varphi)<Y_m(\Omega_1),$$

we reach a contradiction, so (3.4) must holds.

First we verify (3.8). By direct computation, and observing that $\varphi\leqslant u\leqslant 0$,

$$\begin{aligned}\frac{d}{dt}\Phi\big(u+t(\varphi-u)\big)\big|_{t=0}&=\sum_{j=k}^{m}\frac{\alpha^j}{j!}\left\{\frac{\int_{\Omega_2} j\beta|u+t(\varphi-u)|^{j\beta-1}(u-\varphi)\,dx}{[\int_{\Omega_2}(-(u+t(\varphi-u))S_k[u]]^{\frac{j\beta}{k+1}}}\right.\\&\quad\left.-\frac{j\beta}{k+1}\frac{\int_{\Omega_2}|u+t(\varphi-u)|^{j\beta}\,dx\int_{\Omega_2}(u-\varphi)S_k[u]\,dx}{[\int_{\Omega_2}(-(u+t(\varphi-u))S_k[u]\,dx]^{\frac{j\beta}{k+1}+1}}\right\}\Bigg|_{t=0}\\&=\sum_{j=k}^{m}\frac{\alpha^j}{j!}\left\{\frac{\int_{\Omega_1} j\beta|u|^{j\beta-1}(u-\varphi)\,dx}{[\int_{\Omega_1}(-u)S_k[u]]^{\frac{j\beta}{k+1}}}\right.\\&\quad\left.-\frac{j\beta}{k+1}\frac{\int_{\Omega_1}|u|^{j\beta}\,dx\int_{\Omega_2}(u-\varphi)S_k[u]\,dx}{[\int_{\Omega_1}(-u)S_k[u]\,dx]^{\frac{j\beta}{k+1}+1}}\right\}.\end{aligned}$$

Recall that $\int_{\Omega_1}(-u)S_k[u]\,dx=1$, and $S_k[u]=\lambda f_m(u)$ with $\lambda\int(-u)f_m(u)=1$. We have

$$\begin{aligned}\frac{d}{dt}\Phi\big(u+t(\varphi-u)\big)\big|_{t=0}&=\int_{\Omega_1} f_m(u)(u-\varphi)-\frac{1}{k+1}\int_{\Omega_1} f_m(u)|u|\int_{\Omega_2}(u-\varphi)S_k[u]\\&=\int_{\Omega_1} f_m(u)(u-\varphi)-\frac{1}{k+1}\int_{\Omega_1} f_m(u)|u|\int_{\Omega_1}(u-\varphi)\lambda f_m(u)\\&=\left(1-\frac{1}{k+1}\right)\int_{\Omega_1}(u-\varphi)f_m(u)\,dx\geqslant 0.\end{aligned}$$

Next we verify (3.9). Denote

$$\Phi_j(t)=\frac{\int_{\Omega_2}|u+t(\varphi-u)|^{j\beta}}{[\int_{\Omega_2}(-(u+t(\varphi-u))S_k[u]]^{\frac{j\beta}{k+1}}}.$$

It suffices to show that Φ_j satisfies (3.9) for every $k\leqslant j\leqslant m$. Set $p+1=j\beta$ for some fixed j. Then

$$\frac{d}{dt}\Phi_j(t) = \frac{(p+1)\int_{\Omega_2}|u+t(\varphi-u)|^p(u-\varphi)\,dx}{[\int_{\Omega_2}(-(u+t(\varphi-u))S_k[u]]^{\frac{p+1}{k+1}}} - \frac{p+1}{k+1}\frac{\int_{\Omega_2}|u+t(\varphi-u)|^{p+1}\,dx\int_{\Omega_2}(u-\varphi)S_k[u]\,dx}{[\int_{\Omega_2}(-(u+t(\varphi-u))S_k[u]]^{\frac{p+1}{k+1}+1}}$$

$$\begin{aligned}\frac{d^2}{dt^2}\Phi_j(t) &= \frac{p(p+1)\int_{\Omega_2}|u+t(\varphi-u)|^{p-1}(u-\varphi)^2\,dx}{[\int_{\Omega_2}(-(u+t(\varphi-u))S_k[u]]^{\frac{p+1}{k+1}}} \\ &\quad - \frac{2(p+1)^2}{k+1}\frac{\int_{\Omega_2}|u+t(\varphi-u)|^p(u-\varphi)\,dx\int_{\Omega_2}(u-\varphi)S_k[u]\,dx}{[\int_{\Omega_2}(-(u+t(\varphi-u))S_k[u]]^{\frac{p+1}{k+1}+1}} \\ &\quad + \frac{p+1}{k+1}\left(\frac{p+1}{k+1}+1\right)\frac{\int_{\Omega_2}|u+t(\varphi-u)|^{p+1}\,dx[\int_{\Omega_2}(u-\varphi)S_k[u]\,dx]^2}{[\int_{\Omega_2}(-(u+t(\varphi-u))S_k[u]]^{\frac{p+1}{k+1}+2}} \\ &=: \frac{(p+1)G(t)}{[\int_{\Omega_2}(-(u+t(\varphi-u))S_k[u]]^{\frac{p+1}{k+1}+2}}.\end{aligned}$$

Denote

$$a(t) = \int_{\Omega_2}\left|u+t(\varphi-u)\right|S_k[u],$$

$$b(t) = \int_{\Omega_2}(u-\varphi)S_k[u].$$

Then

$$\begin{aligned}G(t) &= pa^2(t)\int_{\Omega_2}\left|u+t(\varphi-u)^2\right|^{p-1}(u-\varphi)^2 \\ &\quad - \frac{2(p+1)}{k+1}a(t)b(t)\int_{\Omega_2}\left|u+t(\varphi-u)\right|^p(u-\varphi) \\ &\quad + \frac{1}{k+1}\left(\frac{p+1}{k+1}+1\right)b^2(t)\int_{\Omega_2}\left|u+t(\varphi-u)\right|^{p+1}.\end{aligned}$$

Observe that when $k=1$, $\|u\|_{\Phi_0^k} = \|u\|_{W_0^{1,2}}$ and the monotonicity inequality is obvious. Hence we may assume $k \geqslant 2$, so that $pk \geqslant (k\beta-1)k \geqslant 1$ (recall that $p+1 = j\beta \geqslant k\beta$). Therefore

$$\sqrt{\frac{p}{k+1}\left(\frac{p+1}{k+1}+1\right)} = \frac{\sqrt{p^2+2p+pk}}{k+1} \geqslant \frac{p+1}{k+1}.$$

By the Cauchy inequality,

$$
\begin{aligned}
&pa^2\big|u+t(\varphi-u)\big|^{p-1}(u-\varphi)^2+\frac{1}{k+1}\left(\frac{p+1}{k+1}+1\right)b^2\big|u+t(\varphi-u)\big|^{p+1}\\
&\geqslant 2ab\sqrt{\frac{p}{k+1}\left(\frac{p+1}{k+1}+1\right)}\big|u+t(\varphi-u)\big|^{p}(u-\varphi)\\
&=2ab\frac{p+1}{k+1}\big|u+t(\varphi-u)\big|^{p}(u-\varphi).
\end{aligned}
$$

Therefore we obtain $G(t)\geqslant 0$ and thus (3.9) holds. □

Lemma 3.2. *Denote*

$$
\begin{aligned}
Y_m^*(B_R)&=\sup\left\{\int_{B_R}F_m\left(\frac{u}{\|u\|_{\Phi_0^k}}\right):\ u\in\Phi_0^k(B_R)\ \textit{is radial}\right\},\\
Y^*(B_R)&=\sup\left\{\int_{B_R}F\left(\frac{u}{\|u\|_{\Phi_0^k}}\right):\ u\in\Phi_0^k(B_R)\ \textit{is radial}\right\},
\end{aligned}
$$

where B_R is a ball of radius R with center at 0. *Then*

$$
\begin{aligned}
Y_m(B_R)&=Y_m^*(B_R),\\
Y(B_R)&=Y^*(B_R).
\end{aligned}
\tag{3.10}
$$

Proof. Obviously $Y_m^*(B_R)\leqslant Y_m(B_R)$. By Theorem 4.1, $Y_m(B_R)$ is attained by a k-admissible function $u\in\Phi_0^k(B_R)$ which satisfies the equation

$$
\begin{aligned}
S_k[u]&=\lambda f_m(u)\quad \text{in } B_R,\\
u&=0\quad \text{on } \partial B_R.
\end{aligned}
$$

By Aleksandrov's moving plane method, we conclude that u is a radial function. Hence $Y_m(B_R)\leqslant Y_m^*(B_R)$. Sending $m\to\infty$ and using monotone convergence theorem, we obtain the second inequality in (3.10). □

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $R>0$ such that $\Omega\subset B_R$. By the monotonicity formula (3.4),

$$
\begin{aligned}
Y_m(\Omega)&=\sup\left\{\int_{\Omega}F_m\left(\frac{u}{\|u\|_{\Phi_0^k}}\right)dx:\ u\in\Phi_0^k(\Omega)\right\},\\
&\leqslant\sup\left\{\int_{B_R}F_m\left(\frac{u}{\|u\|_{\Phi_0^k}}\right)dx:\ u\in\Phi_0^k(B_R)\right\}=Y_m(B_R),
\end{aligned}
$$

where $k = \frac{n}{2}$. By Lemma 3.2, the supremum on the right-hand side is attained by a radial function $u \in \Phi_0^k(B_R)$,

$$Y_m(B_R) = \int_{B_R} F_m\left(\frac{u}{\|u\|_{\Phi_0^k}}\right) dx.$$

Since u is radial,

$$\int_{B_R} F_m(u)\,dx = \omega_{n-1} \int_0^R r^{n-1} F_m(u)\,dr,$$

$$\int_{B_R} (-u) S_k[u]\,dx = \frac{\omega_{n-1}}{k} \binom{n-1}{k-1} \int_0^R r^{\frac{n}{2}} |u'|^{\frac{n}{2}+1}\,dr \quad \left(k = \frac{n}{2}\right).$$

We introduce a new variable $s = r^{\frac{2n}{n+2}}$ and set $v(s) = u(s^{\frac{n+2}{2n}})$. Then

$$\int_0^R r^{\frac{n}{2}} \left|u'(r)\right|^{\frac{n}{2}+1} dr = \left(\frac{2n}{n+2}\right)^{\frac{n}{2}} \int_0^{\hat{R}} s^{\frac{n}{2}} \left|v'(s)\right|^{\frac{n}{2}+1} ds, \tag{3.11}$$

$$\int_0^R r^{n-1} F_m(u)\,dr = \frac{n+2}{2n} \int_0^{\hat{R}} s^{\frac{n}{2}} F_m(v)\,ds, \tag{3.12}$$

where $\hat{R} = R^{\frac{2n}{n+2}}$. Now we regard v as a radial function of $y \in \mathbb{R}^{\frac{n}{2}+1}$ with $|y| = s$. Then

$$\begin{aligned}\left\|u(x)\right\|^{k+1}_{\Phi_0^k(B_R)} &= \frac{\omega_{n-1}}{k} \binom{n-1}{k-1} \left(\frac{2n}{n+2}\right)^{\frac{n}{2}} \int_0^{\hat{R}} s^{\frac{n}{2}} \left|v'(s)\right|^{\frac{n}{2}+1} ds \\ &= \frac{\omega_{n-1}}{k\omega_{n/2}} \binom{n-1}{k-1} \left(\frac{2n}{n+2}\right)^{\frac{n}{2}} \|Dv\|^{k+1}_{L^{k+1}(B_{\hat{R}})}.\end{aligned}$$

Hence

$$\left\|u(x)\right\|_{\Phi_0^k(B_R)} = c_0 \|Dv\|_{L^{k+1}(B_{\hat{R}})},$$

where

$$c_0 = \left[\frac{\omega_{n-1}}{k\omega_{n/2}} \binom{n-1}{k-1} \left(\frac{2n}{n+2}\right)^{\frac{n}{2}}\right]^{\frac{1}{k+1}}.$$

By (3.12) we have

$$
\begin{aligned}
\int\limits_{B_R} F_m\left(\frac{u}{\|u\|_{\Phi_0^k}}\right) dx &= \omega_{n-1}\int\limits_0^R r^{n-1} F_m\left(\frac{u}{\|u\|_{\Phi_0^k}}\right) dr \\
&= \frac{n+2}{2n}\omega_{n-1}\int\limits_0^{\hat{R}} s^{\frac{n}{2}} F_m\left(\frac{v}{\|u\|_{\Phi_0^k}}\right) ds \\
&= \frac{n+2}{2n}\frac{\omega_{n-1}}{\omega_{\frac{n}{2}}}\int\limits_{B_{\hat{R}}} F_m\left(\frac{v}{c_0\|Dv\|_{L^{k+1}(B_{\hat{R}})}}\right) dy \\
&\leqslant \frac{n+2}{2n}\frac{\omega_{n-1}}{\omega_{\frac{n}{2}}}\int\limits_{B_{\hat{R}}} \exp\left\{\frac{\alpha}{c_0^\beta}\left(\frac{|v|}{\|Dv\|_{L^{\frac{n}{2}+1}(B_{\hat{R}})}}\right)^\beta\right\} dy.
\end{aligned}
$$

By the Moser–Trudinger inequality (1.15) and (1.16) in the space $\mathbb{R}^{\frac{n}{2}+1}$, we see that

$$Y_m(B_R) \leqslant C$$

for some $C > 0$ independent of m if

$$
\begin{aligned}
\beta \leqslant \beta_0 = \frac{n+2}{n}, \\
\frac{\alpha}{c_0^\beta} \leqslant \left(\frac{n}{2}+1\right)(\omega_{n/2})^{2/n},
\end{aligned}
$$

i.e.,

$$\alpha \leqslant \left(\frac{n}{2}+1\right)(\omega_{n/2})^{2/n} c_0^\beta = n\left[\frac{\omega_{n-1}}{k}\binom{n-1}{k-1}\right]^{\frac{2}{n}} = \alpha_0.$$

This completes the proof. □

Remark 3.1. It is easy to verify that the exponent $\beta_0 = \frac{n+2}{n}$ is optimal, and when $\beta = \beta_0$, the exponent α_0 is also optimal. Indeed, a truncation of the function $u(x) = \log|x|$ shows that the exponents in the Moser–Trudinger inequality are optimal. As the function is k-admissible, it also implies the exponents in (1.12) are optimal.

4. Gradient flow

In this section we prove that for any fixed $m \geqslant k$, $k = \frac{n}{2}$, there is a maximizer of (3.3) which satisfies the associated Euler equation. By using the Hessian measure in [23], it is easy to prove that there is a (nonsmooth) maximizer of (3.3), but we are unable to show that the maximizer satisfies the associated Euler equation. Here we use the gradient flow method to obtain a smooth maximizer. Let f_m and F_m be as in Section 3. For simplicity, we omit the subscript m.

Theorem 4.1. *Let $k = \frac{n}{2}$ and Ω be a $(k-1)$-convex domain with smooth boundary. There exists a smooth maximizer $u^* \in \Phi_0^k(\Omega)$ of*

$$Y = \sup\{J(u)\colon\ u \in \Phi_0^k(\Omega)\} \tag{4.1}$$

which satisfies the equation

$$\begin{aligned} S_k[u^*] &= \lambda f(u^*),\\ \|u^*\| &= 1,\\ \lambda &= \left[\int_\Omega f(u^*)|u^*|\,dx\right]^{-1}, \end{aligned} \tag{4.2}$$

where $\|u\| = \|u\|_{\Phi_0^k(\Omega)}$, and ($F = F_m$ for a fixed $m \geqslant k$)

$$J(u) = \int_\Omega F\left(\frac{|u|}{\|u\|}\right).$$

We will introduce a gradient flow to prove Theorem 4.1. There are different gradient flows for the maximizers of (4.1). One is to keep the norm $\|u\| \equiv 1$ constant, see Remark 4.1 below. Here we introduce a different one. Instead of the functional J in (4.1), we consider a modified one with a constraint η,

$$Y_{\delta,\eta} = \sup\{J_{\delta,\eta}(u)\colon\ u \in \Phi_0^k(\Omega)\}, \tag{4.3}$$

where $\delta \in (0,1]$ is a small constant,

$$\begin{aligned} J_{\delta,\eta}(u) &= \int_\Omega F_\delta\left(\frac{|u|}{\eta(\|u\|)}\right),\\ F_\delta(t) &= F_m(t) + \delta|t|, \end{aligned} \tag{4.4}$$

and

$$\eta(t) = e^{t-1}$$

is a uniformly convex function. Since $\eta(t) \geqslant t$ and $\eta(t) = t$ only at $t = 1$, a maximizer u of $J_{\delta,\eta}$ necessarily satisfies $\|u\| = 1$ and is also a maximizer of J (when $\delta = 0$). Conversely, if u is a maximizer of J, then $u/\|u\|$ is a maximizer of $J_{\delta,\eta}$ (when $\delta = 0$). See Lemma 4.1 below. The purpose of introducing the constant δ is such that the associated Euler equation is non-degenerate. By the Sobolev type inequality (1.7),

$$0 < Y \leqslant Y_{\delta,\eta} \leqslant Y + C\delta < \infty,$$

where the constant C is the one in (1.7) with $p = 1$.

To get a gradient flow, let $u(\cdot,t)$ be a k-admissible function with parameter t. Differentiating the functional $J_{\delta,\eta}$ to get

$$\begin{aligned}
\frac{d}{dt}J_{\delta,\eta}\big(u(\cdot,t)\big) &= \int_\Omega f_\delta\left(\frac{|u|}{\eta}\right)\left[\frac{-u_t}{\eta} - \frac{u\eta'}{\eta^2\|u\|^k}\int u_t S_k[u]\right] \\
&= \left(\int\left[S_k[u] - \left(\frac{\eta\|u\|^k}{\int(-u)\eta' f_\delta(\frac{|u|}{\eta})}\right) f_\delta\left(\frac{|u|}{\eta}\right)\right]u_t\right)\frac{\int(-u)\eta' f_\delta(\frac{|u|}{\eta})}{\eta^2\,\|u\|^k}, \\
&= \frac{1}{\eta\lambda}\int\left[S_k[u] - \lambda f_\delta\left(\frac{|u|}{\eta}\right)\right]u_t\,dx,
\end{aligned}$$

where $\eta = \eta(\|u\|)$ and (noticing that $\eta'(t) = \eta(t)$)

$$\lambda = \lambda(t) = \frac{\|u\|^k}{\int(-u) f_\delta(\frac{|u|}{\eta})}.$$

Then for any monotone increasing function μ, if $u(x,t)$ is a solution of

$$\begin{aligned}
&u_t = \mu\big(S_k[u]\big) - \mu\left(\lambda(t) f_\delta\left(\frac{u}{\eta}\right)\right) \quad \text{in } \Omega\times[0,\infty), \\
&u = 0 \quad \text{on } \partial\Omega\times[0,\infty), \\
&u = u_0 \quad \text{on } \Omega\times\{t=0\},
\end{aligned} \tag{4.5}$$

we have

$$\frac{d}{dt}J_{\delta,\eta}\big(u(\cdot,t)\big) \geqslant 0. \tag{4.6}$$

In this section we choose the function μ such that $\mu\in C^\infty(0,\infty)$, $\mu'(t)>0$, μ is concave, and

$$\mu(t) = \begin{cases} \log t & \text{when } t < \frac{1}{100}, \\ t^{1/q} & \text{when } t \geqslant 1, \end{cases} \tag{4.7}$$

where $q = m\beta_0$ ($\beta_0 = \frac{n+2}{n}$).

Remark 4.1. To derive a gradient flow which keeps $\|u(\cdot,t)\|$ constant, let $\eta(t)\equiv t$ and $w(\cdot,t) = \frac{u(\cdot,t)}{\|u\|}$. Then

$$\frac{d}{dt}J\big(w(\cdot,t)\big) = \frac{d}{dt}J\big(u(\cdot,t)\big) = \frac{1}{\widetilde{\lambda}(t)}\int_\Omega\big[S_k[w] - \widetilde{\lambda}(t) f(w)\big]\frac{u_t}{\|u\|}\,dx,$$

where $\widetilde{\lambda} = [\int_\Omega(-w) f_\delta(w)\,dx]^{-1}$. Hence for any monotone increasing function μ,

$$u_t = \mu\big(S_k[w]\big) - \mu\big(\widetilde{\lambda}(t) f(w)\big)$$

is an ascent gradient flow. Noting that

$$w_t = \frac{u_t}{\|u\|} - \frac{u}{\|u\|^{k+2}} \int_\Omega (-u_t) S_k[u],$$

we obtain an ascent gradient flow

$$w_t = \big\{\mu\big(S_k[w]\big) - \mu\big(\widetilde{\lambda}(t) f(w)\big)\big\} + w \int S_k[w]\big[\mu\big(S_k[w]\big) - \mu\big(\widetilde{\lambda}(t) f(w)\big)\big], \tag{4.8}$$

which keeps $\|w\| \equiv 1$. However it seems the a priori estimates for the flow (4.8) is difficult. For example, our estimate for $\sup |u_t|$ (Lemma 4.2) and the estimate for higher order derivatives (Theorem 4.2) do not apply to (4.8).

Remark 4.2. Instead of the flow (4.8) which keeps the norm $\|w(\cdot,t)\|_{\Phi_0^k}$ invariant, in this section we consider the flow (4.5). A trick is to introduce the auxiliary function η such that a maximizer of (4.4) must satisfies $\|u\|_{\Phi_0^k} = 1$, as shown in Lemma 4.1 below.

For the parabolic equation (4.5), we say a function $u(x,t)$ is k-admissible if for any time t, $u(\cdot,t)$ is k-admissible.

Choose an initial function $u_0 \in \Phi_0^k(\Omega)$ such that $J_{\delta,\eta}(u_0) > Y_{\delta,\eta} - \varepsilon$, for some small $\varepsilon > 0$. By modifying u_0 slightly (see [5,27,28]), we may assume that u_0 satisfies the compatibility condition

$$S_k[u] = \lambda(t) f_\delta\left(\frac{u}{\eta}\right) \quad \text{on } \partial\Omega \times \{t = 0\}.$$

Let u be a smooth k-admissible solution to initial-boundary value problem (4.5). By the monotonicity (4.6),

$$J_{\delta,\eta}\big(u(\cdot,t)\big) > Y_{\delta,\eta} - \varepsilon$$

for all $t > 0$. First we show

Lemma 4.1. *Denote*

$$\Theta^*(\varepsilon) = \sup\big\{\|u\|\colon\ u \in \Phi_0^k(\Omega),\ J_{\delta,\eta}(u) \geqslant Y_{\delta,\eta} - \varepsilon\big\},$$
$$\Theta_*(\varepsilon) = \inf\big\{\|u\|\colon\ u \in \Phi_0^k(\Omega),\ J_{\delta,\eta}(u) \geqslant Y_{\delta,\eta} - \varepsilon\big\}.$$

Then

$$\Theta^*(\varepsilon), \Theta_*(\varepsilon) \to 1 \quad as\ \varepsilon \to 0.$$

Proof. Let $g(t) = \frac{t}{\eta(t)} = te^{1-t}$. Then $g(t) < g(1) = 1$ for any $t \neq 1$. For any $t > 0, t \neq 1$ and any $u \in \Phi_0^k(\Omega)$ with $\|u\| = t$, we have

$$J_{\delta,\eta}(u) = \int_\Omega F_{\delta,\eta}\left(\frac{u}{\eta(\|u\|)}\right) = \int_\Omega F_{\delta,\eta}\left(g(t)\frac{u}{\|u\|}\right).$$

By the Taylor expansion $F(t) = \sum_{j=k}^m \frac{\alpha^j}{j!}|t|^{\beta j}$, we obtain

$$J_{\delta,\eta}(u) \leqslant g(t)\int_\Omega F_{\delta,\eta}\left(\frac{u}{\|u\|}\right).$$

Therefore

$$\sup\left\{J_{\delta,\eta}(u)\colon\ u \in \Phi_0^k(\Omega),\ \|u\| = t\right\} \leqslant g(t)Y_{\delta,\eta}.$$

Lemma 4.1 follows immediately. □

By Lemma 4.1, we can choose $\varepsilon > 0$ small such that the solution u satisfies

$$\frac{1}{2} \leqslant \left\|u(\cdot,t)\right\| \leqslant \frac{3}{2},$$
$$e^{-1/2} \leqslant \eta\left(\|u\|\right) \leqslant e^{1/2}, \tag{4.9}$$

for all $t > 0$.

Next, observe that

$$\int_\Omega \frac{|u|}{\eta} f_\delta\left(\frac{u}{\eta}\right)dx \geqslant \int_\Omega F_\delta\left(\frac{u}{\eta}\right)dx \geqslant Y_{\delta,\eta} - \varepsilon \geqslant Y - \epsilon. \tag{4.10}$$

By the Sobolev type inequality (1.7) we also have

$$\int_\Omega |u| f_\delta\left(\frac{u}{\eta}\right)dx \leqslant C.$$

Hence

$$C_1 \leqslant \lambda(t) \leqslant C_2. \tag{4.11}$$

In the above, the constants C may depend on m and δ, but do not depend on M_T. We denote

$$M_T = \sup\nolimits_{Q_T}\left|u(x,t)\right|,$$
$$U_T = \sup\nolimits_{Q_T}|u_t|,$$

and $Q_T = \Omega \times (0, T]$.

As a preliminary to the estimates for u_t, we first compute $\frac{d}{dt}\|u(\cdot,t)\|$ and $\frac{d}{dt}\lambda(t)$. We have

$$\begin{aligned}\int_\Omega u_t S_k[u](x,t)\,dx &= \int_\Omega u_t\left(S_k[u](x,t)-\lambda(t)f_\delta\left(\frac{u}{\eta}\right)\right)dx+\lambda(t)\int_\Omega f_\delta\left(\frac{u}{\eta}\right)u_t\,dx\\ &= \psi(t)+\lambda(t)\int_\Omega f_\delta\left(\frac{u}{\eta}\right)u_t\,dx,\end{aligned}$$

where we denote

$$\psi(t)=\int_\Omega u_t\left(S_k[u](x,t)-\lambda(t)f_\delta\left(\frac{u}{\eta}\right)\right)dx\geqslant 0.$$

Denote $\Psi(t)=\int_0^t\psi(t)$. Noting that $\psi(t)=\lambda(t)\eta\frac{d}{dt}J_{\delta,\eta}(u(\cdot,t))$, we have $\Psi(t)\leqslant C$.

By (4.9) and the Sobolev type inequality (1.7),

$$\begin{aligned}\left|\int_\Omega f_\delta\left(\frac{u}{\eta}\right)u_t\,dx\right| &\leqslant CU_T,\\ \left|\int_\Omega |u|f_\delta\left(\frac{u}{\eta}\right)u_t\,dx\right| &\leqslant CU_T,\\ \left|\int_\Omega |u|^2 f'_\delta\left(\frac{u}{\eta}\right)dx\right| &\leqslant C,\end{aligned}\tag{4.12}$$

where C depends on m,δ. Hence

$$\left|\int_\Omega u_t S_k[u](x,t)\,dx\right|\leqslant C\big(\psi(t)+U_T\big).$$

Therefore

$$\begin{aligned}\left|\frac{d}{dt}\big\|u(\cdot,t)\big\|\right| &\leqslant \|u\|^{-k}\left|\int_\Omega(-u_t)S_k[u]\,dx\right|\\ &\leqslant C\big(\psi(t)+U_T\big).\end{aligned}$$

It follows

$$|\eta_t|=\left|\frac{d}{dt}\eta\big(\big\|u(\cdot,t)\big\|\big)\right|\leqslant C\big(\psi(t)+U_T\big)\tag{4.13}$$

for a different C, depending on m but not on M_T.

Next we compute $\frac{d}{dt}\lambda(t)$. Recall that

$$\lambda(t)=\frac{\|u\|^k}{\int(-u)f_\delta(\frac{|u|}{\eta})}.$$

We have

$$\begin{aligned}\frac{\lambda'(t)}{\lambda(t)}&=\frac{k}{\|u\|}\frac{d}{dt}\big\|u(\cdot,t)\big\|-\frac{\int\frac{d}{dt}[(-u)f_\delta(\frac{u}{\eta})]}{\int(-u)f_\delta(\frac{u}{\eta})}\\&=\frac{k}{\|u\|}\frac{d}{dt}\big\|u(\cdot,t)\big\|+\frac{\int f_\delta(\frac{u}{\eta})u_t}{\int(-u)f_\delta(\frac{u}{\eta})}-\frac{\int|u|f_\delta'(\frac{u}{\eta})u_t}{\eta\int(-u)f_\delta(\frac{u}{\eta})}-\frac{\int|u|^2f_\delta'(\frac{u}{\eta})\|u\|_t}{\eta\int(-u)f_\delta(\frac{u}{\eta})}.\end{aligned}$$

In the last integral we have used $\eta'=\eta$. By our previous estimates (4.10)–(4.13) and by the Sobolev type inequality (1.7), we obtain

$$\big|\lambda'(t)\big|\leqslant C\big(\psi(t)+U_T\big).\tag{4.14}$$

With the above preparation, we can compute furthermore

$$\frac{d}{dt}\mu\bigg(\lambda(t)f_\delta\bigg(\frac{u}{\eta}\bigg)\bigg)=\mu'\bigg\{\lambda'f_\delta+\frac{\lambda f_\delta'}{\eta^2}(u_t\eta-u\eta_t)\bigg\},$$

where $C^{-1}\leqslant\lambda,\ \eta\leqslant C$. Note that when $t\geqslant1$, $\mu(t)=t^{1/q}$ with $q=m\beta_0$. Hence when $\lambda(t)f_\delta(\frac{u}{\eta})\geqslant1$,

$$\begin{gathered}\big|\mu'\lambda'f_\delta\big|\leqslant C(1+M_T)\big(\psi(t)+U_T\big),\\\bigg|\mu'\frac{\lambda f_\delta'}{\eta^2}(u_t\eta)\bigg|\leqslant CU_T,\\\bigg|\mu'\frac{\lambda f_\delta'}{\eta^2}(u\eta_t)\bigg|\leqslant C(1+M_T)\big(\psi(t)+U_T\big).\end{gathered}$$

We obtain

$$\bigg|\frac{d}{dt}\mu\bigg(\lambda(t)f_\delta\bigg(\frac{u}{\eta}\bigg)\bigg)\bigg|\leqslant c^*(1+M_T)\big(\psi(t)+U_T\big),\tag{4.15}$$

where c^* depends on m,n,δ,Ω but not on T. When $t>0$ is small, $\mu(t)=\log(t)$. Note that $f_\delta\geqslant\delta>0$. Hence when $\lambda(t)f_\delta(\frac{u}{\eta})\leqslant1$, $\mu'\leqslant C$ and we also obtain estimate (4.15). Therefore (4.15) holds for any u.

Lemma 4.2. *Let $u\in C^4(Q_T)\cap C^3(\overline{Q}_T)$ be a k-admissible solution of* (4.5) *satisfying* (4.9). *Then $\exists C>0$, which depends on n,m,δ,Ω, and the initial function u_0, but is independent of T, such that*

$$\big|u_t(x,t)\big|\leqslant C\big(1+M_T^2\big).\tag{4.16}$$

Proof. First we prove

$$\inf_{Q_T} u_t \geqslant -C(1+M_T) - C(1+M_T)U_T^{\frac{1}{2}}. \tag{4.17}$$

Let

$$G(x,t) = \frac{u_t}{M-u} + c^*\Psi(t)$$

where $M = 2M_T + 1$, the constant $c^* > 0$ is given in (4.15), and Ψ is defined before (4.12), $\Psi' = \psi$. Let $G(x_0,t_0) = \inf_{Q_T} G(x,t)$ and $u_t(x^*,t^*) = \inf_{Q_T} u_t(x,t)$, then from $G(x_0,t_0) \leqslant G(x^*,t^*)$ and $Y_{\delta,\eta} \geqslant J_{\delta,\eta}(u) \geqslant Y_{\delta,\eta} - \varepsilon$ it follows that

$$u_t\big(x^*,t^*\big) \geqslant 3u_t(x_0,t_0) - C(1+M_T). \tag{4.18}$$

If G attains its minimum at the parabolic boundary $\partial^* Q_T$, we have $u_t \geqslant -C$ for some $C > 0$ depending on the initial value u_0. Hence we may assume G attains its minimum at some interior point (x_0,t_0) of Q_T. We may also assume $u_t \leqslant 0$ at (x_0,t_0). Then at (x_0,t_0), $G_t \leqslant 0$, $G_j = 0$ for $1 \leqslant j \leqslant n$ and matrix $\{G_{ij}(x_0,t_0)\} \geqslant 0$, namely

$$u_{tt} + \frac{u_t^2}{M-u} + c^*(M-u)\psi(t) \leqslant 0, \tag{4.19}$$

$$u_{tj} + \frac{u_t u_j}{M-u} = 0 \quad \text{for } 1 \leqslant j \leqslant n,$$

$$\left\{u_{ijt} + \frac{u_t u_{ij}}{M-u}\right\} \geqslant 0. \tag{4.20}$$

Differentiating Eq. (4.5) in t gives

$$u_{tt} = \frac{d}{dt}\mu\big(S_k[u]\big) - \frac{d}{dt}\mu\left(\lambda(t) f_\delta\left(\frac{u}{\eta}\right)\right). \tag{4.21}$$

From (4.20) we have at (x_0,t_0)

$$\frac{d}{dt}\mu\big(S_k[u]\big) = \mu'\big(S_k[u]\big)S_k^{ij}[u]u_{ijt} \geqslant -\mu'\big(S_k[u]\big)\frac{kS_k[u]u_t}{M-u} \geqslant 0, \tag{4.22}$$

where $S_k^{ij}[u] = \frac{\partial}{\partial u_{ij}} S_k[u]$ and we have used the relation

$$\sum\nolimits_{i,j} S_k^{ij}[u]u_{ij} = kS_k[u].$$

From (4.19), (4.21) and (4.22), we obtain at (x_0,t_0),

$$\begin{aligned}\frac{u_t^2}{M-u} &\leqslant -u_{tt} - c^*(M-u)\psi(t) \\ &= -\frac{d}{dt}\mu\big(S_k[u]\big) + \frac{d}{dt}\mu\left(\lambda(t) f_\delta\left(\frac{u}{\eta}\right)\right) - c^*(M-u)\psi(t) \\ &\leqslant \frac{d}{dt}\mu\left(\lambda(t) f_\delta\left(\frac{u}{\eta}\right)\right) - c^*(M-u)\psi(t).\end{aligned}$$

By (4.15) we have

$$\frac{u_t^2}{M-u} \leqslant C(1+M_T)U_T.$$

We obtain (4.17) from (4.18).

Next we prove

$$\sup_{Q_T} u_t \leqslant C(1+M_T) + C(1+M_T)U_T^{\frac{1}{2}}. \tag{4.23}$$

Similarly as above, let

$$G(x,t) = \frac{u_t}{M+u} - c^*\Psi(t).$$

If G attains its maximum on the parabolic boundary $\partial^* Q_T$, then $u_t \leqslant C$ for some $C > 0$ depending only on the initial value u_0. So we may assume that G attains its maximum at some point (x_0, t_0) in Q_T and $u_t(x_0, t_0) \geqslant 0$. Let $u_t(x^*, t^*) = \sup_{Q_T} u_t$. Then as above,

$$u_t\big(x^*, t^*\big) \leqslant 3u_t(x_0, t_0) + C(1+M_T). \tag{4.24}$$

At (x_0, t_0) we have $G_t \geqslant 0$, $G_j = 0$ for $1 \leqslant j \leqslant n$ and matrix $\{G_{ij}(x_0, t_0)\} \leqslant 0$, namely

$$\begin{gathered} u_{tt} - \frac{u_t^2}{M+u} - c^*(M+u)\psi(t) \geqslant 0, \\ u_{tj} - \frac{u_t u_j}{M+u} = 0 \quad \text{for } 1 \leqslant j \leqslant n, \\ \left\{u_{ijt} - \frac{u_t u_{ij}}{M+u}\right\} \leqslant 0. \end{gathered} \tag{4.25}$$

Hence by (4.21),

$$\begin{aligned}\frac{u_t^2}{M+u} &\leqslant u_{tt} - c^*(M+u)\psi(t) \\ &= \frac{d}{dt}\mu\big(S_k[u]\big) - \frac{d}{dt}\mu\left(\lambda(t) f_\delta\left(\frac{u}{\eta}\right)\right) - c^*(M+u)\psi(t).\end{aligned} \tag{4.26}$$

If $S_k[u] \leqslant 1$, by Eq. (4.5), estimate (4.11), and noting that $f_\delta \geqslant \delta$, we have

$$u_t \leqslant \mu(1) - \mu\left(\lambda(t) f_\delta\left(\frac{u}{\eta}\right)\right) \leqslant \mu(1) - \mu\big(\lambda(t)\delta\big) \leqslant C,$$

we are through. If $S_k[u] \geqslant 1$, we have $\mu'(S_k[u]) = \frac{1}{q}(S_k[u])^{\frac{1}{q}-1}$ and therefore we obtain by (4.25) and (4.5), at (x_0, t_0),

$$\begin{aligned} \frac{d}{dt}\mu\big(S_k[u]\big) &= \mu'\big(S_k[u]\big) S_k^{ij} u_{ijt} \\ &\leqslant \mu'\big(S_k[u]\big) \frac{u_t S_k^{ij} u_{ij}}{M+u} \\ &= \frac{1}{q}\big(S_k[u]\big)^{\frac{1}{q}-1} \frac{k u_t S_k[u]}{M+u} \\ &= \frac{k u_t}{q(M+u)}\big(S_k[u]\big)^{\frac{1}{q}} \\ &= \frac{k u_t}{q(M+u)}\left[u_t + \mu\left(\lambda(t) f_\delta\left(\frac{u}{\eta}\right)\right)\right]. \end{aligned} \tag{4.27}$$

By our choice $q = m\beta_0$, $\mu(\lambda(t) f_\delta(\frac{u}{\eta}))$ is of linear growth in u. Hence by (4.11),

$$\frac{1}{M+u}\mu\left(\lambda(t) f_\delta\left(\frac{u}{\eta}\right)\right) \leqslant C.$$

Therefore we obtain from (4.26)

$$\left(1 - \frac{k}{q}\right)\frac{u_t^2}{M+u} \leqslant C u_t - \frac{d}{dt}\mu\left(\lambda(t) f_\delta\left(\frac{u}{\eta}\right)\right) - c^*(M+u)\psi(t).$$

Using (4.15) again, we obtain

$$\frac{q-k}{q}\frac{u_t^2}{M+u} \leqslant C(1+M_T)U_T,$$

and as above, (4.23) follows from (4.24).

Combining (4.17) and (4.23) we obtain

$$U_T \leqslant C(1+M_T) + C(1+M_T)U_T^{1/2}.$$

Hence $U_T \leqslant C(1+M_T^2)$ for a different C. □

Remark 4.3. The proof of Lemma 4.2 is inspired by the argument in [4,5,27]. But to control the terms arising in computing $\frac{d}{dt}\lambda(\|u(\cdot,t)\|)$ and $\frac{d}{dt}\eta(\|u(\cdot,t)\|)$, we have to change the auxiliary function G by adding $c^*\Psi(t)$ to it, making use of the boundedness of $Y_{\delta,\eta}$ and (4.6).

Lemma 4.3. *Let $u \in C^4(Q_T) \cap C^3(\overline{Q}_T)$ be a k-admissible solution satisfying* (4.9). *Then* $\exists C > 0$*, which depends on n, m, δ, Ω, and the initial function u_0, but is independent of T, such that*

$$\sup_{Q_T} \left|Du(x,t)\right| \leqslant C\left(1 + M_T^{\frac{2q}{k}}\right), \tag{4.28}$$

where Du is the derivative of u in x.

Lemma 4.3 can be obtained in a similar way as in [5], see Theorem 5.1 of [5]. Note that in Eq. (4.5), λ and η are functions of t. They do not give us any trouble for the estimation of $\sup_{Q_T} |Du|$. We omit the details here. We remark that in the case $k < \frac{n}{2}$ in [5], a precise power of M_T is required for the L^∞ estimate for u. In this paper, we consider the case $k = \frac{n}{2}$ only. Any positive power of M_T in (4.28) is sufficient for the L^∞ estimate below.

From the gradient estimate, we obtain the estimate for $\sup_{Q_T} |u(\cdot, t)|$, uniformly in t.

Lemma 4.4. *Let $u \in C^4(Q_T) \cap C^3(\overline{Q}_T)$ be a solution of* (4.5) *satisfying* (4.9). *Then* $\exists C > 0$*, which depends on n, m, δ, Ω, and the initial function u_0, but is independent of T, such that*

$$\sup_{Q_T} \left|u(x,t)\right| \leqslant C. \tag{4.29}$$

Proof. Suppose $|u|$ attains its maximum at (x_0, t_0). By the gradient estimate (4.28), we have $B(x_0, \rho) \subset \Omega$ ($\rho = \frac{M_T}{2CM^{\frac{2q}{k}}}$) and

$$\left|u(x, t_0)\right| \geqslant \frac{1}{2} M_T \quad \text{for } |x - x_0| < \rho.$$

Therefore

$$\int_\Omega |u|^r \geqslant \int_{B(x_0,\rho)} |u|^r \geqslant C M_T^{r + (1 - \frac{2q}{k})n} = C M_T$$

if we choose $r = n(\frac{2q}{k} - 1) + 1$. By (4.9) and the Sobolev type inequality (1.7),

$$\left\|u(\cdot, t_0)\right\|_{L^r(\Omega)} \leqslant C\|u\|_{\Phi_0^k} \leqslant C.$$

Hence (4.29) is proved. □

Lemma 4.5. *Let $u \in C^4(Q_T) \cap C^3(\overline{Q}_T)$ be a k-admissible solution of* (4.5) *satisfying* (4.9). *Then* $\exists C > 0$*, which depends on n, m, δ, Ω, and the initial function u_0, but is independent of T, such that*

$$\sup_{Q_T} \left(\left|D_x^2 u(x,t)\right| + \left|\partial_t u(x,t)\right|\right) \leqslant C. \tag{4.30}$$

Proof. By Lemmas 4.3 and 4.4, $\sup(|u| + |Du|)$ is bounded. The estimate for $\partial_t u$ was already established in Lemma 4.2. The estimate for $D_x^2 u$ is similar to that in [5], as the function λ, η are independent of the variable x. We omit the details of the proof here. □

Theorem 4.2. *Let $u \in C^\infty(\overline{Q}_T)$ be a k-admissible solution of* (4.5) *satisfying* (4.9). *Then $\exists C > 0$, which depends on n, m, k, l, δ and Ω, and the initial function $u_0 \in C^\infty(\overline{\Omega}$, but is independent of T, such that*

$$\|u\|_{C^{k,l}_{x,t}(\overline{Q}_T)} \leqslant C. \tag{4.31}$$

Proof. Multiplying both sides of (4.5) by $S_k[u]$ and then integrating over Ω, by Lemma 4.5 we obtain

$$\left| \int_\Omega u_t S_k[u]\, dx \right| \leqslant C.$$

Since

$$\frac{d}{dt}\|u\|^{k+1}_{\Phi^k_0} = (k+1)\int_\Omega -u_t S_k[u]\, dx,$$

by (4.9) we see that $\|u(\cdot,t)\|_{\Phi^k_0}$ is uniformly Lipschitz in t. Hence λ and η in (4.5) are uniformly Lipschitz in t. Note that by (4.30), Eq. (4.5) is uniformly parabolic. Hence by the regularity theory of Krylov it follows

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}_{x,t}} \leqslant C. \tag{4.32}$$

Estimate (4.32) then implies that λ and η are uniformly $C^{1,\alpha}$ smooth in t. By iteration we obtain (4.31) for any $k, l \geqslant 2$. □

Remark 4.4. By estimate (4.30), Eq. (4.5) becomes uniformly parabolic. However, since the norm $\|u\|_{\Phi^k_0}$ involves second derivatives, (4.30) does not imply λ and η are Hölder or Lipschitz continuous, and we cannot apply Krylov's regularity theory [11] to get higher regularity directly. We were stuck at the point for long time. We did find a proof of the interior $C^{2+\alpha,1+\alpha/2}_{x,t}$ estimate for the more general parabolic equation

$$u_t = F\big(D^2u, Du, u, x\big) - f\left(x, u, Du, \int G\right), \tag{4.33}$$

where $G = G(D^2u, Du, u, x)$ is a constraint involving second derivatives. But at the moment we were not able to prove the boundary estimate and the higher regularity for Eq. (4.32). Most recently we realized that for Eq. (4.5), $\|u(\cdot,t)\|_{\Phi^k_0}$ is indeed Lipschitz, which implies the global regularity immediately, as shown above.

5. Proof of Theorem 4.1

With the global regularity, Theorem 4.2, we are in position to prove Theorem 4.1. First we prove the local existence of solutions to the initial boundary value problem (4.5).

Lemma 5.1. *Suppose the initial function* $u_0 \in C^{4+\alpha}(\overline{\Omega}) \cap \Phi_0^k(\Omega)$ $(\alpha \in (0,1))$, *satisfies the compatibility condition at* $\partial\Omega \times \{t=0\}$. *Then for* $T > 0$ *small, there is a unique local solution* $u \in C_{x,t}^{4+\alpha,2+\alpha/2}(\overline{Q}_T)$ *to the problem* (4.5).

There are several papers dealing with fully nonlinear parabolic equations with constraints involving the second derivatives, but in most papers the proof of the local existence is very vague. A natural idea is to introduce the mapping in the standard Banach space $B := C_{x,t}^{4+\alpha,2+\alpha/2}(\overline{Q}_T)$ such that for any $v \in B$, $u = \mathcal{M}(v)$ is the solution of

$$\begin{aligned} u_t = \mu\big(S_k[u]\big) - \mu\left(\lambda_v(t) f_\delta\left(\frac{v}{\eta_v}\right)\right) &\quad \text{in } Q_T, \\ u = 0 &\quad \text{on } \partial\Omega \times [0,\infty), \\ u = u_0 &\quad \text{on } \Omega \times \{t=0\}, \end{aligned} \tag{5.1}$$

where λ_v and η_v are the functions in (4.5) with u replaced by v. But since λ and η are integrals involving second derivatives, we cannot prove the mapping $\mathcal{M}$ is precompact or contractive if we work in the space B. In other words, the usual methods for the local existence of solutions to parabolic equations do not apply to fully nonlinear equations with constraints involving second derivatives.

Our trick here is to introduce the Banach space

$$B^* = \big\{u, u_x, u_{xx} \in C_{x,t}^{2+\alpha,1+\alpha/2}(\overline{Q}_T)\big\} \tag{5.2}$$

equipped with the usual norm for u, u_x, u_{xx} in $C_{x,t}^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ [15]. We will prove the mapping $\mathcal{M}$ is contractive for sufficiently small $T > 0$. Therefore the local existence of solutions follows from the contraction mapping theorem. Note that in the a priori estimates in Section 4, we assume the initial function u_0 satisfies $J_{\delta,\eta}(u_0) > Y_{\delta,\eta} - \varepsilon$. But for local existence, this assumption is not needed.

Proof. To prove the local existence we use the contraction mapping theorem for the mapping $\mathcal{M}$ in the ball

$$B_r^*(u_0) = \big\{u \in B^*\colon \|u - u_0\|_{B^*} < r\big\}, \tag{5.3}$$

where $r > 0$ is a small constant, and u_0 is regarded as a function in B^*. For any $v \in B_r^*(u_0)$, let $u = \mathcal{M}(v)$. As in Section 4 we can establish the global regularity for u, namely

$$\|u\|_{C_{x,t}^{4+\alpha,2+\alpha/2}(\overline{Q}_T)} \leqslant C \tag{5.4}$$

for some C depending on $\|\widetilde{f}(v)\|_{C_{x,t}^{2+\alpha,1+\alpha/2}(\overline{Q}_T)}$, which is uniformly bounded when $v \in B_r^*(u_0)$, where

$$\widetilde{f}(v) = \mu\left(\lambda_v(t) f_\delta\left(\frac{v}{\eta_v}\right)\right).$$

Note that the global regularity implies $\|u(\cdot,t)\|_{\Phi_0^k} = \|u_0\|_{\Phi_0^k} + o(1)$ as $t \to 0$, and for $T > 0$ small, $u = \mathcal{M}(v) \in B_r^*(u_0)$.

Estimate (5.4) is not enough such that the mapping $\mathcal{M}$ is contractive. But we can raise the regularity of u in x, since the function $\widetilde{f}$ has better smooth condition in x than in t. Differentiate Eq. (5.1) in x, we get

$$(u_{x_i})_t = L[u_{x_i}] - \partial_{x_i}\widetilde{f}, \tag{5.5}$$

where L is the linearized operator of $\mu(S_k[u])$, which is uniformly elliptic due to the estimate (5.4). The least and largest eigenvalues of L depend only on the estimate (5.4). Hence by the regularity theory of linear parabolic equations [11],

$$\|u_{x_i}\|_{C_{x,t}^{4+\alpha,2+\alpha/2}(\overline{Q}_T)} \leqslant C. \tag{5.6}$$

Differentiating (5.1) in x again, we obtain estimate for $\|u_{x_ix_j}\|_{C_{x,t}^{4+\alpha,2+\alpha/2}(\overline{Q}_T)}$.

We show that the mapping $\mathcal{M}$ is contractive when $T > 0$ is small. For any given $v_1, v_2 \in B_r^*(u_0)$, let $u_1 = \mathcal{M}(v_1)$ and $u_2 = \mathcal{M}(u_2)$ be the corresponding solutions to (5.1). Then by the a priori estimate (5.4), $u_1 - u_2$ satisfies a linear, uniformly parabolic equation

$$\partial_t(u_1 - u_2) = \hat{L}[u_1 - u_2] - \big(\widetilde{f}(v_1) - \widetilde{f}(v_2)\big). \tag{5.7}$$

One can easily verify that

$$\big|\widetilde{f}(v_1) - \widetilde{f}(v_2)\big| \leqslant C_1\|v_1 - v_2\|_{B^*}.$$

On the other hand, by constructing proper sub and super solutions, we have

$$\sup_{Q_T}|u_1 - u_2| \leqslant C_2T\|v_1 - v_2\|_{B^*}. \tag{5.8}$$

Here the constants C_1, C_2 may look like depending on u and v. But since $v \in B_r^*(u_0)$ and by the estimate (5.4), C_1, C_2 are uniformly bounded with an upper bound depending on r and C in (5.4) but independent of u and v. Similarly to (5.6) we have the estimate

$$\big\|(u_1 - u_2)_{x_i}\big\|_{C_{x,t}^{4+\alpha,2+\alpha/2}(\overline{Q}_T)} \leqslant C\|v_1 - v_2\|_{B^*}. \tag{5.9}$$

Hence

$$\big\|(u_1 - u_2)_{xxx}\big\|_{C_{x,t}^{2+\alpha,1+\alpha/2}(\overline{Q}_T)} \leqslant C\|v_1 - v_2\|_{B^*}.$$

By the interpolation,

$$\big\|(u_1 - u_2)_{xx}\big\|_{C_{x,t}^{2+\alpha,1+\alpha/2}(\overline{Q}_T)} \leqslant C\big\|(u_1 - u_2)_{xxx}\big\|^{2/3}_{C_{x,t}^{2+\alpha,1+\alpha/2}(\overline{Q}_T)}\|u_1 - u_2\|^{1/3}_{C_{x,t}^{2+\alpha,1+\alpha/2}(\overline{Q}_T)}.$$

Similarly to (5.9) we have

$$\|u_1 - u_2\|_{C_{x,t}^{4+\alpha,2+\alpha/2}(\overline{Q}_T)} \leqslant C\|v_1 - v_2\|_{B^*}.$$

Hence by (5.8) and the interpolation again,

$$\|u_1 - u_2\|_{C^{2+\alpha,1+\alpha/2}_{x,t}(\overline{Q}_T)} \leqslant CT^{\sigma} \|v_1 - v_2\|_{B^*}$$

for some $\sigma > 0$. We obtain

$$\big\|(u_1 - u_2)_{xx}\big\|_{C^{2+\alpha,1+\alpha/2}_{x,t}(\overline{Q}_T)} \leqslant C_3 T^{\sigma/3} \|v_1 - v_2\|_{B^*}. \tag{5.10}$$

Therefore when $T > 0$ is small, we obtain the existence of a local solution by the contraction mapping theorem. □

By the local existence and the global estimate (4.31), the solution u to (5.1) exists on the maximal time $[0, \infty)$. Now we are in a position to prove Theorem 4.1.

Proof of Theorem 4.1. By the global estimate (4.31) and since $\frac{d}{dt} J_{\delta,\eta}(u(\cdot,t)) \geqslant 0$, $Y_{\delta,\eta} - \varepsilon \leqslant J_{\delta,\eta}(u) \leqslant Y_{\delta,\eta}$, we have

$$\frac{d}{dt} J_{\delta,\eta}\big(u(\cdot,t)\big) \to 0$$

uniformly as $t \to \infty$. Hence

$$\left\| S_k\big[u(\cdot,t)\big] - \lambda(t) f_\delta\left(\frac{u}{\eta}\right)\right\|_{C^3(\overline{\Omega})} \to 0$$

as $t \to \infty$. Choosing a subsequence t_j such that $\|u(\cdot,t_j)\|$ and $\lambda(t_j)$ converge, we conclude that $u(\cdot,t_j)$ converges to a smooth solution $u = u_{\varepsilon,\delta,\eta}$ of

$$\begin{aligned} S_k[u] &= \lambda f_\delta\left(\frac{u}{\eta}\right) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5.11}$$

and u satisfies by (4.29) the uniform estimate

$$\sup_{\Omega} \big|u(x)\big| \leqslant C, \tag{5.12}$$

where λ is given in (4.5). Note that the solution depends on the choice of the initial function u_0 and so also depends on ε.

We claim that the constant C in (5.12) is independent of ε. Indeed, similarly to (4.28) we can establish a gradient estimate $|Du| \leqslant C(1 + \sup_{\Omega} |u|)^{2q/k}$ for some constant C independent of ε, then apply the proof of Lemma 4.4 to obtain (5.12).

By the monotonicity formula (4.6), we have

$$Y_{\delta,\eta} - \varepsilon \leqslant J_{\delta,\eta}(u_{\varepsilon,\delta,\eta}) \leqslant Y_{\delta,\eta}.$$

For fixed $\delta > 0$, by the uniform estimate (5.12) and global regularity of solutions to the k-Hessian equation [1,27], we may assume that $u_{\varepsilon,\delta,\eta}$ converges in $C^3(\overline{\Omega})$ as $\varepsilon \to 0$ to a solution $u_{\delta,\eta}$ of (5.11) with $\varepsilon = 0$, which satisfies

$$\begin{cases} \|u_{\delta,\eta}\|_{\Phi_0^k} = 1 \quad \text{(by Lemma 4.1)}, \\ J_{\delta,\eta}(u_{\delta,\eta}) = Y_{\delta,\eta}, \\ \lambda = \left[\int\limits_{\Omega} f_\delta(u_{\delta,\eta})|u_{\delta,\eta}|\,dx \right]^{-1}. \end{cases} \tag{5.13}$$

Note that when $\|u\|_{\Phi_0^k} = 1$, $\eta(\|u\|) = \|u\| = 1$. Hence Eq. (5.11) and the quantities in (5.13) are independent of η. So let us drop the subscript η below.

Now for fixed m, recall that $F(t) = \sum_{j=k}^m \frac{\alpha^j}{j!}|t|^{j\beta}$ and $F_\delta(t) = F(t) + \delta|t|$. We have

$$F_\delta(t) \leqslant t f_\delta(t) \leqslant m\beta F_\delta(t).$$

Hence

$$[m\beta Y_\delta]^{-1} \leqslant \lambda \leqslant [Y_\delta]^{-1}.$$

By the Sobolev type inequality (1.7), we have $Y_\delta \to Y$ as $\delta \to 0$, where Y is given in (4.1).

By the a priori estimates in [1], see also [27], we have

$$\|u_\delta\|_{C^{1,1}(\overline{\Omega})} \leqslant C \tag{5.14}$$

for some $C > 0$ independent of $\delta \in (0,1)$. Therefore in any subdomain $\Omega' \subset\subset \Omega$, by (5.13), (5.14) and the subharmonicity of u_δ, there is a constant $C = C_{\Omega'}$ independent of $\delta \in (0,1)$ such that $u_\delta \leqslant -C$ in Ω'. Hence Eq. (5.11) is non-degenerate, uniformly in δ, in Ω'. By Evans and Krylov's regularity theory for fully nonlinear, uniformly elliptic equation [6], we obtain the interior estimate for high order derivatives, namely $\|u_\delta\|_{C^3(\Omega')} \leqslant C_1(\Omega')$. See also the interior estimates in [5]. Therefore by passing to a subsequence we assume that $u_\delta \to u^*$ as $\delta \to 0$. Then $u^* \in C^3(\Omega) \cap C^{1,1}(\overline{\Omega})$ is a maximizer of (4.1) and satisfies the Euler equation (4.2). This completes the proof. □

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References

[1] L. Caffarelli, L. Nirenberg, J. Spruck, Dirichlet problem for nonlinear second order elliptic equations III. Functions of the eigenvalues of the Hessian, Acta Math. 155 (1985) 261–301.
[2] K.S. Chou, On symmetrization and Hessian equations, J. Anal. Math. 52 (1989) 94–116.
[3] K.S. Chou, Remarks on the critical exponents for the Hessian operators, Ann. Inst. H. Poincaré Anal. Non linéaire 7 (1990) 113–122.
[4] K.S. Chou, On a real Monge–Ampere functional, Invent. Math. 101 (1990) 425–448.

[5] K.S. Chou, X.-J. Wang, Variational theory for Hessian equations, Comm. Pure Appl. Math. 54 (2001) 1029–1064.

[6] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, 1983.

[7] B. Guan, The Dirichlet problem for a class of fully nonlinear elliptic equations, Comm. Partial Differential Equations 19 (1994) 399–416.

[8] B. Guan, The Dirichlet problem for Hessian equations on Riemannian manifolds, Calc. Var. Partial Differential Equations 8 (1999) 45–69.

[9] Tian Guji, X.-J. Wang, A class of Sobolev type inequalities, Methods Appl. Anal. 15 (2) (2008) 263–276.

[10] N. Ivochkina, Solution of the Dirichlet problem for some equations of the Monge–Ampere type, Mat. Sb. 128 (1985) 403–415.

[11] N.V. Krylov, Nonlinear Elliptic and Parabolic Equations of Second Order, Nauka, Moscow, 1985, English translation: Reidel Publishing Co., Dordrecht, 1987.

[12] N.V. Krylov, On the general notion of fully nonlinear second-order elliptic equations., Trans. Amer. Math. Soc. 347 (1995) 857–895.

[13] D. Labutin, Potential estimates for a class of fully nonlinear elliptic equations, Duke Math. J. 111 (2002) 1–49.

[14] Y.Y. Li, Some existence results for fully nonlinear elliptic equations of Monge–Ampère type, Comm. Pure Appl. Math. 43 (1990) 233–271.

[15] G. Lieberman, Second Order Parabolic Differential Equations, World Scientific, Singapore, 1996.

[16] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J. 20 (1971) 1077–1092.

[17] B. Opic, A. Kufner, Hardy-Type Inequalities, Pitman Res. Notes Math., vol. 219, Longman Scientific and Technical, Harlow, 1990.

[18] N.C. Phuc, I.E. Verbitsky, Local integral estimates and removable singularities for quasilinear and Hessian equations with nonlinear source terms, Comm. Partial Differential Equations 31 (2006) 1779–1791.

[19] N.C. Phuc, I.E. Verbitsky, Quasilinear and Hessian equations of Lane–Emden type, Ann. of Math. 168 (2008) 859–914.

[20] N.S. Trudinger, On imbedding into Orlicz spaces and some applications, J. Math. Mech. 17 (1967) 473–484.

[21] N.S. Trudinger, On the Dirichlet problem for Hessian equations, Acta Math. 175 (1995) 151–164.

[22] N.S. Trudinger, X.-J. Wang, A Poincaré type inequality for Hessian integrals, Calc. Var. Partial Differential Equations 6 (1998) 315–328.

[23] N.S. Trudinger, X.-J. Wang, Hessian measures II, Ann. of Math. 150 (1999) 579–604.

[24] J. Urbas, On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations, Indiana Univ. Math. J. 39 (1990) 355–382.

[25] J. Urbas, An interior second derivative bound for solutions of Hessian equations, Calc. Var. Partial Differential Equations 12 (2001) 417–431.

[26] I.E. Verbitsky, The Hessian Sobolev inequality and its extensions, preprint.

[27] X.-J. Wang, A class of fully nonlinear elliptic equations and related functionals, Indiana Univ. Math. J. 43 (1994) 25–54.

[28] X.-J. Wang, The k-Hessian Equation, Lecture Notes in Math., vol. 1977, Springer, 2009, pp. 177–252.