Linear Algebra and its Applications 431 (2009) 746–759

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Perturbation of frames and Riesz bases in Hilbert C^*-modules

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ARTICLE INFO

Article history:
Received 11 January 2008
Accepted 18 March 2009
Available online 28 April 2009

Submitted by H. Schneider

AMS classification:
Primary 46L99
Secondary 42C15, 46H25

Keywords:
Perturbations
Frames
Riesz bases
Hilbert C^*-modules

ABSTRACT

We extend the Casazza–Christensen general perturbation theorem for Hilbert space frames to modular frames in Hilbert C^*-modules. In the Hilbert space setting, under the same perturbation condition, the perturbation of any Riesz basis remains to be a Riesz basis. However, this result is no longer true for Riesz bases in Hilbert C^*-modules. We obtain a necessary and sufficient condition under which the perturbation (under Casazza–Christensen's perturbation condition) of Riesz bases of Hilbert C^*-modules remains to be Riesz bases.

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1. Introduction

Let H be a Hilbert space with Riesz basis $\{f_j\}_{j=1}^{\infty}$, and let $\{g_j\}_{j=1}^{\infty}$ be a sequence of vectors in H. If there exists a constant $\lambda \in [0, 1)$ such that

$$\left\| \sum c_j (f_j - g_j) \right\| \leqslant \lambda \left\| \sum c_j f_j \right\| \tag{1}$$

for all finite sequences $\{c_j\}$ of scalars, then $\{g_j\}_{j=1}^{\infty}$ is also a Riesz basis for H. This result is the well-known classical Paley–Wiener Theorem on perturbation of Riesz bases in Hilbert spaces [14]. Note

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[1] The second named author is partially supported by the Faculty Development Research Grant of Fayetteville State University.

0024-3795/$ - see front matter © Published by Elsevier Inc.
doi:10.1016/j.laa.2009.03.025

that condition (1) implies that there exists a bounded invertible operator T such that $Tf_i = g_i$ (see [16]). Therefore, this observation enables us to investigate the perturbation of bases and frames from the operator perturbation point of view (see [3,4]). In the last decade, several authors have generalized the Paley–Wiener perturbation theorem to the perturbation of frames in Hilbert spaces (see [2–5]). The most general result of these was the following obtained by Casazza and Christensen [3].

Theorem 1.1 [3]. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a frame for a Hilbert space H with frame bounds C and D. Assume that $\{y_j\}_{j\in J}$ is a sequence of H and that there exist $\lambda_1, \lambda_2, \mu \geqslant 0$ such that* $\max\left\{\lambda_1 + \frac{\mu}{\sqrt{C}}, \lambda_2\right\} < 1$. *Suppose one of the following conditions holds for any finite scalar sequence $\{c_j\}$ and every $x \in H$. Then $\{y_j\}_{j\in\mathbb{J}}$ is also a frame for H.*

(i) $\left(\sum_{j\in\mathbb{J}} |\langle x, x_j - y_j\rangle|^2\right)^{\frac{1}{2}} \leqslant \lambda_1 \left(\sum_{j\in\mathbb{J}} |\langle x, x_j\rangle|^2\right)^{\frac{1}{2}} + \lambda_2 \left(\sum_{j\in\mathbb{J}} |\langle x, y_j\rangle|^2\right)^{\frac{1}{2}} + \mu\|x\|;$

(ii) $\left\|\sum_{j=1}^{n} c_j(x_j - y_j)\right\| \leqslant \lambda_1 \left\|\sum_{j=1}^{n} c_j x_j\right\| + \lambda_2 \left\|\sum_{j=1}^{n} c_j y_j\right\| + \mu \left(\sum_{j=1}^{n} |c_j|^2\right)^{\frac{1}{2}}.$

Moreover, if $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis for H and $\{y_j\}_{j\in\mathbb{J}}$ satisfies (ii), then $\{y_j\}_{j\in\mathbb{J}}$ is also a Riesz basis for H.

Frames for Hilbert spaces have natural generalizations in Hilbert C^*-modules that are generalizations of Hilbert spaces by allowing the inner product to take values in a more general C^*-algebra than $\mathbb{C}$ (see Definition 2.1). Note that the theory of Hilbert C^*-modules is quite different from that of Hilbert spaces. Unlike Hilbert space cases, not every closed submodule of a Hilbert C^*-module is complemented. Moreover, the well-known Riesz representation theorem for continuous functionals in Hilbert spaces does not hold in Hilbert C^*-modules, which implies that not all bounded linear operators on Hilbert C^*-modules are adjointable. It should also be remarked that, due to the complexity of the C^*-algebras involved in the Hilbert C^*-modules and the fact that some useful techniques available in Hilbert spaces are either absent or unknown in Hilbert C^*-modules, these are many essential differences between Hilbert space frames and Hilbert C^*-module frames. To name a few: in Hilbert spaces every Riesz basis has a unique dual which is also a Riesz basis. But in Hilbert C^*-modules, due to the existence of zero-divisors, not all Riesz bases have unique duals, and not every dual is a Riesz basis (see [9]). Also, there could exist a nonzero element a in the underlying C^*-algebra such that $ax_j = 0$ for each vector x_j in a modular Riesz basis $\{x_j\}_{j\in\mathbb{J}}$ (see Remark 3.6) which never occurs in Hilbert spaces. One of the striking differences is the recent result of Hanfeng Li who proved that not every Hilbert C^*-module admits a frame [13]. This shows that the famous Kasparov stabilization theorem for countably generated Hilbert C^*-modules can not be extended to arbitrary Hilbert C^*-modules. We refer to [7–9,12] for more discussions on some essential differences between Hilbert space frames and Hilbert C^*-modular frames.

In this paper we examine the perturbation of frames and Riesz bases in Hilbert C^*-modules. We will show that while the Casazza–Christensen general perturbation theorem (Theorem 1.1) for frames in Hilbert spaces remains valid for Hilbert C^*-modular frames (Theorem 3.2), the perturbation theory for Riesz bases (under the similar perturbation condition of Theorem 1.1) no longer holds for Riesz bases in Hilbert C^*-modules (Example 3.4). We obtain a necessary and sufficient condition under which the perturbation (under Casazza–Christensen's perturbation condition) of a Hilbert C^*-modular Riesz basis remains to be a Riesz basis (Theorem 3.5).

2. Preliminaries

We first recall some definitions and results about Hilbert C^*-modules, frames and Riesz bases in Hilbert C^*-modules.

Definition 2.1. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{H}$ be a (left) $\mathcal{A}$-module. Suppose that the linear structures given on A and $\mathcal{H}$ are compatible, i.e. $\lambda(ax) = a(\lambda x)$ for every $\lambda \in \mathbb{C}, a \in \mathcal{A}$ and $x \in \mathcal{H}$. Assume that there exists a mapping $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ with the following properties:

(i) $\langle x, x\rangle \geqslant 0$ for every $x \in \mathcal{H}$,
(ii) $\langle x, x\rangle = 0$ if and only if $x = 0$,
(iii) $\langle x, y\rangle = \langle y, x\rangle^*$ for every $x, y \in \mathcal{H}$,
(iv) $\langle ax, y\rangle = a\langle x, y\rangle$ for every $a \in \mathcal{A}$, and every $x, y \in \mathcal{H}$,
(v) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for every $x, y, z \in \mathcal{H}$.

Then the pair $\{\mathcal{H}, \langle\cdot,\cdot\rangle\}$ is called a (left-) pre-Hilbert $\mathcal{A}$-module. The map $\langle\cdot,\cdot\rangle$ is said to be an $\mathcal{A}$-valued inner product. If the pre-Hilbert $\mathcal{A}$-module $\{\mathcal{H}, \langle\cdot,\cdot\rangle\}$ is complete with respect to the induced norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$, then it is called a *Hilbert $\mathcal{A}$-module*.

Definition 2.2 [8]. Let $\mathcal{A}$ be a unital C^*-algebra and $\mathbb{J}$ be a finite or countable index set. A (countable or finite) sequence $\{x_j\}_{j\in\mathbb{J}}$ of elements in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be a (standard) *frame* for $\mathcal{H}$ if there exist two constants $C, D > 0$ such that the *frame inequality*

$$C\langle x, x\rangle \leqslant \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \leqslant D\langle x, x\rangle$$

holds for every $x \in \mathcal{H}$, where the sum in the middle of the inequality is convergent in norm. The numbers C and D are called *frame bounds*. The sequence $\{x_j\}_{j\in\mathbb{J}}$ is called a (standard) *Bessel sequence* with Bessel bound D if we only require the right-hand side of the frame inequality.

Definition 2.3 [8]. A frame $\{x_j\}_{j\in\mathbb{J}}$ for a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be a (standard) *Riesz basis* for $\mathcal{H}$ if it satisfies:

(i) $x_j \neq 0$ for all j;
(ii) if an $\mathcal{A}$-linear combination $\sum_{j\in S} a_j x_j$ with coefficients $\{a_j : j \in S\} \subseteq \mathcal{A}$ and $S \subseteq \mathbb{J}$ is equal to zero, then every summand $a_j x_j$ is zero.

In this paper we focus on finitely and countably generated Hilbert C^*-modules over unital C^*-algebra $\mathcal{A}$. A Hilbert $\mathcal{A}$-module $\mathcal{H}$ is (*algebraically*) *finitely generated* if there exists a finite set $\{x_1, \ldots, x_n\} \subseteq \mathcal{H}$ such that every element $x \in \mathcal{H}$ can be expressed as an $\mathcal{A}$-linear combination $x = \sum_{i=1}^n a_i x_i, a_i \in \mathcal{A}$. A Hilbert $\mathcal{A}$-module $\mathcal{H}$ is *countably generated* if there exists a countable set $\{x_i\} \subseteq \mathcal{H}$ such that $\mathcal{H}$ equals the norm-closure of the linear span (over $\mathbb{C}$ and $\mathcal{A}$) of this set.

From the definition of frames (resp. Bessel sequences) in Hilbert C^*-modules, it is clear that we need to compare positive elements in the underlying C^*-algebra in order to test whether a sequence is a frame (resp. Bessel sequence) or not. This usually is not a trivial task. The following characterization of modular Bessel sequences and frames, which was obtained independently by Arambašić [1] and Jing [11], enables us to verify whether a sequence is a modular frame (resp. Bessel sequence) in terms of norms. It also allows us to characterize modular frames from the operator theory point of view, and it is needed in proving our main results of this paper.

Proposition 2.4 [11]. *Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$ and $\{x_j\}_{j\in\mathbb{J}} \subseteq \mathcal{H}$ a sequence. Then*

(i) *$\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence of $\mathcal{H}$ with Bessel bound D if and only if*

$$\left\|\sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle\right\| \leqslant D\|x\|^2$$

for all $x \in \mathcal{H}$.

(ii) $\{x_j\}_{j\in\mathbb{J}}$ *is a frame of* $\mathcal{H}$ *with frame bounds C and D if and only if*

$$C\|x\|^2 \leqslant \left\|\sum_{j\in\mathbb{J}}\langle x,x_j\rangle\langle x_j,x\rangle\right\| \leqslant D\|x\|^2$$

for all $x \in \mathcal{H}$.

We now introduce a few more notations. For a unital C^*-algebra $\mathcal{A}$, let $l^2(\mathcal{A})$ be the Hilbert $\mathcal{A}$-module defined by

$$l^2(\mathcal{A}) = \left\{\{a_j\}_{j\in\mathbb{J}} \subseteq \mathcal{A} : \sum_{j\in\mathbb{J}} a_j a_j^* \text{ converges in } \|\cdot\|\right\}.$$

Let $\{e_j\}_{j=1}^{\infty}$ denote the standard orthonormal basis of $l^2(\mathcal{A})$, where e_j takes value $1_{\mathcal{A}}$ at j and $0_{\mathcal{A}}$ everywhere else. For any Bessel sequence $\{x_j\}_{j\in J}$ of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$, the associated *analysis operator* $T_X : \mathcal{H} \to l^2(\mathcal{A})$ is defined by

$$T_X x = \sum_{j\in\mathbb{J}}\langle x,x_j\rangle e_j, \quad x \in \mathcal{H}.$$

Note that the analysis operator T_X is adjointable and fulfills $T_X^* e_j = x_j$ for all j. The operator $S_X : \mathcal{H} \to \mathcal{H}$ defined by

$$S_X x = T_X^* T_X x = \sum_{j\in\mathbb{J}}\langle x,x_j\rangle x_j$$

is called the *frame operator*.

In [9] we obtained the following characterization for Riesz bases in Hilbert C^*-modules.

Theorem 2.5 [9]. *Let* $\{x_j\}_{j\in\mathbb{J}}$ *be a frame for a finitely or countably generated Hilbert* C^**-module* $\mathcal{H}$. *Then* $\{x_j\}_{j\in\mathbb{J}}$ *is a Riesz basis if and only if the range space of its analysis operator* T_X *is* P_n*-invariant for each n, where* P_n *is the projection on* $l^2(\mathcal{A})$ *that maps each element to its nth component*.

Following the definition of Riesz bases in Hilbert C^*-modules, to test whether a frame $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis, one needs to show that if $\sum_{j\in\mathbb{J}} c_j x_j = 0$ for some sequence $\{c_j\}_{j\in\mathbb{J}} \subseteq \mathcal{A}$, then $c_j x_j = 0$ for each j. The following result allows us to consider the sequence $\{c_j\}_{j\in\mathbb{J}}$ only in $l^2(\mathcal{A})$.

Proposition 2.6 [9]. *Suppose that* $\{x_j\}_{j\in\mathbb{J}}$ *is a frame of* $\mathcal{H}$, *then* $\{x_j\}_{j\in\mathbb{J}}$ *is a Riesz basis if and only if*

(i) $x_j \neq 0$ *for each* $j \in \mathbb{J}$;
(ii) *if* $\sum_{j\in\mathbb{J}} c_j x_j = 0$ *for some sequence* $\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$, *then* $c_j x_j = 0$ *for each* $j \in \mathbb{J}$.

In the following we give some characterizations of Bessel sequences, frames and Riesz bases in Hilbert C^*-modules from the operator-theoretic point of view. Note that these results are just modifications of their analogues in the Hilbert space setting and the proofs follow the similar line of reasonings as those in Hilbert spaces (see Theorems 3.2.3, 5.5.1, and Lemma 5.5.4 in [6]).

We begin with the following lemma which is due to Heuser [10]. Heuser only considered the $l^2(\mathbb{C})$-sequence case, but his proof also works in a more general setting.

Lemma 2.7. *Let* $\mathcal{A}$ *be a* C^**-algebra and* $\{c_j\}_{j\in\mathbb{J}}$ *a sequence in* $\mathcal{A}$. *If* $\sum_{j\in\mathbb{J}} c_j\xi_j^*$ *converges for all* $\{\xi_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$, *then* $\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$.

Proof. We define a sequence of operators F_n and an operator F by

$$F_n(\{\xi_j\}) = \sum_{j=1}^{n} c_j\xi_j^* \quad \text{and} \quad F(\{\xi_j\}) = \sum_{j=1}^{\infty} c_j\xi_j^* \quad \forall\{\xi_j\} \in l^2(\mathcal{A}).$$

Observe that

$$\|F_n(\{\xi_j\})\|^2 = \left\|\sum_{j=1}^{n} c_j\xi_j^*\right\|^2 \leqslant \left\|\sum_{j=1}^{n} c_jc_j^*\right\| \cdot \left\|\sum_{j=1}^{n} \xi_j\xi_j^*\right\| \leqslant \|\{c_j\}\|^2 \cdot \left\|\sum_{j=1}^{n} \xi_j\xi_j^*\right\|.$$

It follows that F_n is bounded for each n. Clearly, $F_n \to F$ pointwise as $n \to \infty$, so F is bounded by the Uniform Boundedness Theorem. Therefore $\|F(\{\xi_j\})\| \leqslant \|F\| \cdot \|\{\xi_j\}\|$ for each $\{\xi_j\} \in l^2(\mathcal{A})$.

Now fix n, and let

$$\xi_j = \begin{cases} c_j^*, & \text{if } 1 \leqslant j \leqslant n; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{\xi_j\} \in l^2(\mathcal{A})$.

We compute

$$\begin{aligned} \left\|\sum_{j=1}^{n} c_jc_j^*\right\| &= \left\|\sum_{j=1}^{n} c_j\xi_j\right\| \leqslant \|F\| \cdot \|\{\xi_j\}\| \\ &= \|F\| \cdot \left\|\sum_{j=1}^{\infty} \xi_j\xi_j^*\right\|^{\frac{1}{2}} = \|F\| \cdot \left\|\sum_{j=1}^{n} \xi_j\xi_j^*\right\|^{\frac{1}{2}} \\ &= \|F\| \cdot \left\|\sum_{j=1}^{n} c_jc_j^*\right\|^{\frac{1}{2}}. \end{aligned}$$

Therefore $\left\|\sum_{j=1}^{n} c_jc_j^*\right\|^{\frac{1}{2}} \leqslant \|F\|$, and hence $\{c_j\} \in l^2(\mathcal{A})$. □

The following is elementary and well known in Hilbert space setting, and will be used in the next section. We include a proof for completeness.

Proposition 2.8. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a sequence of a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$. We define an operator $U : l^2(\mathcal{A}) \to \mathcal{H}$ by*

$$U\{c_j\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} c_jx_j.$$

Then

(i) *$\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence with Bessel bound D if and only if operator U is a well-defined bounded operator from $l^2(\mathcal{A})$ into $\mathcal{H}$ with $\|U\| \leqslant \sqrt{D}$.*
Moreover, $\{x_j\}_{j\in\mathbb{J}}$ is a frame if and only if U is a bounded operator from $l^2(\mathcal{A})$ onto $\mathcal{H}$.

(ii) *$\{x_j\}_{j\in\mathbb{J}}$ is a frame of $\mathcal{H}$ with bounds C and D if and only if $\overline{span\{x_j : j \in \mathbb{J}\}} = \mathcal{H}$ and operator U is bounded and satisfies*

$$\sqrt{C}\|\{c_j\}\| \leqslant \|U\{c_j\}\| \leqslant \sqrt{D}\|\{c_j\}\| \quad \forall\{c_j\} \in (Ker\ U)^{\perp}. \tag{2}$$

Furthermore, $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis with unique dual frame if and only if $\overline{span\{x_j : j \in \mathbb{J}\}} = \mathcal{H}$ and there exist $C, D \geqslant 0$ such that

$$\sqrt{C}\|\{c_j\}\| \leqslant \|U\{c_j\}\| \leqslant \sqrt{D}\|\{c_j\}\| \quad \forall\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A}).$$

Proof. (i) We first consider the case of Bessel sequences.

"⇒". Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence with bound D. We first show that U is well-defined. For arbitrary $n > m$, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{m} c_j x_j\right\|^2 &= \left\|\sum_{j=m+1}^{n} c_j x_j\right\|^2 \\ &= \sup_{\|x\|=1}\left\|\left\langle \sum_{j=m+1}^{n} c_j x_j, x\right\rangle\right\|^2 = \sup_{\|x\|=1}\left\|\sum_{j=m+1}^{n} c_j \langle x_j, x\rangle\right\|^2 \\ &\leqslant \sup_{\|x\|=1}\left\|\sum_{j=m+1}^{n}\langle x, x_j\rangle\langle x_j, x\rangle\right\| \cdot \left\|\sum_{j=m+1}^{n} c_j c_j^*\right\| \\ &\leqslant D\left\|\sum_{j=m+1}^{n} c_j c_j^*\right\|,\end{aligned}$$

which implies that $\sum_{j\in\mathbb{J}} c_j x_j$ converges. Therefore U is well-defined.

For the boundedness of U, since

$$\begin{aligned}\|U\{c_j\}\|^2 = \sup_{\|x\|=1}\|\langle U\{c_j\}, x\rangle\|^2 &= \sup_{\|x\|=1}\left\|\sum_{j\in\mathbb{J}} c_j\langle x_j, x\rangle\right\|^2 \\ &\leqslant \sup_{\|x\|=1}\left\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\right\| \cdot \left\|\sum_{j\in\mathbb{J}} c_j c_j^*\right\| \\ &\leqslant D\left\|\sum_{j\in\mathbb{J}} c_j c_j^*\right\| = D\|\{c_j\}\|^2,\end{aligned}$$

we have that $\|U\| \leqslant \sqrt{D}$.

"⇐". For arbitrary $x \in \mathcal{H}$ and $\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$, we have

$$\langle x, U\{c_j\}\rangle = \left\langle x, \sum_{j\in\mathbb{J}} c_j x_j\right\rangle = \sum_{j\in\mathbb{J}}\langle x, x_j\rangle c_j^*. \tag{3}$$

By Lemma 2.7, we see that $\{\langle x, x_j\rangle\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$. From (3), we get

$$\langle x, U\{c_j\}\rangle = \langle\{\langle x, x_j\rangle\}, \{c_j\}\rangle,$$

which implies that U is adjointable and $U^*x = \{\langle x, x_j\rangle\}_{j\in\mathbb{J}}$. Observe that

$$\left\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\right\| = \|U^*x\|^2 \leqslant \|U^*\|^2 \cdot \|x\|^2 = \|U\|^2 \cdot \|x\|^2 \leqslant D\|x\|^2.$$

Hence, from Proposition 2.4, $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence.

For the case of modular frames, we only need to show that if U is bounded and onto then $\{x_j\}_{j\in\mathbb{J}}$ is a frame. We already know that $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence. Let D be the Bessel bound of $\{x_j\}_{j\in\mathbb{J}}$. Note that for each $x \in \mathcal{H}$, we have

$$x = UU^*(UU^*)^{-1}x = \sum_{j\in\mathbb{J}}\langle (UU^*)^{-1}x, x_j\rangle x_j.$$

So we get

$$\begin{aligned}\|x\|^4 = \|\langle x,x\rangle\|^2 &= \left\|\sum_{j\in\mathbb{J}}\langle (UU^*)^{-1}x, x_j\rangle\langle x_j, x\rangle\right\|^2\\
&\leqslant \left\|\sum_{j\in\mathbb{J}}\langle (UU^*)^{-1}x, x_j\rangle\langle x_j, (UU^*)^{-1}x\rangle\right\| \cdot \left\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\right\|\\
&\leqslant D\|\langle (UU^*)^{-1}x, (UU^*)^{-1}x\rangle\| \cdot \left\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\right\|\\
&= D\|(UU^*)^{-1}x\|^2 \cdot \left\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\right\|\\
&\leqslant D\|(UU^*)^{-1}\|^2 \cdot \|x\|^2 \cdot \left\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\right\|,\end{aligned}$$

which leads to the lower bound in the frame inequality, that is

$$\frac{1}{D\|(UU^*)^{-1}\|^2}\|x\|^2 \leqslant \left\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\right\|.$$

(ii) We only prove the case of frames, and the case of Riesz bases follows easily.

"⇒". Suppose first that $\{x_j\}_{j\in\mathbb{J}}$ is a frame. Let S be the frame operator of $\{x_j\}_{j\in\mathbb{J}}$. Then we have $S = UU^*$. By (i), it is enough to show that

$$\sqrt{C}\|\{c_j\}\| \leqslant \|U\{c_j\}\|$$

holds for all $\{c_j\} \in (Ker\, U)^\perp$. Since $\{x_j\}_{j\in\mathbb{J}}$ is a frame, it follows that $Rang(U^*)$ is closed. Therefore we have

$$(Ker\, U)^\perp = \overline{Rang(U^*)} = Rang(U^*).$$

As a sequence, $(Ker\, U)^\perp = \{\{\langle x, x_j\rangle\}_{j\in\mathbb{J}} : x \in \mathcal{H}\}$. Now for any $x \in \mathcal{H}$, we have

$$\begin{aligned}\left\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\right\|^2 &= \|\langle Sx, x\rangle\|^2 \leqslant \|Sx\|^2 \cdot \|x\|^2\\
&\leqslant \|Sx\|^2 \cdot \frac{1}{C}\left\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\right\|.\end{aligned}$$

Therefore $C\left\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\right\| \leqslant \|Sx\|^2 = \|UU^*x\| = \|U\{\langle x, x_j\rangle\}\|^2$, as desired.

"⇐". To show that $\{x_j\}_{j\in\mathbb{J}}$ is a frame, by (i), it suffices to show that $Rang(U) = \mathcal{H}$. Since $span\{x_j : j \in \mathbb{J}\} \subseteq Rang(U)$, it only needs to prove that $Rang(U)$ is closed. Suppose that $\{u_n\} \subseteq Rang(U)$ and $u_n \to u$ as $n \to \infty$. Then we can find $\{v_n\} \subseteq (Ker\, U)^\perp$ such that $Uv_n = u_n$. It follows from inequality (2) that $\{v_n\}$ is a Cauchy sequence. Suppose that $v_n \to v$ as $n \to \infty$. Therefore $u_n = Uv_n \to Uv = u$ as $n \to \infty$. This completes the proof. □

3. Perturbation of frames and Riesz bases

Our first result of this paper is to show that the Casazza–Christensen's perturbation theorem of Hilbert space frames still holds for Hilbert C^*-module frames. Although the proof is based on modification of the proof in [3], we include the proof for the sake of completeness. We need the following lemma due to Casazza and Christensen [3]. It is a generalization of the classical result that an operator U on a Banach space is invertible if $\|I-U\|<1$.

Lemma 3.1 [3]. *Let X be a Banach space, and $U: X \to X$ a linear operator. Assume that there exist constants $\lambda_1, \lambda_2 \in (0,1)$ such that*

$$\|Ux-x\| \leqslant \lambda_1\|x\| + \lambda_2\|Ux\| \quad \forall x \in X.$$

Then U is bounded and invertible with

$$\|U\| \leqslant \frac{1+\lambda_1}{1-\lambda_2} \quad \text{and} \quad \|U^{-1}\| \leqslant \frac{1+\lambda_2}{1-\lambda_1}.$$

Theorem 3.2. *Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$, and $\{x_j\}_{j\in\mathbb{J}}$ be a frame for $\mathcal{H}$ with frame bounds C and D. Suppose that $\{y_j\}_{j\in\mathbb{J}}$ is a sequence of $\mathcal{H}$ and that there exist $\lambda_1, \lambda_2, \mu \geqslant 0$ such that $\max\left\{\lambda_1 + \frac{\mu}{\sqrt{C}}, \lambda_2\right\} < 1$. Then $\{y_j\}_{j\in\mathbb{J}}$ is also a frame for $\mathcal{H}$ with frame bounds*

$$\left(\frac{(1-\lambda_1)\sqrt{C}-\mu}{1+\lambda_2}\right)^2 \quad \text{and} \quad \left(\frac{(1+\lambda_1)\sqrt{D}+\mu}{1-\lambda_2}\right)^2,$$

if one of the following conditions is fulfilled for any finite sequence $\{c_j\}_{j=1}^n \subseteq \mathcal{A}$ and all $x \in \mathcal{H}$:

$$\left\|\sum_{j\in\mathbb{J}}\langle x, x_j-y_j\rangle\langle x_j-y_j, x\rangle\right\|^{\frac{1}{2}} \leqslant \lambda_1\left\|\sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle\right\|^{\frac{1}{2}} + \lambda_2\left\|\sum_{j\in\mathbb{J}}\langle x, y_j\rangle\langle y_j, x\rangle\right\|^{\frac{1}{2}} + \mu\|x\|; \tag{4}$$

or

$$\left\|\sum_{j=1}^n c_j(x_j-y_j)\right\| \leqslant \lambda_1\left\|\sum_{j=1}^n c_jx_j\right\| + \lambda_2\left\|\sum_{j=1}^n c_jy_j\right\| + \mu\left\|\sum_{j=1}^n c_jc_j^*\right\|^{\frac{1}{2}}. \tag{5}$$

Proof. Let T_X and S_X denote the analysis operator and frame operator of $\{x_j\}$, respectively.

Assume first that condition (4) holds for all $x \in \mathcal{H}$. We define an operator $T_Y: \mathcal{H} \to l^2(\mathcal{A})$ by

$$T_Yx = \sum_{j\in\mathbb{J}}\langle x, y_j\rangle e_j.$$

Then condition (4) turns to be

$$\|T_Xx - T_Yx\| \leqslant \lambda_1\|T_Xx\| + \lambda_2\|T_Yx\| + \mu\|x\|.$$

On one hand we have

$$(1-\lambda_2)\|T_Yx\| \leqslant (1+\lambda_1)\|T_Xx\| + \mu\|x\|,$$

which implies that

$$\|T_Yx\| \leqslant \frac{1}{1-\lambda_2}[(1+\lambda_1)\|T_Xx\| + \mu\|x\|] \leqslant \frac{(1+\lambda_1)\sqrt{D}+\mu}{1-\lambda_2}\|x\|.$$

Therefore $\{y_j\}_{j\in\mathbb{J}}$ is a Bessel sequence with the Bessel bound $\left(\frac{(1+\lambda_1)\sqrt{D}+\mu}{1-\lambda_2}\right)^2$. On the other hand, we also have

$$(1-\lambda_1)\|T_X x\| - \mu\|x\| \leqslant (1+\lambda_2)\|T_Y x\|.$$

Therefore

$$\|T_Y x\| \geqslant \frac{1}{1+\lambda_2}[(1-\lambda_1)\|T_X x\| - \mu\|x\|] \geqslant \frac{(1-\lambda_1)\sqrt{C}-\mu}{1+\lambda_2}\|x\|,$$

which implies that $\{y_j\}_{j\in\mathbb{J}}$ is a frame.

Suppose now that condition (5) holds. Then for each $\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$ we have that

$$\left\|\sum_{j=1}^{n} c_j y_j\right\| \leqslant \frac{1}{1-\lambda_2}\left[(1+\lambda_1)\left\|\sum_{j=1}^{n} c_j x_j\right\| + \mu\left\|\sum_{j=1}^{n} c_j c_j^*\right\|^{\frac{1}{2}}\right],$$

which yields that

$$\left\|\sum_{j=1}^{n} c_j y_j\right\| \leqslant \frac{1}{1-\lambda_2}\left[(1+\lambda_1)\left\|\sum_{j=1}^{\infty} c_j x_j\right\| + \mu\left\|\sum_{j=1}^{\infty} c_j c_j^*\right\|^{\frac{1}{2}}\right].$$

Furthermore, we obtain

$$\left\|\sum_{j=1}^{\infty} c_j y_j\right\| \leqslant \frac{1}{1-\lambda_2}\left[(1+\lambda_1)\left\|\sum_{j=1}^{\infty} c_j x_j\right\| + \mu\left\|\sum_{j=1}^{\infty} c_j c_j^*\right\|^{\frac{1}{2}}\right].$$

Therefore we can define a bounded operator $U : \mathcal{H} \to l^2(\mathcal{A})$ by

$$U\{c_j\} = \sum_{j\in\mathbb{J}} c_j y_j,$$

which satisfying

$$\|U\{c_j\}\| \leqslant \frac{1}{1-\lambda_2}\left[(1+\lambda_1)\|T_X^*\{c_j\}\| + \mu\|\{c_j\}\|\right] \leqslant \frac{(1+\lambda_1)\sqrt{D}+\mu}{1-\lambda_2}\|\{c_j\}\|.$$

By Proposition 2.8, $\{y_j\}_{j\in\mathbb{J}}$ is a Bessel sequence with Bessel bound $\left(\frac{(1+\lambda_1)\sqrt{D}+\mu}{1-\lambda_2}\right)^2$.

Note that for each $\{c_j\}_{j\in\mathbb{J}} \in l^2(\mathcal{A})$ we also have

$$\left\|\sum_{j\in\mathbb{J}} c_j(x_j - y_j)\right\| \leqslant \lambda_1\left\|\sum_{j\in\mathbb{J}} c_j x_j\right\| + \lambda_2\left\|\sum_{j\in\mathbb{J}} c_j y_j\right\| + \mu\left\|\sum_{j\in\mathbb{J}} c_j c_j^*\right\|^{\frac{1}{2}}.$$

Then for each $x \in \mathcal{H}$, letting $\{c_j\} = T_X S_X^{-1}x$, we get

$$\begin{aligned}\|x - UT_X S_X^{-1}x\| &\leqslant \lambda_1\|x\| + \lambda_2\|UT_X S_X^{-1}x\| + \mu\|T_X S_X^{-1}x\|\\ &\leqslant \lambda_1\|x\| + \frac{\mu}{\sqrt{C}}\|x\| + \lambda_2\|UT_X S_X^{-1}x\|.\end{aligned}$$

By Lemma 3.1, $UT_X S_X^{-1}$ is invertible and we also have

$$\|UT_X S_X^{-1}\| \leqslant \frac{1+\lambda_1+\frac{\mu}{\sqrt{C}}}{1-\lambda_2} \quad\text{and}\quad \|(UT_X S_X^{-1})^{-1}\| \leqslant \frac{1+\lambda_2}{1-\left(\lambda_1+\frac{\mu}{\sqrt{C}}\right)}.$$

Now for arbitrary $x \in \mathcal{H}$, we get

$$x = UT_X S_X^{-1}\left(UT_X S_X^{-1}\right)^{-1}x = \sum_{j\in\mathbb{J}}\left\langle\left(UT_X S_X^{-1}\right)^{-1}x, S_X^{-1}x_j\right\rangle y_j.$$

Therefore

$$\begin{aligned}
\|x\|^4 &= \|\langle x,x\rangle\|^2 \\
&= \left\|\sum_{j\in\mathbb{J}}\langle (UT_XS_X^{-1})^{-1}x, S_X^{-1}x_j\rangle\langle y_j,x\rangle\right\|^2 \\
&\leqslant \left\|\sum_{j\in\mathbb{J}}\left\langle (UT_XS_X^{-1})^{-1}x, S_X^{-1}x_j\right\rangle\left\langle S_X^{-1}x_j, (UT_XS_X^{-1})^{-1}x\right\rangle\right\| \cdot \left\|\sum_{j\in\mathbb{J}}\langle x,y_j\rangle\langle y_j,x\rangle\right\| \\
&\leqslant \frac{1}{C}\|\langle (UT_XS_X^{-1})^{-1}x, (UT_XS_X^{-1})^{-1}x\rangle\| \cdot \left\|\sum_{j\in\mathbb{J}}\langle x,y_j\rangle\langle y_j,x\rangle\right\| \\
&\leqslant \frac{1}{C}\left(\frac{1+\lambda_2}{1-\left(\lambda_1+\frac{\mu}{\sqrt{C}}\right)}\right)^2\|x\|^2 \cdot \left\|\sum_{j\in\mathbb{J}}\langle x,y_j\rangle\langle y_j,x\rangle\right\|,
\end{aligned}$$

where in the second inequality we apply the fact that $\left\{S_X^{-1}x_j\right\}_{j\in\mathbb{J}}$ is a frame with frame bounds $\frac{1}{D}$ and $\frac{1}{C}$. Hence we have obtained the claimed lower frame bound condition:

$$\left(\frac{(1-\lambda_1)\sqrt{C}-\mu}{1+\lambda_2}\right)^2\|x\|^2 \leqslant \left\|\sum_{j\in\mathbb{J}}\langle x,y_j\rangle\langle y_j,x\rangle\right\|. \qquad \square$$

With regard to the extension to Riesz bases part of Theorem 1.1, we first point out that if $\mu = 0$ in the condition (5) of Theorem 3.2, then $\{y_j\}_{j\in\mathbb{J}}$ is a Riesz basis provided that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis.

Theorem 3.3. *Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$ and $\{x_j\}_{j\in\mathbb{J}}$ be a Riesz basis for $\mathcal{H}$. Suppose that $\{y_j\}_{j\in\mathbb{J}}$ is a sequence of $\mathcal{H}$ and there exist $\lambda_1,\lambda_2 \in [0,1)$. If*

$$\left\|\sum_{j\in\mathbb{J}} c_j(x_j-y_j)\right\| \leqslant \lambda_1\left\|\sum_{j\in\mathbb{J}} c_jx_j\right\| + \lambda_2\left\|\sum_{j\in\mathbb{J}} c_jy_j\right\| \tag{6}$$

holds for all finite sequence $\{c_j\}_{j=1}^n \subseteq \mathcal{A}$, then $\{y_j\}_{j\in\mathbb{J}}$ is also a Riesz basis.

Proof. We first claim that $y_j \neq 0$ for each j. Assume to the contrary that there exists j_0 such that $y_{j_0} = 0$. Choose $\{c_j\} = e_{j_0}$, then we have

$$\|x_{j_0}\| \leqslant \lambda_1\|x_{j_0}\|,$$

which implies that $x_{j_0} = 0$, a contradiction. By Theorem 3.2, we see that $\{y_j\}_{j\in\mathbb{J}}$ is also a frame of $\mathcal{H}$. Let us denote the analysis operators of $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in\mathbb{J}}$ by T_X and T_Y, respectively. In order to show that $\{y_j\}_{j\in\mathbb{J}}$ is a Riesz basis, it suffices to show that $Rang(T_X) = Rang(T_Y)$.

If $\{c_j\} \in Ker\, T_X^*$, then we have

$$\|T_Y^*\{c_j\}\| \leqslant \lambda_2\|T_Y^*\{c_j\}\|,$$

which leads to $\{c_j\} \in Ker\, T_Y^*$. In the same manner we can show that $Ker\, T_Y^* \subseteq Ker\, T_X^*$, and so $Ker\, T_X^* = Ker\, T_Y^*$. It follows from Proposition 2.8 that both $Rang\,(T_X^*)$ and $Rang\,(T_Y^*)$ are closed, and hence both $Rang(T_X)$ and $Rang(T_Y)$ are closed. Now applying Theorem 15.3.8 in [15] we see that $Rang(T_X) = Rang(T_Y)$, as claimed. Thus, by Theorem 2.5, we can infer that $\{y_j\}$ is also a Riesz basis of $\mathcal{H}$. $\square$

The following is an example showing the analogue of the second part of Theorem 1.1 in Hilbert C^*-modules is no longer true in general for Hilbert C^*-module Riesz bases.

Example 3.4. Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathbb{N}}$ and $v = \{v_j\}_{j\in\mathbb{N}}$ in l^∞, we define

$$uv = \{u_j v_j\}_{j\in\mathbb{N}}, \quad u^* = \{\bar{u}_j\}_{j\in\mathbb{N}} \quad \text{and} \quad \|u\| = \max_{j\in\mathbb{N}} |u_j|.$$

Then $\mathcal{A} = \{l^\infty, \|\cdot\|\}$ is a C^*-algebra.

Let $\mathcal{H} = c_0$ be the set of all sequences converging to zero. For any $u, v \in \mathcal{H}$ we define

$$\langle u, v\rangle = uv^* = \{u_j \bar{v}_j\}_{j\in\mathbb{N}}.$$

Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module.

For each j, let $x_j = e_j$. Obviously, $\{x_j\}_{j\in\mathbb{N}}$ is a Parseval Riesz basis of $\mathcal{H}$.

Now let

$$y_j = \begin{cases} e_1 + e_2 & \text{if } j = 1, 2; \\ e_j & \text{if } j \neq 1, 2, \end{cases}$$

and $\lambda_1 = \frac{1}{8}, \lambda_2 = \frac{15}{16}$ and $\mu = \frac{3}{4}$.

Then one can check that condition (5) in Theorem 3.2 is satisfied. But $\{y_j\}_{j\in\mathbb{J}}$ is not a Riesz basis.

We obtain the following necessary and sufficient condition under which every perturbation $\{y_j\}_{j\in\mathbb{J}}$ of a Riesz basis $\{x_j\}_{j\in J}$ is also a Riesz basis in Hilbert C^*-modules.

Theorem 3.5. *Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis of $\mathcal{H}$ with frame bounds C and D, where $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$. Assume that there exist $\lambda_1, \lambda_2 \geqslant 0$ and $\mu > 0$ such that*

$$\max\left\{\lambda_1 + \frac{\mu}{\sqrt{C}}, \lambda_2\right\} < 1.$$

Then the following are equivalent:

(i) *Every sequence $\{y_j\}_{j\in\mathbb{J}}$ in $\mathcal{H}$ satisfying the following perturbation condition is again a Riesz basis*:

$$\left\|\sum_{j=1}^n c_j(x_j - y_j)\right\| \leqslant \lambda_1 \left\|\sum_{j=1}^n c_j x_j\right\| + \lambda_2 \left\|\sum_{j=1}^n c_j y_j\right\| + \mu \left\|\sum_{j=1}^n c_j c_j^*\right\|^{\frac{1}{2}} \tag{7}$$

for any $c_1, c_2, \ldots, c_n \in \mathcal{A}$.

(ii) *$Ker\, T_X^* = l^2(\mathcal{B})$, where T_X is the analysis operator of $\{x_j\}_{j\in\mathbb{J}}$ and $\mathcal{B} = \{a \in \mathcal{A} : a\mathcal{H} = \{0\}\}$.*

In case that the above equivalent conditions are satisfied, we also have $Ker\, T_Y^* = Ker\, T_X^*$ and $Rang(T_Y) = Rang(T_X)$, where T_Y is the analysis operator of $\{y_j\}_{j\in\mathbb{J}}$.

Proof. From Theorem 3.2 and its proof we can infer that $\{y_j\}_{j\in\mathbb{J}}$ is a frame and satisfies the condition

$$\left\|\sum_{j\in\mathbb{J}} c_j(x_j - y_j)\right\| \leqslant \lambda_1 \left\|\sum_{j\in\mathbb{J}} c_j x_j\right\| + \lambda_2 \left\|\sum_{j\in\mathbb{J}} c_j y_j\right\| + \mu \left\|\sum_{j\in\mathbb{J}} c_j c_j^*\right\|^{\frac{1}{2}}$$

for all $\{c_j\} \in l^2(\mathcal{A})$.

"(i) $\Rightarrow$ (ii)". Suppose first that any sequence $\{y_j\}_{j\in\mathbb{J}}$ satisfying condition (7) is a Riesz basis. We now show that $Ker\, T_X^* = l^2(\mathcal{B})$. Obviously, $l^2(\mathcal{B}) \subseteq Ker\, T_X^*$. Now pick an arbitrary $\{a_j\}_{j\in\mathbb{J}} \in Ker\, T_X^*$. We need to prove that $a_j\mathcal{H} = \{0\}$ for each j. Assume to the contrary that there exists $j_0 \in \mathbb{J}$ such that $a_{j_0}\mathcal{H} \neq \{0\}$. We have two cases:

Case 1. There exists $j_1 \in \mathbb{J}$ such that $a_{j_0}x_{j_1} \neq 0$.

Choose $M > 0$ such that $\frac{\|x_{j_1}\|}{M} \leqslant \mu$. Consider sequence $\{z_j\}_{j\in\mathbb{J}}$ given by

$$z_j = \begin{cases} x_{j_0} - \frac{1}{M}x_{j_1}, & \text{if } j = j_0; \\ x_j, & \text{otherwise.} \end{cases}$$

One can check that $\{z_j\}_{j\in\mathbb{J}}$ satisfies condition (7). Now let $\{c_j\}$ be a sequence such that

$$c_j = \begin{cases} Ma_{j_0}, & \text{if } j = j_0; \\ a_{j_0}, & \text{if } j = j_1; \\ a_j, & \text{otherwise.} \end{cases}$$

Observe that

$$\sum_{j\in\mathbb{J}} c_j z_j = \sum_{j\in\mathbb{J}} a_j x_j = 0.$$

But

$$c_{j_0} z_{j_0} = -a_{j_0} x_{j_1} \neq 0.$$

Thus $\{z_j\}_{j\in\mathbb{J}}$ is not a Riesz basis, a contradiction.

Case 2. $a_{j_0}x_j = 0$ for all $j \in \mathbb{J}$.

We pick $z \in \mathcal{H}$ such that $a_{j_0}z \neq 0$, and $N > 0$ such that $\frac{\sqrt{2}}{N}\|z\| \leqslant \mu$. Consider a sequence $\{z_j\}_{j\in\mathbb{J}}$ defined by

$$z_j = \begin{cases} x_1 + \frac{1}{N}z, & \text{if } j = 1; \\ x_2 - \frac{1}{N}z, & \text{if } j = 2; \\ x_j, & \text{otherwise.} \end{cases}$$

Note that $\{z_j\}_{j\in\mathbb{J}}$ also satisfies condition (7). By letting $c_j = a_{j_0}$ for all j, we have

$$\sum_{j\in\mathbb{J}} c_j z_j = \sum_{j\in\mathbb{J}} a_{j_0} x_j = 0.$$

But

$$c_1 z_1 = -c_2 z_2 = \frac{a_{j_0}}{N} z \neq 0,$$

which contradicts the fact that $\{z_j\}_{j\in\mathbb{J}}$ is a Riesz basis.

"(ii) $\Rightarrow$ (i)". Suppose now that $Ker\ T_X^* = l^2(\mathcal{B})$ and $\{y_j\}_{j\in\mathbb{J}}$ is an arbitrary sequence satisfying condition (7). By Proposition 2.6, we consider any sequence $\{a_j\} \in l^2(\mathcal{A})$ such that $\sum_{j\in\mathbb{J}} a_j y_j = 0$. We claim that $\{a_j\} \in l^2(\mathcal{B})$. Assume to the contrary that $\{a_j\} \notin l^2(\mathcal{B})$. By Theorem 15.3.8 in [15] we have

$$l^2(\mathcal{A}) = Ker\ T_X^* \oplus \left(Ker\ T_X^*\right)^\perp = l^2(\mathcal{B}) \oplus (l^2(\mathcal{B}))^\perp.$$

Thus $\{a_j\}$ has a unique decomposition

$$\{a_j\} = \left\{a_j^{(1)}\right\} \oplus \left\{a_j^{(2)}\right\},$$

where $\left\{a_j^{(1)}\right\} \in l^2(\mathcal{B})$ and $\left\{a_j^{(2)}\right\}$ is a nonzero sequence in $(l^2(\mathcal{B}))^\perp$. So we have

$$\left\|\sum_{j\in\mathbb{J}} a_j y_j\right\| = \left\|\sum_{j\in\mathbb{J}} (a_j^{(1)} + a_j^{(2)}) y_j\right\| = \left\|\sum_{j\in\mathbb{J}} a_j^{(2)} y_j\right\|$$
$$= \left\|\sum_{j\in\mathbb{J}} a_j^{(2)} x_j - \sum_{j\in\mathbb{J}} a_j^{(2)} (x_j - y_j)\right\|$$

$$
\begin{aligned}
&\geqslant \left\|\sum_{j\in\mathbb{J}} a_j^{(2)} x_j\right\| - \left\|\sum_{j\in\mathbb{J}} a_j^{(2)} (x_j - y_j)\right\| \\
&\geqslant \left\|\sum_{j\in\mathbb{J}} a_j^{(2)} x_j\right\| - \lambda_1 \left\|\sum_{j\in\mathbb{J}} a_j^{(2)} x_j\right\| - \lambda_2 \left\|\sum_{j\in\mathbb{J}} a_j^{(2)} y_j\right\| - \mu \left\|\left\{a_j^{(2)}\right\}\right\| \\
&= (1-\lambda_1) \left\|\sum_{j\in\mathbb{J}} a_j^{(2)} x_j\right\| - \lambda_2 \left\|\sum_{j\in\mathbb{J}} a_j^{(2)} y_j\right\| - \mu \left\|\left\{a_j^{(2)}\right\}\right\| \\
&\geqslant \left[(1-\lambda_1)\sqrt{C}\right] \left\|\left\{a_j^{(2)}\right\}\right\| - \lambda_2 \left\|\sum_{j\in\mathbb{J}} a_j^{(2)} y_j\right\| - \mu \left\|\left\{a_j^{(2)}\right\}\right\| \\
&= \left[(1-\lambda_1)\sqrt{C} - \mu\right] \left\|\left\{a_j^{(2)}\right\}\right\| - \lambda_2 \left\|\sum_{j\in\mathbb{J}} a_j^{(2)} y_j\right\|,
\end{aligned}
$$

where in the last inequality we apply Proposition 2.8 (ii).

Hence

$$
0 = \left\|\sum_{j\in\mathbb{J}} a_j^{(2)} y_j\right\| \geqslant \frac{(1-\lambda_1)\sqrt{C} - \mu}{1+\lambda_2} \left\| \left\{a_j^{(2)}\right\} \right\|,
$$

and therefore $a_j^{(2)} = 0$ for each j, a contradiction. Thus we can infer that $Ker\, T_Y^* = l^2(\mathcal{B})$.

To show that $\{y_j\}_{j\in\mathbb{J}}$ is a Riesz basis, it remains to show that $y_j \neq 0$ for each j. Assume to the contrary that $y_{j_0} = 0$ for some $j_o \in \mathbb{J}$. For any $a \in \mathcal{A}$, let

$$
c_j = \begin{cases} a, & \text{if } j = j_0; \\ 0, & \text{otherwise.} \end{cases}
$$

Then $\sum_{j\in\mathbb{J}} c_j y_j = 0$, i.e. $\{c_j\}_{j\in\mathbb{J}} \in Ker\, T_Y^*$. Since $Ker\, T_X^* = Ker\, T_Y^*$, we see that $ax_{j_0} = 0$ for any $a \in \mathcal{A}$. Therefore $x_{j_0} = 0$ which leads to a contradiction with the assumption that $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis. This completes the proof. $\square$

Remark 3.6. Case 2 in the above proof states that there may exist an element $a \in \mathcal{A}$ such that $ax_j = 0$ for all j but $a\mathcal{H} \neq \{0\}$, where $\{x_j\}_{j\in\mathbb{J}}$ is a Riesz basis of a Hilbert $\mathcal{A}$-module $\mathcal{H}$. Though this never occurs in Hilbert spaces, it may happen in Hilbert C^*-modules. For example, we consider the C^*-algebra $\mathcal{A} = M_{2\times 2}(\mathbb{C})$ of all 2×2 complex matrices. Let $\mathcal{H} = \mathcal{A}$ and for any $x, y \in \mathcal{H}$ define

$$
\langle x, y\rangle = xy^*.
$$

Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module. Choose

$$
x_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad x_2 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.
$$

One can check that $\{x_1, x_2\}$ is a Riesz basis of $\mathcal{H}$. Pick

$$
a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
$$

Then we have

$$
ax_1 = ax_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
$$

But, it is obvious that

$$
a\mathcal{H} \neq \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.
$$

Remark 3.7. Finally we remark that all the above results remain valid if we replace the conditions (4)–(7) by the corresponding conditions stated in the forms without norms. For example, we can drop the norms in (5) by writing as the following:

$$\left(\left\langle\sum_{j=1}^{n} c_j(x_j-y_j),\sum_{j=1}^{n} c_j(x_j-y_j)\right\rangle\right)^{\frac{1}{2}}$$
$$\leqslant\lambda_1\left(\left\langle\sum_{j=1}^{n} c_jx_j,\sum_{j=1}^{n} c_jx_j\right\rangle\right)^{\frac{1}{2}}+\lambda_2\left(\left\langle\sum_{j=1}^{n} c_jy_j,\sum_{j=1}^{n} c_jy_j\right\rangle\right)^{\frac{1}{2}}+\mu\left(\sum_{j=1}^{n} c_jc_j^*\right)^{\frac{1}{2}}.$$

Acknowledgments

The authors are thankful to the referees and the editor for their valuable comments and suggestions that improved the presentation of this paper.

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