Asymptotic Behavior of Solutions of Emden–Fowler Difference Equations with Oscillating Coefficients

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It is known that if $\sum^{\infty} j\,|p_j| < \infty$ then the Emden–Fowler difference equation (A) $\Delta^2 y_{n-1} = p_n y_n^{\gamma}$ ($\gamma > 0$) has a positive solution $\{y_n\}$, defined for n sufficiently large, such that $\lim_{n\to\infty} y_n = c > 0$, while if $\sum^{\infty} j^{\gamma}\,|p_j| < \infty$ then (A) has a positive solution $\{y_n\}$, defined for n sufficiently large, such that $\lim_{n\to\infty} \Delta y_n = c > 0$. Here it is shown that these conclusions hold if the series converge (perhaps conditionally) and satisfy secondary conditions which do not imply absolute convergence. Estimates of $\{y_n\}$ and $\{\Delta y_n\}$ as $n \to \infty$ are also given. Moreover, γ can be any real number other than 0 or 1. © 1993 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

We consider the Emden–Fowler difference equation

$$\Delta^2 y_{n-1} = p_n y_n^{\gamma}, \tag{1.1}$$

where $\{p_n\}_1^{\infty}$ is a sequence of real numbers, γ ($\neq 0, 1$) is a real number, and Δ is the forward difference operator with unit spacing; i.e., $\Delta u_j = u_{j+1} - u_j$. We say that a sequence $\{y_n\}_N^{\infty}$ is a solution of (1.1) if $\{y_n\}_N^{\infty}$ satisfies (1.1) for $n \geqslant N+1$. We are interested in the existence of positive solutions of (1.1) such that

$$\lim_{n\to\infty} y_n = c \tag{1.2}$$

or

$$\lim_{n\to\infty} \frac{y_n}{n} = c, \tag{1.3}$$

where c is a given positive constant. This problem has been studied by other authors (see, e.g., [1–3]) under the assumption that $\gamma > 0$ and under stronger summability conditions on $\{p_n\}$ than we will impose here. For

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example, the usual sufficient conditions guaranteeing the existence of solutions of (1.1) satisfying (1.2) imply that

$$\sum^{\infty} j\,|p_j| < \infty, \tag{1.4}$$

while those implying the existence of solutions satisfying (1.3) imply that

$$\sum^{\infty} j^{\gamma}\,|p_j| < \infty. \tag{1.5}$$

Our summability conditions are weaker than these, and we give precise estimates of the asymptotic behavior of $\{y_n\}$ and $\{\Delta y_n\}$.

We assume throughout this paper that c and θ are given constants, with $c > 0$ and $0 < \theta < 1$. Our results deal with the existence of solutions $\{y_n\}_N^{\infty}$ of (1.1) that either satisfy the inequality

$$|y_n - c| \leqslant \theta c, \qquad n \geqslant N \geqslant 0$$

and have the asymptotic behavior (1.2) or satisfy the inequality

$$|y_n - cn| \leqslant \theta cn, \qquad n \geqslant N \geqslant 1$$

and have the asmptotic behavior (1.3). In both instances the results take two forms. In the first, N is specified and the solution exists provided that c is sufficiently small if $\gamma > 1$ or sufficiently large if $\gamma < 1$, i.e., if $c^{\gamma-1}$ is sufficiently small. In the second, c is specified and the solution exists if N is sufficiently large.

We will write

$$z_n = O(q_n) \quad \text{if} \quad \overline{\lim_{n\to\infty}} \frac{z_n}{q_n} < \infty \qquad \text{and} \qquad z_n = o(q_n) \quad \text{if} \quad \lim_{n\to\infty} \frac{z_n}{q_n} = 0.$$

The following two theorems are our main results.

THEOREM 1. *Suppose that the series* $\sum^{\infty} jp_j$ *converges (perhaps conditionally) and there is a nonincreasing sequence of positive numbers* $\{\rho_n\}_1^{\infty}$ *such that* $\lim_{n\to\infty} \rho_n = 0$,

$$\rho_n \geqslant \alpha_n = \sup_{m \geqslant n} \left\{ \left| \sum_{j=m}^{\infty} jp_j \right| \right\} \tag{1.6}$$

and

$$\overline{\lim_{n\to\infty}} \frac{1}{\rho_n} \sum_{j=n}^{\infty} |P_j|\,\rho_j = D < \infty, \tag{1.7}$$

where

$$P_n = \sum_{j=n}^{\infty} p_j. \tag{1.8}$$

Then (1.1) *has a solution* $\{y_n\}_N^\infty$ *such that*

$$|y_n - c| \leqslant \frac{\theta c \rho_{n+1}}{\rho_1} \quad \text{and} \quad |\Delta y_n| \leqslant \frac{\theta c \rho_{n+1}}{(n+1)\rho_1}, \quad n \geqslant N \geqslant 0,$$

provided that either (a) $c^{\gamma-1}$ *is sufficiently small or* (b) $D = 0$, $\alpha_n = o(\rho_n)$, *and* N *is sufficiently large.*

THEOREM 2. *Suppose that the series* $\sum^\infty j^\gamma p_j$ *converges (perhaps conditionally) and there is a nonincreasing sequence of positive numbers* $\{\sigma_n\}_1^\infty$ *such that* $\lim_{n\to\infty} \sigma_n = 0$ *and*

$$\sigma_n \geqslant \beta_n = \sup_{m \geqslant n} \left\{ \left| \sum_{j=m}^{\infty} j^\gamma p_j \right| \right\}.$$

Let

$$\bar{\sigma}_n = \frac{1}{n} \sum_{j=1}^{n} \sigma_j \quad \text{and} \quad \bar{\beta}_n = \frac{1}{n} \sum_{j=1}^{n} \beta_j. \tag{1.9}$$

Suppose also that either $\gamma > 0$ *and*

$$\overline{\lim_{n\to\infty}} \frac{1}{\sigma_n} \sum_{j=n+1}^{\infty} j^{\gamma-1} |P_j| \bar{\sigma}_{j-1} = E < \infty \tag{1.10}$$

with P_n *as in* (1.8), *or* $\gamma < 1$ ($\gamma \neq 0$) *and*

$$\overline{\lim_{n\to\infty}} \frac{1}{\sigma_n} \sum_{j=n+1}^{\infty} |W_j| \bar{\sigma}_{j-1} = E < \infty, \tag{1.11}$$

where

$$W_n = \sum_{j=n}^{\infty} j^{\gamma-1} p_j. \tag{1.12}$$

Then (1.1) *has a solution* $\{y_n\}_N^\infty$ *such that*

$$|y_n - cn| \leqslant \frac{\theta c n \bar{\sigma}_n}{\sigma_1} \quad \text{and} \quad |\Delta y_n - c| \leqslant \frac{\theta c \sigma_{n+1}}{\sigma_1}, \quad n \geqslant N \geqslant 1,$$

provided that either (a) $c^{\gamma-1}$ *is sufficiently small or* (b) $E = 0$, $\beta_n = o(\sigma_n)$, $\lim_{n\to\infty} n\bar{\sigma}_n = \infty$, *and* N *is sufficiently large.*

Remark 1. Note that $\{\bar{\sigma}_n\}$ and $\{\bar{\beta}_n\}$ are nonincreasing and $\lim_{n\to\infty}\bar{\sigma}_n = \lim_{n\to\infty}\bar{\beta}_n = 0$. Moreover, if $\beta_n = O(\sigma_n)$ and $\lim_{n\to\infty} n\bar{\sigma}_n = \infty$, then $\bar{\beta}_n = o(\bar{\sigma}_n)$.

Before proving Theorems 1 and 2 we will show that their assumptions are weaker than (1.4) and (1.5), respectively. First suppose that (1.5) holds for some $\gamma > 0$ and let $H_n = \sum_{j=n}^{\infty} j^{\gamma} |p_j|$ and $\bar{P}_n = \sum_{j=n}^{\infty} |p_j|$. Then summation by parts shows that

$$\bar{P}_n = \sum_{j=n}^{\infty} \frac{H_j - H_{j+1}}{j^{\gamma}} \leqslant \frac{2H_n}{n^{\gamma}} \tag{1.13}$$

and

$$\begin{aligned}\sum_{j=n}^{M} j^{\gamma} |p_j| &= \sum_{j=n}^{M} j^{\gamma}(\bar{P}_j - \bar{P}_{j+1}) \\ &= n^{\gamma}\bar{P}_n - M^{\gamma}\bar{P}_{M+1} + \sum_{j=n+1}^{M} [j^{\gamma} - (j-1)^{\gamma}]\bar{P}_j.\end{aligned}$$

Since the left side here converges as $M \to \infty$, it follows that $\lim_{M\to\infty} M^{\gamma} P_{M+1} = 0$, and therefore

$$\sum^{\infty} [j^{\gamma} - (j-1)^{\gamma}]\, \bar{P}_j < \infty.$$

An argument using the mean value theorem shows that

$$j^{\gamma-1} \leqslant \frac{j^{\gamma} - (j-1)^{\gamma}}{\gamma} + \frac{|\gamma - 1|}{2} j^{\gamma-2}.$$

Since (1.13) implies that $\sum^{\infty} j^{\gamma-2}\bar{P}_j < \infty$, we can now conclude that $\sum_{j=n}^{\infty} j^{\gamma-1}\bar{P}_j < \infty$, and therefore

$$\sum^{\infty} j^{\gamma-1} |P_j| < \infty.$$

This implies (1.11) (with $E = 0$) for any nondecreasing sequence $\{\sigma_n\}$ such that

$$\overline{\lim_{n\to\infty}} \frac{1}{n\sigma_n} \sum_{j=1}^{n} \sigma_j < \infty. \tag{1.14}$$

Moreover, if we let $\gamma = 1$, we see that (1.4) implies (1.7) for any nondecreasing sequence $\{\rho_n\}$.

Now suppose that (1.5) holds for some $\gamma < 1$ and define $\bar{W}_n = \sum_{j=n}^{\infty} j^{\gamma-1} |p_j|$. Then summation by parts shows that

$$\bar{W}_n = \sum_{j=n}^{\infty} \frac{H_j - H_{j+1}}{j} \leqslant \frac{2H_n}{n}$$

and

$$\sum_{j=n}^{M} j\,|p_j| = \sum_{j=n}^{M} j^{\gamma}(\bar{W}_j - \bar{W}_{j+1}) = n\bar{W}_n - M\bar{W}_{M+1} + \sum_{j=n+1}^{M} \bar{W}_j.$$

Since the left side here converges as $M \to \infty$, it follows that $\lim_{M \to \infty} MP_{M+1} = 0$, and therefore $\sum^{\infty} \bar{W}_j < \infty$, which implies that

$$\sum^{\infty} |W_j| < \infty.$$

Hence, (1.11) holds (with $E = 0$) for any nonincreasing sequence $\{\sigma_n\}$ which satisfies (1.14).

We have now shown that (1.4) and (1.5) imply the hypotheses of Theorems 1 and 2. The converse is false. For example, suppose that

$$p_n = \frac{(-1)^n}{n(\log n)^{\alpha}}, \qquad n \geqslant 2,$$

where $\alpha \leqslant 1$. Then (1.4) does not hold, but $\sum^{\infty} jp_j$ converges,

$$|P_n| \leqslant \frac{1}{n(\log n)^{\alpha}}, \qquad \alpha_n \leqslant \rho_n = \frac{1}{(\log n)^{\alpha}},$$

and (1.7) holds with $D = 1$ if $\alpha = 1$ or $D = 0$ if $\alpha > 1$. Hence, the assumptions of Theorem 1 do not imply (1.4). Similar examples show that the assumptions of Theorem 2 do not imply (1.5).

2. Proof of Theorem 1

It is straightforward to verify that if $\{y_n\}_N^{\infty}$ is a sequence of positive numbers such that

$$y_n = c + \sum_{j=n+1}^{\infty} (j - n)\, p_j y_j^{\gamma}, \qquad n \geqslant N \tag{2.1}$$

then $\{y_n\}_N^\infty$ satisfies (1.1) for $n \geq N+1$ and has the asymptotic behavior (1.2). To put this more conveniently for our purposes, let

$$f_n = \sum_{j=n+1}^{\infty} (j-n) p_j \tag{2.2}$$

and suppose that

$$g_n = c^\gamma f_n + \sum_{j=n+1}^{\infty} (j-n) p_j [(c+g_j)^\gamma - c^\gamma], \qquad n \geq N. \tag{2.3}$$

Then the sequence $y_n = c + g_n$ satisfies (2.1) and therefore (1.1) for $n \geq N$, and has the asymptotic behavior (1.2). Motivated by this, we will obtain a sequence g which satisfies (2.3) as a fixed point of the transformation $v = Tg$, where

$$v_n = c^\gamma f_n + \sum_{j=n+1}^{\infty} (j-n) p_j [(c+g_j)^\gamma - c^\gamma]. \tag{2.4}$$

We will show that T is a contraction mapping of a subset of a certain Banach space $\mathscr{G}_N$ into itself, and g will be the sequence left fixed by T. In order to motivate the definition of $\mathscr{G}_N$, we need the following lemma.

LEMMA 1. *With f as defined by* (2.2),

$$|f_n| \leq \alpha_{n+1} \tag{2.5}$$

and

$$|\Delta f_n| \leq \frac{2\alpha_{n+1}}{n+1}. \tag{2.6}$$

Proof. Let $Q_n = \sum_{j=n}^{\infty} jp_j$. From (1.6)

$$|Q_n| \leq \alpha_n; \tag{2.7}$$

moreover, from (1.8),

$$P_n = \sum_{j=n}^{\infty} \frac{Q_j - Q_{j+1}}{j}$$

and summation by parts shows that

$$|P_n| \leq \frac{2\alpha_n}{n}. \tag{2.8}$$

We can rewrite (2.2) as

$$f_n = \sum_{j=n+1}^{\infty} \left(1 - \frac{n}{j}\right)(Q_j - Q_{j+1}) = n \sum_{j=n+1}^{\infty} \left(\frac{1}{j-1} - \frac{1}{j}\right) Q_j.$$

This and (2.7) imply (2.5). To verify (2.6) we have only to note that $\Delta f_n = -P_{n+1}$ and recall (2.8).

DEFINITION 1. For each $N \geqslant 0$ let $\mathscr{G}_N$ be the Banach space of sequences $g = \{g_n\}_{n=N}^{\infty}$ such that

$$g_n = O(\rho_{n+1}) \qquad \text{and} \qquad \Delta g_n = O\left(\frac{\rho_{n+1}}{n+1}\right),$$

with norm

$$\|g\| = \sup_{n \geqslant N} \left\{ \max\left(\frac{|g_n|}{\rho_{n+1}}, \frac{(n+1)\,|\Delta g_n|}{\rho_{n+1}} \right) \right\}, \tag{2.9}$$

and let $\bar{\mathscr{G}}_N$ be the set of sequences g in $\mathscr{G}_N$ such that

$$\|g\| \leqslant \frac{\theta c}{\rho_1}. \tag{2.10}$$

We will prove Theorem 1 by showing that its hypotheses imply the existence of a sequence in $\bar{\mathscr{G}}_N$ which is left fixed by the transformation $v = Tg$ defined by (2.4).

Since $\alpha_n \leqslant \rho_n$, Lemma 1 implies that $\{f_n\}_N^{\infty} \in \mathscr{G}_N$. Now let K be the transformation defined by $z = Kg$, where

$$z_n = \sum_{j=n+1}^{\infty} (j-n)\, p_j[(c+g_j)^{\gamma} - c^{\gamma}], \qquad n \geqslant N;$$

thus, $v_n = c^{\gamma} f_n + z_n$. (See (2.4).) The following lemma shows that the sequence $z = \{z_n\}_N^{\infty}$ is in $\bar{\mathscr{G}}_N$ and that K is a contraction of $\bar{\mathscr{G}}_N$ under the assumptions of Theorem 1.

LEMMA 2. *Suppose that* $g, h \in \bar{\mathscr{G}}_N$, *and let*

$$F_j = (c+g_j)^{\gamma} - (c+h_j)^{\gamma}.$$

Then the sequence $s(g,h) = \{s_n(g,h)\}_N^{\infty}$ *defined by*

$$s_n(g,h) = \sum_{j=n+1}^{\infty} (j-n)\, p_j F_j, \qquad n \geqslant N, \tag{2.11}$$

is in $\mathscr{G}_N$, *and*

$$\|s(g,h)\| \leqslant \Omega_N c^{\gamma-1} \|g-h\|, \tag{2.12}$$

where Ω_N *is independent of* g *and* h, *and* $\lim_{N\to\infty} \Omega_N = 0$ *if* $D=0$ *in* (1.7).

Proof. It will be convenient to define

$$k_1 = |\gamma|\,(1 \pm \theta)^{\gamma-1} \qquad \text{and} \qquad k_2 = |\gamma(\gamma-1)|\,\theta(1 \pm \theta)^{\gamma-2}, \tag{2.13}$$

where the $\pm$ in each case is $+$ if the exponent is positive or $-$ if the exponent is negative. First, observe that if $g \in \mathscr{G}_N$, then

$$0 < c(1-\theta) \leqslant (c+g_j) \leqslant c(1+\theta) \tag{2.14}$$

because of (2.9) and the definition (2.10) of $\mathscr{G}_N$. Consider the finite sum

$$s_n(M; g, h) = \sum_{j=n+1}^{M} (j-n)\, p_j F_j, \qquad M > n \geqslant N.$$

From (1.8),

$$s_n(M; g, h) = \sum_{j=n+1}^{M} (j-n)(P_j - P_{j+1}) F_j,$$

and summation by parts yields

$$\begin{aligned} s_n(M; g, h) = {} & -(M-n)\, P_{M+1} F_M + \sum_{j=n+1}^{M} P_j F_j \\ & + \sum_{j=n+2}^{M} (j-n-1)\, P_j\, \Delta F_{j-1}. \end{aligned} \tag{2.15}$$

The mean value theorem implies that $F_j = \gamma(c+t_j)^{\gamma-1}(g_j - h_j)$, where t_j is between g_j and h_j. Therefore, from (2.13), (2.14), and (2.9) with g replaced by $g-h$,

$$|F_j| \leqslant k_1 c^{\gamma-1} \|g-h\|\, \rho_{j+1}. \tag{2.16}$$

This and (2.8) imply that

$$\lim_{M\to\infty} (M-n)\, P_{M+1} F_M = 0. \tag{2.17}$$

By applying the mean value theorem to the function

$$B_j(u) = (c + ug_j + (1-u)h_j)^{\gamma} - (c + ug_{j-1} + (1-u)\, h_{j-1})^{\gamma},$$

we see that

$$\Delta F_{j-1} = \gamma(c+\xi_j)^{\gamma-1}(g_j - h_j) - \gamma(c+\eta_j)^{\gamma-1}(g_{j-1} - h_{j-1}), \quad (2.18)$$

where

$$\xi_j = u_j g_j + (1-u_j)h_j, \qquad \eta_j = u_j g_{j-1} + (1-u_j)h_{j-1}, \quad (2.19)$$

and $0 < u_j < 1$. Now using (2.18) and applying the mean value theorem to

$$C_j(v) = \gamma(c + v\xi_j + (1-v)\eta_j)^{\gamma-1}\left[v(g_j - h_j) + (1-v)(g_{j-1} - h_{j-1})\right]$$

shows that

$$\begin{aligned}\Delta F_{j-1} = {} & \gamma(\gamma-1)(c+\delta_j)^{\gamma-2}\left[v_j(g_j - h_j) + (1-v_j)(g_{j-1} - h_{j-1})\right](\xi_j - \eta_j) \\ & + \gamma(c+\delta_j)^{\gamma-1}\Delta(g_{j-1} - h_{j-1}), \quad (2.20)\end{aligned}$$

where $0 < v_j < 1$ and $\delta_j = v_j\xi_j + (1-v_j)\eta_j$. From (2.9), (2.10), and (2.19),

$$|\xi_j - \eta_j| \leqslant u_j\,|\Delta g_{j-1}| + (1-u_j)\,|\Delta h_{j-1}| \leqslant \frac{\theta c}{\rho_1}\frac{\rho_j}{j} \leqslant \frac{\theta c}{j}. \quad (2.21)$$

Moreover, from (2.9),

$$|v_j(g_j - h_j) + (1-v_j)(g_{j-1} - h_{j-1})| \leqslant \|g-h\|\,\rho_j \quad (2.22)$$

and

$$|\Delta(g_{j-1} - h_{j-1})| \leqslant \|g-h\|\,\frac{\rho_j}{j}. \quad (2.23)$$

From (2.13), (2.14), (2.20), (2.21), (2.22), and (2.23),

$$|\Delta F_{j-1}| \leqslant (k_1 + k_2)\,c^{\gamma-1}\,\|g-h\|\,\frac{\rho_j}{j}. \quad (2.24)$$

This and (1.7), (2.16), and (2.17) imply that we can let $M \to \infty$ in (2.15) to infer that $s_n(g, h)$ exists and satisfies the inequality

$$|s_n(g, h)| \leqslant R_n c^{\gamma-1}\,\|g-h\|\,\rho_{n+1}, \quad (2.25)$$

where

$$R_n = (2k_1 + k_2)\frac{1}{\rho_{n+1}}\sum_{j=n+1}^{\infty} |P_j|\,\rho_j,$$

which is finite, because of (1.7).

From (2.11),

$$\Delta s_n(g, h) = -\sum_{j=n+1}^{\infty} p_j F_j,$$

which can be rewritten by means of summation by parts as

$$\Delta s_n(g, h) = -P_{n+1} F_{n+1} - \sum_{j=n+1}^{\infty} P_{j+1} \Delta F_j.$$

This, (2.8), (2.16), and (2.24) imply that

$$|\Delta s_n(g, h)| \leqslant S_n c^{\gamma-1} \|g-h\| \frac{\rho_{n+1}}{n+1},$$

where

$$S_n = 2k_1 \rho_{n+2} + \frac{(k_1+k_2)}{\rho_{n+1}} \sum_{j=n+1}^{\infty} |P_{j+1}| \rho_{j+1}, \tag{2.26}$$

which is also finite, because of (1.7). Now (2.25) and (2.26) imply (2.12) with $\Omega_N = \sup_{n \geqslant N}\{\max(R_n, S_n)\}$, and $\lim_{N\to\infty} \Omega_N = 0$ if $D=0$ in (1.7). This completes the proof of Lemma 2.

We can now complete the proof of Theorem 1. From (2.4), the sequence $v = Tg$ is given by $v_n = c^{\gamma} f_n + s_n(g, 0)$. From (2.5), (2.6), and (2.9), $\|\{f_n\}_n^{\infty}\| \leqslant 2\lambda_N$, where

$$\lambda_N = \sup_{n \geqslant N+1} \{\alpha_n/\rho_n\}.$$

Therefore, if $g \in \bar{\mathscr{G}}_N$ then $v = Tg \in \mathscr{G}_N$ and

$$\|Tg\| \leqslant 2\lambda_N c^{\gamma} + \Omega_N c^{\gamma-1} \|g\| \leqslant (2\lambda_N + \Omega_N \theta/\rho_1) c^{\gamma}.$$

(Recall (2.10).) Moreover, if $g, h \in \mathscr{G}_N$ then (2.12) implies that

$$\|Tg - Th\| = \|s(g, h)\| \leqslant \Omega_N c^{\gamma-1} \|g-h\|.$$

Consequently, T is a contraction mapping of $\bar{\mathscr{G}}_N$ into itself if

$$(2\lambda_N + \Omega_N \theta/\rho_1)\, c^{\gamma-1} \leqslant \theta \qquad \text{and} \qquad \Omega_N c^{\gamma-1} < 1.$$

This implies the conclusions of Theorem 1.

3. Proof of Theorem 2

It is straightforward to verify that if $y_n > 0$ for $n \geqslant N$ and

$$y_n = cn - \sum_{i=N}^{n} \sum_{j=i}^{\infty} p_j y_j^{\gamma},$$

then $\{y_n\}_N^{\infty}$ satisfies (1.1) for $n \geqslant N+1$ and has the asymptotic behavior (1.3). For our purposes it is convenient to reformulate this. If

$$f_n = \sum_{i=N}^{n} \sum_{j=i}^{\infty} p_j j^{\gamma} \tag{3.1}$$

and

$$h_n = -c^{\gamma} f_n + \sum_{i=N}^{n} \sum_{j=i}^{\infty} p_j [(cj)^{\gamma} - (h_j + cj)^{\gamma}], \tag{3.2}$$

then $y_n = h_n + cn$ has the desired properties. We will obtain a sequence h which satisfies (3.2) as a fixed point of the transformation $v = Th$, where

$$v_n = -c^{\gamma} f_n + \sum_{i=N}^{n} \sum_{j=i}^{\infty} p_j [(cj)^{\gamma} - (h_j + cj)^{\gamma}]. \tag{3.3}$$

As in the proof of Theorem 1, we will show that T is a contraction mapping of a subset of a certain Banach space $\mathscr{H}_N$ into itself, and h will be the sequence left fixed by T.

We omit the routine proof of the following lemma, which will motivate our choice of $\mathscr{H}_N$.

Lemma 3. *With f as defined by (3.1), $|f_n| \leqslant n\bar{\beta}_n$ and $|\Delta f_n| \leqslant \beta_{n+1}$.*

Definition 2. For each $N \geqslant 1$ let $\mathscr{H}_N$ be the Banach space of sequences $h = \{h_n\}_{n=N}^{\infty}$ such that $h_n = O(n\bar{\sigma}_n)$ and $\Delta h_n = O(\sigma_{n+1})$, with norm

$$\|h\| = \sup_{n \geqslant N} \left\{ \max \left(\frac{|h_n|}{n\bar{\sigma}_n}, \frac{|\Delta h_n|}{\sigma_{n+1}} \right) \right\}, \tag{3.4}$$

and let $\bar{\mathscr{H}}_N$ be the set of sequences h in $\mathscr{H}_n$ such that

$$\|h\| \leqslant \frac{\theta c}{\sigma_1}. \tag{3.5}$$

Now let K be the transformation defined by $w = Kh$, where

$$w_n = \sum_{i=N}^{n} \sum_{j=i}^{\infty} p_j[(cj)^\gamma - (cj+h_j)^\gamma], \qquad n \geqslant N;$$

thus, $v_n = -c^\gamma f_n + w_n$. (See (3.3).)

LEMMA 4. *Suppose that* $g, h \in \mathscr{H}_N$, *and define*

$$\tilde{g}_j = \frac{g_j}{j}, \qquad \tilde{h}_j = \frac{h_j}{j},$$

and

$$\tilde{F}_j = (c+\tilde{g}_j)^\gamma - (c+\tilde{h}_j)^\gamma.$$

Then the sequence $s(g,h) = \{s_n(g,h)\}_N^\infty$ *defined by*

$$s_n(g,h) = \sum_{i=N}^{n} \sum_{j=i}^{\infty} j^\gamma p_j \tilde{F}_j, \qquad n \geqslant N \tag{3.6}$$

is in $\mathscr{H}_N$, *and*

$$\|s(g,h)\| \leqslant \Phi_N c^{\gamma-1} \|g-h\|, \tag{3.7}$$

where Φ_N *is independent of* g *and* h, *and* $\lim_{N\to\infty} \Phi_N = 0$ *if* $E = 0$ *in* (1.10) *or* (1.11) (*whichever applies*).

Proof. We first show that the series

$$t_m(g,h) = \sum_{j=m}^{\infty} j^\gamma p_j \tilde{F}_j, \qquad m \geqslant N, \tag{3.8}$$

converges and satisfies the inequality

$$|t_m(g,h)| \leqslant \phi_m c^{\gamma-1} \|g-h\| \, \sigma_m, \tag{3.9}$$

where $\overline{\lim}_{m\to\infty} \phi_m = \phi < \infty$ and $\phi = 0$ if $E = 0$.

The arguments used in the proof of Lemma 1 imply that

$$\tilde{F}_j = \gamma(c+\tilde{l}_j)^{\gamma-1}(\tilde{g}_j - \tilde{h}_j),$$

where $\tilde{l}_j$ is between $\tilde{g}_j$ and $\tilde{h}_j$. Hence

$$|\tilde{F}_j| \leqslant k_1 c^{\gamma-1} |g_j - h_j|/j \leqslant k_1 c^{\gamma-1} \|g-h\| \, \tilde{\sigma}_j. \tag{3.10}$$

(Recall (3.4).) Replacing g_j and h_j by $\tilde{g}_j$ and $\tilde{h}_j$ in (2.19) and (2.20) yields

$$\tilde{\xi}_j = u_j \tilde{g}_j + (1-u_j)\tilde{h}_j, \qquad \tilde{\eta}_j = u_j \tilde{g}_{j-1} + (1-u_j)\tilde{h}_{j-1},$$

where $0 < u_j < 1$ and

$$\Delta\tilde{F}_{j-1} = \gamma(\gamma-1)(c+\tilde{\delta}_j)^{\gamma-2}\,[v_j(\tilde{g}_j - \tilde{h}_j) + (1-v_j)(\tilde{g}_{j-1} - \tilde{h}_{j-1})](\tilde{\xi}_j - \tilde{\eta}_j) + \gamma(c+\tilde{\delta}_j)^{\gamma-1}\,\Delta(\tilde{g}_{j-1} - \tilde{h}_{j-1}), \tag{3.11}$$

where $0 < v_j < 1$ and $\tilde{\delta}_j = v_j \tilde{\xi}_j + (1-v_j)\tilde{\eta}_j$. Now,

$$|\tilde{\xi}_j - \tilde{\eta}_j| \leqslant u_j\,|\Delta\tilde{g}_{j-1}| + (1-u_j)\,|\Delta\tilde{h}_{j-1}|, \tag{3.12}$$

and easy manipulations using (3.4) show that

$$|\Delta\tilde{g}_{j-1}| \leqslant \frac{2\,\|g\|\,\bar{\sigma}_{j-1}}{j} \qquad \text{and} \qquad |\Delta\tilde{h}_{j-1}| \leqslant \frac{2\,\|h\|\,\bar{\sigma}_{j-1}}{j};$$

therefore, since $g, h \in \overline{\mathscr{H}_N}$, (3.5) and (3.12) imply that

$$|\tilde{\xi}_j - \tilde{\eta}_j| \leqslant \frac{2\theta c}{j}, \qquad j \geqslant N+1. \tag{3.13}$$

Moreover,

$$|v_j(\tilde{g}_j - \tilde{h}_j) + (1-v_j)(\tilde{g}_{j-1} - \tilde{h}_{j-1})| \leqslant \|g-h\|\,\bar{\sigma}_j \tag{3.14}$$

and

$$|\Delta(\tilde{g}_{j-1} - \tilde{h}_{j-1})| \leqslant 2\,\|g-h\|\,\frac{\bar{\sigma}_{j-1}}{j}. \tag{3.15}$$

From (3.11), (3.13), (3.14), and (3.15),

$$|\Delta\tilde{F}_{j-1}| \leqslant 2(k_1+k_2)\,c^{\gamma-1}\,\|g-h\|\,\frac{\bar{\sigma}_{j-1}}{j}. \tag{3.16}$$

Now consider the finite sum

$$t_m(M; g, h) = \sum_{j=m}^{M} j^{\gamma} p_j \tilde{F}_j, \qquad M \geqslant m \geqslant n. \tag{3.17}$$

If $\gamma > 1$ we use (1.8) to rewrite this as

$$t_m(M; g, h) = \sum_{j=m}^{M} j^{\gamma}(P_j - P_{j+1})\tilde{F}_j,$$

and summation by parts yields

$$t_m(M; g, h) = m^\gamma P_m \tilde{F}_m - M^\gamma P_{M+1} \tilde{F}_M + \sum_{j=m+1}^{M} P_j \Delta[(j-1)^\gamma \tilde{F}_{j-1}]. \quad (3.18)$$

From (3.10), (3.16), and the mean value theorem,

$$\begin{aligned} |\Delta[(j-1)^\gamma \tilde{F}_{j-1}]| &\leq j^\gamma |\Delta \tilde{F}_{j-1}| + |\gamma| j^{\gamma-1} |\tilde{F}_{j-1}| \\ &\leq ((2+\gamma)k_1 + 2k_2) c^{\gamma-1} \|g-h\| j^{\gamma-1} \bar{c}_{j-1}. \end{aligned}$$

Since $\sum^\infty j^\gamma p_j$ converges, summation by parts shows that $|P_n| \leq 2n^{-\gamma}\sigma_n$. This, (3.10), and (3.16) imply that we can let $M \to \infty$ in (3.18) to infer that $t_m(g, h)$ exists and satisfies (3.9) with

$$\phi_m = 2k_1 \bar{\sigma}_m + \frac{(2+\gamma)k_1 + 2k_2}{\sigma_m} \sum_{j=m+1}^{\infty} j^{\gamma-1} |P_j| \bar{\sigma}_{j-1},$$

which is finite, because of (1.10). Moreover, $\lim_{m\to\infty} \phi_m = 0$ if $E = 0$ in (1.10).

If $\gamma < 1$ we use (1.12) to rewrite (3.17) as

$$t_m(M; g, h) = \sum_{j=m}^{M} j(W_j - W_{j+1}) \tilde{F}_j,$$

and summation by arts yields

$$t_m(M; g, h) = mW_m \tilde{F}_m - MW_{M+1} \tilde{F}_M + \sum_{j=m+1}^{M} (\tilde{F}_{j-1} + j\Delta \tilde{F}_{j-1}) W_j.$$

Since $\sum^\infty j^\gamma p_j$ converges, summation by parts shows that $|W_n| \leq 2\sigma_n/n$. This, (3.10), and (3.16) imply that we can let $M \to \infty$ in (3.18) to infer that $t_m(g, h)$ exists and satisfies (3.9) with

$$\phi_m = 2k_1 \bar{\sigma}_m + \frac{3k_1 + 2k_2}{\sigma_m} \sum_{j=m+1}^{\infty} |W_j| \bar{\sigma}_{j-1},$$

which is finite, because of (1.11). Moreover, $\lim_{m\to\infty} \phi_m = 0$ if $E = 0$ in (1.11).

From (3.6) and (3.8),

$$s_n(g, h) = \sum_{i=N}^{n} t_i(g, h)$$

and

$$\Delta s_n(g, h) = t_{n+1}(g, h);$$

therefore, (1.9) and (3.9) imply that

$$|s_n(g, h)| \leqslant \Phi_N \, \|g-h\| \, c^{\gamma-1} \, n\bar{\sigma}_n,$$

where $\Phi_N = \sup_{i \geqslant N}\{\phi_i\}$, and that

$$|\Delta s_n(g, h)| \leqslant \|g-h\| \, c^{\gamma-1} \, \phi_{n+1}\sigma_{n+1}.$$

This implies (3.7). Since $\lim_{n \to \infty} \Phi_n = 0$ if $E = 0$, this completes the proof of Lemma 4.

We can now complete the proof of Theorem 2. From (3.3), the sequence $v = Th$ is given by $v_n = -c^\gamma f_n + s_n(0, h)$. From Lemma 3 and (3.4), $\|\{f_n\}_N^\infty\| \leqslant \lambda_N$, where

$$\lambda_N = \sup_{n \geqslant N} \{\max(\bar{\beta}_n/\bar{\sigma}_n, \beta_n/\sigma_n)\}.$$

Therefore, if $h \in \bar{\mathscr{H}}_N$ then $v = Th \in \mathscr{H}_N$ and

$$\|Tg\| \leqslant \lambda_N c^\gamma + \Phi_N c^{\gamma-1} \, \|h\| \leqslant (\lambda_N + \Phi_N \theta/\sigma_1) c^\gamma.$$

Moreover, if $g, h \in \mathscr{G}_N$ then (3.7) implies that

$$\|Tg - Th\| = \|s(g, h)\| \leqslant \Phi_N c^{\gamma-1} \, \|g-h\|.$$

Consequently, T is a contraction mapping of $\overline{\mathscr{H}_N}$ into itself if

$$(\lambda_N + \Phi_N \theta/\sigma_1) \, c^{\gamma-1} \leqslant \theta \qquad \text{and} \qquad \Phi_N c^{\gamma-1} < 1.$$

This implies the conclusions of Theorem 2. (Recall Remark 1.)

4. Special Cases and Examples

The results in this section are motivated by Abel's convergence test for series. It is convenient to rewrite (1.1) as

$$\Delta^2 y_{n-1} = a_n b_n \, y_n^\gamma. \tag{4.1}$$

THEOREM 3. *Suppose that*

$$|a_1 + a_2 + \cdots + a_n| \leqslant \Psi < \infty, \qquad n = 1, 2, ..., \tag{4.2}$$

$\{nb_n\}$ is nonincreasing and positive, and $\lim_{n \to \infty} nb_n = 0$. *Suppose also that there is a nonincreasing sequence of positive numbers $\{\omega_n\}$ such that $\omega_n \geqslant nb_n$,* $\lim_{n \to \infty} \omega_n = 0$, *and*

$$\overline{\lim_{n \to \infty}} \frac{1}{\omega_n} \sum_{j=n}^{\infty} b_j \omega_j = K < \infty. \tag{4.3}$$

Then (4.1) *has a solution* $\{y_n\}_N^\infty$ *such that*

$$|y_n - c| \leqslant \frac{\theta c \omega_{n+1}}{\omega_1} \qquad \textit{and} \qquad |\Delta y_n| \leqslant \frac{\theta c \omega_{n+1}}{(n+1)\,\omega_1}, \quad n \geqslant N \geqslant 0,$$

provided that either (a) $c^{\gamma-1}$ *is sufficiently small or* (b) $K=0$, $b_n = o(\omega_n/n)$, *and N is sufficiently large.*

Proof. Summation by parts as in the proof of Abel's test shows that $\sum^\infty ja_j b_j$ converges, that

$$\left|\sum_{j=n}^{\infty} ja_j b_j\right| \leqslant 2\Psi n b_n, \tag{4.4}$$

and that

$$\left|\sum_{j=n}^{\infty} a_j b_j\right| \leqslant 2\Psi b_n. \tag{4.5}$$

In the notation of Theorem 1, (4.4) and (4.5) imply that $\alpha_n \leqslant 2\Psi n b_n$ and $|P_n| \leqslant 2\Psi b_n$. Therefore, we can take $\rho_n = 2\Psi\omega_n$ and (4.3) implies (1.7) with $D \leqslant 2\Psi K$. Now the conclusion follows from Theorem 1.

THEOREM 4. *Suppose that* (4.2) *holds,* $\{n^\gamma b_n\}$ *is nonincreasing and positive, and* $\lim_{n\to\infty} n^\gamma b_n = 0$. *Suppose also that there is a nonincreasing sequence of positive numbers* $\{\mu_n\}$ *such that* $\mu_n \geqslant n^\gamma b_n$, *and* $\lim_{n\to\infty} \mu_n = 0$. *Define*

$$\bar{\mu}_n = \frac{1}{n}\sum_{j=1}^{n} \mu_j \qquad \textit{and} \qquad \bar{v}_n = \frac{1}{n}\sum_{j=1}^{n} j^\gamma b_j.$$

Finally, suppose that

$$\overline{\lim_{n\to\infty}}\ \frac{1}{\mu_n}\sum_{j=n+1}^{\infty} j^{\gamma-1} b_j \bar{\mu}_{j-1} = J < \infty. \tag{4.6}$$

Then (4.1) *has a solution* $\{y_n\}_N^\infty$ *such that*

$$|y_n - cn| \leqslant \frac{\theta c n \bar{\mu}_n}{\mu_1} \qquad \textit{and} \qquad |\Delta y_n - c| \leqslant \frac{\theta c \mu_{n+1}}{\mu_1}, \quad n \geqslant N \geqslant 1, \tag{4.7}$$

provided that either (a) $c^{\gamma-1}$ *is sufficiently small or* (b) $J=0$, $b_n = o(n^{-\gamma}\mu_n)$, $\lim_{n\to\infty} n\bar{\mu}_n = \infty$, *and N is sufficiently large.*

Proof. Summation by parts shows that $\sum^{\infty} j^{\gamma} a_j b_j$ converges, that

$$\left| \sum_{j=n}^{\infty} j^{\gamma} a_j b_j \right| \leqslant 2\Psi n^{\gamma} b_n, \tag{4.8}$$

and that

$$\left| \sum_{j=n}^{\infty} a_j b_j \right| \leqslant 2\Psi b_n \tag{4.9}$$

if $\gamma > 0$, or that

$$\left| \sum_{j=n}^{\infty} j^{\gamma-1} a_j b_j \right| \leqslant 2\Psi n^{\gamma-1} b_n \tag{4.10}$$

if $\gamma < 1$. In the notation of Theorem 2, (4.8), (4.9), and (4.10) imply that $\beta_n \leqslant 2\Psi n^{\gamma} b_n$, $|P_n| \leqslant 2\Psi b_n$ (if $\gamma > 0$), and $|W_n| \leqslant 2\Psi n^{\gamma-1} b_n$ (if $\gamma < 1$). Therefore, we can take $\sigma_n = 2\Psi \mu_n$, and (4.6) implies (1.10) if $\gamma > 0$ or (1.11) if $\gamma < 1$, in both cases with $E \leqslant 2\Psi K$. Now the conclusion follows from Theorem 2.

EXAMPLE 1. Consider the difference equation

$$\Delta^2 y_{n-1} = \frac{a_n}{n^{\alpha+1}} y_n, \tag{4.11}$$

where $\{a_n\}$ satisfies (4.2) and $\alpha > 0$. Theorem 3 with $\omega_n = n^{-\alpha}$ implies that if $N \geqslant 0$ is given, then (4.11) has a solution $\{y_n\}_N^{\infty}$ such that

$$y_n = c + O(n^{-\alpha}) \qquad \text{and} \qquad \Delta y_n = O(n^{-\alpha-1}), \tag{4.12}$$

provided that $c^{\gamma-1}$ is sufficiently small. Theorem 3 with $\omega_n = n^{-\beta}$, where $0 < \beta < \alpha$, implies that if $c > 0$ is given, then (4.11) has a solution $\{y_n\}_N^{\infty}$ such that

$$y_n = c + O(n^{-\beta}) \qquad \text{and} \qquad \Delta y_n = O(n^{-\beta-1}), \tag{4.13}$$

provided that N is sufficiently large. (We could obviously obtain estimates more precise that (4.12) and (4.13) from (4.7).)

Notice that if $0 < \alpha \leqslant 1$, then (4.11) with, for example, $a_n = (-1)^n$ fails to satisfy (1.4).

EXAMPLE 2. Consider the difference equation

$$\Delta^2 y_{n-1} = \frac{a_n}{(n+1)(\log(n+1))^{\alpha}} y_n^{\gamma}, \tag{4.14}$$

where $\alpha \geqslant 1$ and $\{a_n\}$ satisfies (4.2). By elementary arguments similar to those used to prove the integral test for convergence of improper integrals, it can be shown that if $\alpha + \beta \geqslant 2$ then

$$\sum_{j=n+1}^{\infty} \frac{1}{(j+1)(\log(j+1))^{\alpha+\beta}} < \frac{1}{(\alpha+\beta-1)(\log(n+2))^{\alpha+\beta-1}} + \frac{1}{(n+2)(\log(n+2))^{\alpha+\beta}}.$$

Therefore, Theorem 3 with $\omega_n = (\log(n+1))^{-\alpha}$ implies that if $N \geqslant 0$ is given, then (4.14) has a solution $\{y_n\}_N^{\infty}$ such that

$$y_n = 1 + O\left(\frac{1}{(\log n)^{\alpha}}\right), \qquad \Delta y_n = O\left(\frac{1}{n(\log n)^{\alpha}}\right),$$

provided that $c^{\gamma-1}$ is sufficiently small. If $\alpha > 1$ and $1 < \beta < \alpha$, then Theorem 3 with $\omega_n = (\log(n+1))^{-\beta}$ implies that if $c > 0$ is given, then (4.14) has a solution $\{y_n\}_N^{\infty}$ such that

$$y_n = 1 + O\left(\frac{1}{(\log n)^{\beta}}\right), \qquad \Delta y_n = O\left(\frac{1}{n(\log n)^{\beta}}\right),$$

provided that N is sufficiently large.

Notice that (4.14) fails to satisfy (1.4) for any α with, for example, $a_n = (-1)^n$.

EXAMPLE 3. Consider the difference equation

$$\Delta^2 y_{n-1} = \frac{a_n}{n^{\alpha+\gamma}} y_n^{\gamma}, \tag{4.15}$$

where $\{a_n\}$ satisfies (4.2) and $\alpha > 0$. We apply Theorem 4 with $\mu_n = n^{-\beta}$, where $0 < \beta < \alpha$. Then

$$\bar{\mu}_n = \begin{cases} O(n^{-\beta}) & \text{if } 0 < \beta < 1, \\ O(\log n/n) & \text{if } \beta = 1, \\ O(n^{-1}) & \text{if } \beta > 1. \end{cases}$$

In all three cases (4.6) holds with $J = 0$. Theorem 4 implies that (4.15) has a solution $\{y_n\}_N^{\infty}$ with the asymptotic behavior

$$y_n = \begin{cases} cn(1 + O(n^{-\beta})) \\ cn(1 + O(\log n/n)) \\ cn(1 + O(n^{-1})) \end{cases} \text{ and } \quad \Delta y_n = \begin{cases} c(1 + O(n^{-\beta})) & \text{if } 0 < \beta < 1, \\ c(1 + O(\log n/n)) & \text{if } \beta = 1, \\ c(1 + O(n^{-1})) & \text{if } \beta > 1, \end{cases}$$

if $0<\beta<\alpha$ and N is sufficiently large, or if $\beta=\alpha$ and $c^{\gamma-1}$ is sufficiently small.

Notice that if $0<\alpha\leqslant 1$, then (4.15) with, for example, $a_n=(-1)^n$ fails to satisfy (1.5).

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