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# **Hyperovals in Hall Planes**

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In this paper we construct two classes of translation hyperovals in any Hall plane of even order  $q^2 \ge 16$ . Two hyperovals constructed in the same Hall plane are equivalent under the action of the automorphism group of that Hall plane iff they are in the same class.

## 1. INTRODUCTION AND PRELIMINARIES

A *k*-arc in a (finite) projective plane  $\pi_n$  of order *n* is a set of *k* points, no three of which are collinear. The maximum *k* for which there exist *k*-arcs in  $\pi_n$  is n + 1 if *n* is odd, and n + 2 if *n* is even. An (n + 1)-arc is an *oval* and an (n + 2)-arc is a *hyperoval*. For more details on these and other definitions and results needed, see [4] or [6], but note that these references use the term 'oval' for a hyperoval.

In the following we denote the automorphism group of  $\pi_n$  by Aut $(\pi_n)$  and the stabiliser of a k-arc  $\mathcal{X}$  in Aut $(\pi_n)$  by Aut $(\mathcal{X})$ . If  $\pi_q = PG(2, q)$ , the desarguesian projective plane of order q, where  $q = p^h$  and p is prime, then Aut(PG(2, q)) is  $P\Gamma L(3, q)$ .

Let *l* be a line of  $\pi_n$ . An elation of  $\pi_n$  with axis *l* is an automorphism of  $\pi_n$  which fixes *l* pointwise and fixes a point  $C \in l$  linewise. A translation *n*-arc  $\mathcal{X}$  in  $\pi_n$  is an *n*-arc the stabiliser of which contains a group of elations acting transitively on its points. (A permutation group G acts transitively on a set X if whenever  $x_1, x_2 \in X$ , there exists an element  $g \in G$  such that  $gx_1 = x_2$ .)

Furthermore, a *translation hyperoval* of  $\pi_n$  is a hyperoval which contains a translation *n*-arc. Let  $\mathcal{X}$  be a translation *n*-arc with group T of elations, and suppose that  $\mathcal{X}$  is contained in a (translation) hyperoval  $\mathcal{X}$ . Then the two points P and Q of  $\mathcal{H}\backslash\mathcal{X}$  lie on the axis of each non-identity element of T. This is because  $\mathcal{X}$  is fixed by each element of T, and hence  $\mathcal{H}$  is fixed by each element of T (for two hyperovals meet in at most half their number of points). Thus the two points P and Q are fixed or interchanged by each element of T, and the centre of each element of T must lie on PQ. Since all elements of T cannot have the same centre, as T is transitive on the points of  $\mathcal{H}$ , it follows that PQ is the axis of each element of T. Thus PQ is the axis of T.

In PG(2, q), q even, the translation hyperovals are exactly those hyperovals which are images under an element of  $P\Gamma L(3, q)$  of a hyperoval

$$\mathscr{D}(\alpha) = \{(1, t, t^{\alpha}): t \in GF(q)\} \cup \{(0, 1, 0), (0, 0, 1)\},\$$

where  $\alpha$  is a generator of Aut(GF(q)) (see [12]). The regular hyperovals are the images of the hyperoval  $\mathcal{D}(2)$ . A regular hyperoval therefore consists of the points of a non-degenerate conic together with its nucleus N, which is often also called the nucleus of the hyperoval.

The Hall plane of order  $q^2$ , for  $q \ge 3$ , is constructed from the desarguesian plane of order  $q^2$  as follows. Let  $l_{\infty}$  be a line of  $PG(2, q^2)$ , and let  $AG(2, q^2)$  denote the affine plane  $PG(2, q^2) \setminus l_{\infty}$ . Let  $\mathcal{D}$  be a *derivation set* for  $AG(2, q^2)$ ; that is, a set of q + 1 points of  $l_{\infty}$  such that, for every pair of points X, Y of  $AG(2, q^2)$  for which the line XY meets  $l_{\infty}$  in  $\mathcal{D}$ , there is a Baer subplane of  $PG(2, q^2)$  containing X, Y and  $\mathcal{D}$ . We define

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a new incidence structure  $\mathcal{D}AG(2, q^2)$ : a point of  $\mathcal{D}AG(2, q^2)$  is a point of  $AG(2, q^2)$ ; a line of  $\mathcal{D}AG(2, q^2)$  is either a line of  $AG(2, q^2)$  the ideal point of which is not in  $\mathcal{D}$ , or is the set of affine points of a Baer subplane of  $PG(2, q^2)$  which contains  $\mathcal{D}$ ; and incidence is the natural containment relation. The incidence structure  $\mathcal{D}AG(2, q^2)$  is an affine plane of order  $q^2$ , and is uniquely completable to a projective plane of the same order, by the addition of an *ideal line*. This is the Hall plane  $Hall(q^2)$  of order  $q^2$  (see [7, 10]).

The construction of ovals in Hall planes of odd order has been considered in [1, 8, 9, 11]. Korchmáros [9] also gives a family of parabolas in  $AG(2, 2^{2m})$ , each of which is still an arc in Hall $(2^{2m})$  and, in fact, is completable to a hyperoval in Hall $(2^{2m})$  (see the remarks following Theorem 2.2 below). This construction seems to have been discovered independently of the earlier result by Crismale [3]. Cherowitzo [2] conducted a complete computer search for hyperovals in all translation planes of order 16. The hyperovals that he found in Hall(16) fall into 15 equivalence classes of translation hyperovals, each with two points on the ideal line and with one of the three possible abstract automorphism groups (see also [5]).

We will give two constructions of translation hyperovals in any Hall plane  $\text{Hall}(q^2)$  of even order  $q^2 \ge 16$ . In both cases the hyperovals constructed have two points on the ideal line of the Hall plane, and the collection of hyperovals arising by the second construction includes the examples of Crismale [3] and Korchmáros [9].

## 2. The Constructions

Throughout this section we assume that  $q \ge 4$ . The affine points of each hyperoval that we will construct in a Hall plane of even order is the set of affine points of a regular hyperoval in the corresponding desarguesian plane. This restricts the position of the nucleus of the hyperoval, as we now demonstrate.

Let  $\mathscr{C}$  be a conic of  $PG(2, q^2)$  with nucleus N and let  $l_{\infty}$  be a secant of the regular hyperoval  $\bar{\mathscr{H}} = \mathscr{C} \cup \{N\}$ . Let  $AG(2, q^2) = PG(2, q^2) \setminus l_{\infty}$  and let  $\mathscr{H} = \bar{\mathscr{H}} \setminus l_{\infty}$ . Let  $\mathscr{D}$  be a derivation set contained in  $l_{\infty}$ . If  $\mathscr{H}$  is a translation  $q^2$ -arc in  $\mathscr{D}AG(2, q^2)$  then  $N \in l_{\infty}$  in  $PG(2, q^2)$ . To see why this is so, note that  $AG(2, q^2)$  and  $\mathscr{D}AG(2, q^2)$  have the same translation group (with axis  $l_{\infty}$ ) (see [7] or [10]), so that the translation  $q^2$ -arc  $\mathscr{H}$  in  $\mathscr{D}AG(2, q^2)$  is also a translation  $q^2$ -arc in  $AG(2, q^2)$ . But a translation  $q^2$ -arc contained in the regular hyperoval  $\tilde{\mathscr{H}}$  of  $PG(2, q^2)$  cannot contain N, for Aut( $\mathscr{H}$ ) = Aut( $\tilde{\mathscr{H}}$ ) fixes N, while the stabiliser of a translation  $q^2$ -arc is transitive on its points. Thus  $N \in l_{\infty}$ .

THEOREM 2.1. Let  $\mathcal{D}$  be a derivation set contained in a line  $l_{\infty}$  of  $PG(2, q^2)$ , where  $q \ge 4$  is even. Let  $\mathcal{C}$  be a conic in  $PG(2, q^2)$  which contains a point of  $l_{\infty} \setminus \mathcal{D}$  and the nucleus N of which is contained in  $\mathcal{D}$ . Let  $\tilde{\mathcal{H}} = \mathcal{C} \cup \{N\}$  and let  $\mathcal{H} = \tilde{\mathcal{H}} \setminus l_{\infty}$ . Then  $\mathcal{H}$  is a translation  $q^2$ -arc in  $\mathcal{D}AG(2, q^2)$  which is uniquely completable to a translation hyperoval in Hall $(q^2)$ . Furthermore, any two hyperovals in Hall $(q^2)$  arising from this construction are equivalent under the action of Aut $(Hall<math>(q^2)$ ).

**PROOF.** First we show that  $\mathcal{H}$  is a  $q^2$ -arc in  $\operatorname{Hall}(q^2)$ . The ideal line of  $\operatorname{Hall}(q^2)$  contains no point of  $\mathcal{H}$ , so we consider the lines of the corresponding affine plane  $\mathcal{D}AG(2, q^2)$ . A line of  $\mathcal{D}AG(2, q^2)$  which is a line of  $AG(2, q^2)$  the ideal point of which is not in  $\mathcal{D}$  contains at most two points of  $\mathcal{H}$ , for  $\mathcal{H}$  is an arc in  $AG(2, q^2)$ . Now let l be a line of  $\mathcal{D}AG(2, q^2)$  which is the set of points of an affine Baer subplane B of  $AG(2, q^2)$ , and suppose that l contains three points of  $\mathcal{H}$ . Now the projective completion  $\overline{\mathcal{B}}$  of  $\mathcal{B}$  contains three points and the nucleus of  $\overline{\mathcal{H}}$ , and since a regular hyperoval is determined by three of its points plus a nucleus, so  $\overline{\mathcal{H}} \cap \overline{\mathcal{B}}$  is a hyperoval of  $\overline{\mathcal{B}}$ . Since a hyperoval has only 0-secants and 2-secants,  $\overline{\mathcal{B}} \cap l_{\infty} = \mathcal{D}$  must be a

2-secant of  $\mathcal{H} \cap \mathcal{B}$ , which contradicts the hypothesis that  $\mathcal{H}$  contains a point of  $l_{\infty} \setminus \mathcal{D}$ . Hence no line of Hall $(q^2)$  contains three points of  $\mathcal{H}$ , so that  $\mathcal{H}$  is a  $q^2$ -arc of Hall $(q^2)$ .

Let  $P = \mathscr{C} \cap l_{\infty}$  in  $PG(2, q^2)$ . Each line in the parallel class of lines in  $AG(2, q^2)$  with ideal point P is a 1-secant of  $\mathscr{H}$  in  $AG(2, q^2)$ . But these lines form a parallel class of lines in  $\mathscr{D}AG(2, q^2)$  with ideal point P', say. Thus  $\mathscr{H} \cup \{P'\}$  is a  $(q^2 + 1)$ -arc in  $Hall(q^2)$ . Now a  $(q^2 + 1)$  arc in a projective plane of order  $q^2$  is uniquely completable to a  $(q^2 + 2)$ -arc [6], so there exists a unique point Q' on the ideal line of  $Hall(q^2)$  such that  $\mathscr{H} \cup \{P', Q'\}$  is a hyperoval in  $Hall(q^2)$ . This is the unique hyperoval in  $Hall(q^2)$ containing  $\mathscr{H}$ , as two hyperovals have at most half their number of points in common.

By the above arguments, since  $\mathcal{H}$  is a translation hyperoval of  $PG(2, q^2)$ , it is a translation hyperoval of Hall $(q^2)$ .

Next we show that any two hyperovals arising by this construction are equivalent under the action of Aut(Hall( $q^2$ )). First,  $PGL(3, q^2)$  is a subgroup of  $P\Gamma L(3, q^2)$  and is transitive on the set of conics [6, Theorem 7.2.1]. Also, the stabiliser of a conic in  $PGL(3, q^2)$  is transitive on the tangents to that conic [6, Theorem 7.2.3, Corollary 8], and the nucleus of a conic is fixed by the stabiliser of that conic. Let  $\mathscr{C}$  be a conic, let tbe a line tangent to  $\mathscr{C}$  at a point P, and let N denote the nucleus of  $\mathscr{C}$ . We now show that the stabiliser  $PGL(3, q^2)_{\mathscr{C},t}^t$  of  $\mathscr{C}$  and t in  $PGL(3, q^2)$ , in its action on t, is transitive on derivation sets on t containing N but not containing P.

First,  $|PGL(3, q^2)_{\mathscr{C},t}| = q^2(q^2 - 1)$ , and there are  $q^2$  elations with axis t fixing  $\mathscr{C}$ , so  $|PGL(3, q^2)_{\mathscr{C},t}^t| = q^2 - 1$ . Let G denote  $PGL(3, q^2)_{\mathscr{C},t}^t$ . We use the orbit-stabiliser theorem to show that G is transitive on the  $q^2 - 1$  derivation sets containing N but not P, by showing that the stabiliser of such a derivation set in G is trivial. Now G is cyclic of order  $q^2 - 1$  and any non-trivial element  $g \in G$  fixes only N and P. Thus all orbits of G on  $t \setminus \{N, P\}$  have the same length, which must be a divisor of  $q^2 - 1$ . So any union of orbits of G on  $t \setminus \{N, P\}$  has length not coprime to  $q^2 - 1$ .

Let  $\mathscr{D}$  be a derivation set on t containing N but not containing P. Then  $G_{\mathscr{D}} = G_{\mathscr{D}\setminus\{N\}}$ , for any three points of t determine a derivation set. Since  $|\mathscr{D}\setminus\{N\}| = q$ , which is coprime to  $q^2 - 1$ , it follows that  $\mathscr{D}\setminus\{N\}$  is not a union of orbits of G on  $t\setminus\{N, P\}$ , and hence is not stabilised by any non-trivial element of G. Thus  $G_{\mathscr{D}\setminus\{N\}} = 1$ , so that  $G_{\mathscr{D}} = 1$ .

We have shown that  $PGL(3, q^2)$  is transitive on the set of configurations of  $PG(2, q^2)$  formed by a conic  $\mathscr{C}$  with a tangent t at a point P, and a derivation set on t containing the nucleus N of  $\mathscr{C}$  but not containing P. So,  $PGL(3, q^2)_{\mathscr{D}}$  is transitive on the set of hyperovals of  $PG(2, q^2)$  described in the statement of the theorem, and hence Aut(Hall( $q^2$ )) is transitive on the set of  $q^2$ -arcs arising from them, by [7] or [10]. Therefore, Aut(Hall( $q^2$ )) is transitive on the set of hyperovals constructed by the method described in this theorem (by the uniqueness of the completion of the constructed  $q^2$ -arcs to hyperovals of Hall( $q^2$ )).

THEOREM 2.2. Let  $\mathcal{D}$  be a derivation set contained in a line  $l_{\infty}$  of  $PG(2, q^2)$ , where  $q \ge 4$  is even. Let  $\mathcal{C}$  be a conic in  $PG(2, q^2)$  which contains a point of  $\mathcal{D}$  and the nucleus of which is contained in  $l_{\infty} \setminus \mathcal{D}$ . Let  $\tilde{\mathcal{H}} = \mathcal{C} \cup \{N\}$  and let  $\mathcal{H} = \tilde{\mathcal{H}} \setminus l_{\infty}$ . Then  $\mathcal{H}$  is a translation  $q^2$ -arc in  $\mathcal{D}AG(2, q^2)$  which is uniquely completable to a translation hyperoval in  $Hall(q^2)$ . Furthermore, any two hyperovals in  $Hall(q^2)$  arising from this construction are equivalent under the action of  $Aut(Hall(q^2))$ .

**PROOF.** The proof is similar to the proof of Theorem 2.1. The only major difference is in the proof that a line l of  $\mathcal{D}AG(2, q^2)$  which is the set of points of a Baer subplane  $\mathcal{B}$  of  $AG(2, q^2)$  contains at most two points of the  $q^2$ -arc  $\mathcal{H}$ . Suppose that l contains three points of  $\mathcal{H}$ , so that the projective completion  $\overline{\mathcal{B}}$  of  $\mathcal{B}$  contains four points of  $\overline{\mathcal{H}}$ and has  $l_{\infty} \cap \overline{\mathcal{B}} = \mathcal{D}$  as a tangent line. But four points plus a tangent line determine a regular hyperoval, so  $\mathfrak{B} \cap \mathfrak{H}$  is a hyperoval of  $\mathfrak{B}$ . This means that  $\mathfrak{D}$  has two points of  $\mathfrak{H}$ , contrary to the hypothesis. The rest of the argument follows, interchanging the roles of N and P where necessary.

Each member of the family of parabolas in  $AG(2, 2^{2m})$  given in [9] and [3] is still an arc in Hall $(2^{2m})$  and completes to a translation hyperoval of Hall $(2^{2m})$ , since it is an instance of the construction described in Theorem 2.2. The conics used by Korchmáros and Crismale have homogeneous equations of the form  $\mathscr{C}: x^2 + sxz + yz = 0$ , where  $s \in GF(2^{2m}) \setminus GF(2^m)$ , and the derivation set used is  $\{(0, 1, 0)\} \cup \{(1, n, 0): n \in GF(2^m)\}$ . (In the notation of Crismale [3] these are the curves with g = 1 and hence r = 2.) The derivation set  $\mathscr{D}$  is on the line z = 0, which is tangent to  $\mathscr{C}$ . Also,  $\mathscr{C}$  contains a point (0, 1, 0) of  $\mathscr{D}$  and has nucleus (1, -s, 0) which is not contained in  $\mathscr{D}$ .

THEOREM 2.3. A hyperoval in  $Hall(q^2)$  constructed as in Theorem 2.1 is inequivalent under the action of  $Aut(Hall(q^2))$  to a hyperoval constructed in the same plane by the method of Theorem 2.2.

**PROOF.** Let  $\tilde{\mathcal{H}}_1$  and  $\tilde{\mathcal{H}}_2$  be hyperovals of Hall $(q^2)$ , constructed as in Theorem 2.1 and Theorem 2.2, respectively. Let  $\mathcal{H}_i = \tilde{\mathcal{H}}_i \setminus I_\infty$ . Suppose that  $\tilde{\mathcal{H}}_1$  and  $\tilde{\mathcal{H}}_2$  are equivalent under the action of Aut(Hall $(q^2)$ ), so there exists an element  $g \in Aut(Hall<math>(q^2)$ ) such that  $g\tilde{\mathcal{H}}_1 = \tilde{\mathcal{H}}_2$ , and  $g\mathcal{H}_1 = \mathcal{H}_2$ . Since every element of Aut(Hall $(q^2)$ ) is inherited from an element of  $P\Gamma L(3, q^2)_{\mathcal{D}}$ , it follows that there exists an element  $h \in P\Gamma L(3, q^2)_{\mathcal{D}}$  such that  $h\mathcal{H}_1 = \mathcal{H}_2$ , and so  $h\tilde{\mathcal{H}}_1 = \tilde{\mathcal{H}}_2$  since  $h\mathcal{H}_1$  and  $\mathcal{H}_2$  are contained in unique hyperovals  $\tilde{\mathcal{H}}_1$  and  $\tilde{\mathcal{H}}_2$  of  $PG(2, q^2)$ . But the nucleus of  $\tilde{\mathcal{H}}_1$  is contained in  $\mathcal{D}$  and the nucleus of  $\tilde{\mathcal{H}}_2$ is not contained in  $\mathcal{D}$ , while h preserves  $\mathcal{D}$ . Since a regular hyperoval has a unique nucleus, this is a contradiction, and the result is proved.

EXAMPLE 2.4. Let  $l_{\infty}$  be the line z = 0 of PG(2, 16), and let  $\mathcal{D} = \{(0, 1, 0)\} \cup \{(1, x, 0): x \in GF(4)\}$  be a derivation set on  $l_{\infty}$ . Here we give two conics in PG(2, 16) through the same point  $P \in l_{\infty} \setminus \mathcal{D}$  and with the same nucleus in  $\mathcal{D}$  which, under the construction in Theorem 2.1, have only one common point on the ideal line of the corresponding Hall plane Hall(16).

Let  $\omega$  be a primitive element for GF(16) satisfying  $\omega^4 + \omega + 1 = 0$ . The conics are:

$$\mathscr{C}_1: \omega x^2 + \omega^3 y^2 + z^2 + yz = 0$$
 and  $\mathscr{C}_2: \omega^2 x^2 + \omega^4 y^2 + z^2 + yz = 0$ 

Each conic contains the point  $P = (\omega, 1, 0)$  and has nucleus N = (1, 0, 0). By Theorem 2.1, these give rise to hyperovals  $\mathcal{H}_1$  and  $\mathcal{H}_2$  in Hall(16), with a common point P' on the ideal line. However, the other ideal point of  $\mathcal{H}_1$  is distinct from the other ideal point of  $\mathcal{H}_2$ .

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