Discrete Mathematics 309 (2009) 2596–2607

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

The exact domination number of the generalized Petersen graphs

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ARTICLE INFO

Article history:
Received 22 May 2007
Received in revised form 31 March 2008
Accepted 1 April 2008
Available online 29 May 2008

Keywords:
Domination number
Generalized Petersen graphs

ABSTRACT

Let $G = (V, E)$ be a graph. A subset $S \subseteq V$ is a dominating set of G, if every vertex $u \in V - S$ is dominated by some vertex $v \in S$. The domination number, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. For the generalized Petersen graph $G(n)$, Behzad et al. [A. Behzad, M. Behzad, C.E. Praeger, On the domination number of the generalized Petersen graphs, Discrete Mathematics 308 (2008) 603–610] proved that $\gamma(G(n)) \le \lceil \frac{3n}{5} \rceil$ and conjectured that the upper bound $\lceil \frac{3n}{5} \rceil$ is the exact domination number. In this paper we prove this conjecture.

1. Introduction

Let $G = (V, E)$ be a finite, undirected, simple graph. For every $v \in V$, the *open neighborhood* of v is $N(v) = \{u \in V \mid (u, v) \in E\}$, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a subset $S \subseteq V$ is the set $N(S) = \cup_{x \in S} N(x)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The subgraph induced by S is denoted by $G[S]$.

Each vertex v of G *dominates* itself and every vertex adjacent to v, i.e., all vertices in its closed neighborhood. A subset of vertices of G is a *dominating set* if $N[S] = V$ (i.e., S dominates G), and every vertex of S is called a *dominator*. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set of minimum cardinality is called a $\gamma(G)$-set [2]. Let S be a dominating set, we say that a vertex u is *privately dominated* by a vertex $v \in S$ (respectively, a subset $S' \subseteq S$) if $N[u] \cap S = \{v\}$ (respectively, $N[u] \cap S \subseteq S'$). We use $Pr(S')$ to denote the set of vertices that are privately dominated by $S' \subseteq S$. For a more thorough treatment of domination parameters and for terminology not presented here, see [2,3].

For each odd integer $n = 2k + 1 \ge 3$, where k is a positive integer, the *generalized Petersen graph* $G(n)$ is the graph with vertex set $\mathcal{O} \cup \mathcal{I}$, where $\mathcal{O} = \{O_i \mid 1 \le i \le n\}$ and $\mathcal{I} = \{I_i \mid 1 \le i \le n\}$, and edge set $E_1 \cup E_2 \cup E_3$, where $E_1 = \{O_iO_{i+1} \mid 1 \le i \le n\}$, $E_2 = \{I_iI_{i+k} \mid 1 \le i \le n\}$ and $E_3 = \{O_iI_i \mid 1 \le i \le n\}$. Here all the subscripts are to be read as integers modulo n.

In [1], Behzad, Behzad and Praeger proposed two novel procedures that between them produce both upper and lower bounds on the domination number of the generalized Petersen graph $G(n)$. In particular, they obtained the following result.

Theorem 1 ([1]). *For each odd integer $n \ge 3$, $\gamma(G(n)) \le \lceil \frac{3n}{5} \rceil$, and moreover*

$$\gamma(G(n)) \le \gamma(G(n+2)) \le \gamma(G(n)) + 2.$$

Behzad, Behzad and Praeger [1] also conjectured that the upper bound $\lceil \frac{3n}{5} \rceil$ in Theorem 1 is the exact domination number of the generalized Petersen graph $G(n)$.

Our aim in this paper is to prove this conjecture.

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doi:10.1016/j.disc.2008.04.026

2. Main results

Motivated by Behzad, Behzad and Praeger's method, we first give an algorithm which constructs from $G(n)$ a smaller generalized Petersen graph $G(n-10)$.

Algorithm 1.

INPUT: the graph $G(n) = (\mathcal{O} \cup \mathcal{I}, E_1 \cup E_2 \cup E_3)$ with $n = 2k+1 \geq 17$.

OUTPUT: a graph G'' with $2(n-10)$ vertices.

step 1. Choose i such that $1 \leq i \leq k$, delete the two subsets of vertices

$$\{O_j, I_j \mid i \leq j \leq i+5\}, \qquad \{O_j, I_j \mid i+k \leq j \leq i+k+5\}$$

along with their 39 incident edges and denote the resulting graph by G'.

step 2. Add four new vertices O'_i, I'_i, O'_{i+k-5}, I'_{i+k-5}, and define the graph G'' to have vertex set $V(G'') = V(G') \cup \{O'_i, I'_i, O'_{i+k-5}, I'_{i+k-5}\}$ and edge set

$$E(G'') = E(G') \cup \{O_{i-1}O'_i, O'_iO_{i+6}, O'_iI'_i, I'_iI'_{i+k-5}, I'_iI_{i+k+6}, I_{i-1}I'_{i+k-5}, O_{i+k-1}O'_{i+k-5}, O'_{i+k-5}O_{i+k+6}, O'_{i+k-5}I'_{i+k-5}\}.$$

Return G''.

Fig. 1 gives an illustration for Algorithm 1 when $i = 1$. The deleted part of the graph in Fig. 1 can be re-depicted in Fig. 2.

Lemma 2. *For each odd integer $n \geq 17$, the graph G'' returned by Algorithm 1 is isomorphic to $G(n-10)$.*

Proof. It is clear that $|V(G'')| = 2(n-10)$ and $|E(G'')| = 3(n-10)$. Relabel the vertices of G'' as follows. For the chosen index i in step 1, set

$$U_i := O'_i, \qquad U_{i+k-5} := O'_{i+k-5}, \qquad W_i := I'_i, \qquad W_{i+k-5} := I'_{i+k-5}$$

for each j such that $1 \leq j < i$, set

$$U_j := O_j, \qquad W_j := I_j$$

for each j such that $i+6 \leq j < i+k$, set

$$U_{j-5} := O_j, \qquad W_{j-5} := I_j$$

for each j such that $i+k+6 \leq j \leq 2k+1 = n$, set

$$U_{j-10} := O_j, \qquad W_{j-10} := I_j.$$

Then we get the sets $\mathcal{U} = \{U_j \mid 1 \leq j \leq n-10\}$ and $\mathcal{W} = \{U_j \mid 1 \leq j \leq n-10\}$ such that $V(G'') = \mathcal{U} \cup \mathcal{W}$. Note that $V(G(n-10))$ was defined to be $\mathcal{O} \cup \mathcal{I}$ with $|\mathcal{O}| = |\mathcal{I}| = n-10$, and the bijection $f : \mathcal{O} \cup \mathcal{I} \to \mathcal{U} \cup \mathcal{W}$, defined by $f(O_j) = U_j$ and $f(I_j) = W_j$ for $1 \leq j \leq n-10$, maintains adjacency and nonadjacency, the result follows immediately. ■

For a small odd integer n, it may not be too hard to count $\gamma(G(n))$ (for example, in [1] the authors showed that $\gamma(G(3)) = 2$, $\gamma(G(5)) = 3$, $\gamma(G(7)) = 5$). The following lemma shows that $\gamma(G(n)) = \lceil \frac{3n}{5} \rceil$ is true for a small odd integer n.

Lemma 3. *Let n be an odd integer such that $3 \leq n \leq 15$, then $\gamma(G(n)) = \lceil \frac{3n}{5} \rceil$.*

Proof. From the discussion above, we still need to consider the remaining cases $n = 9, 11, 13$ and 15. We only give the argument for case $n = 15$, since arguments for other cases are similar. Consider the generalized Petersen graph $G(15)$ with vertex set $\mathcal{O} \cup \mathcal{I}$, where $\mathcal{O} = \{O_i \mid 1 \leq i \leq 15\}$ and $\mathcal{I} = \{I_i \mid 1 \leq i \leq 15\}$ (see Fig. 3), let S be a $\gamma(G(15))$-set of $G(15)$.

Note that $G(15)$ is 3-regular, each vertex in S dominates at most four vertices (including itself), we have $4|S| \geq |V(G(15))| = 30$, which implies that $|S| \geq 8$ ($|S|$ is an integer). From Theorem 1, we have $|S| = \gamma(G(15)) \leq \lceil \frac{45}{5} \rceil = 9$. Next we show that $|S| > 8$, or equivalently that no 8 vertices of $G(15)$ form a dominating set. Suppose on the contrary that there is a dominating set D of $G(15)$ with $|D| = 8$. Let $D_{\mathcal{O}} = D \cap \mathcal{O}$ and $D_{\mathcal{I}} = D \cap \mathcal{I}$, then $|D_{\mathcal{O}}| + |D_{\mathcal{I}}| = 8$. We use the integer pair (i, j), where $i, j \in \{0, 1, \ldots, 8\}$ and $i + j = 8$, to denote the situation that $|D_{\mathcal{O}}| = i$ and $|D_{\mathcal{I}}| = j$. We show that none of these situations would occur. First, note that $D_{\mathcal{O}}$ dominates at most $3i$ vertices of the outer cycle $G[\mathcal{O}]$, there are at least $15 - 3i$ vertices of $\mathcal{O}$ that need to be dominated by $D_{\mathcal{I}}$, and each of them requires a dominator from $\mathcal{I}$ to dominate it, then we must have $|D_{\mathcal{I}}| = 8 - i \geq 15 - 3i$, which implies $i \geq 4$ (since i is an integer). By symmetry, we have $j \geq 4$, which means that $i \leq 4$. Thus, the situation (i, j), $i \in \{0, 1, 2, 3, 5, 6, 7, 8\}$, does not occur. The situation $(4, 4)$ is not possible to occur, since no 4 vertices of $\mathcal{O}$ together with 4 vertices of $\mathcal{I}$ can form a dominating set (this fact can be found by inspection, see Fig. 3). ■

Next we give an upper bound for $\gamma(G(n))$ in terms of $\gamma(G(n+10))$, upon which our main result is based. The proof is just a clumsy and boring case analysis.

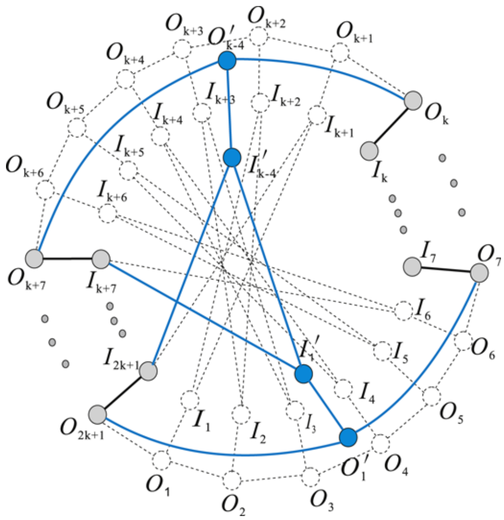

Fig. 1. Algorithm 1 for $i = 1$.

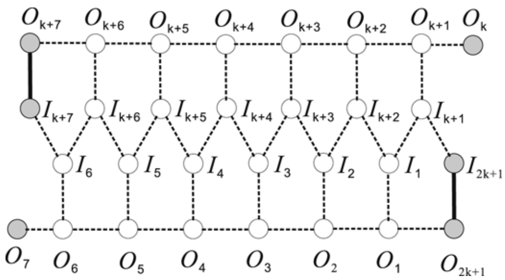

Fig. 2.

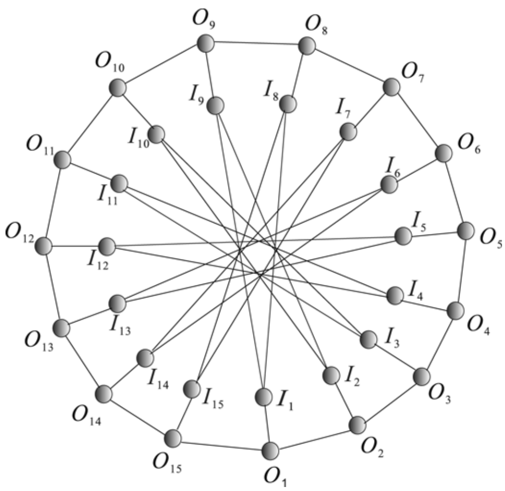

Fig. 3. The generalized Petersen graph $G(15)$.

Lemma 4. *Let n be an odd integer such that $n = 2k + 1 \geq 3$, then $\gamma(G(n)) \leq \gamma(G(n + 10)) - 6$.*

Proof. From Lemma 3, the result holds for $n = 3$ and $n = 5$. Suppose that $n = 2k + 1 \geq 7$. To keep the notation in line with that of Algorithm 1, we may further assume that $n = 2k + 1 \geq 17$, and show $\gamma(G(n - 10)) \leq \gamma(G(n)) - 6$. Let $G = G(n) = (\mathcal{O} \cup \mathcal{I}, E_1 \cup E_2 \cup E_3)$ be defined as before and $S \subseteq V(G)$ be a $\gamma(G)$-set.

Let G'' be the graph returned by Algorithm 1 with the index $i = 1$, then $G'' \cong G(n-10)$. We will identify $V(G(n-10))$ with $V(G'')$ such that $V(G(n-10)) = (\mathcal{O} \cup \mathcal{I} \setminus T) \cup T'$, where $T' = \{O'_1, I'_1, O'_{k-4}, I'_{k-4}\}$ and

$$T = \{O_j, I_j \mid 1 \le j \le 6\} \cup \{O_j, I_j \mid k+1 \le j \le k+6\}.$$

Let G' be the subgraph of G spanned by $V(G) \setminus T$, then G' is also a subgraph of $V(G(n-10))$, and the subset $S' := S \cap V(G')$ dominates all vertices in $V(G')$, except possibly vertices in $R := \{O_{2k+1}, O_7, O_k, O_{k+7}, I_{k+7}, I_{2k+1}\}$. Denote $Q := \{\{O_{2k+1}, O_7\}, \{O_k, O_{k+7}\}, \{I_{k+7}\}, \{I_{2k+1}\}\}$, $A := \{O_1, O_6, I_6, O_{k+1}, I_{k+1}, O_{k+6}\}$. We consider the following several cases.

Case 1. $|S \cap T| \ge 10$.

Since S' dominates all vertices, except possibly vertices in R in $V(G')$, and T' dominates $R \cup T'$ (see Fig. 1), $S' \cup T'$ forms a dominating set of G''. Thus, $\gamma(G(n-10)) = \gamma(G'') \le |S' \cup T'| \le \gamma(G(n)) - 6$, the result follows.

Case 2. $|S \cap T| = 9$.

If there exists at least one element of Q, say X, such that $X \cap Pr(S \cap T) = \emptyset$ (i.e., X is dominated by S' in G), let $x \in T'$ be adjacent to some vertex of X in G''. Then S' dominates all vertices, except possibly vertices in $R \setminus X$ in $V(G')$, and $T' - \{x\}$ dominates $(R \setminus X) \cup T'$ (see Fig. 1). Consequently, $S' \cup (T' - \{x\})$ dominates G'', and we have $\gamma(G(n-10)) = \gamma(G'') \le |S' \cup (T' - \{x\})| = \gamma(G(n)) - 6$. Assume now that $X \cap Pr(S \cap T) \ne \emptyset$ for each $X \in Q$. From now on, in each figure a vertex $\otimes$ indicates a dominator of S and $\oslash$ a vertex that is already dominated by some dominator.

Subcase 2.1. $R \subseteq Pr(S \cap T)$. That is, each vertex of R is privately dominated by some dominator of $S \cap T$, (for example, O_{2k+1} is privately dominated by O_1, O_7 is privately dominated by O_6, I_{2k+1} is privately dominated by I_{k+1}, and so on. see Fig. 1). Then $A \subseteq S$. Denote $Z := T \setminus N[A]$ (i.e. vertices contained in the closed dashed curve in Fig. 4(1)). Note that $G[Z]$ contains two 5-cycles which share a common edge I_3I_{k+4} (see Fig. 4(1)), to dominate the eight vertices on the two 5-cycles, S must contain at least either three vertices (if and only if the three dominators are all on the two 5-cycles) or four vertices (when at least one of the four dominators is not on the two 5-cycles), if it is the former situation, both I_5 and I_{k+2} are at distance two from the two 5-cycles and therefore need to be dominated by other dominators. Thus the vertices in Z cannot be dominated by three or fewer vertices of $T \setminus A = N[Z]$, which contradicts the assumption that $|S \cap T| = 9$.

Throughout the proof, we will always use 'Z' to denote the subset of vertices contained in the closed dashed curve in each corresponding figure. For the convenience of description, when we say that Z cannot be dominated by l or fewer vertices of $N[Z]$, we will omit the formal explanation (since one can enumerate all subsets of cardinality of l of $N[Z]$ and verify that none of them can dominate Z).

Subcase 2.2. $O_k \notin Pr(S \cap T)$. Let $S'' = S' \cup \{O'_1, I'_{k-4}, I_{k+7}\}$, then S'' dominates G'' (see Fig. 1) and we have $\gamma(G(n-10)) = \gamma(G'') \le |S' \cup \{O'_1, I'_{k-4}, I_{k+7}\}| = \gamma(G(n)) - 6$. The result follows.

Subcase 2.3. $O_7 \notin Pr(S \cap T)$. Let $S'' = S' \cup \{O'_{k-4}, I'_1, I_{2k+1}\}$, then S'' dominates G'' (see Fig. 1) and we have $\gamma(G(n-10)) = \gamma(G'') \le |S' \cup \{O'_{k-4}, I'_1, I_{2k+1}\}| = \gamma(G(n)) - 6$. The result follows.

Next assume that $O_k, O_7 \in Pr(S \cap T)$. We have

Subcase 2.4. $R \setminus \{O_{k+7}\} \subseteq Pr(S \cap T)$. Since I_{k+7} is privately dominated by $I_6 \in S \cap T$, we have $O_{k+7} \notin S$ and O_{k+6} can only be dominated by some vertex in $S \cap T$. However vertices in $Z := T \setminus N[A \setminus \{O_{k+6}\}]$ cannot be dominated by four or fewer vertices of $N[Z]$ (see Fig. 4(2)). So this case does not happen.

Subcase 2.5. $R \setminus \{O_{2k+1}\} \subseteq Pr(S \cap T)$. Analogously as the above Subcase 2.4 by symmetry.

Subcase 2.6. $R \setminus \{O_{2k+1}, O_{k+7}\} \subseteq Pr(S \cap T)$. As Subcase 2.4, $O_{2k+1}, O_{k+7} \notin S$ and O_1, O_{k+6} can only be dominated by some vertices in $S \cap T$. The vertices in $Z := T \setminus N[A \setminus \{O_1, O_{k+6}\}]$ cannot be dominated by five or fewer vertices of $N[Z]$ (see Fig. 4(3)). Thus this case does not occur.

Case 3. $|S \cap T| = 8$.

If for each element $X \in Q$, $X \cap Pr(S \cap T) = \emptyset$, let y and y' be any two vertices of T'. If there exists exactly one element $X \in Q$, such that $X \cap Pr(S \cap T) \ne \emptyset$, let $y \in T'$ be adjacent to some vertex of X in G'' and $y' \in T'$ be not adjacent to y in G''. Then $S' \cup \{y, y'\}$ dominates G'', and we have $\gamma(G(n-10)) = \gamma(G'') \le |S' \cup \{y, y'\}| = \gamma(G(n)) - 6$.

Assume now that $|\{X \mid X \in Q, X \cap Pr(S \cap T) \ne \emptyset\}| \ge 2$. Consider the following subcases.

Subcase 3.1. There are exactly two elements $X, Y \in Q$ such that $X \cap Pr(S \cap T) \ne \emptyset$ and $Y \cap Pr(S \cap T) \ne \emptyset$. If $X \cup Y = \{I_{2k+1}\} \cup \{I_{k+7}\}$, let $S'' := S' \cup \{I'_1, I'_{k-4}\}$. If $X \cup Y = \{O_{2k+1}, O_7\} \cup \{O_k, O_{k+7}\}$, let $S'' := S' \cup \{O'_1, O'_{k-4}\}$. If $X \cup Y = \{I_{2k+1}\} \cup \{O_{2k+1}, O_7\}$, let $S'' := S' \cup \{O'_1, I'_{k-4}\}$. If $X \cup Y = \{I_{k+7}\} \cup \{O_k, O_{k+7}\}$, let $S'' := S' \cup \{I'_1, O'_{k-4}\}$. Then S'' dominates G'', and we have $\gamma(G(n-10)) = \gamma(G'') \le |S''| = \gamma(G(n)) - 6$. By symmetry we need only consider $X \cup Y = \{O_{2k+1}, O_7\} \cup \{I_{k+7}\}$.

Subcase 3.1.1. $\{O_{2k+1}, O_7\} \cup \{I_{k+7}\} \subseteq Pr(S \cap T)$. If $O_k \in S$, then $S' \cup \{O'_1, I'_1\}$ dominates G'', the result follows. Suppose next $O_k \notin S$. Since O_{2k+1}, I_{k+7} are privately dominated by $S \cap T$, $I_{2k+1}, O_{k+7} \notin S$ and I_{k+1}, O_{k+6} are dominated by $S \cap T$ in G. Then vertices in $T \setminus N[\{O_1, O_6, I_6\}]$ cannot be dominated by five or fewer vertices of $T \setminus \{O_1, O_6, I_6\}$ (see Fig. 4(4)). This case does not happen.

Subcase 3.1.2. $O_{2k+1} \notin Pr(S \cap T)$. If $O_k \in S$, let $S'' = S' \cup \{O'_1, I'_1\}$; If $I_{2k+1} \in S$, let $S'' = S' \cup \{O'_1, O_{k+7}\}$, in both cases S'' dominates G'', the result follows. Suppose that $O_k \notin S$ and $I_{2k+1} \notin S$, then the Z region cannot be dominated by six or fewer vertices (see Fig. 4(5)). This case does not happen.

Subcase 3.1.3. $O_7 \notin Pr(S \cap T)$. If $O_k \in S$, then $S' \cup \{O'_1, I'_1\}$ dominates G'', the result follows. Suppose that $O_k \notin S$, then the Z region cannot be dominated by six or fewer vertices (see Fig. 5(1)). This case does not happen.

Subcase 3.2. There are exactly three elements $X, Y, H \in Q$ such that each of them has a nonempty intersection with $Pr(S \cap T)$. We first claim that $|R \cap Pr(S \cap T)| \le 4$, since vertices in $T \setminus N[A]$ (i.e. vertices contained in the closed dashed curve in Fig. 4(1)) cannot be dominated by three or fewer vertices from $T \setminus A$.

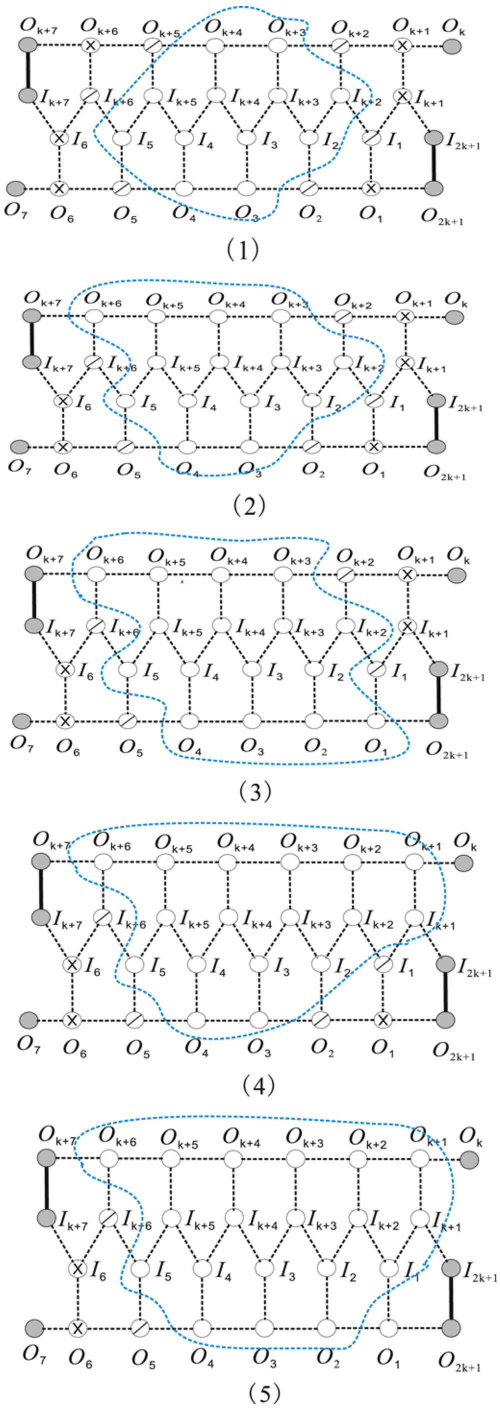

Fig. 4.

By symmetry, we consider only the following two subcases.

Subcase 3.2.1. $X \cup Y \cup H = \{I_{2k+1}\} \cup \{I_{k+7}\} \cup \{O_{2k+1}, O_7\}$.

If $R \cap Pr(S \cap T) = \{I_{2k+1}\} \cup \{I_{k+7}\} \cup \{O_{2k+1}, O_7\}$, then the Z region cannot be dominated by four or fewer vertices (see Fig. 5(2)).

If $R \cap Pr(S \cap T) = \{I_{2k+1}\} \cup \{I_{k+7}\} \cup \{O_{2k+1}\}$, if $O_k \in S$, then $S'' = S' \cup \{O_{2k+1}, I'_1\}$ dominates G'', the result follows. Suppose next that $O_k \notin S$, then the Z region cannot be dominated by five or fewer vertices (see Fig. 5(3)).

If $R \cap Pr(S \cap T) = \{I_{2k+1}\} \cup \{I_{k+7}\} \cup \{O_7\}$, then the Z region cannot be dominated by five or fewer vertices (see Fig. 5(4)).

Subcase 3.2.2. $X \cup Y \cup H = \{O_{k+7}, O_k\} \cup \{I_{k+7}\} \cup \{O_{2k+1}, O_7\}$.

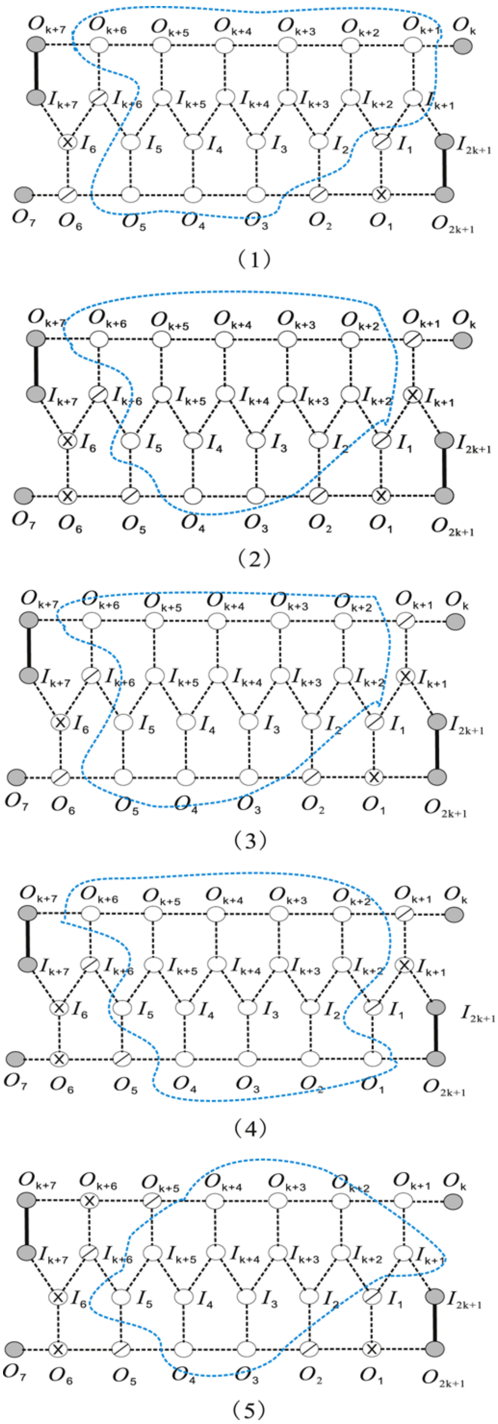

Fig. 5.

Since $|R \cap Pr(S \cap T)| \leq 4$, we look upon the following subcases:
If $|R \cap Pr(S \cap T)| = 4$, we have four possibilities:
(1) $(X \cup Y \cup H) \setminus \{O_k\} \subseteq Pr(S \cap T)$ (see Fig. 5(5));
(2) $(X \cup Y \cup H) \setminus \{O_{k+7}\} \subseteq Pr(S \cap T)$ (see Fig. 6(1));
(3) $(X \cup Y \cup H) \setminus \{O_7\} \subseteq Pr(S \cap T)$ (see Fig. 6(2));
(4) $(X \cup Y \cup H) \setminus \{O_{2k+1}\} \subseteq Pr(S \cap T)$ (see Fig. 6(3)).
In each situation, the Z region cannot be dominated by four or fewer vertices.
If $|R \cap Pr(S \cap T)| = 3$, we have

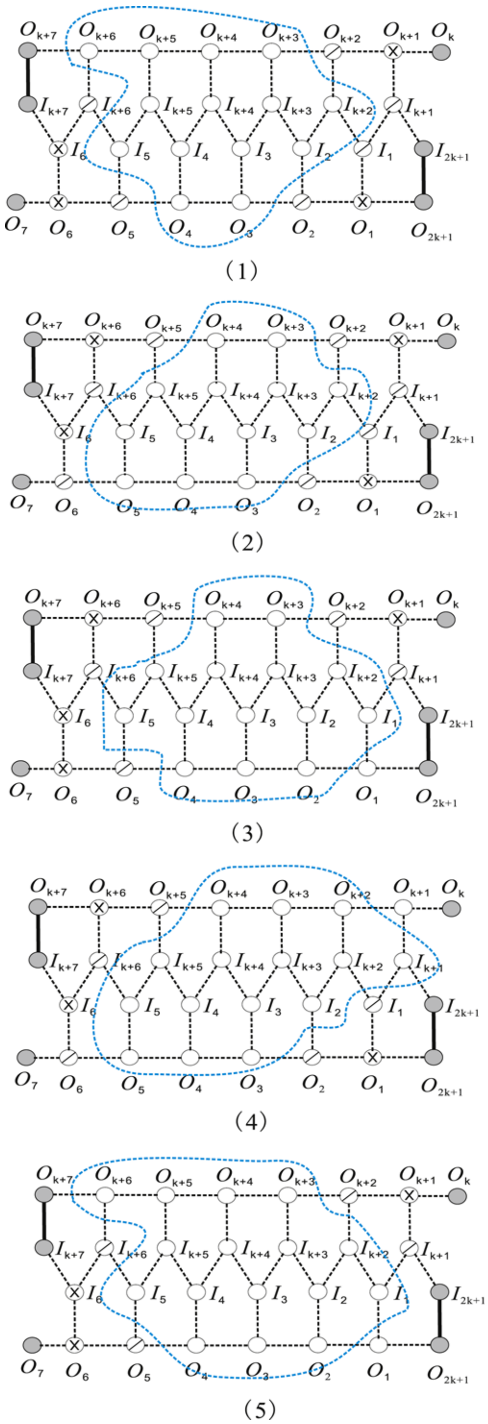

Fig. 6.

(1) $(X \cup Y \cup H) \setminus \{O_7, O_k\} \subseteq Pr(S \cap T)$ (see Fig. 6(4));
(2) $(X \cup Y \cup H) \setminus \{O_{2k+1}, O_{k+7}\} \subseteq Pr(S \cap T)$ (see Fig. 6(5));
(3) $(X \cup Y \cup H) \setminus \{O_{k+7}, O_7\} \subseteq Pr(S \cap T)$ (see Fig. 7(1)).
In each of above three circumstances, the Z region cannot be dominated by five or fewer vertices.

(4) $(X \cup Y \cup Z) \setminus \{O_{2k+1}, O_k\} \subseteq Pr(S \cap T)$. If $I_{2k+1} \in S$, let $S'' = S' \cup \{O'_1, O_{k+7}\}$, then S'' dominates G'' and the result follows. Assume that $I_{2k+1} \notin S$, the Z region cannot be dominated by five or fewer vertices (see Fig. 7(2)).

Subcase 3.3. Every element of Q has a nonempty intersection with $Pr(S \cap T)$. So $I_{k+7}, I_{2k+1} \in S$. If one of O_{k+7}, O_{2k+1}, say O_{k+7}, does not lie in $Pr(S \cap T)$, then $O_{k+7} \notin S$. Thus O_{k+6} and O_{k+5} must be dominated by some vertex of $S \cap T$. However,

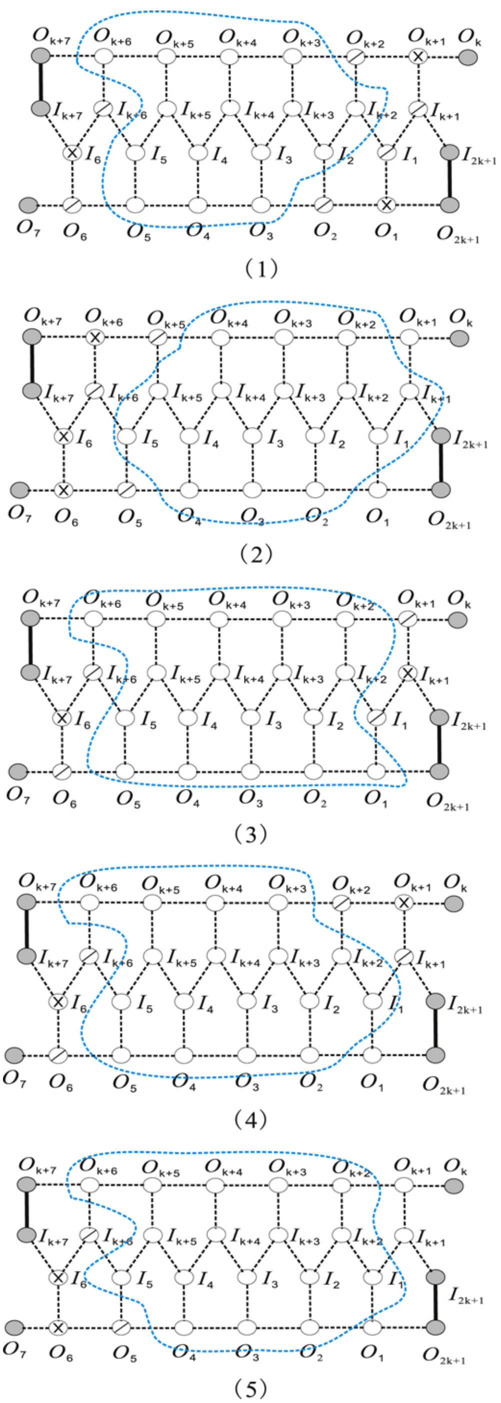

Fig. 7.

$Z = (T \setminus N[A]) \cup \{O_{k+6}, O_{k+5}\}$ cannot be dominated by four or fewer vertices (see Fig. 4(1)). Which is a contradiction. If both O_{k+7} and O_{2k+1} lie in $Pr(S \cap T)$, no matter whether O_7 and/or O_k lie in $Pr(S \cap T)$ or not, it may lead to a contradiction.

Case 4. $|S \cap T| = 7$.

We first observe that $|R \cap Pr(S \cap T)| \geq 3$ does not occur. Then $|R \cap Pr(S \cap T)| = 2, 1, 0$.

Subcase 4.1. $|R \cap Pr(S \cap T)| = 2$

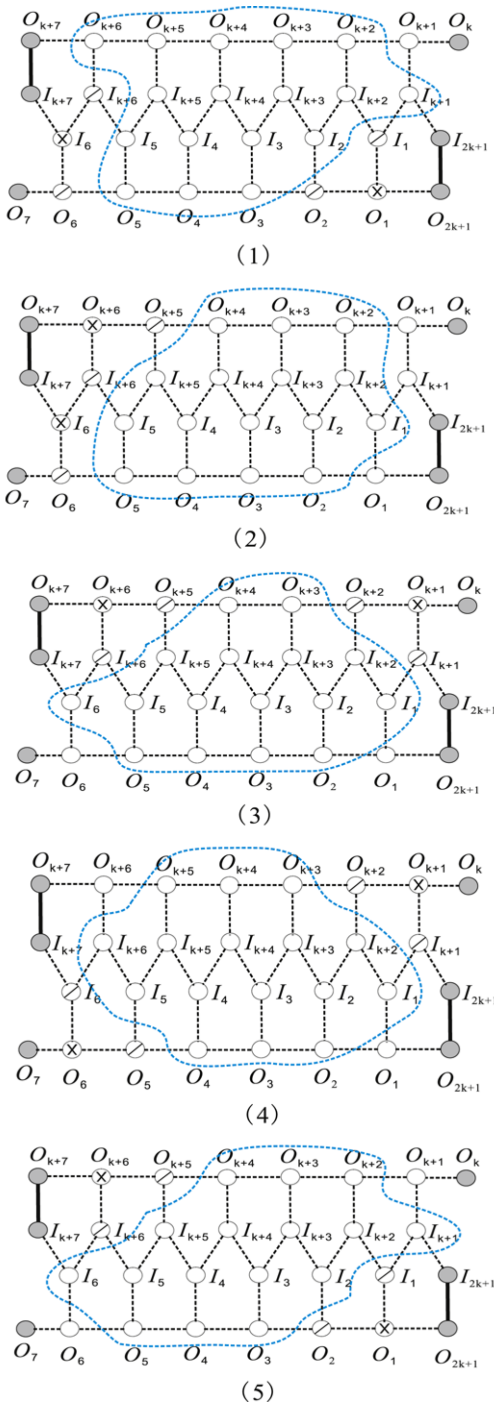

Fig. 8.

If at least one of I_{k+7} and I_{2k+1}, say I_{k+7}, lies in $Pr(S \cap T)$, by symmetry we consider only the following five possibilities:
(1) $R \cap Pr(S \cap T) = \{I_{k+7}, I_{2k+1}\}$ (see Fig. 7(3));
(2) $R \cap Pr(S \cap T) = \{I_{k+7}, O_k\}$ (see Fig. 7(4));
(3) $R \cap Pr(S \cap T) = \{I_{k+7}, O_7\}$ (see Fig. 7(5));
(4) $R \cap Pr(S \cap T) = \{I_{k+7}, O_{2k+1}\}$ (see Fig. 8(1)).
In each of above four circumstances, the Z region cannot be dominated by five or fewer vertices.
(5) $R \cap Pr(S \cap T) = \{I_{k+7}, O_{k+7}\}$. If $O_k, I_{2k+1} \in S$, and $\{O_{2k+1}, O_7\} \cap S \neq \emptyset$, $S'' = S' \cup \{I_{k+7}\}$ dominates G'', and the result follows. Otherwise, each of the three conditions $O_k \notin S$, $I_{2k+1} \notin S$ and $\{O_{2k+1}, O_7\} \cap S = \emptyset$ may lead to a contradiction. Let Z be the

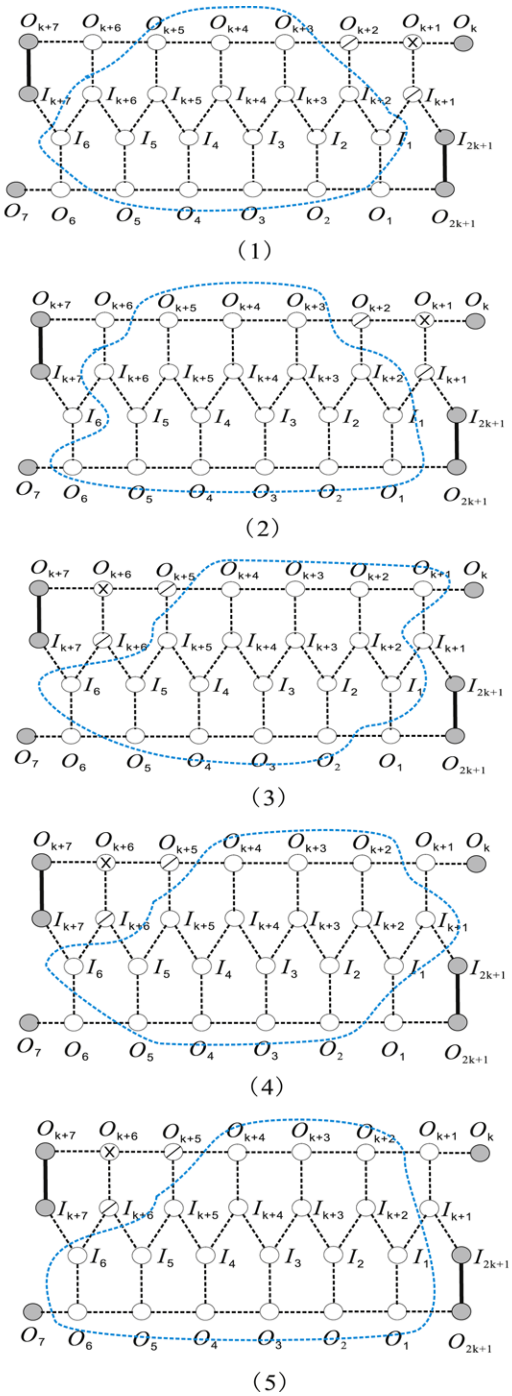

Fig. 9.

vertices contained in the closed dashed curve in Fig. 8(2). Then $Z \cup \{O_{k+1}\}$ (when $O_k \notin S$), $Z \cup \{I_{k+1}\}$ (when $I_{2k+1} \notin S$) or $Z \cup \{O_1\}$ (when $\{O_{2k+1}, O_7\} \cap S = \emptyset$) cannot be dominated by five or fewer vertices.

If both I_{k+7} and I_{2k+1} are not in $Pr(S \cap T)$, by symmetry we consider only the following four possibilities:

(1) $R \cap Pr(S \cap T) = \{O_{k+7}, O_k\}$ (see Fig. 8(3));

(2) $R \cap Pr(S \cap T) = \{O_k, O_7\}$ (see Fig. 8(4));

(3) $R \cap Pr(S \cap T) = \{O_{k+7}, O_{2k+1}\}$ (see Fig. 8(5)).

In each of above four circumstances, the Z region cannot be dominated by five or fewer vertices.

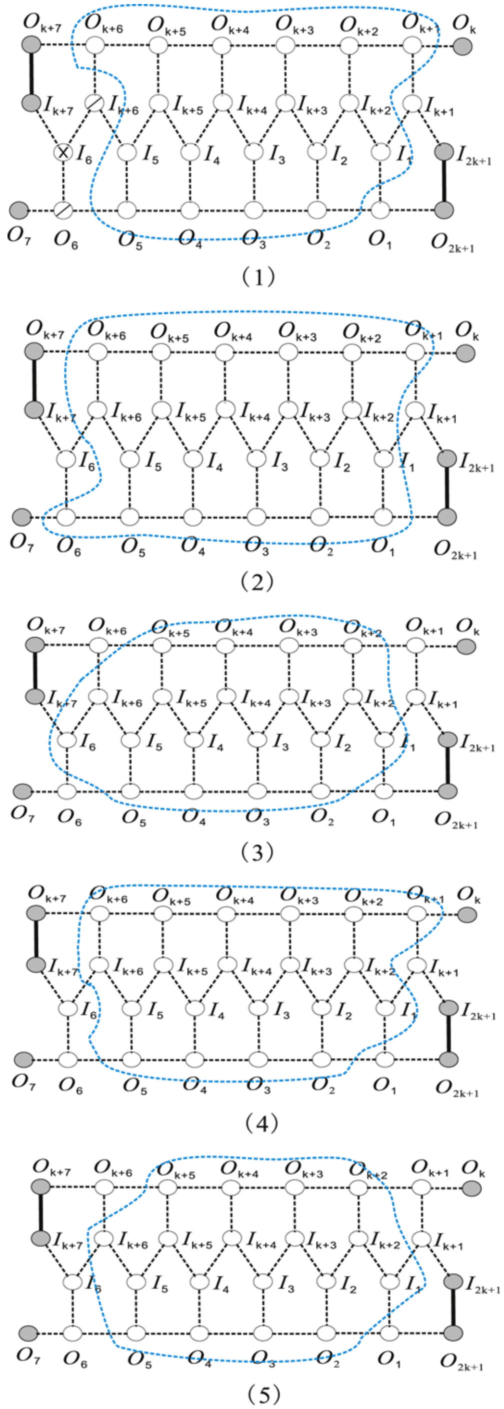

Fig. 10.

(4) $R \cap Pr(S \cap T) = \{O_{k+7}, O_7\}$. This case does not occur, since O_{k+7} and O_7 are privately dominated by O_{k+6} and O_6, respectively, we have $I_{k+7}, I_7 \notin S$, note that I_{k+7} has exactly three neighbors O_{k+7}, I_7, I_6, so I_{k+7} must be dominated by I_6 in G, then I_{k+7} is also privately dominated by $S \cap T$, a contradiction.

Subcase 4.2. $|R \cap Pr(S \cap T)| = 1$.

By symmetry we consider only the following three possibilities:

(1) $O_k \in Pr(S \cap T)$. If $S \cap \{I_{k+7}\} \neq \emptyset$ and $S \cap \{O_{2k+1}, O_7\} \neq \emptyset$, then $S'' = S' \cup \{O'_{k-4}\}$ dominates G'' and the result follows. Otherwise either $S \cap \{I_{k+7}\} = \emptyset$ (see Fig. 9(1)) or $S \cap \{O_{2k+1}, O_7\} = \emptyset$ (see Fig. 9(2)) will mean that the Z region cannot be dominated by six or less vertices.

(2) $O_{k+7} \in Pr(S \cap T)$. If each of the three subsets of $\{O_k\}$, $\{I_{2k+1}\}$ and $\{O_{2k+1}, O_7\}$ has a nonempty intersection with S, $S'' = S' \cup \{I_{k+7}\}$ dominates G'' and the result follows. Otherwise, each of the three conditions $\{O_k\} \cap S = \emptyset$ (see Fig. 9(3)), $\{I_{2k+1}\} \cap S = \emptyset$ (see Fig. 9(4)) and $\{O_{2k+1}, O_7\} \cap S = \emptyset$ (see Fig. 9(5)) will lead to a contradiction.

(3) $I_{k+7} \in Pr(S \cap T)$. If $O_k \in S$, then $S'' = S' \cup \{I'_1\}$ dominates G'' and the result follows. Otherwise, the condition $O_k \notin S$ will mean the that Z region cannot be dominated by six or fewer vertices (see Fig. 10(1)).

Subcase 4.3. $|R \cap Pr(S \cap T)| = 0$. If $S \cap \{O_{k+7}, O_k\} \neq \emptyset$ (respectively, $S \cap \{O_{2k+1}, O_7\} \neq \emptyset$) let $S'' = S' \cup \{I'_1\}$ (respectively, $S'' = S' \cup \{I'_{k-4}\}$). Then S'' dominates G'' and the result follows.

Suppose that $S \cap \{O_{k+7}, O_k\} = \emptyset$ and $S \cap \{O_{2k+1}, O_7\} = \emptyset$. Then it may reach a contradiction no matter which one of the following four possibilities occurs: (1) $I_{k+7} \in S$ and $I_{2k+1} \in S$; (2) $I_{k+7} \in S$ and $I_{2k+1} \notin S$; (3) $I_{k+7} \notin S$ and $I_{2k+1} \in S$; (4) $I_{k+7} \notin S$ and $I_{2k+1} \notin S$. (The Z region in Fig. 10(2) cannot be dominated by seven or fewer vertices.)

Case 5. $|S \cap T| = 6$.

If every element in Q has a nonempty intersection with S, then $S'' = S'$ dominates G'', and the result follows. Otherwise, either $\{I_{k+7}\} \cap Pr(S \cap T) = \emptyset$ (see Fig. 10(3)) or $\{O_{k+7}, O_k\} \cap Pr(S \cap T) = \emptyset$ (see Fig. 10(4)) may lead to a contradiction, since in any case the Z region cannot be dominated by six or fewer vertices.

Case 5. $|S \cap T| \leq 5$.

This case does not happen, since even if all vertices of R lie in S, the Z region (see Fig. 10(5)) cannot be dominated by five or fewer vertices. ■

Theorem 5. *Let $G(n)$ be a generalized Petersen graph with $n = 2k + 1 \geq 3$, then $\gamma(G(n)) = \lceil \frac{3n}{5} \rceil$.*

Proof. By contradiction. Define a graph class $\Omega = \{G(n) \mid \gamma(G(n)) < \lceil \frac{3n}{5} \rceil\}$. If $\Omega = \emptyset$, we are done. Assume that $\Omega \neq \emptyset$. Let $G(n) \in \Omega$ be the graph with minimum order $2n$. Then by Lemma 3 we have $n \geq 17$, and $\gamma(G(j)) = \lceil \frac{3j}{5} \rceil$ for each odd integer $j < n$.

Consider the graph $G(n - 10)$, by Lemma 4 we have

$$\begin{aligned}\gamma(G(n-10)) &\leq \gamma(G(n)) - 6 \\ &< \left\lceil \frac{3n}{5} \right\rceil - 6 \\ &= \left\lceil \frac{3(n-10)}{5} \right\rceil.\end{aligned}$$

Hence we get a graph $G(n - 10) \in \Omega$ with smaller order, which contradicts the choice of $G(n)$. Therefore we conclude that $\Omega = \emptyset$, and the result holds. ■

Acknowledgements

Research was partially supported by the Hong Kong CERG grant PolyU5457/06H, the National Nature Science Foundation of China (Nos. 10571117, 60773078), the ShuGuang Plan of Shanghai Education Development Foundation and Shanghai Education Committee (No. 06SG42) and the Postgraduate Innovation Fund of Shanghai University.

The authors would also like to thank the referees for their valuable suggestions to improve this paper.

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