Perturbations from an Elliptic Hamiltonian of Degree Four
I. Saddle Loop and Two Saddle Cycle

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The paper deals with Liénard equations of the form \( \dot{x} = y, \dot{y} = P(x) + yQ(x) \) with \( P \) and \( Q \) polynomials of degree respectively 3 and 2. Attention goes to perturbations of the Hamiltonian vector fields with an elliptic Hamiltonian of degree 4 and especially to the study of the related elliptic integrals. Besides some general results the paper contains a complete treatment of the Saddle Loop case and the Two Saddle Cycle case. It is proven that the related elliptic integrals have at most two zeros, respectively one zero, the multiplicity taken into account. The bifurcation diagram of the zeros is also obtained.

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1. INTRODUCTION AND STATEMENT OF RESULTS

In describing mathematically natural phenomena one often uses families of planar differential equations, if not directly as the most appropriate models, then indirectly as a simplification that might be tractable for a complete study.

Unfortunately relatively simple nonlinear systems defeat a complete study since no general methods are known to study e.g. the limit cycles and their bifurcations. Already for polynomial planar vector fields there is the famous 16th problem of Hilbert asking for an upper bound on the number of limit cycles depending on the degree of the vector field. It is even not known whether a finite upper bound exists. Also for the limited class of

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(generalized) Liénard equations \( \dot{x} = y, \quad \dot{y} = P(x) + yQ(x) \), with \( P \) and \( Q \) polynomial, Hilbert’s 16th problem is still unsolved. These Liénard equations are representations in phase plane of the second order scalar differential equations \( x'' - x'Q(x) - P(x) = 0 \) and can be met in many constructions and applications. They are e.g. unavoidable in the study of local bifurcations by means of rescaling techniques. We say to have a Liénard equation of type \((m, n)\) if \( \deg P = m \) and \( \deg Q = n \). A complete study has been made for the cases \( m + n \leq 4 \), except for \((m, n) = (1, 3)\); we refer to [DR], [DL1], [DL2] and ref. [LMP]. In all these cases it has been proven that there is at most one limit cycle and for \((1, 3)\) the same has been conjectured (see ref. [LMP]). For \( m + n \geq 5 \) no general results have been obtained, except for local ones, near non-degenerate singularities. We refer to [CL] for a recent account of the known results. An interesting case to consider is definitely \((m, n) = (3, 2)\).

In that case the maximum number of local limit cycles is two. This local analysis is for sure a starting point in a global approach but it can clearly not be expected that the local results will trivially extend, even not in case there is globally only one singularity. For this we can refer to [M] where strong numerical evidence has been given for the existence of systems with four limit cycles; we can also refer to [KKR] where the occurrence of a quadruple limit cycle, together with a full unfolding, has been conjectured.

In a series of papers we intend to study the Liénard equations of type \((3, 2)\) that are small perturbations of Hamiltonian vector fields with an elliptic Hamiltonian of degree four. After linear rescaling the Hamiltonians are given by the functions \( H(x, y) = y^2/2 \pm x^4/4 + a(x^3/3) + b(x^2/2) \), with \((a, b) \in \mathbb{R}^2\). As announced the perturbations are given by adding \( \delta y(x^2 + \beta x + \alpha) \) for small \( \delta > 0 \). It is well known that a first step in studying the limit cycles consists in calculating the zeros of the Abelian integrals, or more precisely the elliptic integrals, obtained by integrating the related 1-form \( y(x^2 + \beta x + \alpha) \, dx \) over the compact level curves of the Hamiltonian \( H \).

The study of the zeros of Abelian integrals obtained by integrating polynomial 1-forms over level curves of polynomial Hamiltonians is called the weak 16th problem of Hilbert or the Hilbert–Arnold problem. In that program it is natural to start looking at Newtonian mechanical problems, i.e. restricting to Hamiltonians of the form \( H_{n+1}(x, y) = y^2/2 + P_{n+1}(x) \), where \( P_{n+1}(x) \) is a polynomial in \( x \) of degree \( n + 1 \geq 3 \). For \( n = 2 \), the level curves of \( H_3(x, y) \) may contain at most a saddle loop or a cuspidal point. Many authors have studied the number of zeros of the elliptic integrals obtained by integrating the 1-forms \( f(x, y) \, dx \), with \( f \) a polynomial of degree \( m + 1 \), over the compact level curves of the Hamiltonians \( H_f \). Results can be found in ref. [B] for \( m = 1 \) and 2, in ref. [DRS] for \( m = 3 \) and in
ref. [LR] for \( m = 4 \). Finally it was proved in ref. [P1], [P2] that for any \( m \geq 1 \) the sharp upper bound of the number of zeros of the corresponding elliptic integrals is \( \left\lfloor \frac{2m+1}{3} \right\rfloor \). Note that the integration of a 1-form \( g(x, y) \, dy \) over the compact level curves of \( H_n+1(x, y) \) can be changed to the integration of \( -yg_x(x, y) \, dx \).

Having this complete study for elliptic integrals in case \( n = 2 \), it is hence very natural to consider the case \( n = 3 \), as we intend to do.

For \( n = 3 \) and if the level curves of \( H_n(x, y) \) contain compact components, then there are five different types, shown in Figs. 1A–1B; they are respectively called the cases of two saddle cycle, saddle loop, global center, cuspidal loop and figure eight-loop. Note that case (A) is a limiting case of (B); and case (D) is a limiting case of (C) and (E).

In all cases we will restrict to the integration of 1-forms \( y(x^2+\beta x + a) \, dx \), aiming at a complete investigation of the exact number of zeros. Our results hence merely deal with the zeros of the elliptic integrals. They however have some consequences on the study of limit cycles for the equations 
\[
\dot{x} = y, \quad \dot{y} = \pm x^3 + ax^2 + bx + y(x^2+\beta x + a)
\]
with \( \beta > 0 \) but small.

Some results on limit cycles are immediate by using the implicit function theorem. Whenever this is the case we will point it out. Other results on limit cycles might require some extra analysis and we will not carry it out.

Since the calculations are quite lengthy and different estimations are required depending on the case under consideration, we prefer to treat different cases separately.

In this paper we first present some generalities on our approach and second tackle the Saddle Loop case (B) and the Two Saddle Cycle case (A). Three forthcoming papers will deal with respectively the Cuspidal Loop case (D), the Global Centre case (C) and the Figure-eight Loop case (E).

In Section 2 we first calculate the Picard-Fuchs equation for \( I_k(h) \), where \( I_k(h) \) is the elliptic integral obtained by integrating the 1-form \( x^k y \, dx \), with
$k = 0, 1, 2$. As is usual in this theory we also consider the related differential equation on $P(h) = \frac{h_1^{(0)}}{e^{h_0}}$ and $Q(h) = \frac{h_2^{(0)}}{e^{h_0}}$. It provides a first order differential equation in $(h, P, Q)$-space, which unfortunately is not easy to deal with. If we combine it with a study of $\omega(h) = \frac{v_1^{(0)}}{v_0^{(0)}}$ and $\nu(h) = \frac{v_2^{(0)}}{v_0^{(0)}}$ it will be possible to get the necessary results, since $\omega$ satisfies a Ricatti-equation.

This observation has first been made in ref. [Z] and we intend to use it in the treatment of all cases (A)–(E).

After these general calculations we turn to the cases (A) and (B). Up to linear coordinate changes there is a 1-parameter family $H_l(x, y) = y^2 - \frac{1}{4}x^4 - \frac{\lambda - 1}{3}x^3 + \frac{\lambda}{2}x^2$ of Hamiltonians, with $\lambda \in [1, \infty)$, representing a Two Saddle Cycle for $\lambda = 1$ and Saddle Loops for $\lambda > 1$. The special situation of the TSC, nl. $\lambda = 1$, is rather simple and has been studied completely in ref. [H]. The maximum number of zeros is one. Throughout the paper we will hence not deal with it, except in paragraph 6 where we will link the results that we will get on the Saddle Loops with the knowledge on the Two Saddle Cycle. In Fig. 20 we show how the bifurcation diagram of the zeros for the SL-case degenerates into the one for the TSC-case. In the literature some partial results can be found. Petrov in ref. [P3] proved that the maximal number is at most 4 while Zhao in ref. [Z] obtained a sharp upper bound for the limited range of parameters $\lambda \in \left(\frac{1}{4}(7 + \sqrt{33}), +\infty\right)$.

In the rest of the paper we essentially intend to prove the following theorem:

**Theorem 1.** If we integrate the 1-form $(x^2 + \beta x + \alpha) y \, dx$ over the compact level curves of the Hamiltonians

$$H_l(x, y) = \frac{y^2}{2} - \frac{1}{4}x^4 - \frac{\lambda - 1}{3}x^3 + \frac{\lambda}{2}x^2$$

with $\lambda \in (1, \infty)$, then for all constants $\alpha$ and $\beta$ the maximum number of zeros is two, taking into account the multiplicity. The bifurcation diagram of the zeros is as represented in Fig. 2. In this figure $H$ stands for a line of Hopf bifurcations, $L$ a line of saddle loop bifurcations, $DC$ a curve of double limit cycle bifurcations, and the points $H_2$ and $L_2$ represent respectively a Hopf bifurcation and a saddle loop bifurcation of codimension 2.

In Fig. 2 the line of Hopf bifurcations is always given by $\{x = 0\}$ and it is easy to calculate the precise position of $H_2$ (depending on $\lambda$). The other curves $L$ and $DC$ are represented in a qualitative way. The precise position changes with $\lambda$ and is more difficult to situate numerically.

In Section 6 we will indicate how this result on zeros of the elliptic integrals can be translated to a similar result on the limit cycles of the SADDLE LOOP AND TWO SADDLE CYCLE 117
FIG. 2. Bifurcation diagram in $(\alpha, \beta)$-plane.

equations “$\dot{x} = y, \dot{y} = x^3 + (\lambda - 1)x^2 - \lambda x + \delta y(x^2 + \beta x + \alpha)$” with $\lambda \in (1, \infty)$ and $\delta > 0$ sufficiently small.

The theorem will follow by proving that $(P(h), Q(h))$ is a convex curve in $(P, Q)$-plane having everywhere a non-zero curvature.

Indeed, for $h > 0$, the number of zeros of

$$I(h) = aI_0(h) + bI_1(h) + I_2(h)$$

is the number of intersection points of the straight line

$$L: \{(x, y) \mid ax + by + c = 0\}$$

and the curve $\Sigma = \{(P, Q)(h) \mid h \in (0, h_1)\}$.

We will essentially prove that $\Sigma$ has no inflection points nor quadruple points.

A crucial observation will be proven in lemma 13 of Section 5, namely that when $\mathcal{L}$ and $\Sigma$ have at least two intersection points, multiplicity taken into account, then for the same $\alpha$ and $\beta$, in $(\omega, \nu)$-plane, the straight line $\mathcal{L}' = \{(x, y) \mid x + \beta \omega + \nu = 0\}$ can cut the curve $\Omega = \{(\omega, \nu)(h) \mid h \in [0, h_1]\}$ at most twice, counted the multiplicity.
These geometric ideas can be kept in mind when going through the proof of Theorem 1; the proof is spread over the next four paragraphs.

In the second part of Section 2 we obtain the necessary expressions for the differential equations on $(P, Q)$ and $(\omega, \nu)$. In Section 3 we obtain some interesting properties on $P, Q$ by direct calculation. In Section 4 a study is made in the $(h, \omega)$- and $(h, \nu)$-plane. This is the central part of the proof, providing information that cannot be obtained directly on $(P, Q)$. In Section 5 we finish the proof by simultaneously considering the curves $(P(h), Q(h))$ and $(\omega(h), \nu(h))$. In Section 6 we link our results to the known results on the Two Saddle Cycle.

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2. PICARD-FUCHS EQUATION AND RICCATI EQUATION

We consider the general form of the elliptic Hamiltonian function of degree four

$$H(x, y) = \frac{y^2}{2} + \frac{a}{4} x^4 + \frac{b}{3} x^3 + \frac{c}{2} x^2, \quad (a \neq 0)$$

(1)

associated to a Newtonian mechanical system. We restrict our attention to the values $h$ for which $C_h = \{(x, y) \mid H(x, y) = h\} \text{ or at least some component is compact. Let } I_k(h) = \int_{r_h} x^k y dx, k = 0, 1, 2, \ldots \text{ Since } I_k'(h) = \int_{r_h} y dx, \text{ we get}

$$I_k(h) = \int_{r_h} x^k y dx = \int_{r_h} x^k \left(2h - \frac{a}{2} x^4 - \frac{2}{3} b x^3 - c x^2\right) dx

= 2h I'_k(h) - c I'_{k+2}(h) - \frac{2}{3} b I'_{k+3}(h) - \frac{a}{2} I'_{k+4}(h).$$

(2)

On the other hand, integrating by parts and using $y dy + (ax^3 + bx^2 + cx) dx = 0$, we have

$$I_k(h) = \int_{r_h} x^k y dx = \int_{r_h} \frac{y}{k+1} dx^{k+1} = \frac{1}{k+1} \int_{r_h} \frac{ax^{k+4} + bx^{k+3} + cx^{k+2}}{y} dx

= \frac{1}{k+1} \left[a I'_{k+4}(h) + b I'_{k+3}(h) + c I'_{k+2}(h)\right].$$

(3)
Removing $I_{k+4}(h)$ from (2) and (3), we obtain

$$(k + 3) I_k(h) = 4h I'_k(h) - c I'_{k+2}(h) - \frac{b}{3} I'_{k+3}(h).$$  \hfill (4)$$

Taking $k = 0, 1, 2$, we have:

$$3I_0 = 4h I'_0 - c I'_2 - \frac{b}{3} I'_3,$$

$$4I_1 = 4h I'_1 - c I'_3 - \frac{b}{3} I'_4,$$

$$5I_2 = 4h I'_2 - c I'_4 - \frac{b}{3} I'_5.$$  \hfill (5)$$

Note that along $I_h$ we have $y^2 \, dy + (ax^3 + bx^2 + cx) \, y \, dx = 0$, hence

$$0 \equiv \int_{r_a} (ax^3 + bx^2 + cx) \, y \, dx = aI_1 + bI_2 + cI_3.$$  \hfill (6)$$

Using the derivative of (6) and (3) with $k = 0, 1$, we remove $I'_5, I'_4, I'_3$ from (5), and finally obtain

$$N \begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix} = (4hE + S) \begin{pmatrix} I'_0 \\ I'_1 \\ I'_2 \end{pmatrix},$$  \hfill (7)$$

where $E$ is the identity matrix, and

$$N = \begin{pmatrix} 3 & 0 & 0 \\ \frac{b}{3a} & 4 & 0 \\ \frac{3ac-b^2}{3a^2} & \frac{2b}{3a} & 5 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & bc & \frac{b^2-3ac}{3a} \\ \frac{3ac-b^2}{3a^2} & 0 & \frac{b(4ac-b^2)}{3a^2} \\ \frac{-bc(4ac-b^2)}{3a^3} & \frac{3a^2c^2-5ab^2c+b^4}{3a^4} & 0 \end{pmatrix}.$$  \hfill (8)$$

Differentiating (7), we get

$$(4hE + S) \begin{pmatrix} I''_0 \\ I''_1 \\ I''_2 \end{pmatrix} = (N - 4E) \begin{pmatrix} I'_0 \\ I'_1 \\ I'_2 \end{pmatrix}.$$  \hfill (8)$$
Using the special form of \((N-4E)\), and removing \(I_0'\) from the first two equations of (8), we get

\[
I_2'' = \frac{12a}{2b^2-9ac} I_0'' + \frac{1}{b} \left( \frac{36a^2}{2b^2-9ac} h - c \right) I_1'',
\]

(9)

if \(b(2b^2-9ac) \neq 0\).

From (7), we obtain the Picard-Fuchs equation

\[
G(h) \frac{d}{dh} \begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix},
\]

(10)

where

\[
G(h) = 12h[144a^4h^2 + 12(b^4 - 6ab^2c + 6a^2c^2) h - c^3(2b^2 - 9ac)],
\]
\[
a_{00} = 12[108a^3h^2 + (10b^4 - 61ab^2c + 63a^2c^2) h - c^3(2b^2 - 9ac)],
\]
\[
a_{01} = -2b[12a(b^2 + 3ac) h + 7c^2(2b^2 - 9ac)],
\]
\[
a_{02} = -15a[12a(b^2 - 3ac) h + c^3(2b^2 - 9ac)],
\]
\[
a_{10} = 12b[12a^2h + c(2b^2 - 9ac)] h,
\]
\[
a_{11} = 24[72a^2h + (7b^4 - 34ab^2c + 18a^2c^2)] h,
\]
\[
a_{12} = 180ab(b^2 - 4ac) h,
\]
\[
a_{20} = -12[12a(b^2 - 3ac) h + c^2(2b^2 - 9ac)] h,
\]
\[
a_{21} = 24b[12a^2h - c(7b^2 - 27ac)] h,
\]
\[
a_{22} = 180a[12a^2h - c(b^2 - 3ac)] h.
\]

Let \(P(h) = I_1(h)/I_0(h)\), \(Q(h) = I_2(h)/I_0(h)\); then from (10) we get the following system of equations for \(P(h)\) and \(Q(h)\)

\[
\dot{h} = G(h),
\]
\[
\dot{P} = a_{10} + a_{11} P + a_{12} Q - P(a_{00} + a_{01} P + a_{02} Q),
\]
\[
\dot{Q} = a_{20} + a_{21} P + a_{22} Q - Q(a_{00} + a_{01} P + a_{02} Q).
\]

(11)

Let \(\omega(h) = I_1''(h)/I_0''(h)\) and \(\nu(h) = I_2''(h)/I_0''(h)\); then from (9) we have

\[
\nu(h) = \frac{12a}{2b^2-9ac} h + \frac{1}{b} \left( \frac{36a^2}{2b^2-9ac} h - c \right) \omega(h).
\]

(12)
Differentiating (8) with respect to $h$, we have
\[
(4hE + S) \begin{pmatrix} I''_0 \\
I''_1 \end{pmatrix} = (N - 8E) \begin{pmatrix} I''_0 \\
I''_1 \end{pmatrix}.
\]

Substituting (9) into the above equations, we get
\[
G(h) \frac{d}{dh} \begin{pmatrix} I''_0 \\
I''_1 \end{pmatrix} = \begin{pmatrix} b_{00} & b_{01} \\
b_{10} & b_{11} \end{pmatrix} \begin{pmatrix} I''_0 \\
I''_1 \end{pmatrix},
\]
(13)
where $G(h)$ is the same as in (10) and (11), and
\[
b_{00} = \frac{432a^3(36ac - 7b^2)}{2b^2 - 9ac} h^2 - \left(864a^2c^2 - 996ab^2c + 168b^4\right) h + 12(2b^2 - 9ac) c^3,
\]
\[
b_{01} = \frac{3888a^4(b^2 - 3ac)}{b(2b^2 - 9ac)} h^2 + \frac{12a(54a^2c^2 + 9ab^2c - 2b^4)}{b} h
\]
+ \frac{(2b^2 - 9ac)(10b^2 - 9ac)c^2}{b},
\]
\[
b_{10} = -\frac{144a^2b(7b^2 - 27ac)}{2b^2 - 9ac} h^2 + 12bc(2b^2 - 9ac) h,
\]
\[
b_{11} = \frac{432a^3(72ac - 17b^2)}{2b^2 - 9ac} h^2 - 12(2b^2 - 9ac)(5b^2 - 8ac) h.
\]

From (13) we obtain the Riccati equation
\[
G(h) \omega'(h) = -b_{01} \omega^2 - (b_{00} - b_{11}) \omega + b_{10},
\]
which is equivalent to the system
\[
\dot{h} = G(h),
\]
\[
\dot{\omega} = -b_{01} \omega^2 - (b_{00} - b_{11}) \omega + b_{10} = \varphi(h, \omega),
\]
(14)
where $G(h)$, $b_{ij}$ are the same as above.

By using (10) we see that for $k = 0, 1, 2$
\[
G^2(h) I''_k(h) = A_{10} I_0(h) + A_{31} I_1(h) + A_{32} I_2(h),
\]
where
\[ A_{ij} = G(h) \frac{\partial a_{ij}}{\partial h} + a_{i0}a_{0j} + a_{i1}a_{1j} + a_{i2}a_{2j} - G'(h) a_{ij}, \]
i, j = 0, 1, 2. Hence
\[ \omega(h) = \frac{I_0^*(h)}{I_0(h)} = \frac{A_{00} + A_{01}P(h) + A_{02}Q(h)}{A_{00} + A_{01}P(h) + A_{02}Q(h)}, \]
\[ \nu(h) = \frac{I_1^*(h)}{I_1(h)} = \frac{A_{10} + A_{11}P(h) + A_{12}Q(h)}{A_{10} + A_{11}P(h) + A_{12}Q(h)}. \]

In this paper a lot of attention will go to the number of inflection points of the curve
\[ \Omega = \{(\omega, \nu)(h) \mid h \in (h_1, h_2)\}. \]

Using (12) we have
\[ \frac{d^2 \nu}{d\omega^2} = \frac{\nu''(h) \omega'(h) - \nu'(h) \omega''(h)}{\omega^3(h)} = \frac{12a[6a\omega^2 - (b + 3a\omega) \omega'']}{b(2b^2 - 9ac) \omega^3}. \]

From (14) we know
\[ G\omega' = \varphi(h, \omega), \]
\[ G^2\omega'' = \frac{\partial \varphi}{\partial h} G + \left( \frac{\partial \varphi}{\partial \omega} - G' \right) \varphi. \]

Hence \( \frac{d^2 \nu}{d\omega^2} = 0 \) is equivalent to
\[ F(h, \omega) = 0, \]
where
\[ F(h, \omega) = \left[ 6a\varphi^2 - (b + 3a\varphi) \left( \frac{\partial \varphi}{\partial h} G + \left( \frac{\partial \varphi}{\partial \omega} - G' \right) \varphi \right) \right] / b(2b^2 - 9ab), \]
the function \( \varphi = \varphi(h, \omega) \) is given in (14), and it is a polynomial of \((h, \omega)\).
Now we consider the saddle loop case. Without loss of generality we suppose that the center of the Hamiltonian system

\[ \dot{x} = y = \frac{\partial H}{\partial y}, \]
\[ \dot{y} = -\frac{\partial H}{\partial x} \]

is located at the origin, the saddle point, whose separatrices form a saddle loop, is at \((1, 0)\) and the other saddle point is at \((-\lambda, 0)\); see Fig. 3.

Then we must have \(\lambda > 1\), and the Hamiltonian function (1) becomes

\[ H(x, y) = \frac{y^2}{2} + \Phi(x), \quad (18) \]

where \(\Phi(x) = -\frac{1}{4} x^4 - \frac{\lambda - 1}{6} x^3 + \frac{\lambda}{4} x^2\).

Taking \(a = -1\), \(b = -\frac{\lambda - 1}{2}\) and \(c = \lambda\) in (11), (14) and (15) respectively, we obtain the systems

\[ \dot{h} = G(h), \]
\[ \dot{P} = f(h, P, Q, \lambda), \]
\[ \dot{Q} = g(h, P, Q, \lambda), \quad (19) \]
and
\begin{align}
\dot{h} &= G(h), \\
\dot{\omega} &= \varphi(h, \omega, \lambda),
\end{align}
(20)
where
\begin{align*}
G(h) &= -12^3 h(h-h_1)(h-h_2), \quad h_1 = (2\lambda+1)/12, \quad h_2 = \lambda^3(\lambda+2)/12. \\
f(h, P, Q, \lambda) &= -144(\lambda-1) h^2 - 12(\lambda-1)(\lambda+2)(2\lambda+1) \lambda h \\
&\quad + [ -432h^2 + 12(4\lambda^4 - 9\lambda^3 - 17\lambda^2 - 9\lambda + 4) h + 12\lambda^3(\lambda+2)(2\lambda+1) ] P \\
&\quad + 180(\lambda-1)(\lambda+1)^2 h Q + [ 24(\lambda-1)(\lambda^2 - 5\lambda + 1) h \\
&\quad - 14(\lambda-1)(\lambda+2)(2\lambda+1) \lambda^2 ] P^2 \\
&\quad + [ 180(\lambda^2 + \lambda + 1) h - 15\lambda^2(\lambda+2)(2\lambda+1) ] PQ, \\
g(h, P, Q, \lambda) &= 144(\lambda^2 + \lambda + 1) h^2 - 12\lambda^2(\lambda+2)(2\lambda+1) h \\
&\quad + [ 288(1-\lambda) h^2 + 24\lambda(\lambda-1)(7\lambda^2 + 13\lambda + 7) h ] P \\
&\quad + [ -864h^2 - 24(5\lambda^4 + 3\lambda^3 - 7\lambda^2 + 3\lambda + 5) h + 12\lambda^3(\lambda+2)(2\lambda+1) ] Q \\
&\quad + [ 24(\lambda-1)(\lambda^2 - 5\lambda + 1) h - 14\lambda^2(\lambda-1)(\lambda+2)(2\lambda+1) ] PQ \\
&\quad + [ 180(\lambda^2 + \lambda + 1) h - 15\lambda^2(\lambda+2)(2\lambda+1) ] Q^2, \\
\varphi(h, \omega, \lambda) &= \left[ \frac{3888(\lambda^2 + \lambda + 1)}{(\lambda-1)(\lambda+2)(2\lambda+1)} h^2 + \frac{12(2\lambda^4 + \lambda^3 - 60\lambda^2 + \lambda + 2)}{(\lambda-1)} h \\
&\quad + \frac{\lambda^2(\lambda+2)(2\lambda+1)(10\lambda^2 - 11\lambda + 10)}{(\lambda-1)} \omega^2 \\
&\quad + \frac{864(5\lambda^2 + 8\lambda + 5)}{(\lambda+2)(2\lambda+1)} h^2 + 24(\lambda-1)^2 (2\lambda^2 + 7\lambda + 2) h \\
&\quad - 12\lambda^3(\lambda+2)(2\lambda+1) \right] \omega \\
&\quad + \left[ \frac{144(\lambda-1)(7\lambda^2 + 13\lambda + 7)}{(\lambda+2)(2\lambda+1)} h^2 - 12\lambda(\lambda-1)(\lambda+2)(2\lambda+1) h \right].
\end{align*}
and the relationship between \( \omega, \nu \) and \( P, Q \):

\[
\omega(h) = \frac{-(\lambda - 1)(108h + (\lambda + 2)(2\lambda + 1)(12\lambda - 14(\lambda - 1) P - 15Q))}{324h - (\lambda + 2)(2\lambda + 1)(10\lambda^2 - 11\lambda + 10 + 12(\lambda - 1) P)},
\]

\[
12h[2(5\lambda^2 + 8\lambda + 5) + 30(\lambda - 1) P - 45Q]
\]

\[
\nu(h) = \frac{+(\lambda + 2)(2\lambda + 1)[-12\lambda^2 + 14\lambda(\lambda - 1) P + 15\lambda Q + 324h - (\lambda + 2)(2\lambda + 1)(10\lambda^2 - 11\lambda + 10 + 12(\lambda - 1) P).}
\]

Similarly, the function \( F(h, \omega) \) in (17) becomes

\[
F(h, \omega) = F_3(h) \omega^3 + F_2(h) \omega^2 + F_1(h) \omega + F_0(h),
\]

where

\[
F_3(h) = -\frac{10077696(\lambda^2 + \lambda + 1)}{(\lambda - 1)^2(\lambda + 2)^2(2\lambda + 1)^2} h^4 + \frac{839808\lambda^2(17\lambda^2 + 38\lambda + 17)}{(\lambda - 1)^2(\lambda + 2)^2(2\lambda + 1)^2} h^3
\]

\[
288(44\lambda^2 + 110\lambda^2 - 2386\lambda^2 - 3733\lambda^2 - 1192\lambda^4
\]

\[
+ \frac{30(\lambda - 1)(\lambda + 2)(\lambda - 1)^2 h^2}{(\lambda + 2)(2\lambda + 1)(\lambda - 1)^2}
\]

\[
+ \frac{24(380\lambda^4 + 336\lambda^4 + 393\lambda^4 + 1670\lambda^3 + 393\lambda^2 + 336\lambda + 380)}{(\lambda - 1)^2} \lambda^2 h
\]

\[
- \frac{10(10\lambda^2 - 11\lambda + 10)(\lambda + 2)^2 (2\lambda + 1)^2 \lambda^4}{(\lambda - 1)^2},
\]

\[
F_2(h) = -\frac{1119744(13\lambda^2 + 19\lambda + 13)}{(\lambda - 1)(\lambda + 2)^2(2\lambda + 1)^2} h^4
\]

\[
- \frac{10368(62\lambda^6 + 279\lambda^5 - 675\lambda^4 - 2005\lambda^3 - 675\lambda^2 + 279\lambda + 62)}{(\lambda - 1)(\lambda + 2)^2(2\lambda + 1)^2} h^3
\]

\[
864(32\lambda^6 + 106\lambda^5 - 301\lambda^4 + 89\lambda^3 + 1201\lambda^2 + 89\lambda)
\]

\[
+ \frac{-301\lambda^2 + 106\lambda + 32}{(\lambda - 1)(\lambda + 2)(2\lambda + 1)} h^2
\]

\[
+ \frac{72\lambda^2(2\lambda^2 - \lambda + 2)(10\lambda^4 - 81\lambda^3 - 209\lambda^2 - 81\lambda + 10)}{(\lambda - 1)} h
\]

\[
+ \frac{120\lambda^2(\lambda + 2)^2 (2\lambda + 1)^2}{(\lambda - 1)}.
\]
\[ F_1(h) = -\frac{373248(17\lambda^2 + 29\lambda + 17)}{(\lambda + 2)^2(2\lambda + 1)^2} h^4 - \frac{3456(104\lambda^4 + 438\lambda^3 - 258\lambda^2 - 1297\lambda^3 - 258\lambda^2 + 438\lambda + 104)}{(\lambda + 2)^2(2\lambda + 1)^2} h^3 \\
+ \frac{288(16\lambda^4 - 8\lambda^2 + 289\lambda^3 + 1321\lambda^4 + 289\lambda^3 - 350\lambda^2 - 8\lambda + 16)}{(\lambda + 2)(2\lambda + 1)} h^2 \\
- 24\lambda^3(76\lambda^4 + 80\lambda^3 - 15\lambda^2 + 80\lambda + 76) h, \]

\[ F_0(h) = -\frac{124416(\lambda - 1)(7\lambda^2 + 13\lambda + 7)}{(\lambda + 2)^2(2\lambda + 1)^2} h^4 \\
- \frac{3456(\lambda - 1)^3(\lambda + 1)^2(2\lambda + 7)(7\lambda + 2)}{(\lambda + 2)^2(2\lambda + 1)^2} h^3 \\
- \frac{288\lambda(\lambda - 1)(8\lambda^6 + 26\lambda^5 - 71\lambda^4 - 169\lambda^3 - 71\lambda^2 + 26\lambda + 1 + 8)}{(\lambda + 2)(2\lambda + 1)} h^2 \\
- 144\lambda^4(\lambda - 1)(\lambda + 2)(2\lambda + 1) h. \]

Related to (18), we consider the perturbed Hamiltonian system

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= x(x - 1)(x + \lambda) + \delta(\alpha + \beta x + x^2) y,
\end{align*}
\] (23)

and the Abelian integral

\[
I(h) = \int_{\Gamma_h} (\alpha + \beta x + x^2) y \, dx = \alpha I_0(h) + \beta I_1(h) + I_2(h),
\] (24)

where

\[ \Gamma_h : \{ (x, y) \mid H(x, y) = h, 0 < h < h_1 \}, \]

\[ h_1 = \frac{2\lambda + 1}{3}. \]

When \( h \to 0 + 0 \), \( \Gamma_h \) shrinks to the center \((0, 0)\), and when \( h \to h_1 - 0 \), \( \Gamma_h \) expands to the saddle loop (see Fig. 3).

We intend to prove theorem 1, stated in the introduction.

Let \( P(h) = I_1(h)/I_0(h) \), \( Q(h) = I_2(h)/I_0(h) \), and define \( P(0) \) and \( Q(0) \) as its limits for \( h \to 0 + 0 \).
3. THE PROPERTIES OF $P(h)$ AND $Q(h)$

**Lemma 1.**

(i) $P(0) = Q(0) = 0$;

(ii) $(14\lambda - 2) P(h_1) + 15Q(h_1) - (2\lambda + 1) = 0$;

(iii) $P'(h) > 0, Q'(h) > 0$ for $h \in (0, h_1)$.

**Proof.** Conclusion (i) is obvious by the definition of $P$ and $Q$ and by using the mean value theorem for integrals. Conclusion (ii) comes from $f(h_1, P(h_1), Q(h_1)) = g(h_1, P(h_1), Q(h_1)) = 0$. Conclusion (iii) can be obtained directly from Example 5 of ref. [LZ].

**Lemma 2.**

(i) $\lim_{h \to 0^+} \frac{Q'(h)}{P'(h)} = \frac{\lambda}{\lambda - 1}$;

(ii) $\lim_{h \to 0^+} \frac{Q''(h) P'(h) - Q'(h) P''(h)}{(P'(h))^3} = \frac{20}{3} \frac{\lambda}{(\lambda - 1)^2}$.

**Proof.** At the singular point $(h, P, Q) = (0, 0, 0)$, system (19) has as linear part the matrix

$$12(\lambda + 2)(2\lambda + 1) \begin{pmatrix} -\lambda^2 & 0 & 0 \\ -(\lambda - 1) & \lambda^2 & 0 \\ -\lambda & 0 & \lambda^2 \end{pmatrix}.$$ 

The orbit we are interested in is the stable manifold corresponding to the simple eigenvalue $-\lambda^2$. Calculation shows that an eigenvector corresponding to this eigenvalue is given by

$$\begin{pmatrix} 1 \\ \frac{\lambda - 1}{2\lambda^2} \\ \frac{1}{2\lambda} \end{pmatrix}.$$
Hence $P'(0) = \frac{\lambda - 1}{2\xi^2}$, $Q'(0) = \frac{1}{2\xi}$. For $0 < h \ll 1$, let

$$h = h,$$
$$P = \frac{\lambda - 1}{2\xi^2} h + \frac{a_2}{2} h^2 + \ldots,$$
$$Q = \frac{1}{2\xi} h + \frac{b_2}{2} h^2 + \ldots.$$

Substituting it into (19) we have

$$P''(0) = a_2 = \frac{1}{72} \frac{(\lambda - 1)(110\lambda^2 - 37\lambda + 110)}{\lambda^3},$$
$$Q''(0) = b_2 = \frac{1}{72} \frac{110\lambda^2 - 157\lambda + 110}{\lambda^4}.$$

The conclusions of Lemma 2 follow by these results.

Since by (18) $h_1 = \frac{2\xi + 1}{12}$, the saddle loop $I_{1, h}$ cuts the negative $x$-axis at the point $(x^*, 0)$ with $x^*$ a solution of $3x^2 + 2(2\lambda + 1) x + 2\lambda + 1 = 0$ and hence

$$x^* = -\frac{1}{3} \left[ 2\lambda + 1 - \sqrt{2(\lambda - 1)(2\lambda + 1)} \right];$$

see Fig. 4. Note that

$$\Phi' = -x^2(x - 1)(x + \lambda) > 0 \quad \text{for} \quad x \in (x^*, 1), x \neq 0,$$

hence for any fixed $h \in (0, h_1), x \in (a(h), 0)$, there is a unique $x \in (0, b(h))$ such that $\Phi(x) = \Phi(\bar{x})$, where $a(h)$ and $b(h)$ are abscissa of the intersection points of $I_{1, h}$ with the $x$-axis. For a ratio of two Abelian integrals

$$R(h) = \int_{R_1} \frac{f_2(x)}{f_1(x)} \Phi(x) y \, dx$$

we define a criterion function

$$\zeta(x) = \frac{f_2(x) \Phi'(\bar{x}) - f_2(x) \Phi'(x)}{f_1(x) \Phi'(\bar{x}) - f_1(x) \Phi'(x)},$$

where $\bar{x} = \bar{x}(x)$ is defined by $\Phi(\bar{x}) = \Phi(x), x \in (a(h), 0)$.
**FIG. 4.** Introducing $x$, $\bar{x}$ and $x^*$.  

**Lemma 3.** If $\zeta'(x) > 0$ (resp. $< 0$) for $x \in (x^*, 0)$, then $R'(h) < 0$ (resp. $> 0$) for $h \in (0, h_1)$.

*Proof.* This is a special case of Theorem 1 in ref. [LZ].

**Lemma 4.** $d/dh I_2(h)/I_1(h) < 0$ for $h \in (0, h_1)$.

*Proof.* Substituting $f_2(x) = x^2$, $f_1(x) = x$ and (18) into (26), we get the criterion function

$$
\zeta(x) = \frac{x\bar{x} + \lambda}{\bar{x} + \bar{x} + \lambda - 1};
$$

hence

$$
\zeta'(x) = \frac{1}{(x + \bar{x} + \lambda - 1)^2} \left[ \bar{x}^2 + (\lambda - 1) \bar{x} - \lambda + (x^2 + (\lambda - 1) x - \lambda) \frac{d\bar{x}}{dx} \right].
$$

Since $\frac{dx}{dx} = \frac{\Phi(x)}{\Phi'(x)}$ and $\Phi'(\bar{x}) = -\bar{x}(\bar{x} - 1)(\bar{x} + \lambda) > 0$ for $0 < \bar{x} < b(h) < 1$, $\zeta'(x)$ has the same sign as

$$
\Psi(x, \bar{x}) = (\bar{x}^2 + (\lambda - 1) \bar{x} - \lambda) \Phi'(\bar{x}) + (x^2 + (\lambda - 1) x - \lambda) \Phi'(x)
= -(\bar{x}^2 + x^2) - 2(\lambda - 1)(\bar{x}^2 + x^2) + (4\lambda - 1 - \lambda^2)(\bar{x}^3 + x^3)
+ 2\lambda(\lambda - 1)(\bar{x}^2 + x^2) - \lambda^2(\bar{x} + x).
$$

Let $\bar{x} + x = u$, $\bar{x} x = v$, then from $\Phi(\bar{x}) = \Phi(x)$ we have

$$
u(3u^2 + 4(\lambda - 1) u - 6\lambda) = 2(3u + 2(\lambda - 1)) v. \quad (27)$$
Since $\lambda > 1$ and $v < 0$ (for $x \neq 0$), we have from (27) that $u > 0$ for $0 < |u| \ll 1$. This implies $u = x + \bar{x}$ keeps positive for all $(x, y), (\bar{x}, y) \in T_{\lambda}$ (hence, the function $\zeta(x)$ above is well defined). In fact if this is not true, then there exists $x < 0 < \bar{x}$ such that $u = x + \bar{x} = 0$ and $v = x\bar{x} < 0$, which contradicts to (27). Thus, from (27) we have

$$v = \frac{u(3u^2 + 4(\lambda - 1)u - 6\lambda)}{2(3u + 2(\lambda - 1))}, \quad u > 0. \quad (28)$$

To find the maximum value of $u$, we consider

$$u' = 1 + \frac{d\bar{x}}{dx} = 1 + \frac{\Phi'(x)}{\Phi'(\bar{x})} = 0,$$

which is equivalent to

$$\Phi'(x) + \Phi'(\bar{x}) = 0. \quad (29)$$

By using the relationships

$$\bar{x}^2 + x^2 = u^2 - 2v,$$
$$\bar{x}^3 + x^3 = u^3 - 3uv,$$  

as well as (28), we have

$$\Phi'(x) + \Phi'(\bar{x}) = -(\bar{x}^3 + x^3) - (\lambda - 1)(\bar{x}^2 + x^2) + \lambda(\bar{x} + x)$$
$$= \frac{1}{2} u(u - 2)(u + 2\lambda),$$

which has unique positive root $u = 2$. This means that $u$ keeps increasing for $\bar{x}$ increasing from 0 to 1. Hence $0 < u < u^* = 1 + x^* < 1$ for $x^* < x < 0$. By using (25), we have

$$h(u) = 9u^2 + 12(\lambda - 1)u - 6(\lambda - 1) < 0 \quad \text{for} \quad x^* < x < 0. \quad (31)$$

Now we have enough information to give an estimate of the sign of the function $\Psi(x, \bar{x})$. By using (30) and

$$\bar{x}^4 + x^4 = u^4 - 4u^2v + 2v^2,$$
$$\bar{x}^5 + x^5 = u^5 - 5u^3v + 5uv^2,$$
as well as (28), we obtain
\[ \Psi(x, x) = \frac{u(u-2)(u+2\lambda)\psi(u)}{4(3u+2(\lambda-1))^2}, \]  
(32)
where
\[ \psi(u) = 9u^4 + 30(\lambda-1)u^3 + 38(\lambda-1)^2u^2 + 8(\lambda-1)(2\lambda^2 - 7\lambda + 2)u - 20\lambda(\lambda-1)^2. \]

Since by (31) and \( u > 0, \lambda > 1 \) we have
\[ \psi(u) = \left( u^2 + 2(\lambda-1)\left( u + \frac{7\lambda-4}{9} \right) \right) h(u) \]
\[ -\frac{4}{3}(\lambda-1)(2\lambda+1)((\lambda+5)u + 4(\lambda-1)) < 0; \]
hence \( \Psi(x, x) > 0 \) by (32) and the conclusion of Lemma 4 follows from Lemma 3.

In \( PQ \)-plane; we consider the curve
\[ \Sigma: \{(P, Q)(h) \mid h \in [0, h_1]\}, \]  
(33)
which is parametrized by \( h \). We denote the tangent line of \( \Sigma \) at \( (P, Q)(0) \) by \( L_1 \) and the straight line passing through the two points \( (P, Q)(0) \) and \( (P, Q)(h_1) \), by \( L_2 \). Lemma 4 induces an important property of \( \Sigma \) as follows.

**Lemma 5.** The slope of \( L_2 \) must be smaller than the slope of \( L_1 \), and \( \Sigma \) must be located entirely between \( L_1 \) and \( L_2 \), as shown in Fig. 5.

**FIG. 5.** The curve \( \Sigma \) near 0.
Proof. By Lemma 1(i) and Lemma 2(i), $\Sigma$ has the tangent line $L'\colon Q = \frac{1}{r-1} P$ at the endpoint $(P(0), Q(0))$, and by Lemma 2(ii) $\Sigma$ must stay below $L'$ for $0 < h \ll 1$. Since $Q(h)/P(h) = I_{1}(h)/I_{1}(h) = \tan \alpha(h)$, where $\alpha(h)$ is the polar angle of the point $(P, Q)(h)$ on $\Sigma$, the conclusion of Lemma 5 follows immediately from Lemma 4.

In the next Lemma we prove that the curve $\Sigma$ has the same convexity property near the endpoint $(P(h_{1}), Q(h_{1}))$ as near the endpoint $(P(0), Q(0))$.

**Lemma 6.**

$$Q'(h)P(h) - Q'(h)P(h) = \frac{Q''(h)}{(P'(h))^{2}} < 0 \text{ for } 0 < h - h_{1} \ll 1.$$  

**Proof.** Let $x = \xi + 1$, $y = y$, then the Hamiltonian function becomes

$$H(\xi, y) = \frac{y^{2}}{2} - \left( \frac{\xi^{4}}{4} + \frac{\lambda + 1}{3} \xi^{3} + \frac{\lambda + 1}{2} \xi^{2} \right) = t, \quad (34)$$

where $t = h - \frac{\lambda + 1}{12} < 0$. The saddle loop corresponds to $t = 0$.

For (34), we define $J_{k}(t) = \int r, \xi y d\xi$, $k = 0, 1, 2$, and $P(t) = \frac{J_{1}(t)}{J_{0}(t)}$, $Q(t) = \frac{J_{2}(t)}{J_{0}(t)}$. It is obvious that

$$I_{0}(h) = \int_{r, h} y d\xi = J_{0}(t),$$

$$I_{1}(h) = \int_{r, h} xy d\xi = J_{1}(t) + J_{0}(t),$$

$$I_{2}(h) = \int_{r, h} x^2 y d\xi = J_{2}(t) + 2J_{1}(t) + J_{0}(t).$$

Hence $P'(h) = \bar{P}'(t)$ and $Q'(h) = \bar{Q}'(t) + 2\bar{P}'(t)$, and

$$\frac{Q''(h) P'(h) - Q'(h) P'(h)}{(P'(h))^{3}} = \frac{\bar{Q}''(t) \bar{P}'(t) - \bar{Q}'(t) \bar{P}'(t)}{(\bar{P}'(t))^{3}}. \quad (35)$$

From (34) we have \(\lim_{t \to 0-} \frac{\xi}{J_{0}(t)} = \pm \sqrt{\lambda + 1}\), hence both $\lim_{t \to 0-} J_{1}'(t) = \lim_{t \to 0-} \int_{r, h} \xi d\xi = \infty$, $\lim_{t \to 0-} J_{0}'(t) = \lim_{t \to 0-} \int_{r, h} \xi d\xi = +\infty$. Hence

$$\lim_{t \to 0-} \frac{\bar{Q}'(t)}{\bar{P}'(t)} = \lim_{t \to 0-} \left( \frac{J_{1}(t)}{J_{0}(t)} \right)' = \lim_{t \to 0-} \frac{J_{1}(t) J_{0}(t) - J_{0}(t) J_{1}(t)}{J_{0}(t) J_{0}(t) - J_{1}(t) J_{1}(t)} = \frac{Q(0)}{P(0)}, \quad (36)$$
and \( \lim_{\lambda \to 0} \frac{\partial J(\lambda)}{\partial \lambda} \) has a meaning. Let us prove that

\[
\frac{d}{dt} \frac{J_1(t)}{J_2(t)} < 0 \quad \text{for} \quad t \leq 0. \tag{37}
\]

We use the same technique as in the first part of the proof of Lemma 4, so we consider the criterion function

\[
\eta(\xi) = \frac{\xi^2 - (\lambda + 1)}{\xi + \xi + (\lambda + 2)},
\]

where \( \bar{\xi} = \bar{\xi}(\xi) \) is defined by \( H(\bar{\xi}, y) = H(\xi, y), \xi \in [\xi^*, -1) \), and

\[
\eta'(\xi) = \frac{1}{(\xi + \xi + (\lambda + 2))^2} \left[ \bar{\xi}^2 + (\lambda + 2) \bar{\xi} + \lambda + 1 + (\bar{\xi}^2 + (\lambda + 2) \bar{\xi} + (\lambda + 1)) \frac{d\bar{\xi}}{d\xi} \right].
\]

Since \( -(\lambda + 1) < \xi^* \leq \xi < -1 < \bar{\xi} \leq 0 \) (see Fig. 6), we have

\[
\bar{\xi}^2 + (\lambda + 2) \bar{\xi} + \lambda + 1 = (\bar{\xi} + 1)(\bar{\xi} + \lambda + 1) > 0,
\]

\[
\bar{\xi}^2 + (\lambda + 2) \bar{\xi} + \lambda + 1 = (\xi^* + 1)(\xi^* + \lambda + 1) < 0.
\]

On the other hand, \( \frac{d\xi}{d\xi} < 0 \) is always true. Therefore, \( \eta'(h) > 0 \) which implies (37) by Lemma 3.

---

**Fig. 6.** Introducing \( \xi, \bar{\xi} \) and \( \xi^* \).
FIG. 7. The curve $\Sigma$ for $t \sim 0$.

(36) means that in $\mathcal{PQ}$-plane the curve

$$\tilde{\Sigma}: \{(\tilde{P}, \tilde{Q})(t) | t \in [-h_1, 0]\}$$

has a tangent line $\mathcal{D}: \tilde{Q} = \frac{\tilde{Q}(0)}{\tilde{P}(0)} \tilde{P}$ at the endpoint $(\tilde{P}, \tilde{Q})(0)$. Note also that the functions $G$, $f$ and $g$ in (19) are polynomials in $h$, $P$ and $Q$, hence the right hand side of (35) has at most a finite number of zeros for $t \in [-h_1, 0]$. We can now show that for $t$ near 0 the relative position between $\tilde{\Sigma}$ and $\mathcal{D}$ is as shown in Fig. 7.

For a same $\tilde{P}(t)$, the difference of $\tilde{Q}(t)$ with the corresponding point on $\mathcal{D}$ is given by

$$\tilde{Q}(t) - \frac{\tilde{Q}(0)}{\tilde{P}(0)} \tilde{P}(t) = \tilde{P}(t) \left( \frac{d \tilde{Q}(t)}{d t} \frac{\tilde{J}_1(t)}{\tilde{J}_2(t)} \right)_{t=0} t = \tilde{P}(t) \left( \frac{d \tilde{J}_2(t)}{d t} \right)_{t=0} t$$

which is negative by using $\tilde{P}(t) < 0$, $t < 0$ as well as (37). As such the conclusion of Lemma 6 follows from (35) and $t = h - h_1$.

4. THE STUDY IN THE $(h, \omega)$- AND $(h, \gamma)$-PLANES

It is not difficult to see that for $0 \leq h \leq h_1 = \frac{21\alpha^2}{12}$, system (20) has four singularities: two saddle points at $A(0, \omega_A)$ and $D(h_1, \omega_D)$ and two improper nodes at $O(0, 0)$ and $B(h_1, 1)$, where
FIG. 8. The curve $C_w$ in the $(h, \omega)$-plane.

\[ \omega_A = \frac{12\lambda(\lambda - 1)}{10\lambda^2 - 11\lambda + 10} > 0, \]

\[ \omega_D = -\frac{(\lambda - 1)(2\lambda + 7)}{10\lambda^2 + 31\lambda + 31} < 0; \]

see Fig. 8.

By Lemma 1(i) we have $P(0) = Q(0) = 0$. Substituting $h = P = Q = 0$ into (21), we obtain $\omega(0) = \omega_A$. By using Lemma 1(ii) and substituting $h = h_1$ and $Q = -\frac{1}{\gamma} (14\lambda - 2) P + \frac{1}{\gamma} (2\lambda + 1)$ into (21), we obtain $\omega(h_1) = 1$. Hence the orbit of system (20), that we look for, is the stable manifold from the node $B$ to the saddle $A$, denoted by $C_w$ in Fig. 8.

Using (22) we see that $F(0, \omega) = 0$ has a simple zero at $\omega = \omega_A$ and a double zero at $\omega = 0$; $F(h_1, \omega) = 0$ has a double zero at $\omega = 1$ and a simple zero at $\omega = \omega_D$. The locus of $F(h, \omega) = 0$ for $0 \leq h \leq h_1$, shown in Fig. 9, has three branches.

We denote the branch from $A$ to $B$ by $C_F$. One of the crucial steps in this paper is to study the number of intersection points of $C_w$ and $C_F$.

**Lemma 7.** For $0 < h \ll 1$ $C_w$ is located above $C_F$ if $\lambda < \lambda^* \ (\approx 1.659)$, and $C_w$ is below $C_F$ if $\lambda \geq \lambda^*$, where $\lambda^*$ is the biggest root of $E(\lambda) = 0$.

\[ E(\lambda) = 220\lambda^4 - 628\lambda^3 + 735\lambda^2 - 628\lambda + 220. \]  \hspace{1cm} (38)
Proof. \( C_w \) and \( C_F \) have the common endpoint \( A \). We denote the first three derivatives at the point \( A \) of the function \( \omega = \omega(h) \) defined by \( C_F \) and \( C_w \) respectively by \( \omega'_F_A \), \( \omega''_F_A \), \( \omega'''_F_A \), \( \omega'_W_A \), \( \omega''_W_A \) and \( \omega'''_W_A \). Using symbolic calculation (e.g. “Maple”) it is not difficult to find from (22) that

\[
\begin{align*}
\omega'_F_A &= -\frac{\partial F}{\partial h} \left|_{(0, \omega_A)} \right. = -\frac{\partial F}{\partial \omega} \left|_{(0, \omega_A)} \right. = - \frac{35 (\lambda - 1)(\lambda + 2)(2\lambda + 1)(2\lambda^2 - 7\lambda + 2)}{2 \lambda^2 (10\lambda^2 + 11\lambda + 10)^2}, \\
\omega''_F_A &= \left( \frac{\partial^2 F}{\partial h^2} + \frac{\partial^2 F}{\partial \omega \partial h} \omega'_F_A + \frac{\partial^2 F}{\partial \omega^2} \omega''_F_A \right) \left|_{(0, \omega_A)} \right. \\
&= \left( \frac{\partial^2 F}{\partial h^2} + 2 \frac{\partial^2 F}{\partial \omega \partial h} \omega'_F_A + 2 \frac{\partial^2 F}{\partial \omega^2} \omega''_F_A \right) \left|_{(0, \omega_A)} \right. \\
&= \frac{735 (\lambda - 1)(\lambda + 2)(2\lambda + 1)(2\lambda^2 - 7\lambda + 2)^2}{2 \lambda^4 (10\lambda^2 - 11\lambda + 10)^3}, \\
\omega'''_F_A &= \left[ \frac{\partial^3 F}{\partial t^3} \omega''_F_A \right] \left|_{(0, \omega_A)} \right. \\
&= \frac{35 (440000\lambda^{12} - 2788800\lambda^{11} + 8710224\lambda^{10} - 19251680\lambda^9)}{192 \lambda^8 (10\lambda^2 - 11\lambda + 10)^4}.
\end{align*}
\]
Note that $C_w$ is the stable manifold of system (20) at the saddle point $(0, \omega_A)$, we suppose that for $0 < h \ll 1$,

$$h = h, \omega = \omega_A + \omega'_A h + \frac{1}{2!} \omega''_A h^2 + \frac{1}{3!} \omega'''_A h^3 + \ldots,$$

and put them into

$$G \frac{d\omega}{dh} - \varphi(h, \omega, \lambda) = 0,$$

then we can determine successively

$$\omega'_A = - \frac{35}{2} \frac{(\lambda - 1)(\lambda + 2)(2\lambda + 1)(2\lambda^2 - 7\lambda + 2)}{\lambda^2 (10\lambda^2 - 11\lambda + 10)^2}, \quad (39)$$

$$\omega''_A = - \frac{35}{72} \frac{(\lambda - 1)(\lambda + 2)(2\lambda + 1)(2200\lambda^6 - 11724\lambda^5 + 37626\lambda^4)}{\lambda^3 (10\lambda^2 - 11\lambda + 10)^3},$$

and

$$\omega'''_A = - \frac{35}{1152} \frac{(\lambda - 1)(\lambda + 2)(2\lambda + 1)(2948000\lambda^{10} - 20490800\lambda^9)}{\lambda^5 (10\lambda^2 - 11\lambda + 10)^4}.$$

Hence,

$$\omega'_A - \omega'_{FA} = 0,$$

$$\omega''_A - \omega''_{FA} = - \frac{35}{72} \frac{(\lambda - 1)(\lambda + 2)(2\lambda + 1) E(\lambda)}{\lambda^3 (10\lambda^2 - 11\lambda + 10)^2},$$

where $E(\lambda)$ is given in (38) and has only two real roots: 0.6028... and $\lambda^* = 1.6588...$

If $\lambda = \lambda^*$, we need to compute

$$\omega'''_A - \omega'''_{FA} = - \frac{35}{1152} \frac{(\lambda - 1)(325600\lambda^{10} - 592720\lambda^9 + 417168\lambda^8)}{\lambda^5 (10\lambda^2 - 11\lambda + 10)^3},$$

which is negative for $\lambda = \lambda^*$.  ■
Lemma 8. For $0 < h_1 - h \ll 1$, $C_w$ is located below $C_F$ for all $\lambda > 1$.

Proof. We note that both $C_w$ and $C_F$ are tangent to the straight line $\{h = h_1\}$ at the point $B(h_1, 1)$; see Figs. 8 and 9. At the singularity $B$, system (20) has a linear part with matrix

$$
12 (\lambda - 1) (2 \lambda + 1) (\lambda + 1) \begin{pmatrix} (\lambda + 1)^2 & 0 \\ \lambda + 2 & (\lambda + 1)^2 \end{pmatrix}.
$$

Let $\bar{h} = h - h_1 < 0$, $\bar{\omega} = \omega - 1 > 0$, then the linearized system has the solution

$$
\bar{\omega} = \frac{\lambda + 2}{(\lambda + 1)^2} \bar{h} \ln |\bar{h}| + C_0 \bar{h},
$$

where the constant $C_0$ corresponds to the special solution $C_w$. For $0 < |\bar{h}| \ll 1$, the inverse function $\bar{h} = \bar{h}(\bar{\omega})$ has the derivative

$$
(40)
$$

$$
\bar{h} = \frac{1}{C_0 + (\lambda + 2)/(\lambda + 1)^2 (1 + \ln |\bar{h}|)}.
$$

On the other hand, calculation shows that for $\lambda > 1$

$$
\frac{\partial F}{\partial h} (h_1, 1) = -144 (\lambda - 1)(\lambda + 2)(2 \lambda + 1)(\lambda + 1)^4 < 0,
$$

$$
\frac{\partial F}{\partial \omega} (h_1, 1) = 0,
$$

$$
\frac{\partial^2 F}{\partial h \partial \omega} (h_1, 1) = 24(76 \lambda^4 + 224 \lambda^3 + 201 \lambda^2 - 46 \lambda - 23)(\lambda + 1)^3 > 0,
$$

(there the second factor has only two real roots at approximately $-0.26939$ and $0.3687$),

$$
\frac{\partial^2 F}{\partial \omega^2} (h_1, 1) = -240 \frac{(\lambda - 1)^2 (2 \lambda + 1)^2(\lambda + 1)^5}{\lambda + 2} < 0.
$$

Hence $F(h, \omega) = 0$ gives

$$
\bar{h}(\star + \ldots) + \bar{\omega}^2(\star + \ldots) = 0,
$$

(41)
where \(*\) denotes some non-zero constant and where \(\cdots\) means the higher order terms with respect to \(*\) as \(h \to 0\) or \(\bar{\omega} \to 0\). Similarly we have

\[
\frac{\partial F}{\partial \bar{\omega}}(h, \omega) = \bar{h}(\cdots) + \bar{\omega}(\cdots). \quad (42)
\]

Since \(\frac{\partial F}{\partial h}(h_1, 1) \neq 0\), near the point \(B(h_1, 1)\), the curve \(C_f\) can be expressed as a function \(h = \bar{h}(\bar{\omega})\). By using (42) and (41) we obtain that this function has a derivative

\[
\bar{h}'_a = O(|\bar{h}|^{1/2}). \quad (43)
\]

Comparing (43) with (40), the conclusion of Lemma 8 follows.

**Lemma 9.** We denote by \(\lambda_0\) the biggest root of the equation \(2\lambda^2 - 7\lambda + 2 = 0\) (\(\lambda_0 \approx 3.186\)). If \(\lambda < \lambda_0\), then \(\omega(h) = 0\) has a unique solution for \(h \in (0, h_1)\) corresponding to a maximum of \(C_w\); if \(\lambda \geq \lambda_0\), then \(C_w\) is monotonically decreasing.

**Proof.** The 0-cline of system (20) has two branches. We denote the upper branch, joining the two singularities \(A\) and \(B\), by \(C_0\), and denote by \(\omega'_{6\lambda}, \omega''_{6\lambda}\) and \(\omega'_{6\lambda B}\) the first and second derivatives at point \(A\) and the first derivative at point \(B\) of the function \(\omega = \omega(h)\) defined by \(\varphi(h, \omega, \lambda) = 0\), then

\[
\omega'_{6\lambda} = -\frac{35(\lambda - 1)(\lambda + 2)(2\lambda + 1)(2\lambda^2 - 7\lambda + 2)}{\lambda^2(10\lambda^2 - 11\lambda + 10)^2}, \quad (44)
\]

\[
\omega''_{6\lambda}|_{\lambda_0} = \frac{35(\lambda_0 - 1)(400\lambda_0 - 704\lambda_0^2 - 9464\lambda_0^3 - 4976\lambda_0^4 + 38965\lambda_0^5)}{6\lambda_0^2(10\lambda_0^2 - 11\lambda_0 + 10)^2} < 0
\]

\[
\omega'_{6\lambda B} = -\frac{\lambda + 2}{(\lambda + 1)^2} < 0.
\]

Hence, if \(\lambda \geq \lambda_0\), then near \(A\) and \(B\) \(C_0\) is decreasing. If there is a \(\tilde{\lambda} \geq \lambda_0\) such that \(C_0\) has a minimum for \(0 < h < h_1\), then it must be followed by a maximum; see Fig. 10(c), hence we can find a \(\bar{\omega} \in (1, \omega_A)\), such that the straight line \(\{\omega = \bar{\omega}\}\) intersects \(C_0\) at least at three points.

But for system (20) we know \(\varphi(h, \bar{\omega}, \tilde{\lambda})\) is a polynomial of \(h\) of degree two, which is a contradiction.

Comparing (44) with (39) we have

\[
\omega'_{6\lambda} = 2\omega'_{\lambda}
\]

\[
(45)
\]
and calculation gives $(\omega'_0 - \omega'_w)|_{h=0} < 0$. Hence if $\lambda > \lambda_0$ then $\omega'_0 < 0$, by (45) $C_w$ is located above $C_0$ for $0 < h < 1$. We just proved that $C_0$ is strictly decreasing, hence $C_0$ must stay above $C_0$ for $0 < h < h_1$, and it is also strictly decreasing; see Fig. 10(b).

If $\lambda < \lambda_0$, then $\omega'_0 > 0$, by (45) $C_w$ is located below $C_0$ for $0 < h < 1$. However, $C_w$ must be tangent to $\{h = h_1\}$ at $B$ with slope $-\infty$, hence $C_w$ must cut $C_0$ at some point $M$. By the same argument as shown in Fig. 10(c), $C_0$ has unique maximum which implies that the point $M$ is the unique intersection point of $C_w$ and $C_0$ for $h \in (0, h_1)$; see Fig. 10(a).

**Lemma 10.** For $0 < h < h_1$, the curves $C_w$ and $C_F$ have no intersection point if $\lambda = 4$, and have a unique intersection point if $\lambda = \frac{3}{2}$.

**Proof.** (i) We first consider the case $\lambda = 4$. Since $C_w$ is an orbit of system (20), and $C_w$ is located below $C_F$ for $h$ near 0 and near $h_1$ (Lemmas 7 and 8), if we prove that for $0 < h < h_1$ there is no point on $C_F$ at which the vector field (20) is tangent to $C_F$, then the desired conclusion follows.

Taking $\lambda = 4$ in (22) we have

$$F(h, \omega) = 576[(-14h^4 + 392h^3 - 9765h^2 + 156744h - 181440) \omega^3$$
$$+ (-6h^4 - 471h^3 + 27786h^2 - 125640h + 207360) \omega^2$$
$$- 6h(15h^3 + 248h^2 - 202h + 10992) \omega - h(38h^3 + 625h^2$$
$$+ 3264h + 10368)].$$

Calculation shows that by using (20), eliminating $\omega$ from

$$F(h, \omega) = 0,$$

$$\frac{\partial F}{\partial h} h + \frac{\partial F}{\partial \omega} \theta = \frac{\partial F}{\partial h} \varphi(h, \omega) = 0,$$  \hspace{1cm} (46)
we obtain
\[
F(h, \omega) = \frac{9}{49} \left( (-5318784h^4 + 23567112h^3 - 34187440h^2 + 19138224h \\
- 3457440) \omega^3 + (-4401216h^4 + 4836924h^3 + 3879246h^2 \\
- 7095690h + 1944810) \omega^2 - h(1023840h^3 - 95262h^2 - 600397h \\
+ 360297) \omega - 3h(24336h^3 + 3125h^2 - 21112h + 9261) \right).
\]

(ii) Next we consider the case \( \lambda = \frac{3}{2} \). By Lemmas 7 and 8 we know that near \( h = 0 \) and near \( h = h_1 \), the curve \( C_w \) is located respectively above and below \( C_F \). Hence it is necessary to have at least one intersection point. Since \( C_w \) is the stable manifold of the system at \( A \), along \( C_F \) between the point \( A \) and the first intersection point from \( A \) there is at least one point at which the vector field (20) is tangent to \( C_F \); see Fig. 11.

So, if we prove that such a tangent point on \( C_F \) is unique, then the intersection point must be also unique.

Taking \( \lambda = \frac{3}{2} \) in (22) we have

\[
F(h, \omega) = \frac{9}{49} \left( (-5318784h^4 + 23567112h^3 - 34187440h^2 + 19138224h \\
- 3457440) \omega^3 + (-4401216h^4 + 4836924h^3 + 3879246h^2 \\
- 7095690h + 1944810) \omega^2 - h(1023840h^3 - 95262h^2 - 600397h \\
+ 360297) \omega - 3h(24336h^3 + 3125h^2 - 21112h + 9261) \right).
\]
Similarly to case (i), putting it into (46) and eliminating \( \omega \), we obtain
\[
\begin{align*}
&h^4(3h-1)^4 (64h-63)^4 (38h-21)(664848h^4 - 2945889h^3 + 4273430h^2 \\
&- 2392278h + 432180)(1215202455467681712h^{10} \\
&- 7657795474142703120h^9 + 2099857455227471180h^8 \\
&- 3306494052968398640h^7 + 33151703409048962220h^6 \\
&- 2211042044504829668h^5 + 9908042691828759765h^4 \\
&- 292713816772867210h^3 + 539276696782798905h^2 \\
&- 54804734858498940h + 2240046238035438) = 0,
\end{align*}
\]
which has unique solution \( h = \hat{h} \approx 0.1063 \) for \( h \in (0, \frac{1}{2}) \). Note that \( h_1 = \frac{1}{3} \) for \( \lambda = \frac{3}{2} \).

Taking \( h = \hat{h} \), \( H(\hat{h}, \omega) = 0 \) has a unique solution \( \omega = \hat{\omega} \approx 0.6664 \) for \( \omega > \omega_* = 0.5625 \). This finishes the proof of the Lemma.

**Lemma 11.** For \( 0 < h < h_1 \), the curves \( C_w \) and \( C_F \) have no intersection point if \( \lambda \geq \lambda^* \), and have a unique intersection point if \( \lambda < \lambda^* \), where \( \lambda^* \) is the same as in Lemma 7.

**Proof.** (i) We first consider the case \( \lambda \geq \lambda^* \). We know from Lemmas 7 and 8 that \( C_w \) is below \( C_F \) for \( h \) near 0 and near \( h_1 \). If for some \( \lambda \geq \lambda^* \), \( C_w \) and \( C_F \) have intersection points which are not contact points, then there are at least two of them. Let \( \lambda \) vary from \( \lambda \) to \( \lambda = 4 \) monotonously, by Lemma 10 \( C_w \) and \( C_F \) have no intersection for \( h \in (0, h_1) \) and \( \lambda = 4 \). On the other hand \( C_w \) and \( C_F \) have fixed relative positions near \( h = 0 \) and \( h = h_1 \) for \( \lambda \geq \lambda^* \), and \( C_w \) and \( C_F \) must change continuously and smoothly for \( \lambda \) varying (since \( F(h, \omega) \) and the right hand sides of (20) are polynomials of \( h, \omega \) and \( \lambda \)). As such there is a value \( \lambda = \hat{\lambda} \) between \( \lambda = \lambda \) and \( \lambda = 4 \), such that \( C_w \) and \( C_F \) have a contact point for \( \lambda = \hat{\lambda} \) and \( h \in (0, h_1) \). Let us show that this is impossible.

From (16) and (17) we know that a transverse intersection point of \( C_w \) and \( C_F \) give rise to an inflection point for the curve \( \Omega = \{(\omega, v)(h) | h \in (0, h_1)\} \). In other words, there are constants \( \alpha \) and \( \beta \), such that the straight line \( \alpha + \beta \omega + v = 0 \) in \( (\omega, v) \)-plane has a triple contact point with \( \Omega \). Now, a contact point between \( C_w \) and \( C_F \) implies the existence of \( \alpha \) and \( \beta \), such that the corresponding line \( \alpha + \beta \omega + v = 0 \) has a (at least) quadruple contact point with \( \Omega \).

Taking \( a = -1, b = 1 - \lambda, c = \lambda \) in (12), we have
\[
v(h) = a(h) + (b(h) + c) \omega(h),
\]
(47)
where

\[ a_1 = -\frac{12}{(\lambda + 2)(2\lambda + 1)}, \]

\[ b_1 = -\frac{36}{(\lambda - 1)(\lambda + 2)(2\lambda + 1)}, \] (48)

\[ c_1 = \frac{\lambda}{\lambda - 1}. \]

Hence

\[ \alpha + \beta \omega(h) + v(h) = (b_1 h + c_1 + \beta)(\omega - U(h)), \] (49)

where

\[ U(h) = -\frac{a_1 h + \alpha}{b_1 h + c_1 + \beta}. \] (50)

Here we may suppose that \( b_1 h + c_1 + \beta \neq 0 \); otherwise \( \alpha + \beta \omega(h) + v(h) \) can have only a simple zero at \( h = -\alpha/a_1 \).

The fact that in \((\omega, v)\)-plane the line \( L \) and curve \( \Omega \) have a quadruple contact point is equivalent, because of (49) to the fact that in \((h, \omega)\)-plane the curves \( C_\omega \) and \( C_v \), defined by \( \omega = U(h) \), have a quadruple contact point.

Note that by (50) and (48)

\[ U'(h) = \frac{12((\lambda - 1) \beta - 3\alpha + \lambda)}{(\lambda - 1)(\lambda + 2)(2\lambda + 1)(b_1 h + c_1 + \beta)^2}, \] (51)

and

\[ U''(h) = -\frac{864[(\lambda - 1) \beta - 3\alpha + \lambda]}{(\lambda - 1)^2(\lambda + 2)^2(2\lambda + 1)^2(b_1 h + c_1 + \beta)^3}. \] (52)

Since \( \lim_{h \to \pm \infty} U(h) = -\frac{a_1}{b_1} = -\frac{\lambda - 1}{\lambda} < 0 \), \( C_\omega \) can meet only the branch of \( C_v \) which is above the asymptotic line \( \{\omega = -\frac{\lambda - 1}{\lambda}\} \). On the other hand, by (50) and (20) we have that for \( \omega = U(h) \),

\[ \omega - U'(h) \hat{h} = \varphi(h, \omega) - U'(h) G(h) = \frac{n_1 h^3 + n_2 h^2 + n_1 h + n_0}{(b_1 h + c_1 + \beta)^2}, \] (53)
where \( n_i \) depends only on \( \alpha, \beta \) and \( \lambda \) \((i = 0, \ldots, 3)\), and

\[
n_3 = k \left( (\lambda - 1) \beta - 3\alpha - \frac{5\lambda^2 - \lambda + 5}{6} \right),
\]

(54)

where \( k \) is a positive constant.

(53) shows that contact points between \( C_w \) and \( C_U \) can be at most quadruple. If we suppose that \( C_w \) and \( C_U \) have a quadruple contact point \( \bar{M} \), then they have no other intersection point for \( 0 \leqslant h \leqslant h_1 \). Otherwise, we should find one more contact point \( \bar{M} \) on \( C_U \) with the vector field (20) between \( M \) and the intersection point \( M' \), see Fig. 12.

This gives four contact point on \( C_U \) with the vector field (20) (taking into account the multiplicity), contradicting (53). We note that if \( C_w \) and \( C_U \) intersect at \( A \) or \( B \), then this point has to be a contact point (i.e. \( h = 0 \) or \( h = h_1 \) would be a zero of (53)).

Next, we consider the case that \( C_w \) and \( C_U \) have a quadruple contact point at \( M \), and \( C_U \) cuts the \( \omega \)-axis at a point \( A' \) (by the discussion above, \( A' \) is different from \( A \)). Then by the saddle point property of \( A \), we must find a point \( \bar{M} \) on \( C_U \) between \( M \) and \( A' \) such that the vector field (20) is tangent to \( C_U \) at \( \bar{M} \), and this gives the same contradiction as above; see Figs. 13(a) and 13(b).

The only possibility left is the case that \( C_w \) and \( C_U \) have a quadruple contact point at \( M \), they have no other intersection point for \( 0 \leqslant h \leqslant h_1 \), and \( C_U \) does not cut the \( \omega \)-axis. This means that the zero point of \( h_1 + c_1 + \beta = 0 \), say \( h^* \), is between \( 0 \) and \( h_1 \), \( C_U \) is decreasing, and \( C_U \) must cut the straight line \( \{h = h_1\} \) at a point \( B' \) above \( B \). At \( B' \), the vector field (20) is pointing upward (see Fig. 14).

On the other hand, the fact that \( C_U \) is decreasing implies \( (\lambda - 1) \beta - 3\alpha + \lambda < 0 \) by (51), which gives \( n_3 < 0 \) by (54), hence by (53) for \( h \gg 1 \) the
FIG. 13. Comparing $C_w$ and $C_U$ in the $(h, \omega)$-plane.

Vectorfield (20) is pointing downward with respect to $C_U$, this gives one more contact point on $C_U$ and leads to the same contradiction.

Note that if $(\lambda - 1) \beta - 3 \alpha + \lambda = 0$, then $\omega = U(h)$ actually is the line $\{\omega = -\frac{1}{3} \}$ which has no intersection with $C_w$.

(ii) We next consider the case $\lambda < \lambda^*$. If there is a $\bar{\lambda} < \lambda^*$ such that for $h \in (0, h_1)$ $C_w$ and $C_F$ have more than one intersection points, then we let $\lambda$ vary from $\lambda$ to $\frac{4}{3}$ monotonously. We must find a $\lambda$ between $\bar{\lambda}$ and $\bar{\lambda}^*$ such that $C_w$ and $C_F$ have a contact point for some $h \in (0, h_1)$ which is impossible as shown in (i). Thus the proof of Lemma 11 is finished.

FIG. 14. More about contacts between $C_w$ and $C_U$ in the $(h, \omega)$-plane.
Lemma 12. $v'(h) < 0$ for $h \in (0, h_1)$ and $\lambda \in (1, +\infty)$.

Proof. We first deduce a differential equation for $v(h)$. From (47) we have

$$v'(h) = a_1 + b_1 \omega(h) + (b_1 h + c_1) \omega'(h).$$

Using (48) and (20) we obtain

$$\dot{h} = T(h),$$

$$\dot{v} = \psi(h, v),$$

where

$$T(h) = 36G(h)(h_3 - h),$$

$$h_3 = \frac{\lambda(\lambda + 2)(2\lambda + 1)}{36},$$

$$\psi(h, v) = m_2 v^2 + m_1 v + m_0,$$

and

$$m_2 = 3888(\lambda^2 + \lambda + 1) h^2 + 12(\lambda + 2)(2\lambda + 1)(2\lambda^3 + 6\lambda^2 + \lambda + 2) h$$

$$+ \lambda(10\lambda^2 - 11\lambda + 10)(\lambda + 2)^2 (2\lambda + 1)^2,$$

$$m_1 = 31104 h^4 - 288(22\lambda^4 + 29\lambda^3 + 6\lambda^2 + 29\lambda + 22) h^2$$

$$+ 24\lambda(\lambda + 2)(2\lambda + 1)\lambda(\lambda + 1)(2\lambda^3 + 11\lambda + 2) h - 12\lambda^4(\lambda + 2)^2 (2\lambda + 1)^2,$$

$$m_0 = -1728(5\lambda^2 - \lambda + 5) h^3 - 144\lambda(8\lambda^4 - 11\lambda^3 - 48\lambda^2 - 11\lambda + 8) h^2$$

$$- 12\lambda^4(\lambda + 2)^2 (2\lambda + 1)^2 h.$$

Note that since $h_3 > h_1$ for $\lambda > 1$, the new factor $(h_3 - h)$ in $T(h)$ has no influence on our discussion.

System (55) has four singularities for $0 \leq h \leq h_1$: two saddle points at $A(0, v_A)$ and $D(h_1, v_D)$; two nodes at $O(0, 0)$ and $B(h_1, 1)$, where

$$v_A = \frac{12\lambda^2}{10\lambda^2 - 11\lambda + 10} > 1,$$

$$v_D = -\frac{2\lambda^2 + 17\lambda + 5}{10\lambda^2 + 31\lambda + 31} < 0.$$

Using Lemma 1 and taking $h = P = Q = 0$ in the second equality of (21) we find $v(0) = v_A$; and taking $Q = \frac{-4\lambda^2 + 7\lambda + 2}{15\lambda^2 + 15}$ and $h = h_1$ we get $v(h_1) = 1$.

Hence the orbit in $(h, v)$-plane, which we are interested in, is the unstable manifold from the saddle point $A$ to the node $B$, we denote it by $C_v$. Note
that one branch of the 0-cline defined by $\psi(h, v) = 0$ also joins the two points $\tilde{A}$ and $\tilde{B}$, we denote it by $\tilde{C}_0$. Calculation shows that at the point $\tilde{A}$ the curve $\tilde{C}_0$ has slope

$$v'_0 = -\frac{5(28\lambda^4 + 212\lambda^3 - 453\lambda^2 + 212\lambda + 28)}{\lambda(10\lambda^2 - 11\lambda + 10)^2} < 0$$

and the slope of $C_n$ at the same point $\tilde{A}$ is $v'_A = \frac{\omega}{\lambda}$. Hence, if we prove that $\tilde{C}_0$ is strictly decreasing from $\tilde{A}$ to $\tilde{B}$, then $C_n$ must stay above $\tilde{C}_0$ for $0 < h < h_1$ (see Fig. 15(a)) and the required result follows.

Since the slope of $\tilde{C}_0$ at point $\tilde{B}(h_1, 1)$ is $-\frac{(\lambda + 3)}{(\lambda + 1)^2}$, also negative and hence if $\tilde{C}_0$ is not strictly decreasing for $0 < h < h_1$, then it must have at least one minimum and one maximum, so we can find a value $\bar{v}$ such that the line $\{v = \bar{v}\}$ cuts $\tilde{C}_0$ at least three times (see Fig. 15(b)); on the other hand, we will show that if $\lambda \geq \lambda_1$ ($\lambda_1 \approx 1.259$ is the positive root of equation $10\lambda^2 - 11\lambda - 2 = 0$), then there is one more intersection point of $\tilde{C}_0$ and the line $\{v = \bar{v}\}$, which contradicts the fact that $\psi(h, \bar{v}) = 0$ is a polynomial of $h$ of degree three, see (56) and (57).

It is not difficult to verify that $m_0 < 0$ for $h > 0$ and $\lambda > 1$. Hence $\psi(h, v) = 0$ defines two curves $\tilde{C}_0$: $v = v_+(h)$ and $\tilde{C}_0$: $v = v_-(h)$ for $h \in (0, +\infty)$, satisfying $v_+(h) > 0$ and $v_-(h) < 0$.

Note also that system (55) has two more singularities on the line $\{h = h_2\}$: $E(h_2, \lambda^2)$ and $F(h_2, v_F)$ with $v_F < 0$. By the above argument $\tilde{C}_0$ must pass through the point $E$. If $\lambda \geq \lambda_1$, then

$$\lambda^2 - v_A = \frac{\lambda^2(10\lambda^2 - 11\lambda - 2)}{10\lambda^2 - 11\lambda + 10} \geq 0.$$

Hence $\tilde{C}_0$ must be strictly decreasing for $0 < h < h_1$.

---

**FIG. 15.** The curve $\tilde{C}_0$ in the $(h, \omega)$-plane.
Finally we consider the case \( \lambda < \lambda_i \approx 1.259 \). Eliminating \( \nu \) from
\[
\psi(h, \nu) = 0
\]
\[
\psi'(h, \nu) = 0,
\]
we obtain
\[
\sigma(\lambda, h) = \sum_{k=0}^{2} \sigma_k(\lambda) h^k = 0,
\]
where
\[
\sigma_0(\lambda) = \lambda^5(28\lambda^4 + 212\lambda^3 + 453\lambda^2 + 212\lambda + 28)(\lambda + 2)^7 (2\lambda + 1)^7,
\]
\[
\sigma_1(\lambda) = 12\lambda^7 (\lambda + 2)^5 (2\lambda + 1)^5 (128\lambda^8 + 1976\lambda^7 - 13444\lambda^6 - 6718\lambda^5 + 31013\lambda^4
- 6718\lambda^3 - 13444\lambda^2 + 1976\lambda + 128),
\]
\[
\sigma_2(\lambda) = -144\lambda^8(\lambda + 2)^4 (2\lambda + 1)^4 (128\lambda^{12} + 4080\lambda^{11} + 168\lambda^{10} - 45188\lambda^9
+ 6426\lambda^8 + 50562\lambda^7 - 78279\lambda^6 + 50562\lambda^5 + 6426\lambda^4 - 45188\lambda^3
+ 168\lambda^2 + 4080\lambda + 128),
\]
\[
\sigma_3(\lambda) = 1728\lambda^9(\lambda + 2)^3 (2\lambda + 1)^3 (1600\lambda^{12} - 8880\lambda^{11} - 26256\lambda^{10} + 32984\lambda^9
- 54666\lambda^8 - 127926\lambda^7 + 136653\lambda^6 - 127926\lambda^5 + 54666\lambda^4
+ 32984\lambda^3 - 26256\lambda^2 - 8880\lambda + 1600),
\]
\[
\sigma_4(\lambda) = 20736\lambda(\lambda + 2)^2 (2\lambda + 1)^2 (256\lambda^{14} + 13840\lambda^{13} + 19774\lambda^{12}
- 22945\lambda^{11} + 34334\lambda^{10} + 58628\lambda^9 + 65058\lambda^8 + 351015\lambda^7
+ 65058\lambda^6 + 58628\lambda^5 + 34334\lambda^4 - 22945\lambda^3 + 19774\lambda^2
+ 13840\lambda + 256),
\]
\[
\sigma_5(\lambda) = 248832(\lambda + 2)(2\lambda + 1)(1344\lambda^{14} + 6592\lambda^{13} - 30294\lambda^{12}
- 110871\lambda^{11} - 8494\lambda^{10} + 45636\lambda^9 - 305550\lambda^8 - 436755\lambda^7
- 305550\lambda^6 + 45636\lambda^5 - 8494\lambda^4 - 110871\lambda^3 - 30294\lambda^2
- 6592\lambda + 1344),
\]
\[
\sigma_6(\lambda) = -8957952(\lambda + 2)(2\lambda + 1)(232\lambda^{10} + 893\lambda^9 - 8136\lambda^8
- 20982\lambda^7 + 12\lambda^6 + 10035\lambda^5 + 12\lambda^4 - 20982\lambda^3
- 8136\lambda^2 + 893\lambda + 232),
\]
\[
\sigma_7(\lambda) = -2902376448(\lambda^2 + \lambda + 1)(73\lambda^6 + 147\lambda^5 - 30\lambda^4
- 137\lambda^3 - 30\lambda^2 + 147\lambda + 73).
\]
It is not difficult to show (by “Maple”, for example) that
\[ (-1)^k \frac{\partial^k \sigma}{\partial \ell^k} (\lambda, 0) > 0 \quad \text{and} \quad (-1)^k \frac{\partial^k \sigma}{\partial \ell^k} (\lambda, h_1) > 0 \]
for \( k = 0, 1, \ldots, 7 \) and \( \lambda \in (1, \lambda_1) \) uniformly. By the Fourier-Budan rule, (59) has no solution for \( h \in (0, h_1) \) and \( \lambda \in (1, \lambda_1) \), hence \( C_0 \) has no minimum nor maximum for \( h \in (0, h_1) \).  

5. ESTIMATING THE NUMBER OF ZEROS OF \( I(h) \)

We need one more lemma.

**Lemma 13.** For all constants \( \alpha \) and \( \beta \), if in \((P, Q)\)-plane the straight line \( L: \{ \alpha + \beta P + Q = 0 \} \) and the curve \( \Sigma: \{(P, Q)(h) \mid h \in [0, h_1]\} \) have at least two intersection points, multiplicity taken into account, then for the same \( \alpha \) and \( \beta \), in \((\omega, v)\)-plane the straight line \( \tilde{L}: \{ \alpha + \beta \omega + v = 0 \} \) can cut the curve \( \Omega : \{(\omega, v)(h) \mid h \in [0, h_1]\} \) at most twice, counted the multiplicity.

**Proof.** If \( \lambda \geq \lambda^* \), then by Lemma 11 \( C_w \) and \( C_F \) have no intersection point for \( h \in (0, h_1) \), this means that the curve \( \Omega \) has no inflection point, hence the conclusion is obviously true.

We note by Lemma 9 that \( C_w \) has a unique maximum for \( \lambda < \lambda_0 \) while \( C_w \) is strictly decreasing for \( \lambda \geq \lambda_0 \). On the other hand, by (39) and (58) we have
\[
\left. \frac{dv}{d\omega} \right|_{\omega = \omega_0} = \frac{v_0' / 2}{\omega_0^2} = \frac{C}{2\lambda^2 - 7\lambda + 2}, \tag{60}
\]
where \( C \) is positive for \( \lambda > 1 \). By (47) we have
\[
\frac{dv}{d\omega} = (a_1 + b_1 \omega) \frac{dh}{d\omega} + b_1 h + c_1,
\]
combining with the fact that \( \lim_{\omega \to 1} \frac{dh}{d\omega} = 0 \) we get by using (48)
\[
\left. \frac{dv}{d\omega} \right|_{\omega = 1} = b_1 h_1 + c_1 = \frac{\lambda + 3}{\lambda + 2}, \tag{61}
\]
FIG. 16. The curve $\Omega$ in the $(\omega, \nu)$-plane, $\lambda \geq \lambda^*$. $(60)$ and $(61)$ give the slopes of the curve $\Omega$ at its two endpoints. By using these facts as well as Lemma 11 we obtain the behaviour of $\Omega$ for $\lambda \geq \lambda^*$ shown in Fig. 16.

If $\lambda < \lambda^*$, then by Lemma 11 $C_\omega$ and $C_F$ have a unique intersection point which corresponds to the unique inflection point on $\Omega$. In this case we have $\omega_\lambda < 1$ and there is a unique $\hat{h} > 0$ such that $\omega'(\hat{h}) = 0$ as we have proved in Lemma 9. On the other hand, the fact that $C_\omega$ is located above $C_F$ for $h$ near 0 implies $d^2\omega/d\omega^2 > 0$ for $(\omega, \nu) \in \Omega$ and near $A$. Hence the inflection point of $\Omega$ must take place for a $\hat{h} < \hat{h}$, see Fig. 17.

By Lemma 12 $\nu'(\hat{h}) < 0$, hence if any straight line cuts $\Omega$ at three points, then it must have a negative slope. By Lemma 1 (iii), this line can cut the curve $\Sigma$ at most at one point. This finishes the proof of Lemma 13.

Let us now finish the proof of the main result.

FIG. 17. The curve $\Omega$ in the $(\omega, \nu)$-plane, $\lambda < \lambda^*$. 

The case $\lambda < \lambda^*$
**Proof of Theorem 1.** For $h > 0$, the number of zeros of

$$I(h) = aI_0(h) + bI_1(h) + I_2(h)$$

is the number of intersection points of the straight line

$$\mathcal{L}: \{a + bP + Q = 0\}$$

and the curve $\Sigma: \{(P, Q)(h) | h \in (0, h_1]\}$. Let us prove that $\Sigma$ has no inflection points nor quadruple points inducing the conclusion.

By Lemma 2 and 6, $d^2Q/dP^2 < 0$ for $(P, Q) \in \Sigma$ and near the two endpoints $(P(0), Q(0))$ and $(P(h_1), Q(h_1))$. By Lemma 5 and (37), $\Sigma$ is strictly located in the triangle formed by the straight lines $\mathcal{L}_1$, $\mathcal{L}_3$ and $\mathcal{L}_2$, which are tangent lines of $\Sigma$ at these two endpoints, and the line passing through them. All these facts imply that if $\Sigma$ has a first inflection point, it has to be followed by another inflection point, and there are constants $a^*$ and $b^*$, such that the straight line $\mathcal{L}^*: \{a^* + b^*P + Q = 0\}$ cuts the curve $\Sigma$ at least at four points for $h > 0$; see Fig. 18.

On the other hand, $I(h)$ has always a zero at $h = 0$, hence for these $a^*$ and $b^*$, $I(h)$ has at least 5 zeros for $0 \leq h \leq h_1$. Therefore,

$$I''(h) = I''_0(h)(a^* + b^*\omega(h) + \nu(h))$$
has at least 3 zeros for \( h \in (0, h_1) \). But \( I_2^*(h) \neq 0 \) for \( h \in (0, h_1) \) ([CS] and ref. [G]); hence the straight line \( \{ \alpha^* + \beta^* \omega + \nu = 0 \} \) cuts the curve \( \Omega \) at least at 3 points, and this contradicts Lemma 13.

Suppose now that \( \Sigma \) had a quadruple point at some \( 0 < \bar{h} < h_1 \) and \( \mathcal{L}^* \) denote the tangent line of \( \Sigma \) at \((P, Q)(\bar{h})\). By similar arguments as above the same line, seen in \((\omega, \nu)\)-plane, has to cut \( \Omega \) at 2 points, among which one is a tangent point; again this is not possible by Lemma 13.

6. TWO SADDLE CYCLE

The two saddle cycle case can be seen as a limit of the saddle loop case for \( \lambda \to 1 \). Thus, the Hamiltonian function (18), the perturbed system (23) and the Abelian integral (24) become respectively

\[
\begin{align*}
\bar{H}(x, y) &= \frac{y^2}{2} - \frac{1}{4} x^4 + \frac{1}{2} x^2, \\
\dot{x} &= y, \\
\dot{y} &= x(x^2 - 1) + \delta(\alpha + \beta x + x^2) y,
\end{align*}
\]

and

\[
I(h) = \int_{\Gamma_h} (\alpha + \beta x + x^2) y \, dx,
\]

where

\[
\Gamma_h = \{(x, y) \mid \bar{H}(x, y) = h, 0 < h < \frac{1}{4}\}.
\]

For \( \delta = 0 \), the unperturbed system (63) has a phase portrait as shown in Fig. 19.

**Theorem 2.** For all constants \( \alpha \) and \( \beta \) the least upper bound of the number of zeros of the Abelian integral (64) is one, taking into account the multiplicity.

**Proof.** Since \( I(h) = aI_0(h) + \beta I_1(h) + I_2(h) \), by the symmetry of the Hamiltonian function, \( I_1(h) = 0 \) for \( h \in (0, \frac{1}{4}) \), we have \( I(h) = I_0(h)(\alpha + Q(h)) \), where \( Q(h) = I_2(h)/I_0(h) \).

Let \( \Phi(x) = \bar{H}(x, 0) \), then for each \( x \in (-1, 0) \) there is unique \( \bar{x} = -x \in (0, 1) \) such that \( \Phi(x) = \Phi(\bar{x}) \). The function \( \zeta(x) \) defined by (26) is \( \zeta(x) = x^2 \), hence \( \zeta'(x) = 2x < 0 \). By Lemma 3, \( Q'(h) > 0 \) for \( h \in (0, 1/4) \), and this finishes the proof of the theorem.
Remark. Theorem 2 was first obtained in ref. [H], and there is an alternative study in [CLW]. We give a simple proof here.

It is important to consider the relation between the two cases in this paper, by comparing the bifurcation diagrams in $(\alpha, \beta)$-plane. Although all our results deal with the number of zeros of Abelian integrals, they induce similar results on the number of limit cycles for system (23) and $\delta > 0$ sufficiently small. This is clearly true on the regular part of the Hamiltonian function, because of the implicit function theorem. At the center and near the homoclinic loop, or near the two saddle cycle, we refer to [DFL] to see that the results on the Abelian integrals can be transposed to results on limit cycles. As such the first and second order Hopf bifurcations (resp. homoclinic bifurcations) of system (23) are given by

$$\alpha + \beta P(h) + Q(h) = 0$$

and

$$\alpha + \beta P(h) + Q(h) = 0, \quad P'(h) + Q'(h) = 0$$

for $h = 0$ (resp. $h = h_1$). From Lemmas 1 and 2 we know that $P(0) = Q(0) = 0$ and $\frac{Q'(0)}{P'(0)} = \frac{1}{\pi^2}$. By the definitions of $P$ and $Q$ and Lemma 1 (iii) we get $0 < P(h_1) < 1$, $0 < Q(h_1) < 1$ and $P'(h) > 0$, $Q'(h) > 0$. Hence we obtain the bifurcation diagram for system (23) shown in Figure 20(a), where the curves $H$ and $L$ (resp. the points $H_2$ and $L_2$) correspond to Hopf and homoclinic loop bifurcation of order 1 (resp. order 2).
The coordinates of the points $H_2$, $L_2$, $B$ and $A$ are

$$
\left( 0, -\frac{\hat{\lambda}}{\hat{\lambda}-1} \right), \quad \left( \frac{Q'(h_1)}{P'(h_1)} - Q(h_1), -\frac{Q'(h_1)}{P'(h_1)} \right), \quad \left( 0, -\frac{Q(h_1)}{P(h_1)} \right),
$$

and \((-Q(h_1), 0)\) respectively. From the proof of Lemma 6 and formula (36) we obtain that

$$
\frac{Q'(h_1)}{P'(h_1)} = \frac{Q'(0) + 2P(0)}{P'(0)} = \frac{1 - Q(h_1)}{1 - P(h_1)}.
$$

As $\hat{\lambda} \to 1 + 0$, $P(h_1) \to 0$, $Q(h_1) \to \frac{1}{\hat{\lambda}}$, hence the points $H_2 \to (0, -\infty)$, $L_2 \to L'_2 = (-\frac{1}{\hat{\lambda}}, -\frac{1}{\hat{\lambda}})$, $B \to (0, -\infty)$ and $A \to A' = (-\frac{1}{\hat{\lambda}}, 0)$. Thus the curved triangle $H_2L_2B$ in Fig. 18(a), corresponding to the region of two zeros of the elliptic integrals, tends to the half line \(\{(a, b) | a = -\frac{1}{5}, b < -\frac{4}{5}\}\) which is contained in the bifurcation curve of the two saddle connection $TSC : \{\alpha = -\frac{1}{5}\}$ for system (63); see Figure 20(b). Along the line $TSC = \{\alpha = -1/5\}$, if we calculate the divergence at the saddle $(x, y) = (1, 0)$, we find $\delta(\frac{1}{5} + \beta)$. The expression after $\delta$ changes its sign at $L'_2$. As such the point $L'_2$ does not play a special role in studying the zeros of the elliptic integrals with respect to $H_l(x, y)$ for $l = 1$, but it does when we consider $\hat{\lambda}$ as a changing parameter. There is a similar point $L'_2$ at $\beta = -\frac{4}{5}$, but it only

FIG. 20. Bifurcation diagrams in the $(\alpha, \beta)$-plane.
leads to a possibility of having two zeros if we move to Hamiltonians $H_l(x, y)$ with $l < 1$.

This gives an explanation of the fact that the maximum number of zeros of integral (24) is two, while in the limit, for $\lambda \to 1$, integral (64) has at most 1 zero.

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