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Note

On convex decompositions of a planar point set

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Abstract

Let P be a planar point set in general position. Neumann-Lara et al. showed that there is a convex decomposition of P with at most $\frac{10n-18}{7}$ elements. In this paper, we improve this upper bound to $\lceil \frac{7}{5}(n-3) \rceil + 1$.

Keywords: Convex hull of a point set; Decomposition problem; Combinatorial convexity

1. Introduction

Let P be a set of points in general position in the plane. A family F of convex polygons with vertices in P and with pairwise disjoint interiors is called a *convex decomposition* of P if their union is the convex hull $ch(P)$ of P and no point of P lies in the interior of any polygon in F. We call a convex polygon in a convex decomposition a *cell*. Let $H(P)$ be the minimum number of cells in a convex decomposition of P, and let $h(n)$ represent the maximum value of $H(P)$ over all sets P with n points.

In 1998, Urrutia [5] conjectured that $h(n) \leq n+1$ for any $n \geq 3$, and Neumann-Lara et al. proved that $h(n) \leq \frac{10n-18}{7}$ for any $n \geq 3$ in [4]. As for the lower bound, Aichholzer and Krasser [2] showed that $h(n) \geq n+2$ for any $n \geq 13$. Later, García-López and Nicolás proved that $h(n) > \frac{12}{11}n - 2$ for any $n \geq 4$ in [3]. In addition, Aichholzer et al. [1] discuss a subdivision of the plane that consists of both convex polygons and pseudo-triangles with this problem.

In this paper, we improve on the upper bound of $h(n)$.

Theorem. $h(n) \leq \left\lceil \frac{7}{5}(n-3) \right\rceil + 1$ *for any* $n \geq 3$.

We first introduce the definitions and notation required for the remainder of the paper. Let Q be a subset of a given point set P. Denote the vertices on the boundary of $ch(Q)$ by $V(Q)$. Let R be a region in the plane. The region R is said to be *empty* if R contains no elements of P in the interior. Let $S(R)$ be the elements of P in R, i.e., $S(R) = P \cap R$. We denote a cell in a convex decomposition of Q with an edge $\overline{uv}$ lying in the boundary of $ch(Q)$ by $C(\overline{uv}, Q)$.

Let a, b and c be any three non-collinear points in the plane. Denote by $\gamma(a; b, c)$ the *convex cone* with apex a, determined by a, b and c. If $\gamma(a; b, c)$ is not empty, we define an *attack point* $\alpha(a; b, c)$ from the half-line ab to ac as the element of P in $\gamma(a; b, c)$ such that $\gamma(a; b, \alpha(a; b, c))$ is empty. For $\delta = b$ or c of $\gamma(a; b, c)$, let δ' be a

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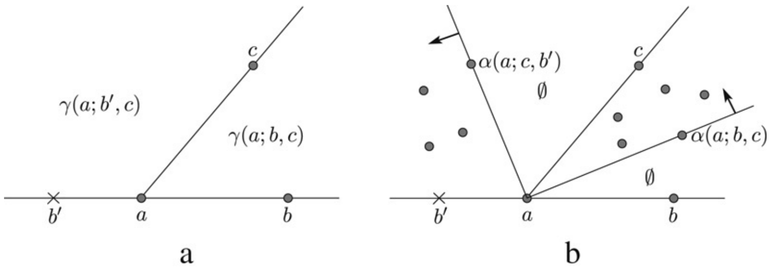


Fig. 1. (a) Convex cones $\gamma(a; b, c)$ and $\gamma(a; b', c)$. (b) Attack points $\alpha(a; b, c)$ and $\alpha(a; c, b')$.

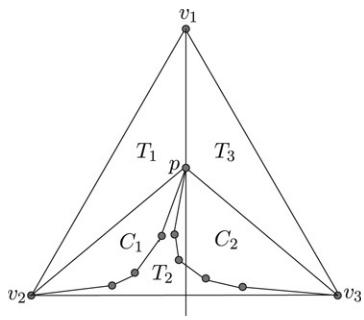


Fig. 2. $T_1 \cup C_1$ or $T_3 \cup C_2$ is a cell in a convex decomposition of P.

point collinear with a and δ so that a lies on the segment $\delta\delta'$. Then we obtain a new convex cone or attack point of $\gamma(a; b', c)$ or $\alpha(a; c, b')$. See Fig. 1.

2. Proof

Let P be a planar n point set in general position. We construct a convex decomposition of P by recursively constructing convex decompositions of subsets of P. Namely, we show that $h(n) \leq \lceil \frac{7}{5}(n-3) \rceil + 1$ for any $n \geq 3$ by induction on n. Since $h(3) = 1 \leq \lceil \frac{7}{5}(3-3) \rceil + 1$, we suppose that $n \geq 4$ and use the induction hypothesis that $h(n') \leq \lceil \frac{7}{5}(n'-3) \rceil + 1$ for any $n' \leq n-1$.

2.1. $|V(P)| \geq 4$

Let $V(P) = \{v_1, v_2, \ldots, v_L\}$ in counter-clockwise order. Consider $K_1 = ch(\{v_1, v_2, v_3\}) = \triangle v_1 v_2 v_3$ and $K_2 = (ch(P) \setminus K_1) \cup \overline{v_1 v_3}$ where $K_1 \cap K_2 = \overline{v_1 v_3}$. By the induction hypothesis, for $i = 1, 2$, there is a convex decomposition of $S(K_i)$ with at most $\lceil \frac{7}{5}(n_i - 3) \rceil + 1$ cells where $n_i = |S(K_i)|$. For the cell of $C_i = C(\overline{v_1 v_3}, S(K_i))$ for $i = 1, 2$, we claim that $C_1 \cup C_2$ is a cell in a convex decomposition of P. Therefore, $h(n) \leq (\lceil \frac{7}{5}(n_1 - 3) \rceil + 1) + (\lceil \frac{7}{5}(n_2 - 3) \rceil + 1) - 1 \leq \lceil \frac{7}{5}(n_1 + n_2 - 5) \rceil + 1 = \lceil \frac{7}{5}(n-3) \rceil + 1$ with $n = n_1 + n_2 - 2$.

2.2. $|V(P)| = 3$

Let $V(P) = \{v_1, v_2, v_3\}$ in counter-clockwise order. We first consider $P' = P \setminus \{v_1\}$ and suppose that $|V(P')| = 3$ where $V(P') = \{p, v_2, v_3\}$. Let $T_1 = \triangle p v_1 v_2$, $T_2 = \triangle p v_2 v_3$ and $T_3 = \triangle p v_3 v_1$ where $ch(P) = T_1 \cup T_2 \cup T_3$ and T_1 and T_3 are empty.

If T_2 is also empty, $H(P) = 3$ and $h(4) \leq \lceil \frac{7}{5}(4-3) \rceil + 1 = 3$ holds. If T_2 is not empty, we have $C_1 = C(\overline{p v_2}, S(T_2))$ and $C_2 = C(\overline{p v_3}, S(T_2))$ with $C_1 \neq C_2$. Then $T_1 \cup C_1$ or $T_3 \cup C_2$ is a cell in a convex decomposition of P since either C_1 is in $\gamma(v_1; v_2, p)$ or C_2 is in $\gamma(v_1; v_3, p)$. See Fig. 2. Since there is a convex decomposition of P' with at most $\lceil \frac{7}{5}(n'-3) \rceil + 1$ cells for $n' = |P'|$, $h(n) \leq (\lceil \frac{7}{5}(n'-3) \rceil + 1) + 2 - 1 \leq \lceil \frac{7}{5}(n'-2) \rceil + 1 = \lceil \frac{7}{5}(n-3) \rceil + 1$ with $n = n' + 1$.

We can now make the following assumption.

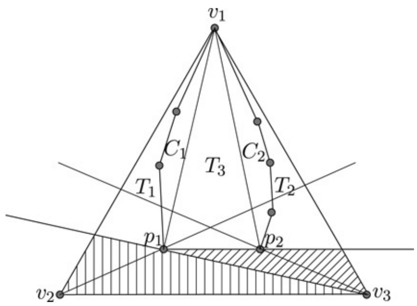


Fig. 3. A cell C_i in a convex decomposition of $S(T_i)$ for $i = 1, 2$.

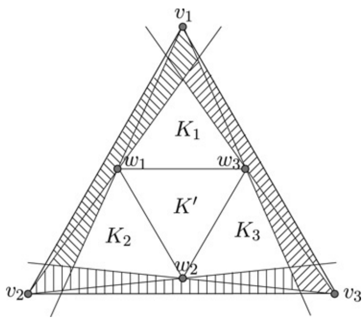


Fig. 4. The shaded portion is empty.

Assumption 1. For any v in $V(P)$, $ch(P \setminus \{v\})$ is not a triangle.

For $p_1 = \alpha(v_3; v_2, v_1)$, we suppose that $\gamma(p_1; v_2', v_3)$ is not empty and consider $p_2 = \alpha(p_1; v_3, v_2')$, where $\{v_2, v_3, p_1, p_2\}$ is an empty convex quadrilateral. Let $T_1 = \triangle v_1 v_2 p_1$, $T_2 = \triangle v_1 p_2 v_3$ and $T_3 = \triangle v_1 p_1 p_2$.

By Assumption 1, neither T_1 nor T_2 is empty. Let $C_i = C(\overline{v_1 p_i}, S(T_i))$ for $i = 1, 2$ where C_1 is in $\gamma(p_1; v_1, p_2')$ and C_2 is in $\gamma(p_2; v_1, p_1')$ as shown in Fig. 3. If T_3 is empty or not, $C_1 \cup T_3 \cup C_2$ is a cell in a convex decomposition of P or both $C_1 \cup C(\overline{v_1 p_1}, S(T_3))$ and $C_2 \cup C(\overline{v_1 p_2}, S(T_3))$ are cells in a convex decomposition of P, respectively. Thus, $h(n) \le (\lceil \frac{7}{5}(n_1 - 3)\rceil + 1) + (\lceil \frac{7}{5}(n_2 - 3)\rceil + 1) + (\lceil \frac{7}{5}(n_3 - 3)\rceil + 1) + 1 - 2 \le \lceil \frac{7}{5}(n_1 + n_2 + n_3 - 7)\rceil + 1 = \lceil \frac{7}{5}(n - 3)\rceil + 1$ where $n_i = |S(T_i)|$ for $i = 1, 2, 3$ and $n = n_1 + n_2 + n_3 - 4$.

For the remaining of the proof we can make the following stronger assumption.

Assumption 2. For any edge $\overline{v_i v_{i+1}}$ on the boundary of $ch(P)$, there exists an element w_i of P such that $\gamma(v_i; v_{i+1}, w_i) \cup \gamma(v_{i+1}; v_i, w_i)$ is empty for $i = 1, 2, 3$, where $v_4 = v_1$.

See Fig. 4 where the shaded portion is empty. We remark that $w_i \ne w_j$ for $i \ne j$ by Assumption 1. Let $K_1 = \triangle v_1 w_1 w_3$, $K_2 = \triangle v_2 w_2 w_1$, $K_3 = \triangle v_3 w_3 w_2$ and $K' = \triangle w_1 w_2 w_3$.

We continue under Assumption 2.

Case 1: K_i is empty for every $i = 1, 2, 3$.

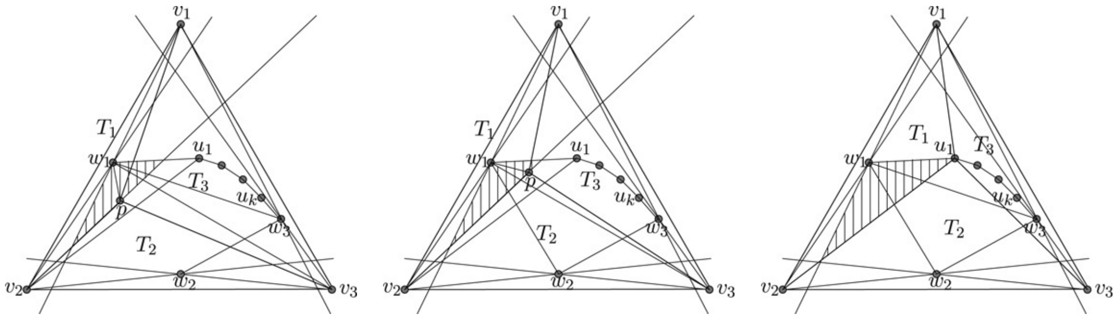
If K' is empty, i.e., $n = 6$, then $H(P) = 6$ and $h(6) \le \lceil \frac{7}{5}(6 - 3)\rceil + 1 = 6$ holds. Let $C_i = C(\overline{w_{i-1} w_i}, S(K'))$ for $i = 1, 2, 3$ where $w_0 = w_3$. If K' is not empty, since $C_i \ne C_j$ for $i \ne j$ and $K_i \cup C_i$ is a cell in a convex decomposition of P for each i, $h(n) \le (\lceil \frac{7}{5}(n' - 3)\rceil + 1) + 6 - 3 \le \lceil \frac{7}{5} n'\rceil + 1 = \lceil \frac{7}{5}(n - 3)\rceil + 1$ with $n = n' + 3$ for $n' = |S(K')|$.

Case 2: There exists a non-empty K_i, say K_1.

Let $V(S(K_1) \setminus \{v_1\}) = \{w_1, u_1, u_2, \ldots, u_k, w_3\}$ in clockwise order.

(A) $\triangle v_2 u_1 w_1$ is not empty.

Consider $p = \alpha(v_2; w_1, u_1)$ and let $T_1 = \triangle v_1 v_2 p$, $T_2 = \triangle v_2 v_3 p$ and $T_3 = \triangle v_3 v_1 p$, where $|S(T_1)| = 4$. There is a convex decomposition of $S(T_i)$ with at most $\lceil \frac{7}{5}(n_i - 3)\rceil + 1$ cells for $n_i = |S(T_i)|$, $i = 2, 3$.



(a) p is in $\gamma(v_3; w_2, w_1)$. (b) p is in $\gamma(v_3; w_1, u_1)$. (c) $\triangle v_2u_1w_1$ is empty.

Fig. 5.

If p is in $\gamma(v_3; w_2, w_1)$ as shown in Fig. 5a, we consider $C_2 = C(\overline{v_3p}, S(T_2))$, $C_3 = C(\overline{v_3p}, S(T_3))$ and $C'_3 = C(\overline{v_1p}, S(T_3))$. Since $\{v_1, v_3, w_1, p\}$ is in convex position, $\triangle v_1w_1p \cup C'_3$ is a cell in a convex decomposition of P. And C_3 is contained in $\gamma(p; v'_2, v_3)$ since, otherwise, $C_3 = T_3$ and C_3 would not be empty, that is, $C_2 \cup C_3$ is also a cell in a convex decomposition of P. Thus, $h(n) \le (\lceil \frac{7}{5}(n_2 - 3) \rceil + 1) + (\lceil \frac{7}{5}(n_3 - 3) \rceil + 1) + 3 - 2 \le \lceil \frac{7}{5}(n_2 + n_3 - 4) \rceil + 1 = \lceil \frac{7}{5}(n - 3) \rceil + 1$ with $n = n_2 + n_3 - 1$ $(*)$.

If p is in $\gamma(v_3; w_1, u_1)$ as shown in Fig. 5b, we let $C_2 = C(\overline{v_3p}, S(T_2))$ and $C'_2 = C(\overline{v_2p}, S(T_2))$ where $C_2 \ne C'_2$ by the existence of w_2. Since both $\triangle v_2pw_1 \cup C'_2$ and $C_2 \cup C(\overline{v_3p}, S(T_3))$ are cells in a convex decomposition of P for the same reason, we obtain the same inequalities as $(*)$.

(B) $\triangle v_2u_1w_1$ is empty.

Let $T_1 = \triangle v_1v_2u_1$, $T_2 = \triangle v_2v_3u_1$ and $T_3 = \triangle v_3v_1u_1$ with $|S(T_1)| = 4$. See Fig. 5c. Since $C(\overline{v_3u_1}, S(T_3))$ is in $\gamma(u_1; v'_2, v_3)$ and both $\triangle v_2u_1w_1 \cup C(\overline{v_2u_1}, S(T_2))$ and $C(\overline{v_3u_1}, S(T_2)) \cup C(\overline{v_3u_1}, S(T_3))$ are cells in a convex decomposition of P, $h(n) \le (\lceil \frac{7}{5}(n_2 - 3) \rceil + 1) + (\lceil \frac{7}{5}(n_3 - 3) \rceil + 1) + 3 - 2 \le \lceil \frac{7}{5}(n - 3) \rceil + 1$ with $n = n_2 + n_3 - 1$ for $n_i = |S(T_i)|$, $i = 2, 3$. $\square$

3. Final remark

There is still a substantial gap between the upper and lower bounds for $h(n)$. We believe that a more complicated approach may be able to prove that $h(n) \le \lceil \frac{4}{3}(n - 2) \rceil$ for any $n \ge 3$.

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