

## Bi-complement Reducible Graphs

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We introduce a new family of bipartite graphs which is the bipartite analogue of the class of *complement reducible graphs* or *cographs*. A *bi-complement reducible graph* or *bi-cograph* is a bipartite graph  $G = (W \cup B, E)$  that can be reduced to single vertices by recursively bi-complementing the edge set of all connected bipartite subgraphs. The *bi-complemented graph*  $\overline{G}^{\text{bip}}$  of  $G$  is the graph having the same vertex set  $W \cup B$  as  $G$ , while its edge set is equal to  $W \times B - E$ . The aim of this paper is to show that there exists an equivalent definition of bi-cographs by three forbidden configurations. We also propose a tree representation for this class of graphs. © 1997 Academic Press

### 1. INTRODUCTION

One of the best known exponents of graphs, discovered and investigated independently by various researchers, is the *complement reducible graphs*, known also as *cographs*. Among the different characterizations established for these graphs, we concentrate our attention here on two of them.

First, a cograph can be defined recursively as follows:

- (i) A graph on a single vertex is a cograph.
- (ii) If  $G_1, G_2, \dots, G_k$  are cographs, then so is their union  $G_1 \cup G_2 \cup \dots \cup G_k$ .
- (iii) If  $G$  is a cograph, then so is its complement  $\overline{G}$ .

Clearly, this definition implies that any cograph can be obtained from single node graphs by performing a finite number of graph operations involving union and complementation. It is also clear that this class of graphs is self-complemented.

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Cographs themselves were first introduced by Lerchs in [12, 13], where he studied their structural and algorithmic properties and proved that these graphs admit a unique up to isomorphism tree representation. This tree representation for a cograph  $G$  is obtained by associating with  $G$  a rooted tree  $T(G)$ , called a *cotree*, whose leaves are precisely the vertices of  $G$  while the internal nodes are labeled by the symbol  $\overline{\cup}$  representing complemented union. In [12] it is proved that from the definition of a cograph we can easily deduce that the complement of a connected cograph is disconnected. Thus, a cograph can be reduced to single vertices by recursively complementing connected subgraphs, and for this reason cographs are also called complement reducible graphs. A top-down traversal of  $T(G)$  clearly describes this decomposition of a cograph  $G$ .

The internal nodes of a cotree  $T(G)$  can also be labeled by 0 and 1 in such way that two vertices are adjacent in  $G$ , if and only if their least common ancestor in  $T(G)$  is labeled by 1. In this form, the cotree was employed as basic data structure into the linear recognition algorithm for cographs, obtained in [6].

Hence, this class of graphs provides an excellent paradigm of graphs possessing a unique tree representation, and, for such graphs, many results confirmed that a great number of intractable problems have efficient algorithmic solutions (see, for example, [5, 6]). Cographs are also interesting in connection with so-called empirical logic (see, for example, [8]).

A second definition for cographs was obtained by Lerchs in [12], where he established that cographs are precisely the graphs which contain no induced subgraph isomorphic to a chordless path of four vertices or  $P_4$ . It comes as no surprise that cographs appeared also in many areas requiring graphs having local density metrics, as in LAN technologies, group based cooperation, scheduling, cluster analysis, and resource allocation (see [11] for a bibliographic summary).

All these wide theoretical and practical applications of cographs motivated us to search for their bipartite analogue, and our research led us to define a new class of bipartite graphs, the *bi-complement reducible* graphs, or, briefly, *bi-cographs*. Our study is in line with bipartite graph theory, which is interested in mirroring the basic phenomena of graph theory within bipartite concepts. A considerable bibliography in this field is available. For example, Harary, Kabell, and McMorris [10] identify bipartite concepts for interval and chordal graphs, Bagga [1] and Beineke [2] compare ordinary with bipartite tournaments, McKee [14] attempts a graphic-to-bigraphic translation, Frost *et al.* [9] propose several possibilities for a bipartite analogue of the concept of split graphs, and A. Branstädt [4] examines connections between chordal, strongly chordal, and split graphs with bipartite graphs.

## 2. TERMINOLOGY

For terms not defined in this paper the reader is referred to [3]. All graphs considered in this report are finite, without loops or multiple edges. The set of vertices  $V$  of a graph  $G = (V, E)$  will also be denoted by  $V(G)$  and the set  $E$  of its edges by  $E(G)$ , while  $n$  will be the number of vertices and  $m$  the number of edges of  $G$ . Let  $X$  be a set of vertices of  $G$ , then the graph induced by  $V - X$  will be denoted by  $G \setminus X$ . The neighbourhood of a vertex  $v$  is  $N(v) = \{w \mid vw \in E\}$ , while  $N(X)$  ( $X \subseteq V$ ) is the set of vertices outside  $X$  which are adjacent to at least one vertex of  $X$ . A graph  $G$  will be bipartite if there is a bipartition of  $V$  into  $W$  (a set of white vertices) and  $B$  (a set of black vertices) such that  $E \subseteq W \times B$ . We shall say that  $v \in B$  is a black vertex or  $v \in W$  is a white vertex. A vertex  $v$  is  $W$ -universal (resp.  $B$ -universal) if and only if  $N(v) = W$  (resp.  $N(v) = B$ ).

A vertex  $x$  will be an articulation point for  $G$  if the number of connected components of  $G - \{x\}$  is greater than that of  $G$ . The union of two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  will be the graph  $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ .

A chordless path of  $k$  vertices is denoted by  $P_k$  and a chordless cycle of  $k$  vertices is denoted by  $C_k$ .

We shall call a graph  $G$   $\mathcal{Z}$ -free, where  $\mathcal{Z}$  is a set of graphs, if no induced subgraph of  $G$  is isomorphic to a graph of  $\mathcal{Z}$ . A set of graphs  $\mathcal{F}$  will be  $\mathcal{Z}$ -free if every graph of  $\mathcal{F}$  is  $\mathcal{Z}$ -free.

The bi-complemented graph  $\overline{G}^{\text{bip}}$  of a bipartite graph  $G = (W \cup B, E)$  is the graph having the same vertex set  $W \cup B$  as  $G$ , while its edge set is equal to  $W \times B - E$ .

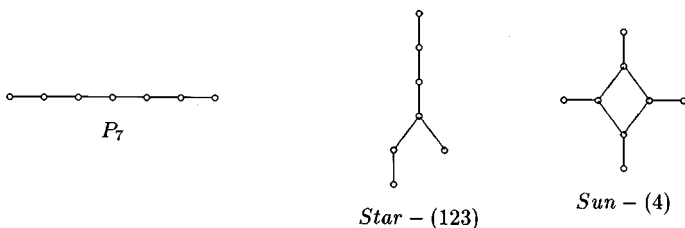
*The bi-cographs.* A bi-cograph or bi-complement reducible graph is a bipartite graph defined recursively as follows:

- (i) A graph on a single black or white vertex is a bi-cograph.
- (ii) If  $G_1, G_2, \dots, G_k$  are bi-cographs, then so is their union  $G_1 \cup G_2 \cup \dots \cup G_k$ .
- (iii) If  $G$  is a bi-cograph, then so is its bi-complement  $\overline{G}^{\text{bip}}$ .

As for cographs, this definition clearly implies that any bi-cograph can be obtained from single black or white node graphs, by performing a finite number of graph operations involving union and bi-complementation. It is also clear that the class of bi-cographs is self-complemented.

## 3. FORBIDDEN CONFIGURATIONS FOR BI-COGRAPHS

We shall show in this section that the class of bi-cographs can be characterized, up to isomorphism, by the following three forbidden configurations:



*Notation.* We shall denote, henceforth, by  $\mathcal{Z}$  the set of the above graphs. For convenience, in the following a  $Star-(123)$  will be denoted by a sequence of its vertices, having as first vertex the vertex of degree 3, which is an endpoint of a  $P_2$ , of a  $P_3$ , and of a  $P_4$ . This vertex will be followed by the remaining vertex of the  $P_2$ , then by the remaining vertices of the  $P_3$ , and finally by the remaining vertices of the  $P_4$ .

*Observation.* Any graph of  $\mathcal{Z}$  is self-bi-complemented.

**THEOREM 3.1.** *A bipartite graph  $G$  is a bi-cograph iff  $G$  is  $\mathcal{Z}$ -free.*

*Proof.* We first show, by contradiction, the only if part of the theorem. Assume that  $G$  is a bi-cograph containing a subgraph  $Z$  isomorphic to a graph of  $\mathcal{Z}$ . Then, when decomposing recursively  $G$  by bi-complementing all connected subgraphs, the self-bi-complemented graph  $Z$  will be entirely contained to a connected subgraph, at each stage of this recursive process. Consequently,  $G$  could never be reduced to single vertices, a contradiction.

Conversely, suppose, by contradiction, that a  $\mathcal{Z}$ -free bipartite graph  $G$  is not a bi-cograph. Then  $G$  cannot be reduced to single vertices by bi-complementing recursively all connected subgraphs and thus when applying the above recursive process we shall find a connected graph  $G_i$  such that  $G_i$  and  $\overline{G}^{\text{bip}}$  are connected. Let  $H$  be a connected subgraph of  $G_i$  having a minimum number of vertices and such that  $\overline{H}^{\text{bip}}$  is connected.

To complete the proof we require the following claims and facts together with their proofs.

CLAIM 1.  $H$  or  $\bar{H}^{\text{bip}}$  has an articulation point.

*Proof.* Indeed, suppose that  $\forall x \in V(H)$ ,  $H \setminus \{x\}$  is connected, then if  $\bar{H}^{\text{bip}} \setminus \{x\}$  is also connected,  $H$  would not be a minimum subgraph of  $G_i$  verifying that  $H$  and  $\bar{H}^{\text{bip}}$  are connected, a contradiction. ■

Without loss of generality, we assume henceforth that  $H$  has an articulation point that will be a white vertex, denoted by  $w$ .

CLAIM 2. *There exists a black vertex  $b$  into a connected component  $C$  of  $H \setminus \{w\}$ , such that  $wb \notin E(H)$ .*

*Proof.* Otherwise,  $w$  would be a  $B$ -universal vertex for  $H$  and thus an isolated vertex of  $\bar{H}^{\text{bip}}$ , a contradiction. ■

*Notation.* In the following, whenever we use  $C$  and  $b$ , they will have the meanings given in Claim 2.

CLAIM 3. *Every chordless chain in  $H$  from  $w$  to a black vertex non-adjacent to  $w$  is of length 3.*

*Proof.* Assume on the contrary that there is a chordless chain  $l$  between  $w$  and a black vertex  $b_i$ , whose length is greater than 3; then since the bipartite graph  $H$  is  $P_7$ -free, the length of  $l$  must be equal to 5. Let  $C_i$  be the connected component of  $H \setminus \{w\}$  whose vertex set contains  $b_i$ . Since  $w$  is an articulation point, we deduce on the one hand that every vertex of  $l$  distinct from  $w$  belongs to  $V(C_i)$ , and on the other hand, that in every connected component of  $H \setminus \{w\}$ , there is a neighbour of  $w$ . Thus the vertices of  $l$  together with a neighbour of  $w$  in a connected component  $C_j \neq C_i$  induce a  $P_7$  in  $H$ , a contradiction. ■

CLAIM 4. *The vertex  $w$  is adjacent to every black vertex of any connected component of  $H \setminus \{w\}$ , distinct from  $C$ .*

*Proof.* Assume the contrary, namely that there exists a connected component  $C' \neq C$  of  $H \setminus \{w\}$  having a black vertex  $b'$  that is non-adjacent to  $w$ . Let  $l$  be a chordless chain from  $w$  to  $b$  in the connected graph induced by  $V(C) \cup \{w\}$  and  $l'$  be a chordless chain from  $w$  to  $b'$  in the connected graph induced by  $V(C') \cup \{w\}$ . Then, since both  $l$  and  $l'$  are of length 3 (Claim 3), there exists an induced  $P_7$  in  $H$ , a contradiction. ■

CLAIM 5. *Every connected component of  $H \setminus \{w\}$  distinct from  $C$  is a complete bipartite graph.*

*Proof.* Assume, by contradiction, that there is a connected component  $C' \neq C$  of  $H \setminus \{w\}$  containing a black vertex  $b'$  and a white vertex  $w'$  such that  $b'w' \notin E(G)$ . Let  $b''$  be a black vertex of  $C'$  adjacent to  $w'$ . By Claim 3 there is a chordless chain of length 3 from  $w$  to  $b$  in the graph induced

by  $V(C) \cup \{w\}$ . Let us denote this chain by  $wb_0w_0b$ , where  $b_0$  and  $w_0$  are respectively black and white vertices of  $C$ . Then, by Claim 4,  $w$  is adjacent to both  $b'$  and  $b''$  and consequently the set of vertices  $\{b', b'', w'\}$  together with the vertices of  $wb_0w_0b$  induces the Star-(123)  $wb'b''w'b_0w_0b$ , a contradiction. ■

CLAIM 6.  $C$  is not a complete bipartite graph.

*Proof.* Assume on the contrary that  $C$  is a complete bipartite graph.

FACT.  $H \setminus \{w\}$  contains at least two connected components distinct from  $C$ , one of which is not a single vertex.

*Proof.* Otherwise, if every connected component of  $H \setminus \{w\} \cup V(C)$  is a single black vertex or if there is only one connected component  $C' \neq C$ , no white vertex of  $C$  would be connected to  $w$  in  $\bar{H}^{\text{bip}}$ , a contradiction. ■

Let  $C'$  be a connected component of  $H \setminus \{w\}$  distinct from  $C$ , that is not a single black vertex and let  $b'w'$  be an edge of  $C'$ , with  $w'$  the white vertex ( $C'$  exists by previous fact). Consider also a chordless chain  $l$  from  $w$  to  $b$  and a black vertex  $b''$  of a component  $C'' \neq C, C'$ . Since by Claim 3,  $l$  induces in  $V(C) \cup \{w\}$  a  $P_4$  say  $wb_0w_0b$ , there would be a Star-(123)  $wb''b'w'b_0w_0b$  in  $H$ , a contradiction. The claim is proved. ■

CLAIM 7. Every connected component distinct from  $C$  is a single black vertex.

*Proof.* Assume, by contradiction, that there is a connected component  $C' \neq C$  of  $H \setminus \{w\}$ , having an edge  $b'w'$ , where  $b'$  is a black vertex. We recall that by Claim 4,  $w$  is adjacent to the vertex  $b'$ . We know, by Claim 6, that there is a  $P_4$  in  $C$ , say  $b_1w_1b_2w_2$ , where  $b_1$  and  $b_2$  are black vertices.

FACT 1.  $w$  is adjacent to both  $b_1$  and  $b_2$ .

*Proof.* Observe that  $w$  cannot be adjacent to only one of the vertices  $b_1$  and  $b_2$ , for otherwise there would be respectively the  $P_7$   $w'b'wb_1w_1b_2w_2$  or the Star-(123)  $b_2w_2w_1b_1wb'w'$  in  $H$ , a contradiction. Assume now that  $w$  is adjacent to neither  $b_1$  nor  $b_2$ , and consider a black vertex, say  $b_3$ , belonging to a chordless chain from  $w$  to  $b_1$  in the graph induced by  $V(C) \cup \{w\}$ . By Claim 3 this chain is isomorphic to a  $P_4$ , say  $wb_3w_3b_1$  (possibly  $w_3 = w_1$ ). Then  $b_3$  must be adjacent to  $w_1$ , for otherwise there would be the induced  $P_7$   $w'b'wb_3w_3b_1w_1$  in  $H$ , a contradiction. But now, if  $b_3w_2 \notin E(G)$ ,  $w'b'wb_3w_1b_2w_2$  is an induced  $P_7$  and if  $b_3w_2 \in E(G)$  we find the Star-(123)  $b_3w_2w_1b_1wb'w'$  in  $H$ , a contradiction. ■

Consider now the shortest chain, say  $l$ , in the connected graph  $C$  between  $b_1$  and the black vertex  $b$  which by assumption is non-adjacent to  $w$ .

FACT 2. *The chain  $l$  is isomorphic to a  $P_3$ .*

*Proof.* Since there is no  $P_7$  in  $H$ , the length of the chain  $l$  can only be 2 or 4. Assume by contradiction that  $l$  is isomorphic to a  $P_5$ , say  $b_1w_3b_3w_4b$ , where  $w_3$  and  $w_4$  are white vertices of  $C$ . Then, by Fact 1,  $w$  must be adjacent to both black vertices  $b_3$  and  $b$  of the  $P_4$   $w_3b_3w_4b$ , which contradicts our assumption that  $b$  is non-adjacent to  $w$ . ■

Denote  $l$  by  $b_1w_3b$ . Then, the vertex  $b$  must be adjacent to  $w_1$  for otherwise there would be the Star-(123)  $b_1w_1w_3bwb'w'$  in  $H$ , a contradiction. But now, if  $bw_2 \in E(G)$ , there would be the induced  $P_7$   $w'b'wb_1w_1bw_2$ , and if  $bw_2 \notin E(G)$  there would be the Star-(123)  $b_2w_2w_1bwb'w'$  in  $H$ , a contradiction. The claim is proved. ■

CLAIM 8. *There is no vertex in  $C$  that is  $W$ -universal for this component.*

*Proof.* Assume on the contrary that there is a black vertex, say  $b_0$ , in  $C$  that is adjacent to every white vertex of  $C$ . Since by assumption  $\bar{H}^{\text{bip}}$  is connected, there is no  $W$ -universal vertex in  $H$  and consequently,  $b_0$  is not adjacent to  $w$ . Let  $wb_1w_1b_0$  be a chordless chain from  $w$  to  $b_0$  (by Claim 3 every chordless chain from  $w$  to  $b_0$  is isomorphic to a  $P_4$ ), then in using for  $b_1$  the same argument as for  $b_0$ , we deduce that  $b_1$  is not a  $W$ -universal vertex for  $H$ . Let  $w_0$  be a vertex of  $C$  that is not adjacent to  $b_1$  and denote by  $W_1$  the common neighbourhood of  $b_1$  and  $b_0$  in  $C$  and by  $W_0$  the remaining set of white vertices of  $C$ . Consider now a shortest chain  $l$  in  $\bar{H}^{\text{bip}}$  from  $b_0$  to a vertex of  $W_0$ . Since  $w$  is the only white vertex of  $H$  that is non-adjacent to  $b_0$  and  $w$  is adjacent to every black vertex that does not belong to  $C$ , the first edge of  $l$  will be  $b_0w$  and the second  $wb_2$ , with  $b_2$  a black vertex of  $C$ . Suppose now that  $l$  is isomorphic to a  $P_6$ ; then the internal white vertices of  $l$  must be in  $W_1$  ( $l$  is a chain of minimum length in  $\bar{H}^{\text{bip}}$  from  $b_0$  to a vertex of  $W_0$ ), and consequently the vertices of  $l$  together with  $\{b_1\}$  induce a  $P_7$  in  $\bar{H}^{\text{bip}}$ , a contradiction. Hence  $l$  induces a  $P_4$  in  $\bar{H}^{\text{bip}}$ , say  $b_0wb_2w'_0$ , with  $w'_0$  a vertex of  $W_0$ . But, if  $b_2$  is adjacent to a vertex  $w'_1$  of  $W_1$  we find the induced Star-(123)  $w'_1b_2b_0w'_0b_1wb'$  in  $H$  with  $b' \notin C$ , and if  $b_2$  is adjacent to no vertex in  $W_1$ , the set  $\{b', w, b_1, w_1, b_0, w'_2, b_2\}$ , with  $w'_2$  a neighbour of  $b_2$  in  $W_0$ , induces a  $P_7$  in  $H$ , a contradiction. ■

CLAIM 9. *Every black vertex having a maximum degree in  $C$  is an internal vertex of a bipartite  $P_5$  having two black vertices.*

*Proof.* Let  $b_1$  be a vertex of  $C$  with maximum degree; then since  $b_1$  is not  $W$ -universal for  $C$  (see Claim 8), there must be a white vertex, say  $w'$  in  $C$  such that  $w' \notin N(b_1)$ . Consider a chordless chain, say  $l$ , from  $b_1$  to  $w'$ ; then clearly  $l$  contains a  $P_4$ , say  $b_1w_1b_2w'_1$ . Then, since  $b_1$  is of

maximum degree in  $C$ , there must be a neighbour of  $b_1$ , say  $w_0$ , that is not adjacent to the black vertex  $b_2$  and hence  $w_0b_1w_1b_2w'_1$  is a bipartite  $P_5$  with two black vertices, as claimed. ■

*Notation.* In the following, the vertices  $b_1$  and  $b_2$  have the meanings given in Claim 9. Moreover, we shall denote by  $W_1$  the set  $N(b_1) - N(b_2)$ , by  $W_0$  the set  $N(b_1) \cap N(b_2)$ , and by  $W_2$  the set  $N(b_2) - N(b_1)$ .

Finally, we shall denote by  $b'$  a black vertex of  $H$  that does not belong to  $C$ .

CLAIM 10.  $w$  is adjacent to both vertices  $b_1$  and  $b_2$ .

*Proof.* Observe that  $w$  cannot be adjacent to only one of the vertices  $b_1$  or  $b_2$ , for otherwise the vertices  $w_0 \in W_0$ ,  $w_1 \in W_1$ , and  $w_2 \in W_2$  together with  $w$  and  $b'$  would induce in  $H$  the Star-(123)  $b_1w_1wb'w_0b_2w_2$  or the Star-(123)  $b_2w_2wb'w_0b_1w_1$ , a contradiction. Assume now that  $w$  is adjacent to neither of  $b_1$  nor  $b_2$  and consider a neighbour of  $w$ , say  $b_3$ , in a chordless chain from  $w$  to  $b_1$  in  $C \cup \{w\}$ . By Claim 3 this chain is isomorphic to a  $P_4$ .

Since the degree of  $b_3$  in  $C$  is at most equal to the degree of  $b_1$  in  $C$  (by assumption  $b_1$  has a maximum degree in  $C$ ),  $b_3$  cannot be adjacent to every vertex in  $W_0 \cup W_1 \cup W_2$ . Suppose that  $b_3$  is adjacent to every vertex  $w_0$  in  $W_0$ ; then there is a vertex, say  $w_3$ , in  $W_1 \cup W_2$  such that  $b_3w_3 \notin E(G)$  and thus the set  $\{b', w, b_3, w_0, b_1, b_2, w_3\}$  induces the Star-(123)  $w_0b_2b_1w_3b_3wb'$  or  $w_0b_1b_2w_3b_3wb'$  according to  $w_3 \in W_1$  or  $w_3 \in W_2$ , a contradiction. Suppose now that  $b_3$  is adjacent to none of the vertices of  $W_0$ ; then since  $b_3$  belongs to a  $P_4$  from  $w$  to  $b_1$ , there must be a vertex, say  $w_1$ , in  $W_1$  that is adjacent to  $b_3$ . Consequently the set  $\{b', w, b_3, w_1, b_1, w_0, b_2\}$  (with  $w_0$  any vertex of  $W_0$ ) induces a  $P_7$ , a contradiction. Hence, there are two vertices, say  $w_3$  and  $w_4$ , in  $W_0$  such that  $b_3w_3 \in E(G)$  and  $b_3w_4 \notin E(G)$ . But now the set  $\{b', w, b_3, w_3, w_4, w_1, b_1, b_2\}$  (with  $w_1$  any vertex of  $W_1$ ) contains the induced  $P_7$   $b'wb_3w_1b_1w_4b_2$  if  $b_3w_1 \in E(H)$  or the induced Star-(123)  $w_3b_2b_1w_1b_3wb'$  if  $b_3w_1 \notin E(H)$ , a contradiction. ■

Consider now the black vertex  $b$  of  $C$  that by assumption is not adjacent to  $w$  (see Claim 2), and denote by  $l$  the shortest chain in  $C$  from  $b$  to  $b_1$ . Since  $H$  is  $P_7$ -free,  $l$  is isomorphic to either a  $P_3$  or a  $P_5$ . Assume that  $l$  is isomorphic to  $P_5$ , say  $bw_3b_3w_4b_1$ ; then  $w$  is adjacent to  $b_3$  for otherwise the set  $\{b', w, b_1, w_4, b_3, w_3, b\}$  induces a  $P_7$  in  $H$ , a contradiction.

But, since  $b_1$  has a maximum degree in  $C$ , there must be a vertex, say  $w_1 \in N(b_1)$ , which is not adjacent to  $b_3$ . But now, since  $l$  is supposed isomorphic to a  $P_5$  and  $bw_1 \notin E(H)$ , there is an induced Star-(123)  $wb'b_1w_1b_3w_3b$  in  $H$ , a contradiction.

Hence,  $l$  is isomorphic to a  $P_3$  denoted in the following by  $b_1w'b$ .



CLAIM 11.  $b$  is adjacent to no vertex of  $W_0$ .

*Proof.* Assume by contradiction that  $b$  is adjacent to a vertex, say  $w_0$ , of  $W_0$ .

FACT 1.  $b$  is adjacent to every vertex of  $W_1$  or  $b$  is adjacent to every vertex of  $W_2$ .

*Proof.* Otherwise, a vertex  $w_1$  in  $W_1$ , a vertex  $w_2$  in  $W_2$  such that  $w_1b \notin E(H)$  and  $w_2b \notin E(H)$ , together with the set  $\{b', w, b, w_0, b_1, b_2\}$ , would induce a Sun-(4) in  $H$ , a contradiction. ■

FACT 2. The degree of  $b$  is less than the degree of  $b_1$ .

*Proof.* Otherwise, if we suppose that  $b$  and  $b_1$  have the same degree, by applying successively to  $b$  Claims 9 and 10, we deduce that  $bw \in E(H)$ , a contradiction. ■

Thus, there exists a vertex  $x \in W_1$  that is not adjacent to  $b$  and by Fact 1 we obtain that  $b$  is adjacent to every vertex of  $W_2$  and consequently there would be the induced Star-(123)  $b_1xwb'w_0bw_2$  with  $w_2$  a vertex of  $W_2$ , a contradiction. The claim is proved. ■

Consider now the vertex  $w'$  of the  $P_3$   $b_1w'b$ . Then from Claim 11 we deduce that  $w' \in W_1$ . Also let  $w_2$  be a vertex of  $W_2$ ; then there is an induced Star-(123)  $b_2w_0wb'w_2bw'$  or  $wb'b_2w_2b_1w'b$  according to whether  $bw_2 \in E(H)$  or not, a contradiction. This concludes the proof of Theorem 3.1. ■

*Remark.* Using an argument similar to that in the above theorem we can establish in a very simple way that cographs are  $P_4$ -free. More precisely, we first show that there is an articulation point  $x$  to a connected cograph  $G$  having a minimum number of vertices and whose complemented graph  $\overline{G}$  (which is also a cograph), is also connected. Then, there must be an edge  $yz$  to a connected component  $C$  of  $G \setminus \{x\}$  such that  $xy \in E(G)$  and  $xz \notin E(G)$  (as in Claim 2). In this way we obtain a contradiction, since  $xyzt$ , with  $t$  a neighbour of  $x$  into a connected component  $C' \neq C$ , would be an induced  $P_4$  in  $G$ .

#### 4. A TREE REPRESENTATION OF BI-COGRAPHS

In order to establish a tree representation for bi-cographs, we use the following property that the reader can easily verify:

LEMMA 4.1. *If a bi-cograph  $G$  of order at least 2 is connected, then  $\overline{G}^{\text{bip}}$  is disconnected.*

The above property implies that, uniquely up to an isomorphism, a bi-cograph  $G$  can be decomposed into single vertices by recursively bi-complementing connected bipartite subgraphs.

Let us associate now with the above decomposition process of  $G$  a rooted tree  $T(G)$ . The leaves of  $T(G)$  will be then the colored vertices of  $G$ , while its internal nodes correspond to the connected bi-cographs obtained during the decomposition of  $G$ . Clearly, a bottom-up traversal of  $T(G)$  describes the fact that any bi-cograph can be obtained as bi-complemented union of bi-cographs.

As for a cotree, we can label each internal node of  $T(G)$  by 0 or 1 as follows: the root is labeled 1 when  $G$  is connected and by 0 if not, and the children of a node with label 1 (resp. 0) are labeled 0 (resp. 1). Thus, 1 and 0 nodes alternate along every path starting from the root. In this way, two vertices of  $G$  having different colours are adjacent (resp. non-adjacent) iff their least common ancestor in  $T(G)$  is labeled 1 (resp. 0). By analogy to a cotree, such a tree will be called a *bi-cotree*.

Unfortunately, as we show in the example in Fig. 1, this tree representation of a bi-cograph is not unique, as is true for cographs. Indeed, the first bi-cotree  $T_1$  is obtained when the bi-cograph  $G$  is recursively decomposed by bi-complementing all connected subgraphs. The reader can easily verify the fact that, when composing bipartite graphs in using bi-complemented union following a bottom-up traversal of  $T_2$ , we also obtain  $G$ .

#### 4.1. Enumerating all Equivalent Bi-cotrees

In order to understand why several bi-cotrees can be associated with a bi-cograph while there is a unique associated cotree with a cograph, we shall look for the differences between bi-complementation and complementation of a graph. Such a difference concerns the bi-complement and the complement of a disconnected graph. Effectively we know that  $\overline{G}$  is connected when  $G$  is disconnected. But, when a bipartite graph  $G$  is

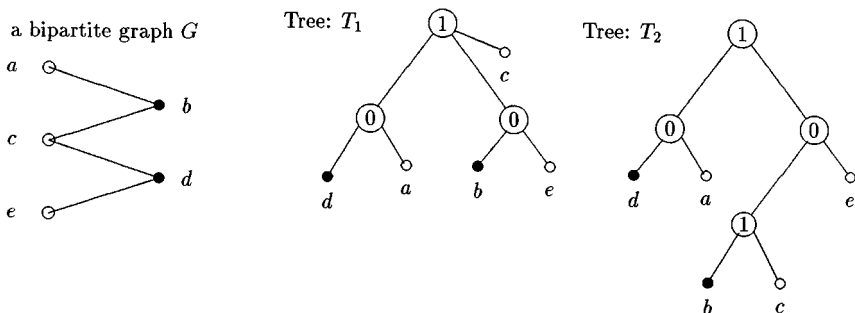


FIGURE 1

disconnected,  $\overline{G}^{\text{bip}}$  is not necessarily connected. In this section, we shall prove that this last fact is the reason for the existence of equivalent bi-cotrees associated with a bi-cograph  $G$ .

Let us first characterize the disconnected bipartite graphs whose bi-complement is disconnected. The reader can easily verify the following result:

**THEOREM 4.1.1.** *Let  $C_1, \dots, C_k$ ,  $k \geq 2$ , be the connected components of a bipartite graph  $G$  and assume that  $G$  does not contain only white or only black vertices. Then  $\overline{G}^{\text{bip}}$  is disconnected if and only if the following conditions are verified:*

- (i) *Every connected component of  $G$  is a complete bipartite graph (a singleton is considered a complete bipartite graph).*
- (ii) *There exists a connected component  $C$  having at least two vertices.*
- (iii) *If  $k > 2$ , then all components of  $G$  distinct from  $C$  are single white or single black vertices.*

**DEFINITION.** Let  $G$  be a non-edgeless bipartite graph and let  $S$  be a proper subgraph of  $G$  obtained as the union of some but not all of the connected components of  $G$ . Then  $S$  will be called a *nice-subgraph* of  $G$  if and only if  $\overline{S}^{\text{bip}}$  is disconnected.

*Notation.* Denote by  $\mathcal{F}$  the family of bipartite graphs that can be reduced to single vertices by applying the following recursive process: If  $G \in \mathcal{F}$  is connected, then we recursively bi-complement  $G$ . Otherwise let  $\pi(V(G)) = V(S_1), \dots, V(S_k)(G)$  be a partition of  $V$  such that  $S_i$ ,  $i = 1, \dots, k$ , is a nice-subgraph of  $G$ ; then we recursively bi-complement  $S_1$ , we recursively bi-complement  $S_2, \dots$ , we recursively bi-complement  $S_k$ .

**THEOREM 4.1.2.** *The family of bipartite graphs  $\mathcal{F}$  is exactly the family of bi-cographs.*

*Proof.* Indeed, by definition, a bi-cograph can be reduced to single vertices by recursively bi-complementing connected subgraphs. Thus, the if part follows by observing that a connected component of a disconnected bi-cograph  $G$ , is a nice-subgraph of  $G$ .

For the only if part, consider a bipartite graph  $G \in \mathcal{F}$  and decompose  $G$  following the previous recursive process. Associate with  $G$  a tree  $T(G)$  in the following manner:

The leaves of  $T(G)$  are labeled by the colored vertices of  $G$  and the internal nodes are labeled by the nice-subgraphs obtained during the decomposition of  $G$  (see Fig. 2). Observe now that the graph associated with an internal node  $f$  of  $T(G)$  can be obtained as a bi-complemented union of the graphs associated with the set of sons of  $f$ . Thus,  $G$  can be obtained from its vertex set by performing a finite number of unions and bi-complementations, and consequently  $G$  is a bi-cograph, as claimed. ■

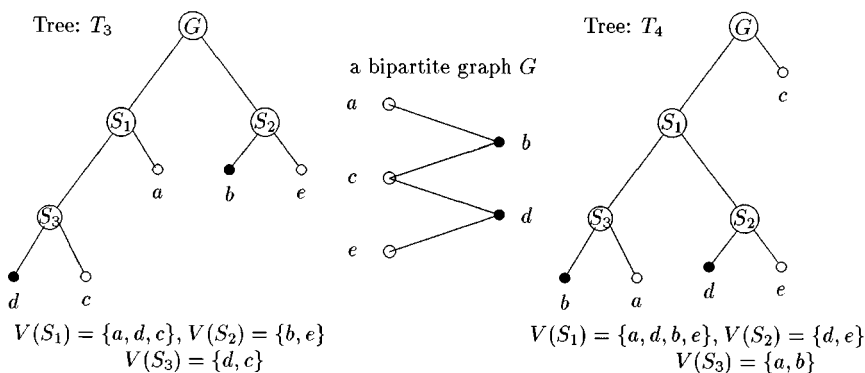


FIGURE 2

Let us illustrate now, using an example, how we can enumerate all the equivalent bi-cotrees associated with a bi-cograph. In Figs. 1 and 2, four equivalent decompositions of a graph  $G$  that is isomorphic to a  $P_5$  are depicted. Observe that by labeling with 1 and 0 the internal nodes of trees  $T_3$  and  $T_4$  (Fig. 2) following the manner previously described, we obtain two equivalent bi-cotrees associated with  $G$ . We can easily see now that the set of four trees  $T_1, T_2$  (Fig. 1) and  $T_3, T_4$  (Fig. 2) corresponds to all equivalent bi-cotrees associated with  $G$ . Indeed, there are three connected components of  $\bar{G}^{\text{bip}}$ , the edge  $ad$ , the edge  $eb$ , and the single vertex  $c$ . Hence, there are four different partitions of  $V(\bar{G}^{\text{bip}})$  following nice-subgraphs, namely  $\pi_1 = (\{a, d\}, \{b, e\}, \{c\})$  (tree  $T_1$ ),  $\pi_2 = (\{a, d\}, \{b, e, c\})$  (tree  $T_2$ ),  $\pi_3 = (\{a, d, c\}, \{b, e\})$  (tree  $T_3$ ), and  $\pi_4 = (\{a, d, b, e\}, \{c\})$  (tree  $T_4$ ). We would obtain a new tree  $T_5$ , if there were a partition into nice-subgraphs of a graph corresponding to an internal node of  $T_i, i = 1, \dots, 4$ , not considered in any of the four trees  $T_1, \dots, T_4$ . We can easily see that no more such partitions are possible since any graph other than  $\bar{G}^{\text{bip}}$  corresponding to an internal node of  $T_i, i = 1, \dots, 4$ , has at most two connected components.

*Observation.* Let us define  $S$  as a nice-subgraph of a disconnected graph  $G$  whenever  $S$  is obtained as the union of connected components of  $G$  and  $\bar{S}$  is disconnected. Then, since the complement of a disconnected graph is connected, there is only one possible partition of  $V(G)$  following nice-subgraphs, the partition induced by the connected components of  $G$ . This can explain why a cotree associated with a cograph is unique.

*Recognition algorithm for bi-cographs.* From the definition of bi-cographs, we deduce an  $O(n^3)$  recognition algorithm, since we must apply the bi-complementation of bipartite graphs  $O(n)$  times.

## CONCLUSION

We believe that this work, could be a start point for further research in the spirit of the ideas exposed in the introduction of our paper. A deeper knowledge of bi-cographs could allow us to understand, for which kind of problems in different fields (as for example in empirical logic) bi-cographs become a powerful tool for their solutions. Moreover, we know that cographs is the family of graphs that are completely decomposable with respect to the *modular decomposition* (see [7] for definitions and an efficient algorithm for modular decomposition). It would be interesting then to research the bipartite analogue for modular decomposition and apply it for recognition and other algorithmic problems.

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