Note

On convex decompositions of a planar point set

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Abstract

Let $P$ be a planar point set in general position. Neumann-Lara et al. showed that there is a convex decomposition of $P$ with at most $10n - 18$ elements. In this paper, we improve this upper bound to $\lceil \frac{7}{5}(n - 3) \rceil + 1$.

Keywords: Convex hull of a point set; Decomposition problem; Combinatorial convexity

1. Introduction

Let $P$ be a set of points in general position in the plane. A family $F$ of convex polygons with vertices in $P$ and with pairwise disjoint interiors is called a convex decomposition of $P$ if their union is the convex hull $ch(P)$ of $P$ and no point of $P$ lies in the interior of any polygon in $F$. We call a convex polygon in a convex decomposition a cell. Let $H(P)$ be the minimum number of cells in a convex decomposition of $P$, and let $h(n)$ represent the maximum value of $H(P)$ over all sets $P$ with $n$ points.

In 1998, Urrutia [5] conjectured that $h(n) \leq n + 1$ for any $n \geq 3$, and Neumann-Lara et al. proved that $h(n) \leq \frac{10n - 18}{7}$ for any $n \geq 3$ in [4]. As for the lower bound, Aichholzer and Krasser [2] showed that $h(n) \geq n + 2$ for any $n \geq 13$. Later, García-López and Nicolás proved that $h(n) > \frac{12}{11}n - 2$ for any $n \geq 4$ in [3]. In addition, Aichholzer et al. [1] discuss a subdivision of the plane that consists of both convex polygons and pseudo-triangles with this problem.

In this paper, we improve on the upper bound of $h(n)$.

Theorem. $h(n) \leq \lceil \frac{7}{5}(n - 3) \rceil + 1$ for any $n \geq 3$.

We first introduce the definitions and notation required for the remainder of the paper. Let $Q$ be a subset of a given point set $P$. Denote the vertices on the boundary of $ch(Q)$ by $V(Q)$. Let $R$ be a region in the plane. The region $R$ is said to be empty if $R$ contains no elements of $P$ in the interior. Let $S(R)$ be the elements of $P$ in $R$, i.e., $S(R) = P \cap R$. We denote a cell in a convex decomposition of $Q$ with an edge $uv$ lying in the boundary of $ch(Q)$ by $C(uv, Q)$.

Let $a$, $b$ and $c$ be any three non-collinear points in the plane. Denote by $\gamma(a; b, c)$ the convex cone with apex $a$, determined by $a$, $b$ and $c$. If $\gamma(a; b, c)$ is not empty, we define an attack point $\alpha(a; b, c)$ from the half-line $ab$ to $ac$ as the element of $P$ in $\gamma(a; b, c)$ such that $\gamma(a; b, \alpha(a; b, c))$ is empty. For $\delta = b$ or $c$ of $\gamma(a; b, c)$, let $\delta'$ be a

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point collinear with $a$ and $\delta$ so that $a$ lies on the segment $\delta \delta'$. Then we obtain a new convex cone or attack point of $\gamma(a; b', c)$ or $\alpha(a; c, b')$. See Fig. 1.

2. Proof

Let $P$ be a planar $n$ point set in general position. We construct a convex decomposition of $P$ by recursively constructing convex decompositions of subsets of $P$. Namely, we show that $h(n) \leq \lceil \frac{7}{8} (n - 3) \rceil + 1$ for any $n \geq 3$ by induction on $n$. Since $h(3) = 1 \leq \lceil \frac{7}{8} (3 - 3) \rceil + 1$, we suppose that $n \geq 4$ and use the induction hypothesis that $h(n') \leq \lceil \frac{7}{8} (n' - 3) \rceil + 1$ for any $n' \leq n - 1$.

2.1. $|V(P)| \geq 4$

Let $V(P) = \{v_1, v_2, \ldots, v_L\}$ in counter-clockwise order. Consider $K_1 = ch([v_1, v_2, v_3]) = \Delta v_1 v_2 v_3$ and $K_2 = (ch(P) \setminus K_1) \cup \overline{v_1 v_3}$ where $K_1 \cap K_2 = \overline{v_1 v_3}$. By the induction hypothesis, for $i = 1, 2$, there is a convex decomposition of $S(K_i)$ with at most $\lceil \frac{7}{8} (n_i - 3) \rceil + 1$ cells where $n_i = |S(K_i)|$. For the cell of $C_i = C(\overline{v_1 v_3}, S(K_i))$ for $i = 1, 2$, we claim that $C_1 \cup C_2$ is a cell in a convex decomposition of $P$. Therefore, $h(n) \leq (\lceil \frac{7}{8} (n_1 - 3) \rceil + 1) + (\lceil \frac{7}{8} (n_2 - 3) \rceil + 1) - 1 \leq \lceil \frac{7}{8} (n_1 + n_2 - 5) \rceil + 1 = \lceil \frac{7}{8} (n - 3) \rceil + 1$ with $n = n_1 + n_2 - 2$.

2.2. $|V(P)| = 3$

Let $V(P) = \{v_1, v_2, v_3\}$ in counter-clockwise order. We first consider $P' = P \setminus \{v_1\}$ and suppose that $|V(P')| = 3$ where $V(P') = \{p, v_2, v_3\}$. Let $T_1 = \Delta p v_1 v_2, T_2 = \Delta p v_2 v_3$ and $T_3 = \Delta p v_3 v_1$ where $ch(P) = T_1 \cup T_2 \cup T_3$ and $T_1$ and $T_3$ are empty.

If $T_2$ is also empty, $H(P) = 3$ and $h(4) \leq \lceil \frac{7}{8} (4 - 3) \rceil + 1 = 3$ holds. If $T_2$ is not empty, we have $C_1 = C(\overline{p v_2}, S(T_2))$ and $C_2 = C(\overline{p v_3}, S(T_2))$ with $C_1 \neq C_2$. Then $T_1 \cup C_1$ or $T_3 \cup C_2$ is a cell in a convex decomposition of $P$ since either $C_1$ is in $\gamma(v_1; v_2, p)$ or $C_2$ is in $\gamma(v_1; v_3, p)$. See Fig. 2. Since there is a convex decomposition of $P'$ with at most $\lceil \frac{7}{8} (n' - 3) \rceil + 1$ cells for $n' = |P'|, h(n) \leq (\lceil \frac{7}{8} (n' - 3) \rceil + 1) + 2 - 1 \leq \lceil \frac{7}{8} (n' - 2) \rceil + 1 = \lceil \frac{7}{8} (n - 3) \rceil + 1$ with $n = n' + 1$.

We can now make the following assumption.
Assumption 1. For any $v$ in $V(P)$, $ch(P \setminus \{v\})$ is not a triangle.

For $p_1 = \alpha(v_3; v_2, v_1)$, we suppose that $\gamma(p_1; v_2, v_3)$ is not empty and consider $p_2 = \alpha(p_1; v_3, v_2)$, where $\{v_2, v_3, p_1, p_2\}$ is an empty convex quadrilateral. Let $T_1 = \Delta v_1 v_2 p_1$, $T_2 = \Delta v_1 p_2 v_3$ and $T_3 = \Delta v_1 p_1 p_2$.

By Assumption 1, neither $T_1$ nor $T_2$ is empty. Let $C_1 = C(v_1 p_1, S(T_1))$ for $i = 1, 2$ where $C_1$ is in $\gamma(p_1; v_1, p_2)$ and $C_2$ is in $\gamma(p_2; v_1, p_1')$ as shown in Fig. 3. If $T_3$ is empty or not, $C_1 \cup T_3 \cup C_2$ is a cell in a convex decomposition of $P$ or both $C_1 \cup C(v_1 p_1, S(T_3))$ and $C_2 \cup C(v_1 p_2, S(T_3))$ are cells in a convex decomposition of $P$, respectively. Thus, $h(n) \leq \sum_{i=7}^{11} (\left\lceil \frac{n_i}{2} \right\rceil - 3) + 1 = \sum_{i=7}^{11} (\left\lceil \frac{n_i}{2} \right\rceil - 3) + 1 = \sum_{i=7}^{11} (\left\lceil \frac{n_i}{2} \right\rceil - 3) + 1 = \sum_{i=7}^{11} (\left\lceil \frac{n_i}{2} \right\rceil - 3) + 1$ where $n_i = \lvert S(T_i) \rvert$ for $i = 1, 2, 3$ and $n = n_1 + n_2 + n_3 - 4$.

For the remaining of the proof we can make the following stronger assumption.

Assumption 2. For any edge $v_iv_{i+1}$ on the boundary of $ch(P)$, there exists an element $w_i$ of $P$ such that $\gamma(v_i; v_{i+1}, w_i) \cup \gamma(v_{i+1}; v_i, w_i)$ is empty for $i = 1, 2, 3$, where $v_4 = v_1$.

See Fig. 4 where the shaded portion is empty. We remark that $w_i \neq w_j$ for $i \neq j$ by Assumption 1. Let $K_1 = \Delta v_1 w_1 w_3$, $K_2 = \Delta v_2 w_2 w_1$, $K_3 = \Delta v_3 w_3 w_2$ and $K' = \Delta w_1 w_2 w_3$.

We continue under Assumption 2.

Case 1: $K_i$ is empty for every $i = 1, 2, 3$.

If $K'$ is empty, i.e., $n = 6$, then $H(P) = 6$ and $h(6) \leq \sum_{i=7}^{11} (\left\lceil \frac{n_i}{2} \right\rceil - 3) + 1 = 6$ holds. Let $C_i = C(w_{i-1} w_i, S(K'))$ for $i = 1, 2, 3$ where $w_0 = w_3$. If $K'$ is not empty, since $C_i \neq C_j$ for $i \neq j$ and $K_i \cup C_i$ is a cell in a convex decomposition of $P$ for each $i$, $h(n) \leq \left\lceil \frac{n_i}{2} \right\rceil - 3 \leq \left\lceil \frac{n'}{2} \right\rceil + 1 = \left\lceil \frac{n'}{2} \right\rceil + 1$ with $n = n' + 3$ for $n' = \lvert S(K') \rvert$.

Case 2: There exists a non-empty $K_i$, say $K_1$.

Let $V(S(K_1) \setminus \{v_1\}) = \{w_1, u_1, u_2, \ldots, u_k, w_3\}$ in clockwise order.

(A) $\Delta v_2 u_1 w_1$ is not empty.

Consider $p = \alpha(v_2; u_1, v_1)$ and let $T_1 = \Delta v_1 v_2 p$, $T_2 = \Delta v_2 v_3 p$ and $T_3 = \Delta v_3 v_1 p$, where $\lvert S(T_1) \rvert = 4$. There is a convex decomposition of $S(T_1)$ with at most $\left\lceil \frac{n_i}{2} \right\rceil + 1$ cells for $n_i = \lvert S(T_i) \rvert$, $i = 2, 3$. 

Fig. 4. The shaded portion is empty.
If \( p \) is in \( \gamma(v_3; w_2, w_1) \) as shown in Fig. 5a, we consider \( C_2 = C(v_3w_2, S(T_2)) \), \( C_3 = C(v_3w_1, S(T_3)) \) and \( C'_3 = C(v_1w_2, S(T_3)) \). Since \( \{v_1, v_3, w_1, p\} \) is in convex position, \( \triangle v_1w_1p \cup C'_3 \) is a cell in a convex decomposition of \( P \). And \( C_3 \) is contained in \( \gamma(p; v'_2, v_3) \) since, otherwise, \( C_1 = T_3 \) and \( C_3 \) would not be empty, that is, \( C_2 \cup C_3 \) is also a cell in a convex decomposition of \( P \). Thus, \( h(n) \leq (\lceil \frac{7}{2}(n_2 - 3) \rceil + 1) + (\lceil \frac{7}{5}(n_3 - 3) \rceil + 1) + 3 - 2 \leq \lceil \frac{7}{5}(n_2 + n_3 - 4) \rceil + 1 = \lceil \frac{7}{5}(n - 3) \rceil + 1 \) with \( n = n_2 + n_3 - 1 \) \((*)\).

If \( p \) is in \( \gamma(v_3; w_1, u_1) \) as shown in Fig. 5b, we let \( C_2 = C(v_3w_1, S(T_2)) \) and \( C'_2 = C(v_2w_1, S(T_2)) \) where \( C_2 \neq C'_2 \) by the existence of \( w_2 \). Since both \( \triangle v_2p_{w_1} \cup C'_2 \) and \( C_2 \cup C(v_2w_1, S(T_3)) \) are cells in a convex decomposition of \( P \) for the same reason, we obtain the same inequalities as \((*)\).

(B) \( \triangle v_2u_1w_1 \) is empty.

Let \( T_1 = \triangle v_1v_2u_1 \), \( T_2 = \triangle v_2v_3u_1 \) and \( T_3 = \triangle v_3v_1u_1 \) with \( |S(T_i)| = 4 \). See Fig. 5c. Since \( C(v_3u_1, S(T_3)) \) is in \( \gamma(u_1; v'_2, v_3) \) and both \( \triangle v_2u_1w_1 \cup C(v_2u_1, S(T_2)) \) and \( C(v_3u_1, S(T_2)) \cup C(v_2u_1, S(T_3)) \) are cells in a convex decomposition of \( P \), \( h(n) \leq (\lceil \frac{7}{2}(n_2 - 3) \rceil + 1) + (\lceil \frac{7}{5}(n_3 - 3) \rceil + 1) + 3 - 2 \leq \lceil \frac{7}{5}(n - 3) \rceil + 1 \) with \( n = n_2 + n_3 - 1 \) for \( n_i = |S(T_i)|, i = 2, 3 \). □

3. Final remark

There is still a substantial gap between the upper and lower bounds for \( h(n) \). We believe that a more complicated approach may be able to prove that \( h(n) \leq \lceil \frac{4}{3}(n - 2) \rceil \) for any \( n \geq 3 \).

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References