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AN APPROACH TO STEADY-STATE SOLUTIONS

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Abstract—The steady-state solution of the nonlinear heat equation is calculated using the decomposition method.

DISCUSSION

We consider the nonlinear heat equation

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \beta f(u),$$

for $0 \le x \le \ell$, $t > 0$ and $u(0,t) = T_0$, $u(\ell,t) = T_1$, $u(x,0) = h(x)$, and $f(u)$ an analytic function. (We assume here that $h(x)$ is a differentiable function but can allow it to be piecewise-differentiable using the technique of [1].)

Let's assume $u(x,t) = w(x,t) + v(x)$, where $\lim_{t\to\infty} u(x,t) = v(x)$ and $\lim_{t\to\infty} w(x,t) = 0$, considering w to be the transient solution and v to be the steady-state solution.

To calculate the steady-state solution, we consider

$$\lim_{t\to\infty} \left\{ \alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \beta f(u) \right\} = \lim_{t\to\infty} \alpha^2 \frac{\partial^2 u}{\partial x^2} = \lim_{t\to\infty} \frac{\partial u}{\partial t} + \lim_{t\to\infty} \beta f(u),$$

$$\alpha^2 \frac{\partial^2 v}{\partial x^2} = 0 + \beta f(v), \quad \text{or} \quad \frac{d^2 v}{dx^2} = 0 + k f(v), \qquad \text{where } k = \frac{\beta}{\alpha^2}.$$

The $\lim_{t\to\infty} u(0,t) = v(0) = T_0$ and $\lim_{t\to\infty} u(\ell,t) = v(\ell) = T_1$. Thus, the steady-state solution is

$$\frac{d^2 v}{dx^2} = k f(v), \quad v(0) = T_0, \quad v(\ell) = T_1.$$

Write $L v = k f(v)$ where $L = d^2/dx^2$. Applying L^{-1} to both sides, we get

$$v = c_0 + c_1 x + I_x^2 k f(v),$$

where I_x^2 is the two-fold pure integration with respect to x. Using decomposition, let

$$v = \sum_{m=0}^{\infty} v_m, \quad f(v) = \sum_{m=0}^{\infty} A_m,$$

where the $A_m (v_0, \dots, v_m)$ are the Adomian polynomials. Using double decomposition, we get

$$v_0 = T_0 + \frac{T_1 - T_0}{\ell} x,$$

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$$v_1 = I_x^2\, k\, A_0,$$
$$v_2 = I_x^2\, k\, A_1,$$
$$\vdots$$
$$v = \sum_{n=0}^{\infty} v_n,$$

then

$$\phi_n[v] = \sum_{i=0}^{n-1} v_i; \qquad c_0 = \sum_{m=0}^{\infty} c_0^{(m)} \quad \text{and} \quad c_1 = \sum_{m=0}^{\infty} c_1^{(m)},$$

$$\sum_{m=0}^{\infty} v_m = \sum_{m=0}^{\infty} c_0^{(m)} + x \sum_{m=0}^{\infty} c_1^{(m)} + k\, I_x^2 \sum_{m=0}^{\infty} A_m,$$

$$v_0 = c_0^{(0)} + x\, c_1^{(0)},$$
$$v_1 = c_0^{(1)} + x\, c_1^{(1)} + k\, I_x^2\, A_0,$$
$$\vdots$$
$$v_m = c_0^{(m)} + x\, c_1^{(m)} + k\, I_x^2\, A_{m-1}.$$

In order to compute the matching coefficients of the boundary conditions, we develop the approximate boundary conditions by using our approximations to the solution

$$\phi_{m+1}[v] = \sum_{n=0}^{m} v_n = \phi_m + v_m.$$

Since $v(0) = T_0$ and $v(\ell) = T_1$, we can write the approximate boundary conditions

$$\phi_{m+1}(0) = T_0 \quad \text{and } \phi_{m+1}(\ell) = T_1.$$

Since $\phi_{m+1} = \phi_m + v_m$, we have

$$v_0(0) = T_0 \quad \text{and } v_0(\ell) = T_1, \qquad \text{for } m = 0, \quad \text{and}$$
$$v_m(0) = 0 \quad \text{and } v_m(\ell) = 0 \qquad \text{for } m > 0.$$

Consequently, writing $T_0 \equiv T_0^{(0)}$ for convenience,

$$c_0^{(0)} = T_0^{(0)} \quad \text{and } c_1^{(0)} = \frac{1}{\ell}\left(T_1^{(0)} - T_0^{(0)}\right), \qquad v_m(0) = 0 \quad \text{and } v_m(\ell) = 0;$$
$$c_0^{(m)} + 0 = -k\, I_x^2\, A_{m-1}\big|_{x=0} \equiv T_0^{(m)}, \quad \text{by definition},$$
$$c_0^{(m)} + \ell\, c_1^{(m)} = -k\, I_x^2\, A_{m-1}\big|_{x=\ell} \equiv T_1^{(m)}, \quad \text{by definition},$$

or

$$\begin{pmatrix} 1 & 0 \\ 1 & \ell \end{pmatrix} \begin{pmatrix} c_0^{(m)} \\ c_1^{(m)} \end{pmatrix} = \begin{pmatrix} T_0^{(m)} \\ T_1^{(m)} \end{pmatrix};$$

so $c_0^{(m)}$ and $c_1^{(m)}$ are determined and we can write $v_0, v_1 \ldots$; and finally $\phi_n[v] = \sum_{i=0}^{n-1} v_i$.

REMARKS. This work is not limited to the heat equation; it is generic. The objective of the decomposition method is the physically correct solution of nonlinear equations. Closed form solutions are generally obtained at a price. The decomposition series converges quite rapidly, generally in a few terms, and no linearization is necessary. The definition of the L and L^{-1} operators makes integration simpler. The α and the β used in the above example are not limited to constants and f can be any function for which the A_n polynomials can be found.

REFERENCES

1. R. Rach and G. Adomian, Smooth polynomial expansions of piecewise-differentiable functions, *Appl. Math. Lett.* **2** (4), 377–379 (1989).