International Journal of Approximate Reasoning 50 (2009) 1259–1278

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Building a binary outranking relation in uncertain, imprecise and multi-experts contexts: The application of evidence theory

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ARTICLE INFO

Article history:
Received 27 November 2008
Received in revised form 31 May 2009
Accepted 1 June 2009
Available online 7 June 2009

Keywords:
Multicriteria decision aid
Evidence theory
First belief dominance
ELECTRE I
Multi-experts context

ABSTRACT

We consider multicriteria decision problems where the actions are evaluated on a set of ordinal criteria. The evaluation of each alternative with respect to each criterion may be uncertain and/or imprecise and is provided by one or several experts. We model this evaluation as a basic belief assignment (BBA). In order to compare the different pairs of alternatives according to each criterion, the concept of first belief dominance is proposed. Additionally, criteria weights are also expressed by means of a BBA. A model inspired by ELECTRE I is developed and illustrated by a pedagogical example.

1. Introduction

Evidence theory [9,10], also called Dempster–Shafer theory or belief functions theory, is a convenient framework for modelling imperfection in data and for combining information. This formalism has been widely used in many fields. For instance, one may cite: classification [36], data mining [38], knowledge elicitation [26], etc. In multicriteria decision analysis [6], it has been incorporated in the analytic hierarchy process (AHP) method to permit a decision maker to make judgements on groups of decision alternatives and to allow for additional analysis including levels of uncertainty and conflict in the decisions made [23–25]. In addition, the evidential reasoning algorithm, which is an approach used to aggregate multiple attributes in uncertain context, has been developed on the basis of evidence theory [12,16,17]. Furthermore, a sorting method called DISSET has been modelled by this theory in the context where information about categories is represented by a set of alternatives of which their related labels are either precise or imprecise [22].

In this paper, we propose another application of evidence theory within the framework of multicriteria decision making. We consider multicriteria problems for which different experts collaborate to estimate the evaluation of alternatives according to ordinal criteria. The information provided can be uncertain and/or imprecise. For example, in risk assessment of investment projects, an expert may hesitate between two or more successive evaluation levels. He may be sure that the project is either "safe" or "extremely safe" without being able to refine his judgment.

The inter-criteria information, usually represented by "weights", may also be imprecise. For example, one could accept that the total importance allocated to the coalition of financial criteria, including investment costs and operational costs,

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0888-613X/$ - see front matter
doi:10.1016/j.ijar.2009.06.001

is equal to 0.6 while the importance related to each of the single criteria is not determined or is such that their sum is lower than 0.6.

Finally, information provided by several experts about the same alternative and the same criterion has to be combined. Evidence theory offers tools to combine information issued from several experts for instance the Dempster's rule [9] and the normalized cautious rule [34]. However, we will show that these combination operators do not respect the unanimity property which is a natural condition of an aggregation operator. Therefore, we have adapted the algorithm AL3 proposed by Jabeur and Martel [19].

In this work, we propose a model inspired by the ELECTRE I [5,13] method to address these kinds of problems. Evidence theory will be used to represent imprecise and/or uncertain data and to represent the inter-criteria information.

This paper is organized as follows: in Section 2 we introduce the key concepts of evidence theory, then the notion of first belief dominance is presented in Section 3 and a model inspired by ELECTRE I is proposed in Section 4. The model is illustrated by a pedagogical example in Section 5.

2. Evidence theory: some concepts

Evidence theory has been initially developed by Arthur Dempster in 1967 [4], formalized by Glenn Shafer in 1976 [9], and axiomatically justified by Philippe Smets in his transferable belief model [31]. This theory has been proposed as a generalization of the Bayesian theory. It allows representing uncertainty and imprecision in situations where the available information is imperfect. In this section, the basic notations of evidence theory are introduced and the main concepts that are necessary to understand the rest of the paper are briefly recalled.

2.1. Knowledge model

Let Θ be a finite set of mutually exclusive and exhaustive hypotheses, called the frame of discernment. Let 2^Θ be the set of all subsets of Θ. A basic belief assignment (BBA) [9] is a function m from 2^Θ to [0, 1] verifying $\sum_{A \subseteq \Theta} m(A) = 1$. The quantity $m(A)$ represents the belief that is committed exactly to A. When $m(A) \neq 0$, A is called focal set.

A BBA m is said to be:

- Normal if $\emptyset$ is not a focal set, i.e., $m(\emptyset) = 0$. The initial works [4,9] on evidence theory requires that $m(\emptyset) = 0$, but this condition is not imposed in the transferable belief model [31]. In this paper, we will only consider normal BBAs.
- Dogmatic if Θ is not a focal set, i.e., $m(\Theta) = 0$.
- Vacuous if Θ is the only focal set, i.e., if $m(\Theta) = 1$ and $m(A) = 0$ for all $A \neq \Theta$. This type of BBA is used to represent the state of total ignorance.
- Simple if $m(\Theta) = w$ and $m(A) = 1 - w$ for some $A \neq \Theta$ and $0 \leqslant w \leqslant 1$. When $w > 1$, m is not a BBA since it is no longer a function from 2^Θ to $[0,1]$. Such a function can be referred as an inverse simple BBA. Both simple and inverse simple BBAs are known as generalized simple BBAs and denoted as denoted A^w. w is interpreted as the weight of evidence (these notions will be used later in the definition of the normalized cautious rule of combination).

A BBA can equivalently be represented by its associated belief (or credibility), plausibility and commonality functions [9] defined, respectively, as:

$$Bel(A) = \sum_{B \subseteq A} m(B) \tag{1}$$

$$Pl(A) = \sum_{A \cap B \neq \emptyset} m(B) \tag{2}$$

$$Q(A) = \sum_{A \subseteq B} m(B) \tag{3}$$

The quantity $Bel(A)$ measures the total belief associated to A. The plausibility $Pl(A)$ measures the belief that can potentially be placed in A whereas the commonality $Q(A)$ quantifies the belief that can be transferred to any element of A. The belief and plausibility functions can be connected by the equation $Pl(A) = 1 - Bel(\overline{A})$, where $\overline{A}$ denotes the complement of A. Let us note that the weight of evidence [32] introduced above can be obtained from the commonality function using the following formula:

$$w(A) = \prod_{A \subseteq B} Q(B)^{(-1)^{|B|-|A|+1}} \tag{4}$$

Evidence theory has been proposed as a generalization of the Bayesian theory. In fact, if all the belief masses of a given BBA are associated to singletons, the induced belief function is nothing else than a probability function and it called a Bayesian belief function. In his transferable belief model, Smets has proposed a technique called the pignistic transformation [29] for translating the belief functions models to probability models in order to make decisions [30]. This transformation consists in distributing equally each belief mass $m(A)$ among the elements of A. This leads to the pignistic probability function $BetP$ defined as follows:

$$BetP(H_i) = \sum_{A \subseteq \Theta / H_i \in A} \frac{m(A)}{|A|}, \quad \forall H_i \in \Theta \tag{5}$$

where $|A|$ is the cardinal of the subset A. This function is considered as a measure of probability used to make decisions. However, it is not a probability function suitable for representing beliefs.

2.2. Combination

The combination is an operation that plays a central role in evidence theory. The BBAs induced by several sources are aggregated using a combination rule in order to yield a global BBA that synthesizes the knowledge of the different sources. Within this context, several combinations rules have been proposed to aggregate independent and dependant sources. Among others, we can mention the Dempster's rule, the Dubois and Prade's rule, the Yager's rule, the cautious rule (normalized and unnormalized), etc. [28,34]. In this section, we will only present the Dempster's and the normalized cautious rules that are used, respectively, to combine independent and dependant sources.

The Dempster's rule [9], also known as the normalized conjunctive rule, has been the first combination operator proposed in evidence theory. This rule allows the combination of BBAs provided by independent sources, i.e., distinct BBAs. Let m_1 and m_2 be two distinct BBAs to combine, the Dempster's rule is defined as follows:

$$m(A) = \frac{\sum_{B \cap C = A} m_1(B) \cdot m_2(C)}{1 - k}, \quad \forall A \subseteq \Theta \tag{6}$$

where k is defined by $k = \sum_{B \cap C = \emptyset} m_1(B) \cdot m_2(C)$. The coefficient k represents the mass that the combination assigns to $\emptyset$ and reflects the conflict between the sources. The quotient $(1 - k)^{-1}$ is a term of normalization that guaranties $m(\emptyset) = 0$ and $\sum_{A \subseteq \Theta} m(A) = 1$.

The Dempster's rule has several interesting mathematical proprieties. It can be proved to be both commutative and associative. That is why it has been used in expert systems [24]. However, it should not be applied to combine BBAs given by dependant sources, i.e., nondistinct BBAs. To perform the combination in such situations, Denoeux introduced the normalized cautious rule [34]. This operator combines nondogmatic BBAs. In practice, the cautious combination of two nondogmatic BBAs m_1 and m_2 is computed as follows:

- Compute the commonality functions Q_1 and Q_2 using Eq. (3).
- Compute the weight functions w_1 and w_2 that are obtained from the commonalities using Eq. (4).
- Determine the generalized simple BBAs $A^{w_1 \wedge w_2}$ for all $A \subset \Theta$ such that $w_1 \wedge w_2 \neq 1$.
- Combine the induced generalized simple BBAs using the normalized Dempster's rule.

The normalized cautious rule is also commutative and associative and it is has been also used to combine expert opinions. The interesting reader can refer to [26] that gives an application of this rule to climate sensitivity assessment.

3. First belief dominance

The stochastic dominance is an approach used to perform comparisons between probability distributions. This concept has been addressed fundamentally in [8,11,14,27,37]. It has been applied in several domains especially in the field of multicriteria decision aid to compare evaluations of alternatives with respect to criteria.

Initially, the stochastic dominance concept has been used in comparing the evaluations expressed by probability distributions in order to build some outranking relations that reflect the decision maker's preferences [15]. Then, the use of stochastic dominance has been extended to ambiguous probability distributions in ranking problem [3]. Recently, this concept has been employed to compare mixed evaluations, i.e., evaluations expressed by probability distributions, fuzzy membership functions, possibility measures and belief masses [33]. However, the use of this concept necessitates the transformation of these functions to others of which the proprieties are similar to those of probability functions. For instance, the pignistic transformation [29] developed essentially to make decisions in evidence theory [30] is used to transform a BBA into a pignistic probability function.

In this section, we present a new concept called the first belief dominance [21] in the context of multicriteria decision problems. This approach allows comparing evaluations expressed by BBAs and generalizes the first stochastic dominance concept which is the simplest case of the stochastic dominance approach. Before presenting this concept, it is worth mentioning that similar extensions of stochastic ordering to belief functions, called credal orderings, have been developed by Thierry Denoeux and have been published recently in [35]. Some of these orderings have been introduced, without development, in [1,2] in the context of novelty detection. Let us note that the credal orderings and the concept of first belief dominance have been developed simultaneously and independently, i.e., we have been not aware of the Denoeux's orderings when our concept has been developed. In Section 3.2, we will present the relationship between our approach and two of the credal orderings.

3.1. Problem formulation

We consider a multicriteria problem which can be represented by three elements: the actions, the ordinal criteria and the criteria assessment grades. At first, we will consider situations where only one expert evaluates the alternatives. In what follows, let:

- $A = \{a_1, a_2, \ldots, a_n\}$ be the set of actions.
- $G = \{g_1, g_2, \ldots, g_q\}$ be the set of ordinal criteria.
- $X^h = \{x_1^h, x_2^h, \ldots, x_{r_h}^h\}$ be the set of the ordinal assessment grades of the criterion g_h.

The n alternatives are assessed, on each criterion g_h, using the same set of the r_h ordinal assessment grades x_j^h (with $j = 1, 2, \ldots, r_h$) which are required to be mutually exclusive and exhaustive. The r_h ordinal assessment grades constitute the frame of discernment in evidence theory and are defined such as the first ordinal grade x_1^h is less preferred than the second ordinal grade x_2^h and so on, i.e., $x_1^h \prec x_2^h \prec \cdots \prec x_{r_h}^h$. Thus, we have a total order of the ordinal assessment grades on each criterion g_h.

The evaluation of each action a_i with respect to each criterion g_h is given by a normal BBA of which the focal sets are subsets of the set X^h. Formally, this BBA is defined as a function $m_i^h : 2^{X^h} \to [0, 1]$ such as $m_i^h(\emptyset) = 0$ and $\sum_{C \subseteq X^h} m_i^h(C) = 1$. In the case where the expert is unable to express the assessment of action a_i on criterion g_h, the evaluation of a_i is represented by a vacuous BBA, i.e., the total mass is assigned to the frame X^h.

In order to compare the BBAs that represent the evaluations of alternatives with respect to each criterion, we propose a new concept called "the first belief dominance". This is naturally inspired by the concept of first stochastic dominance. Before describing it, some notations should be clarified.

Let a_i and $a_{i'}$ be two different actions and let m_i^h and $m_{i'}^h$ be, respectively, the BBAs of a_i and $a_{i'}$ defined on the frame X^h of each criterion g_h.

For all $h \in \{1, 2, \ldots, q\}$ and for all $k \in \{0, 1, \ldots, r_h\}$, let:

$$A_k^h = \begin{cases} \emptyset & \text{if } k = 0 \\ \{x_1^h, \ldots, x_k^h\} & \text{otherwise} \end{cases} \tag{7}$$

and let $\vec{S}(X^h)$ denote the set $\left\{A_1^h, A_2^h, \ldots, A_{r_h}^h\right\}$. Similarly, for all $h \in \{1, 2, \ldots, q\}$ and for all $l \in \{0, 1, \ldots, r_h\}$ such as $l = r_h - k$, let:

$$B_l^h = \begin{cases} \emptyset & \text{if } l = 0 \\ \left\{x_{r_h - l + 1}^h, \ldots, x_{r_h}^h\right\} & \text{otherwise} \end{cases} \tag{8}$$

and let $\overleftarrow{S}(X^h)$ denote the set $\left\{B_1^h, B_2^h, \ldots, B_{r_h}^h\right\}$.

k and l represent, respectively the number of elements of the sets A_k^h and B_l^h. Obviously, $|\vec{S}(X^h)| = |\overleftarrow{S}(X^h)| = r_h$, $\overline{A_k^h} = B_{r_h - k}^h = B_l^h$ for all $k \in \{0, 1, \ldots, r_h\}$ and $\overline{B_l^h} = A_{r_h - l}^h = A_k^h$ for all $l \in \{0, 1, \ldots, r_h\}$.

3.2. Definitions

Definition 1. The ascending belief function noted $\overrightarrow{Bel_i^h}$, induced by m_i^h and associating to the evaluation of action a_i with respect to criterion g_h, is a function $\overrightarrow{Bel_i^h} : \vec{S}(X^h) \to [0, 1]$ defined such as $\overrightarrow{Bel_i^h}\left(A_k^h\right) = \sum_{C \subseteq A_k^h} m_i^h(C)$ for all $A_k^h \in \vec{S}(X^h)$.

Definition 2. The descending belief function noted $\overleftarrow{Bel_i^h}$, induced by m_i^h and associating to the evaluation of action a_i with respect to criterion g_h, is a function $\overleftarrow{Bel_i^h} : \overleftarrow{S}(X^h) \to [0, 1]$ defined such as $\overleftarrow{Bel_i^h}\left(B_l^h\right) = \sum_{C \subseteq B_l^h} m_i^h(C)$ for all $B_l^h \in \overleftarrow{S}(X^h)$.

The third definition is that of the first belief dominance. This condition holds between two BBAs m_i^h and $m_{i'}^h$ whenever the two following conditions are verified simultaneously:

- The ascending belief function $\overrightarrow{Bel_i^h}$ lies, entirely or partly, below the ascending belief function $\overrightarrow{Bel_{i'}^h}$.
- The descending belief function $\overleftarrow{Bel_i^h}$ lies, entirely or partly, above the descending belief function $\overleftarrow{Bel_{i'}^h}$.

Definition 3. m_i^h is said to dominate $m_{i'}^h$ according the first belief dominance if and only if the following two conditions are verified simultaneously:

- For all $A_k^h \in \vec{S}(X^h), \overrightarrow{Bel_i^h}\left(A_k^h\right) \leqslant \overrightarrow{Bel_{i'}^h}\left(A_k^h\right)$.
- For all $B_l^h \in \overleftarrow{S}(X^h), \overleftarrow{Bel_i^h}(B_l^h) \geqslant \overleftarrow{Bel_{i'}^h}\left(B_l^h\right)$.

The first condition means that there is greater belief mass under $\overrightarrow{Bel_{i'}^h}$ than $\overrightarrow{Bel_i^h}$ for all $A_k^h \in \vec{S}(X^h)$. On the contrary, the second condition means that there is greater belief mass under $\overleftarrow{Bel_i^h}$ then $\overleftarrow{Bel_{i'}^h}$ for all $B_l^h \in \overleftarrow{S}(X^h)$.

In the case where the two conditions are not verified simultaneously, then m_i^h does not dominate $m_{i'}^h$ according the first belief dominance concept. As a conclusion, two situations are identified in our approach: FBD identifies first belief dominance situations consistent with the conditions imposed by Definition 3, and $\overline{\text{FBD}}$ designates those which are not consistent with these conditions. In what follows, we denote:

- $m_i^h \text{FBD} m_{i'}^h$ if m_i^h dominates $m_{i'}^h$ according the first belief dominance approach,
- $m_i^h \overline{\text{FBD}} m_{i'}^h$ if m_i^h does not dominate $m_{i'}^h$ according the first belief dominance approach.

There is a relationship between the first belief dominance concept and two of Denoeux's orderings. Denoting A_k^h and B_l^h, respectively as $[x_1^h, x_k^h]$ and $\left[x_{r_h-l+1}^h, x_{r_h}^h\right]$, we have:

- $\overrightarrow{Bel_i^h}\left(A_k^h\right) = Bel_i^h\left([x_1^h, x_k^h]\right) = \begin{cases} 1 - Pl_i^h\left(\left[x_{k+1}^h, x_{r_h}^h\right]\right) & \text{if } k < r_h \\ 1 & \text{otherwise,} \end{cases}$
- $\overleftarrow{Bel_i^h}\left(B_l^h\right) = Bel_i^h\left(\left[x_{r_h-l+1}^h, x_{r_h}^h\right]\right) = Bel_i^h\left(\left[x_{k+1}^h, x_{r_h}^h\right]\right)$ if $k < r_h$.

Consequently, m_i^h dominates $m_{i'}^h$ according to the first belief dominance if and only if for all $k \in \{1, 2, \ldots, r_h\}, Pl_i^h\left(\left[x_k^h, x_{r_h}^h\right]\right) \geqslant Pl_{i'}^h\left(\left[x_k^h, x_{r_h}^h\right]\right)$ and $Bel_i^h\left(\left[x_k^h, x_{r_h}^h\right]\right) \geqslant Bel_{i'}^h\left(\left[x_k^h, x_{r_h}^h\right]\right)$. Using Denoeux's notations, m_i^h dominates $m_{i'}^h$ if and only if $m_i^h \gtrsim m_{i'}^h$ and $m_i^h \geqslant m_{i'}^h$.

3.3. Partial preferences between alternatives

As indicated above, the performances of the alternatives with respect to each criterion are expressed by BBAs. In multicriteria decision problems, it is always necessary to determine the preferences between alternatives on each criterion in order to apply a multicriteria procedure.

The first belief dominance concept allows concluding if m_i^h dominates $m_{i'}^h$ or not and consequently if the evaluation of a_i is at least as good as the evaluation of $a_{i'}$ or not. Furthermore, it permits establishing four partial preference situations between the actions performances:

- If $m_i^h \text{FBD} m_{i'}^h$ and $m_{i'}^h \text{FBD} m_i^h$, then m_i^h is indifferent to $m_{i'}^h$. So, a_i is indifferent to $a_{i'}$ on criterion g_h.
- If $m_i^h \text{FBD} m_{i'}^h$ and $m_{i'}^h \overline{\text{FBD}} m_i^h$, then m_i^h is strictly preferred to $m_{i'}^h$. So, a_i is strictly preferred to $a_{i'}$ on criterion g_h.
- If $m_i^h \overline{\text{FBD}} m_{i'}^h$ and $m_{i'}^h \text{FBD} m_i^h$, then $m_{i'}^h$ is strictly preferred to m_i^h. So, $a_{i'}$ is strictly preferred to a_i on criterion g_h.
- If $m_i^h \overline{\text{FBD}} m_{i'}^h$ and $m_{i'}^h \overline{\text{FBD}} m_i^h$, then m_i^h and $m_{i'}^h$ are incomparable. So, a_i and $a_{i'}$ are incomparable on criterion g_h.

The incomparability situation appears between two alternatives when their evaluations given by BBAs differ significantly. This information may be important especially in decision problems where incomparability is allowed.

Example 1. In order to illustrate the first belief dominance concept, let us consider the following simple example in which three actions are evaluated on the basis of two ordinal criteria g_1 and g_2. Let $X^1 = \{x_1^1, x_2^1, x_3^1\}$ and $X^2 = \{x_1^2, x_2^2, x_3^2\}$ be, respectively the assessment grades sets of g_1 and g_2 such as $x_1^1 \prec x_2^1 \prec x_3^1$ and $x_1^2 \prec x_2^2 \prec x_3^2$. The evaluations are expressed by BBAs and given in Table 1.

To apply the first belief dominance approach, it is first necessary to determine on each criterion the ascending and descending belief functions:

Table 1
BBA's characterizing the actions performances.

	g_1	g_2
a_1	$m_1^1(x_1^1) = 0.2$ $m_1^1(x_1^1, x_2^1) = 0.4$ $m_1^1(x_3^1) = 0.4$	$m_1^2(x_1^2, x_2^2) = 0.7$ $m_1^2(x_2^2, x_3^2) = 0.3$
a_2	$m_2^1(x_1^1) = 0.2$ $m_2^1(x_2^1) = 0.4$ $m_2^1(x_1^1, x_2^1, x_3^1) = 0.4$	$m_2^2(x_1^2) = 0.64$ $m_2^2(x_1^2, x_2^2, x_3^2) = 0.36$
a_3	$m_3^1(x_2^1, x_3^1) = 1$	$m_3^2(x_1^2) = 0.6$ $m_3^2(x_1^2, x_2^2) = 0.4$

- On criterion g_1:

$$a_1\begin{cases}\overrightarrow{Bel_1^1}\left(A_1^1\right)=\overrightarrow{Bel_1^1}(\{x_1^1\})=0.2\\ \overrightarrow{Bel_1^1}\left(A_2^1\right)=\overrightarrow{Bel_1^1}(\{x_1^1,x_2^1\})=0.6\\ \overrightarrow{Bel_1^1}\left(A_3^1\right)=\overrightarrow{Bel_1^1}(\{x_1^1,x_2^1,x_3^1\})=1\end{cases}\quad a_2\begin{cases}\overrightarrow{Bel_2^1}\left(A_1^1\right)=\overrightarrow{Bel_2^1}(\{x_1^1\})=0.2\\ \overrightarrow{Bel_2^1}\left(A_2^1\right)=\overrightarrow{Bel_2^1}(\{x_1^1,x_2^1\})=0.6\\ \overrightarrow{Bel_2^1}\left(A_3^1\right)=\overrightarrow{Bel_2^1}(\{x_1^1,x_2^1,x_3^1\})=1\end{cases}$$

$$a_3\begin{cases}\overrightarrow{Bel_3^1}\left(A_1^1\right)=\overrightarrow{Bel_3^1}(\{x_1^1\})=0\\ \overrightarrow{Bel_3^1}\left(A_2^1\right)=\overrightarrow{Bel_3^1}(\{x_1^1,x_2^1\})=0\\ \overrightarrow{Bel_3^1}\left(A_3^1\right)=\overrightarrow{Bel_3^1}(\{x_1^1,x_2^1,x_3^1\})=1\end{cases}$$

$$a_1\begin{cases}\overleftarrow{Bel_1^1}\left(B_1^1\right)=\overleftarrow{Bel_1^1}(\{x_3^1\}1)=0.4\\ \overleftarrow{Bel_1^1}\left(B_2^1\right)=\overleftarrow{Bel_1^1}(\{x_2^1,x_3^1\})=0.4\\ \overleftarrow{Bel_1^1}\left(B_3^1\right)=\overleftarrow{Bel_1^1}(\{x_1^1,x_2^1,x_3^1\})=1\end{cases}\quad a_2\begin{cases}\overleftarrow{Bel_2^1}\left(B_1^1\right)=\overleftarrow{Bel_2^1}(\{x_3^1\})=0\\ \overleftarrow{Bel_2^1}\left(B_2^1\right)=\overleftarrow{Bel_2^1}(\{x_2^1,x_3^1\})=0.4\\ \overleftarrow{Bel_2^1}\left(B_3^1\right)=\overleftarrow{Bel_2^1}(\{x_1^1,x_2^1,x_3^1\})=1\end{cases}$$

$$a_3\begin{cases}\overleftarrow{Bel_3^1}\left(B_1^1\right)=\overleftarrow{Bel_3^1}(\{x_3^1\})=0\\ \overleftarrow{Bel_3^1}\left(B_2^1\right)=\overleftarrow{Bel_3^1}(\{x_2^1,x_3^1\})=1\\ \overleftarrow{Bel_3^1}\left(B_3^1\right)=\overleftarrow{Bel_3^1}(\{x_1^1,x_2^1,x_3^1\})=1\end{cases}$$

- On criterion g_2:

$$a_1\begin{cases}\overrightarrow{Bel_1^2}\left(A_1^2\right)=\overrightarrow{Bel_1^2}(\{x_1^2\})=0\\ \overrightarrow{Bel_1^2}\left(A_2^2\right)=\overrightarrow{Bel_1^2}(\{x_1^2,x_2^2\})=0.7\\ \overrightarrow{Bel_1^2}\left(A_3^2\right)=\overrightarrow{Bel_1^2}(\{x_1^2,x_2^2,x_3^2\})=1\end{cases}\quad a_2\begin{cases}\overrightarrow{Bel_2^2}\left(A_1^2\right)=\overrightarrow{Bel_2^2}(\{x_1^2\})=0.64\\ \overrightarrow{Bel_2^2}\left(A_2^2\right)=\overrightarrow{Bel_2^2}(\{x_1^2,x_2^2\})=0.64\\ \overrightarrow{Bel_2^2}\left(A_3^2\right)=\overrightarrow{Bel_2^2}(\{x_1^2,x_2^2,x_3^2\})=1\end{cases}$$

$$a_3\begin{cases}\overrightarrow{Bel_3^2}\left(A_1^2\right)=\overrightarrow{Bel_3^2}(\{x_1^2\})=0.6\\ \overrightarrow{Bel_3^2}\left(A_2^2\right)=\overrightarrow{Bel_3^2}(\{x_1^2,x_2^2\})=1\\ \overrightarrow{Bel_3^2}\left(A_3^2\right)=\overrightarrow{Bel_3^2}(\{x_1^2,x_2^2,x_3^2\})=1\end{cases}$$

$$a_1\begin{cases}\overleftarrow{Bel_1^2}\left(B_1^2\right)=\overleftarrow{Bel_1^2}(\{x_3^2\})=0\\ \overleftarrow{Bel_1^2}\left(B_2^2\right)=\overleftarrow{Bel_1^2}(\{x_2^2,x_3^2\})=0.3\\ \overleftarrow{Bel_1^2}\left(B_3^2\right)=\overleftarrow{Bel_1^2}(\{x_1^2,x_2^2,x_3^2\})=1\end{cases}\quad a_2\begin{cases}\overleftarrow{Bel_2^2}\left(B_1^2\right)=\overleftarrow{Bel_2^2}(\{x_3^2\})=0\\ \overleftarrow{Bel_2^2}\left(B_2^2\right)=\overleftarrow{Bel_2^2}(\{x_2^2,x_3^2\})=0\\ \overleftarrow{Bel_2^2}\left(B_3^2\right)=\overleftarrow{Bel_2^2}(\{x_1^2,x_2^2,x_3^2\})=1\end{cases}$$

$$a_3\begin{cases}\overleftarrow{Bel_3^2}\left(B_1^2\right)=\overleftarrow{Bel_3^2}(\{x_3^2\})=0\\ \overleftarrow{Bel_3^2}\left(B_2^2\right)=\overleftarrow{Bel_3^2}(\{x_2^2,x_3^2\})=0\\ \overleftarrow{Bel_3^2}\left(B_3^2\right)=\overleftarrow{Bel_3^2}(\{x_1^2,x_2^2,x_3^2\})=1\end{cases}$$

Then, the first belief dominance concept is applied to compare on each criterion the actions performances. The observed belief dominances on each criterion are illustrated on Tables 2 and 3.

Finally, based on the observed belief dominances on each criterion, the partial preference situations between the actions are established:

- On criterion g_1:
  - $m_1^1\mathrm{FBD}m_2^1$ and $m_2^1\overline{\mathrm{FBD}}m_1^1$, then a_1 is strictly preferred to a_2;
  - $m_1^1\overline{\mathrm{FBD}}m_3^1$ and $m_3^1\overline{\mathrm{FBD}}m_1^1$, then a_1 and a_3 are incomparable;

Table 2
Observed belief dominances between the alternatives on criterion g_1.

	a_1	a_2	a_3
a_1	–	FBD	$\overline{\text{FBD}}$
a_2	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$
a_3	$\overline{\text{FBD}}$	FBD	–

Table 3
Observed belief dominances between the alternatives on criterion g_2.

	a_1	a_2	a_3
a_1	–	$\overline{\text{FBD}}$	FBD
a_2	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$
a_3	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–

  – $m_2^1 \overline{\text{FBD}} m_3^1$ and $m_3^1 \text{FBD} m_2^1$, then a_3 is strictly preferred to a_2.

- On criterion g_2:
  – $m_1^2 \overline{\text{FBD}} m_2^2$ and $m_2^2 \overline{\text{FBD}} m_1^2$, then a_1 and a_2 are incomparable;
  – $m_1^2 \text{FBD} m_3^2$ and $m_3^2 \overline{\text{FBD}} m_1^2$, then a_1 is strictly preferred to a_3;
  – $m_2^2 \overline{\text{FBD}} m_3^2$ and $m_3^2 \overline{\text{FBD}} m_2^2$, then a_2 and a_3 are incomparable.

3.4. First stochastic dominance: a particular case of first belief dominance

Of course, it seems natural that the first belief dominance definition coincides with the first stochastic dominance definition in the context of Bayesian belief functions. This result is contained in Denoeux's paper [35].

The first stochastic dominance is defined as follows. Let p_i^h and $p_{i'}^h$ denote the probability functions associating to the evaluations of actions a_i and $a_{i'}$ with respect to criterion g_h, and let P_i^h and $P_{i'}^h$ be, respectively their cumulative distributions. According to [14], p_i^h is said to dominate $p_{i'}^h$ according the first stochastic dominance if and only if for all $x_j^h \in X^h, P_i^h\left(x_j^h\right) \leqslant P_{i'}^h\left(x_j^h\right)$.

Proposition 1. *If Bel_i^h and $Bel_{i'}^h$ are two Bayesian belief functions over the frame X^h, then the two conditions of Definition 3 are equivalent.*

Proof 1. If Bel_i^h is a Bayesian belief function over the frame X^h, then $\overrightarrow{Bel_i^h}\left(A_k^h\right) + \overrightarrow{Bel_i^h}\left(\overline{A_k^h}\right) = 1$ for all $k = 0,1,\ldots,r_h$. Or, we have $\overline{A_k^h} = B_{r_h-k}^h = B_l^h$ for all $k = 0,1,\ldots,r_h$, thus $\overrightarrow{Bel_i^h}\left(A_k^h\right) + \overleftarrow{Bel_i^h}\left(B_l^h\right) = 1$. The first condition of Definition 3 can be written as $1 - \overrightarrow{Bel_i^h}\left(A_k^h\right) \geqslant 1 - \overrightarrow{Bel_{i'}^h}\left(A_k^h\right)$ for all $k = 1,2,\ldots,r_h$, then $\overleftarrow{Bel_i^h}\left(B_l^h\right) \geqslant \overleftarrow{Bel_{i'}^h}\left(B_l^h\right)$ for all $l = 0,1,\ldots,r_h - 1$. As a result, the two conditions of Definition 3 are equivalent when Bel_i^h and $Bel_{i'}^h$ are Bayesian belief functions.

Proposition 2. *If Bel_i^h and $Bel_{i'}^h$ are two Bayesian belief functions over the frame X^h, then the first stochastic dominance is a particular case of the first belief dominance.*

Proof 2. According to Proposition 1, since the two conditions of Definition 3 are equivalent, then m_i^h dominates $m_{i'}^h$ according the first belief dominance if and only if $\overrightarrow{Bel_i^h}\left(A_k^h\right) \leqslant \overrightarrow{Bel_{i'}^h}\left(A_k^h\right)$ for all $A_k^h \in \vec{S}(X^h)$. Bel_i^h is a Bayesian belief function, then all the focal sets of m_i^h are singletons. As a result, $\overrightarrow{Bel_i^h}\left(A_k^h\right) = \sum_{j=1}^k m_i^h\left(x_j^h\right) = P_i^h(x_k^h)$ for all $A_k^h \in \vec{S}(X^h)$. Therefore, m_i^h dominates $m_{i'}^h$ according our approach if and only if $P_i^h(x_j^h) \leqslant P_{i'}^h\left(x_j^h\right)$ for all $x_j^h \in X^h$. So, the first stochastic dominance is a particular case of the first belief dominance.

Before ending this section, let us note that the first belief dominance approach provides a uniform treatment not only for evaluations expressed by BBAs and probability functions, but also for evaluations given by possibility measures. In this case, the close connection between possibility distributions and consonant belief structures is exploited to transform the distribution containing the possible evaluations into a consonant belief structure, i.e., a BBA with nested focal sets [7].

Similarly, it is worth mentioning that the dominance (the non dominance, resp.) according the first belief dominance approach does not imply necessarily the dominance (the non dominance, resp.) according the stochastic one when applying the second approach on the pignistic probability functions associated to the considered BBAs. Indeed, it is possible to have two BBAs such that the first one dominates (does not dominate, resp.) the second one according the first belief dominance approach whereas this does not hold according the stochastic ordering. Example 2 illustrates these situations.

Example 2. The objective of this example is to show the difference between the first belief and stochastic dominance approaches when using the second approach to compare the pignistic probability functions derived from the considered BBAs.

Let us consider again the data of Example 1. It is easy to build for each BBA its related pignistic probability function $BetP_i^h$ using the pignistic transformation. Table 4 presents the pignistic probability distributions related to the BBAs that describe the actions performances.

The first stochastic dominance concept is used to compare the actions according to each criterion. In order to apply this approach, the cumulative distributions associated to the pignistic probability functions are computed. Then, the observed stochastic dominances between alternatives are determined on each criterion. Two situations are identified: FSD identifies the situations consistent with the first stochastic dominance theorem, and $\overline{\text{FSD}}$ designates those which are not consistent with this theorem. The observed stochastic dominances on each criterion are given respectively on Tables 5 and 6.

Based on these results and those achieved in Example 1, we remark that the first belief and stochastic dominance approaches lead to similar results except in the following cases:

- $m_1^1 \text{FBD} m_2^1$ whereas $BetP_1^1 \overline{\text{FSD}} BetP_2^1$,
- $m_3^1 \overline{\text{FBD}} m_1^1$ whereas $BetP_3^1 \text{FSD} BetP_1^1$,
- $m_1^2 \overline{\text{FBD}} m_2^2$ whereas $BetP_1^2 \text{FSD} BetP_2^2$,
- $m_2^2 \overline{\text{FBD}} m_3^2$ whereas $BetP_2^2 \text{FSD} BetP_3^2$.

Therefore, the dominance (the non dominance, resp.) according the first belief dominance approach does not imply necessarily the dominance (the non dominance, resp.) according the stochastic one when using the last approach on the pignistic probability functions.

4. The model

In this section, we propose a model inspired by ELECTRE I method [5,13] to build a binary outranking relation in uncertain imprecise and multi-experts contexts.

Table 4
Pignistic probability functions associated to the BBA's characterizing the actions performances.

	g_1	g_2
a_1	$BetP_1^1(x_1^1) = 0.4$ $BetP_1^1(x_2^1) = 0.2$ $BetP_1^1(x_3^1) = 0.4$	$BetP_1^2(x_1^2) = 0.35$ $BetP_1^2(x_2^2) = 0.5$ $BetP_1^2(x_3^2) = 0.15$
a_2	$BetP_2^1(x_1^1) = 0.333$ $BetP_2^1(x_2^1) = 0.533$ $BetP_2^1(x_3^1) = 0.133$	$BetP_2^2(x_1^2) = 0.76$ $BetP_2^2(x_2^2) = 0.12$ $BetP_2^2(x_3^2) = 0.12$
a_3	$BetP_3^1(x_2^1) = 0.5$ $BetP_3^1(x_3^1) = 0.5$	$BetP_3^2(x_1^2) = 0.8$ $BetP_3^2(x_2^2) = 0.2$

Table 5
Observed stochastic dominances between the alternatives on criterion g_1.

	a_1	a_2	a_3
a_1	–	$\overline{\text{FSD}}$	$\overline{\text{FSD}}$
a_2	$\overline{\text{FSD}}$	–	$\overline{\text{FSD}}$
a_3	FSD	FSD	–

Table 6
Observed stochastic dominances between the alternatives on criterion g_2.

	a_1	a_2	a_3
a_1	–	FSD	FSD
a_2	$\overline{\text{FSD}}$	–	FSD
a_3	$\overline{\text{FSD}}$	$\overline{\text{FSD}}$	–

ELECTRE I is a multicriteria outranking method that consists in choosing a subset of actions considered as the best according to the criteria set. It is close to the Condorcet rule that has been developed in the context of voting theory. The underlying idea is first to build a binary relation denoted S between pairs of alternatives and then to exploit this relation to select the subset of "best" actions. Intuitively, $a_iSa_{i'}$ is interpreted as "a_i is at least as good as $a_{i'}$". Therefore, a_i must be at least as good as $a_{i'}$ on a sufficiently important coalition of criteria without being much worst on any criterion. More formally, we assume that:

- All the criteria have to be maximized.
- The decision maker has provided a set of weights, denoted w_h, representing the relative importance of the criteria. Furthermore, we assume that$w_h \geqslant 0(h = 1,2, \ldots ,q)$ and $\sum_{h=1}^{q} w_h = 1$.
- The decision maker has provided an acceptability threshold denoted c and a veto threshold denoted v.

Following the intuition explained before, both a concordance index denoted $C(a_iSa_{i'})$, and a discordance index denoted $D(a_iSa_{i'})$, are computed as follows:

$$C(a_iSa_{i'}) = \sum_{\{h:g_h(a_i) \geqslant g_h(a_{i'})\}} w_h \tag{9}$$

$$D(a_iSa_{i'}) = \max_{\{h:g_h(a_i) < g_h(a_{i'})\}} (g_h(a_{i'}) - g_h(a_i)) \tag{10}$$

Based on these two indices, one is able to define the S relation in the following way:

$$a_iSa_{i'} \iff \begin{cases} C(a_iSa_{i'}) \geqslant c \\ D(a_iSa_{i'}) \leqslant v \end{cases} \tag{11}$$

The selection of "best" alternatives is based on the kernel of the graph induced by all the binary outranking relations.

In what follows, we will describe our model inspired by ELECTRE I. For the sake of simplicity, we will only consider the concordance index when the binary outranking relations are built. At first, we assume that the experts are non-equivalent in their importance within the group. We further assume that several experts have expressed their assessments for actions set with respect to criteria set. This information is provided on the form of BBAs. In what follows, let:

- w_f be the importance of expert E_f within the group (with $f = 1,2, \ldots ,s$);
- $m^h_{i|f}$ be the BBA that represents the evaluation of action a_i according criterion g_h and given by expert E_f.

Furthermore, we propose to model the criteria weights by means of a BBA. Let m_G be the BBA that defines the inter-criteria information and Bel_G its associated belief function. Intuitively, $m_G(T)$ represents the weight committed exactly to the criteria set T and $Bel_G(T)$ represents the total weight assigned to this set. Thus, in addition to the usual weights on criteria taken separately, weights on coalitions of criteria can also be defined. The weight assigned to a criteria set quantifies the synergy between the criteria that compose this set. Of course, if Bel_G is as Bayesian belief function, m_G is nothing else than the weighted sum. In this case, we consider that there is no synergy between criteria.

The steps of the proposed model are the following. At first, the first belief dominance approach is applied by each expert to compare his/her individual BBAs. Based on the observed individual belief dominances on each criterion, each expert builds his/her individual binary outranking relations between the actions and so his/her individual outranking graph. Then, the algorithm AL3 proposed by Jabeur and Martel [19] is used to aggregate the individual outranking graphs in order to establish a collective one which is at minimum distance from those. This algorithm takes into account the coefficients of the experts' relative importance. Finally, the collective outranking graph is exploited to determine the subset of the "best" alternatives.

4.1. Comparison between the individual BBAs

In the first step of the model, the individual BBAs given by each expert and characterizing the actions performances on each criterion are compared. The first belief dominance concept is used by each expert to perform comparisons between the pairs of alternatives on each criterion.

As indicated previously, this approach allows concluding clearly if a given action is at least as good as another action on a given criterion or not, i.e., if there is a dominance according the first belief dominance approach or not. Once the belief dominances between the alternatives on each criterion are determined by each expert, they are exploited in the next step to build the individual binary outranking relations.

4.2. Building the individual outranking graphs

The use of the first belief dominance concept by each expert E_f allows identifying the coalition of criteria, denoted D_f, for which a given action a_i is at least as good as another action $a_{i'}$. More formally:

$$D_f = \left\{ g_h \in G | m^h_{i|f} \mathrm{FBD} m^h_{i'|f} \right\} \tag{12}$$

Intuitively, $Bel_G(D_f)$ quantifies the total belief associated to the fact that the action a_i is at least as good as the action $a_{i'}$ for all the criteria in D_f according expert E_f. Thus, we will call $Bel_G(D_f)$ the concordance index.

Of course, if the BBAs are such that the total belief is restricted to a unique singleton and if Bel_G is as Bayesian belief function, $Bel_G(D_f)$ is nothing else than the concordance index computed as in the ELECTRE I method.

In the same way, the application of the first belief dominance approach by each expert E_f permits identifying the coalition of criteria, denoted Q_f, for which the actions a_i and $a_{i'}$ are incomparable (i.e., $m^h_{i|f}$ and $m^h_{i'|f}$ do not dominate each other). More formally:

$$Q_f = \left\{ g_h \in G | m^h_{i|f} \overline{\mathrm{FBD}} m^h_{i'|f} \text{ and } m^h_{i'|f} \overline{\mathrm{FBD}} m^h_{i|f} \right\} \tag{13}$$

Therefore, we define a new degree called the incomparability index and denoted $Bel_G(Q_f)$. This degree quantifies the total belief associated to the fact that the actions a_i and $a_{i'}$ are incomparable for all the criteria in Q_f according expert E_f.

The concordance and incomparability indexes are exploited to build the individual binary outranking relations. These indexes are compared, respectively to concordance and incomparability thresholds denoted, respectively c and r and which are fixed according the type of the application. Therefore, a_i outranks $a_{i'}$ (denoted $a_i S a_{i'}$) if and only if $Bel_G(D_f)$ is greater than or equal to the concordance threshold and $Bel_G(Q_f)$ is lesser than or equal to the incomparability threshold. More formally:

$$a_i S a_{i'} \iff \begin{cases} Bel_G(D_f) \geqslant c \\ Bel_G(Q_f) \leqslant r \end{cases} \tag{14}$$

The experts can agree on the values of thresholds or cannot. In such situation, each expert can propose and use his/her own values of thresholds. Finally, based on his/her individual binary outranking relations, each expert builds his/her individual outranking graph.

4.3. Aggregation of the individual outranking graphs

In this step, the individual outranking graphs are aggregated in order to establish a collective one from which the "best" alternatives subset is determined. The aggregation is performed using the algorithm AL3 that has been applied in the context of group decision making [18].

The algorithm AL3 takes into account the coefficients of the experts' relative importance within the group. Furthermore, it allows determining for each pair of alternatives the collective preference relation $H^* \in \{\succ, \prec, \approx, ?\}$ where $\succ$ is the strict preference relation, $\prec$ is the inverse strict preference relation, $\approx$ the indifference relation and ? is the incomparability relation. So before aggregating, the individual outranking graphs should be transformed into individual preference graphs where the alternatives are represented by nodes and individual binary preference relations [18]. The only difference between these graphs is in the representation of the indifference relation. In fact, in an outranking graph, the indifference is represented by two arrows which have opposite direction while it is represented by an arrow with double directions in a preference graph [18]. The algorithm AL3 determines, for each pair of alternatives $(a_i, a_{i'})$, the nearest collective preference relation $H^* \in \{\succ, \prec, \approx, ?\}$ to the individual ones. For this purpose, a divergence index $\Phi^H(a_i, a_{i'})$ that measures the deviation between the collective preference relation $H \in \{\succ, \prec, \approx, ?\}$ and each individual one $H_f(a_i, a_{i'})$ is calculated as follows:

$$\Phi^H(a_i, a_{i'}) = \sum_{f=1}^{s} w_f \cdot \Delta(H, H_f(a_i, a_{i'})) \tag{15}$$

where w_f is the importance of expert E_f within the group and $\Delta(H, H_f(a_i, a_{i'}))$ is the distance measure between the collective preference relation H and the individual one $H_f(a_i, a_{i'})$ suggested in [20]. The numerical values of the distance measure are given in Table 7.

Then, we identify the collective relation $H^*(a_i, a_{i'})$ that minimizes the divergence indexes, i.e.:

$$H^* = \underset{H \in \{\succ, \prec, \approx, ?\}}{Arg\min}\, \Phi^H(a_i, a_{i'}) \tag{16}$$

Table 7
Numerical values of the distance measure Δ.

Δ	$a_i \approx a_{i'}$	$a_i \succ a_{i'}$	$a_i ? a_{i'}$	$a_i \prec a_{i'}$
$a_i \approx a_{i'}$	$\Delta(\approx,\approx) = 0$	$\Delta(\approx,\succ) = 1$	$\Delta(\approx,?) = 4/3$	$\Delta(\approx,\prec) = 1$
$a_i \succ a_{i'}$	$\Delta(\succ,\approx) = 1$	$\Delta(\succ,\succ) = 0$	$\Delta(\succ,?) = 4/3$	$\Delta(\succ,\prec) = 5/3$
$a_i ? a_{i'}$	$\Delta(?,\approx) = 4/3$	$\Delta(?,\succ) = 4/3$	$\Delta(?,?) = 0$	$\Delta(?,\prec) = 4/3$
$a_i \prec a_{i'}$	$\Delta(\prec,\approx) = 1$	$\Delta(\prec,\succ) = 5/3$	$\Delta(\prec,?) = 4/3$	$\Delta(\prec,\prec) = 0$

Once the collective preference relations between all pairs of alternatives are determined, we build the collective preference graph. Therefore, it is easy to deduce the collective outranking graph.

Finally, let us note that the application of the algorithm AL3 may produce several collective preference relations at minimum distance from the individual ones. Then, several collective outranking graphs can be obtained.

4.4. Determination of the "best" alternatives subset

In the last step of the model, the collective outranking graph is exploited to identify the "best" alternatives. Since several collective outranking graphs can be deducted, several subsets of "best" alternatives can be proposed. For both situations, the experts can either not accept the unique subset obtained by a mathematical processing or fail to build, from several subsets, a single subset of the "best" alternatives. In order to treat both of these situations, Jabeur and Martel propose an interactive and iterative procedure that helps the experts to reach a consensus on the "best" alternatives subset. More details can be found in [18].

Finally, it is worth mentioning that another manner to the aggregation problem is the use of a combination rule offered by evidence theory to aggregate the individual BBAs given by the experts. Let m_i^h be the BBA resulting from the combination of $m_{i|1}^h, m_{i|2}^h, \ldots$ and $m_{i|s}^h$. Obviously, m_i^h represents the collective evaluation of action a_i according criterion g_h.

The combination rule used for the aggregation of the individual BBAs should be commutative and associative. Therefore:

- If the BBAs are distinct, we suggest the Dempster's rule of combination to take into account independencies between experts.
- If the BBAs are nondistinct, we suggest the normalized cautious rule of combination to take into account dependencies between experts.

Once the collective BBAs are determined, the first belief dominance approach is used to compare these BBAs and so to build the collective outranking graph of which the "best" alternatives subset is identified. However, both combination rules that we have suggested do not respect in some situations the unanimity property which is a natural condition of an aggregation operator. Formally, this property means that:

- If $m_{i|f}^h \mathrm{FBD} m_{i'|f}^h$ for all $f \in \{1,2,\ldots,s\}$, then $m_i^h \mathrm{FBD} m_{i'}^h$.
- If $m_{i|f}^h \overline{\mathrm{FBD}} m_{i'|f}^h$ for all $f \in \{1,2,\ldots,s\}$, then $m_i^h \overline{\mathrm{FBD}} m_{i'}^h$.

That's why we have adapted the algorithm AL3 for the aggregation.

Two counter-examples to the unanimity property are introduced below (see Examples 3 and 4). In the first one, we have used the Dempster's rule to combine distinct BBAs. In the second one, we have used the normalized cautious rule to aggregate nondistinct BBAs.

Example 3. Let us consider two alternatives of which the evaluations are expressed by distinct BBAs and given by two experts. In this example, we will consider only the evaluations on criterion g_1. Let x_1^1, x_2^1, x_3^1 and x_4^1 be the assessment grades set of g_1 such as $x_1^1 \prec x_2^1 \prec x_3^1 \prec x_4^1$.

The Dempster's rule of combination is used in this example to aggregate the BBAs induced by the experts for every action on criterion g_1. The objective is to yield a combined BBA that represents the performance of every action on criterion g_1. Then, the first belief dominance approach is applied to compare the BBAs given by each expert and the combined BBAs. The results are given in Table 8.

As can be noticed, the experts agree that a_1 dominates a_2 on criterion g_1 according the first belief dominance concept. However, when we apply this approach on the combined BBAs, we obtain: a_1 does not dominate a_2 on g_1. Then, the Dempster's rule does not respect the unanimity property.

Table 8
Counter-example to the unanimity property: case of distinct BBAs.

	a_1	a_2	The observed belief dominances
The BBA's given by expert 1	$m_{1\|1}^1(x_2^1, x_3^1) = 0.5$ $m_{1\|1}^1(x_4^1) = 0.5$	$m_{2\|1}^1(x_2^1) = 0.5$ $m_{2\|1}^1(x_3^1) = 0.5$	$m_{1\|1}^1 \mathrm{FBD} m_{2\|1}^1$
The BBA's given by expert 2	$m_{1\|2}^1(x_2^1, x_3^1) = 1$	$m_{2\|2}^1(x_1^1, x_2^1) = 0.5$ $m_{2\|2}^1(x_2^1, x_3^1) = 0.5$	$m_{1\|2}^1 \mathrm{FBD} m_{2\|2}^1$
The combined BBA's given by the two experts	$m_1^1(x_2^1, x_3^1) = 1$	$m_2^1(x_2^1) = 0.67$ $m_2^1(x_3^1) = 0.33$	$m_1^1 \overline{\mathrm{FBD}} m_2^1$

Table 9
Counter-example to the unanimity property: case of nondistinct BBAs.

	a_1	a_2	The observed belief dominances
The BBA's given by expert 1	$m^1_{1\|1}(x^1_2) = 0.1$ $m^1_{1\|1}(x^1_1, x^1_2) = 0.9$	$m^1_{2\|1}(x^1_1) = 0.6$ $m^1_{2\|1}(x^1_2) = 0.2$ $m^1_{2\|1}(x^1_1, x^1_2) = 0.2$	$m^1_{1\|1}\overline{\text{FBD}}m^1_{2\|1}$
The BBA's given by expert 2	$m^1_{1\|2}(x^1_1) = 0.5$ $m^1_{1\|2}(x^1_2) = 0.2$ $m^1_{1\|2}(x^1_1, x^1_2) = 0.3$	$m^1_{2\|2}(x^1_2) = 0.1$ $m^1_{2\|2}(x^1_1, x^1_2) = 0.9$	$m^1_{1\|2}\overline{\text{FBD}}m^1_{2\|2}$
The combined BBA's given by the two experts	$m^1_1(x^1_1) = 0.5$ $m^1_1(x^1_2) = 0.2$ $m^1_1(x^1_1, x^1_2) = 0.3$	$m^1_2(x^1_1) = 0.6$ $m^1_2(x^1_2) = 0.2$ $m^1_2(x^1_1, x^1_2) = 0.2$	$m^1_1\text{FBD}m^1_2$

Example 4. Let us consider two alternatives of which the evaluations are expressed by nondistinct BBAs and given by two experts. In this example, we will consider only the evaluations on criterion g_1. Let x^1_1 and x^1_2 be the assessment grades set of g_1 such as $x^1_1 \prec x^1_2$.

The normalized cautious rule of combination is used in this example to aggregate the BBAs induced by the experts for every action on criterion g_1. The objective is to yield a combined BBA that represents the performance of every action on criterion g_1. Then, the first belief dominance approach is applied to compare the BBAs given by each expert and the combined BBAs. The results are given in Table 9.

As can be noticed, the experts agree that a_1 does not dominate a_2 on criterion g_1 according the first belief dominance concept. However, when we apply this approach on the combined BBAs, we obtain: a_1 dominates a_2 on g_1. Then, the normalized cautious rule does not respect the unanimity property.

5. Illustrative example

In order to illustrate the model, let us consider the following example. A public establishment, which has for mission to sustain the innovation and the growth of the SME (small and medium enterprises), announces its call for bids destined to the industrial enterprises.

The projects transmitted by the enterprises are submitted to a committee composed of three experts referred to as experts 1, 2 and 3. The coefficients of their relative importance within the group are, respectively 0.5, 0.3 and 0.2.

The projects are evaluated on the basis of four ordinal criteria to maximize which are the following:

- Originality of the project or the activity, i.e., the innovation degree brought by the project (new products, new procedures, improved products, improved procedures).
- Competitiveness, i.e., the capacity of the enterprise to face with success the concurrence on the national and/or international markets. The evaluation of the project on this criterion is determined by analysing the product supply and its adequacy in relation to the market, the market and the concurrence related to the activity, ...
- Profitability, i.e., the capacity of the enterprise to generate positive results by using rationally the resources.

Table 10
Criteria and the assessment grades.

Criterion	Assessment grades
g_1: Originality of the project or the activity	x^1_1: No innovation in the project
	x^1_2: There is the improvement in the project
	x^1_3: There is the innovation in the project
	x^1_4: Absolutely an original project
g_2: Competitiveness	x^2_1: Not competitive
	x^2_2: Competitive
g_3: Profitability	x^3_1: Very low
	x^3_2: Low
	x^3_3: Average
	x^3_4: High
	x^3_5: Very High
g_4: Creation of employment	x^4_1: Limited number of jobs
	x^4_2: Average number of jobs
	x^4_3: High number of jobs

- Creation of employment, i.e., the capacity of the enterprise to generate jobs on several levels (strategic, intermediate, operational).

The criteria are proposed by the public establishment. Table 10 presents the assessment grades of each criterion which are given from the least preferred to the most preferred.

Finally, the inter-criteria information is given by a BBA defined as follows:

$$m_G(g_1) = 0.2$$
$$m_G(g_2) = 0.2$$
$$m_G(g_1, g_2) = 0.2$$
$$m_G(g_3) = 0.3$$
$$m_G(g_4) = 0.1$$

As can be noticed, the total importance allocated to the coalition of criteria g_1 and g_2 (the originality of the project or the activity and the competitiveness) is equal to 0.6 while the importance related to each of the single criteria is equal to 0.2, i.e., their sum is lower than 0.6.

We propose to consider 5 projects. The evaluation of each project with respect to each criterion is given by a BBA. We suppose that each expert can express individually the assessments for all the projects with respect to all the criteria. Tables 11–13 present the BBAs characterizing the evaluations of the projects given, respectively by experts 1, 2 and 3. For instance, the evaluations of project a_1 are established by the experts as follows:

- On criterion g_1, experts 1 and 3 hesitate between the third and the fourth assessment grades. They are sure that the project is either innovating or absolutely original without being able to refine their judgment whereas expert 2 is certain that the project is innovating.

Table 11
BBA's characterizing the evaluations of the projets given by expert 1.

	g_1	g_2	g_3	g_4
a_1	$m^1_{1\|1}(x^1_3) = 0.8$ $m^1_{1\|1}(x^1_3, x^1_4) = 0.2$	$m^2_{1\|1}(x^2_2) = 1$	$m^3_{1\|1}(x^1_1) = 0.1$ $m^3_{1\|1}(x^1_1, x^1_2, x^1_3) = 0.9$	$m^4_{1\|1}(x^4_1) = 1$
a_2	$m^1_{2\|1}(x^1_1) = 0.6$ $m^1_{2\|1}(x^1_1, x^1_2) = 0.4$	$m^2_{2\|1}(x^2_1) = 1$	$m^3_{2\|1}(x^3_4) = 0.6$ $m^3_{2\|1}(x^3_4, x^3_5) = 0.4$	$m^4_{2\|1}(x^4_2) = 1$
a_3	$m^1_{3\|1}(x^1_2) = 1$	$m^2_{3\|1}(x^2_1) = 0.3$ $m^2_{3\|1}(x^2_1, x^2_2) = 0.7$	$m^3_{3\|1}(x^3_2) = 0.6$ $m^3_{3\|1}(x^3_3) = 0.4$	$m^4_{3\|1}(x^4_2) = 0.7$ $m^4_{3\|1}(x^4_2, x^4_3) = 0.3$
a_4	$m^1_{4\|1}(x^1_1, x^1_2, x^1_3, x^1_4) = 1$	$m^2_{4\|1}(x^2_1, x^2_2) = 1$	$m^3_{4\|1}(x^3_1, x^3_2) = 0.7$ $m^3_{4\|1}(x^3_2, x^3_3) = 0.3$	$m^4_{4\|1}(x^4_2) = 0.8$ $m^4_{4\|1}(x^4_3) = 0.2$
a_5	$m^1_{5\|1}(x^1_2) = 1$	$m^2_{5\|1}(x^2_2) = 1$	$m^3_{5\|1}(x^3_3) = 0.6$ $m^3_{5\|1}(x^3_3, x^3_4) = 0.4$	$m^4_{5\|1}(x^4_1) = 0.9$ $m^4_{5\|1}(x^4_1, x^4_2) = 0.1$

Table 12
BBA's characterizing the evaluations of the projets given by expert 2.

	g_1	g_2	g_3	g_4
a_1	$m^1_{1\|2}(x^1_3) = 1$	$m^2_{1\|2}(x^2_2) = 1$	$m^3_{1\|2}(x^3_3, x^3_4) = 0.8$ $m^3_{1\|2}(x^3_5) = 0.2$	$m^4_{1\|2}(x^4_1, x^4_2, x^4_3) = 1$
a_2	$m^1_{2\|2}(x^1_1, x^1_2) = 1$	$m^2_{2\|2}(x^2_1, x^2_2) = 1$	$m^3_{2\|2}(x^3_1, x^3_2, x^3_3, x^3_4, x^3_5) = 1$	$m^4_{2\|2}(x^4_2) = 1$
a_3	$m^1_{3\|2}(x^1_2) = 0.6$ $m^1_{3\|2}(x^1_3) = 0.4$	$m^2_{3\|2}(x^2_1) = 0.2$ $m^2_{3\|2}(x^2_2) = 0.8$	$m^3_{3\|2}(x^3_3) = 1$	$m^4_{3\|2}(x^4_2) = 0.7$ $m^4_{3\|2}(x^4_2, x^4_3) = 0.3$
a_4	$m^1_{4\|2}(x^1_1) = 1$	$m^2_{4\|2}(x^2_2) = 1$	$m^3_{4\|2}(x^3_3) = 0.9$ $m^3_{4\|2}(x^3_3, x^3_4) = 0.1$	$m^4_{4\|2}(x^4_2) = 0.8$ $m^4_{4\|2}(x^4_2, x^4_3) = 0.2$
a_5	$m^1_{5\|2}(x^1_2) = 1$	$m^2_{5\|2}(x^2_2) = 1$	$m^3_{5\|2}(x^3_3, x^3_4) = 1$	$m^4_{5\|2}(x^4_1) = 0.5$ $m^4_{5\|2}(x^4_2) = 0.5$

Table 13
BBA's characterizing the evaluations of the projets given by expert 3.

	g_1	g_2	g_3	g_4
a_1	$m^1_{1\|3}(x^1_3) = 0.94$ $m^1_{1\|3}(x^1_3, x^1_4) = 0.06$	$m^2_{1\|3}(x^2_2) = 1$	$m^3_{1\|3}(x^3_1) = 0.04$ $m^3_{1\|3}(x^3_3) = 0.6$ $m^3_{1\|3}(x^3_1x^3_2, x^3_3) = 0.36$	$m^4_{1\|3}(x^4_1) = 1$
a_2	$m^1_{2\|3}(x^1_1) = 0.6$ $m^1_{2\|3}(x^1_1, x^1_2) = 0.4$	$m^2_{2\|3}(x^2_1) = 1$	$m^3_{2\|3}(x^3_4) = 0.6$ $m^3_{2\|3}(x^3_4, x^3_5) = 0.4$	$m^4_{2\|3}(x^4_2) = 1$
a_3	$m^1_{3\|3}(x^1_2) = 1$	$m^2_{3\|3}(x^2_1) = 0.28$ $m^2_{3\|3}(x^2_2) = 0.44$ $m^2_{3\|3}(x^2_1, x^2_2) = 0.28$	$m^3_{3\|3}(x^3_2) = 0.33$ $m^3_{3\|3}(x^3_3) = 0.67$	$m^4_{3\|3}(x^4_2) = 0.84$ $m^4_{3\|3}(x^4_2, x^4_3) = 0.16$
a_4	$m^1_{4\|3}(x^1_1) = 0.67$ $m^1_{4\|3}(x^1_1, x^1_2, x^1_3, x^1_4) = 0.33$	$m^2_{4\|3}(x^2_2) = 0.67$ $m^2_{4\|3}(x^2_1, x^2_2) = 0.33$	$m^3_{4\|3}(x^3_1, x^3_2) = 0.44$ $m^3_{4\|3}(x^3_2, x^3_3) = 0.19$ $m^3_{4\|3}(x^3_3) = 0.37$	$m^4_{4\|3}(x^4_2) = 0.9$ $m^4_{4\|3}(x^4_3) = 0.1$
a_5	$m^1_{5\|3}(x^1_2) = 1$	$m^2_{5\|3}(x^2_2) = 1$	$m^3_{5\|3}(x^3_3) = 0.6$ $m^3_{5\|3}(x^3_3, x^3_4) = 0.4$	$m^4_{5\|3}(x^4_1) = 0.9$ $m^4_{5\|3}(x^4_2) = 0.05$ $m^4_{5\|3}(x^4_1, x^4_2) = 0.05$

- On criterion g_2, the three experts are certain that the enterprise is competitive.
- On criterion g_3, experts 1 and 3 hesitate between the first, the second and the third assessment grades whereas expert 2 hesitates between the third, the fourth and the fifth ones. The formers are sure that the project profitability is very low, low or average whereas the latter is sure that it is average, high or very high. The three experts are unable to refine their judgments.

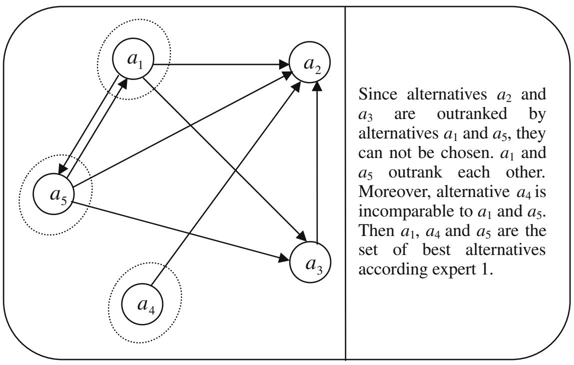

Fig. 1. The outranking graph induced by expert 1.

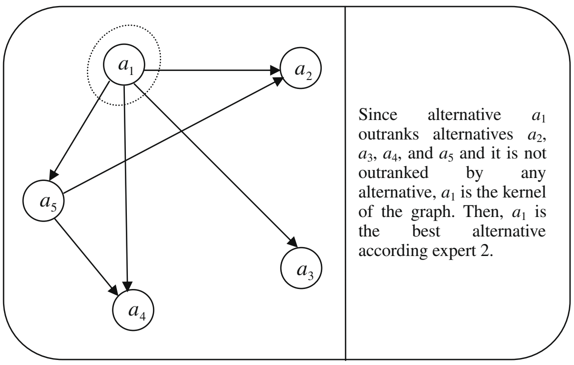

Fig. 2. The outranking graph induced by expert 2.

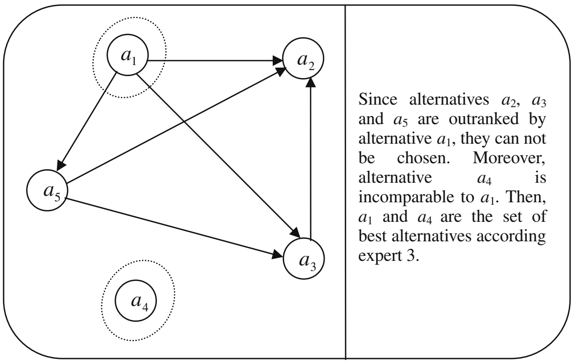

Fig. 3. The outranking graph induced by expert 3.

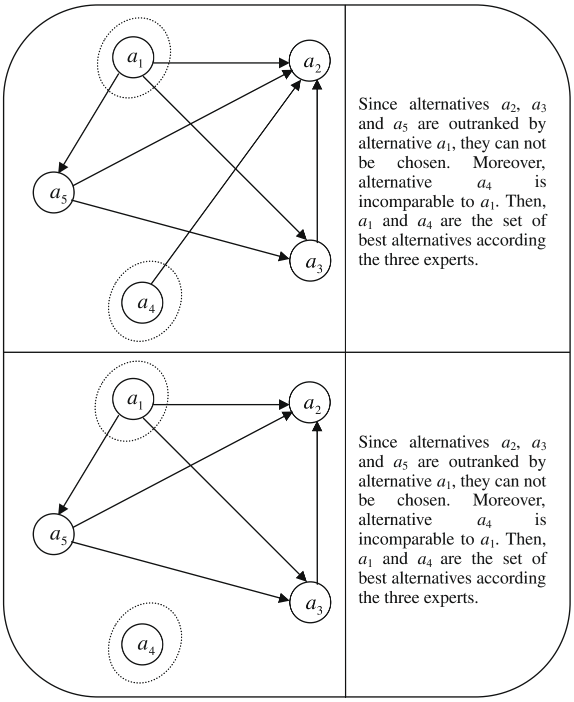

Fig. 4. Collective outranking graphs.

- On criterion g_4, experts 1 and 3 are certain that the project creates a limited number of jobs whereas expert 2 does not express his/her assessments on the project. That's why the total masse is assigned to the set of assessment grades, i.e., the frame X^4 (total ignorance case).

The first belief dominance approach is applied by each expert to compare, on each criterion, the different pairs of alternatives. Based on the observed individual belief dominances on each criterion, the concordance and incomparability indexes are computed by each expert for each pair of alternatives using the BBA representing the inter-criteria information. Appendixs A, B and C illustrate the partial belief dominances between alternatives as well as the concordance and incomparability indexes given, respectively by experts 1, 2 and 3.

Therefore, each expert builds his/her individual binary outranking relations between the actions and so his/her individual outranking graph. In this example, we suppose that the experts agree on the values of the concordance and incomparability thresholds which are equal, respectively to 0.6 and 0.4. Figs. 1–3 give the outranking graphs built, respectively by experts 1, 2 and 3.

Then, the individual outranking graphs are aggregated using the algorithm AL3 to obtain the collective outranking graph. The application of this algorithm produces for the pair of alternatives (a_2,a_4) two distinct preference relations which are at minimum distance from the individual ones. In fact, we can consider a_2 and a_4 as incomparable or we can accept that a_4 is strictly preferred to a_2. So, we obtain two collective outranking graphs that differ between them on the arrow (a_4,a_2). Fig. 4 shows these two collective outranking graphs.

Finally, the collective outranking graphs are exploited to identify the subset of the "best" alternatives. In both situations, the collective decision is to choose alternatives a_1 and a_4.

As we remark, the collective decision reflects well the individual ones since alternative a_1 is chosen individually by each expert and alternative a_4 is chosen by experts 1 and 3. Alternative a_5 is among the best alternatives according expert 1 whereas it is not chosen by experts 2 and 3.

6. Conclusion

In this paper, we have addressed the question of building a binary outranking relation in uncertain, imprecise and multi-experts contexts. Evidence theory offers convenient tools to tackle such kind of problems. At first, the concept of BBA allows experts to express freely their assessments and even to represent the total ignorance. In order to compare the BBAs associated to two actions for a given criterion, the concept of the first belief dominance has been introduced. We have shown that it is a natural extension of the first stochastic dominance approach. In addition, the concept of BBA has been used to represent the criteria weights. Finally, a model inspired by ELECTRE I has been proposed and illustrated on a pedagogical example.

At this point, we have illustrated the benefits of using evidence theory in these kinds of problems. Of course there are still many directions for future research. Among others, we can mention the extension of the first belief dominance concept to second and third degrees or the development of combination rules that respect the unanimity principle.

Acknowledgement

The authors thank the anonymous referees for their valuable comments and suggestions that helped to improve this paper. Thanks also to Professor Thierry Denoeux for his constructive comments regarding the concept of first belief dominance.

Appendix A. Expert 1: The partial belief dominances between alternatives/The concordance and incomparability indexes

Tables 14–19.

Appendix B. Expert 2: The partial belief dominances between alternatives/The concordance and incomparability indexes

Tables 20–25.

Table 14
The observed belief dominances between alternatives on criterion g_1.

	a_1	a_2	a_3	a_4	a_5
a_1	–	FBD	FBD	$\overline{\text{FBD}}$	FBD
a_2	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_3	$\overline{\text{FBD}}$	FBD	–	$\overline{\text{FBD}}$	FBD
a_4	$\overline{\text{FBD}}$	FBD	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$
a_5	$\overline{\text{FBD}}$	FBD	FBD	$\overline{\text{FBD}}$	–

Table 15
The observed belief dominances between alternatives on criterion g_2.

	a_1	a_2	a_3	a_4	a_5
a_1	–	FBD	FBD	FBD	FBD
a_2	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_3	$\overline{\text{FBD}}$	FBD	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_4	$\overline{\text{FBD}}$	FBD	FBD	–	$\overline{\text{FBD}}$
a_5	FBD	FBD	FBD	FBD	–

Table 16
The observed belief dominances between alternatives on criterion g_3.

	a_1	a_2	a_3	a_4	a_5
a_1	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_2	FBD	–	FBD	FBD	FBD
a_3	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–	FBD	$\overline{\text{FBD}}$
a_4	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$
a_5	FBD	$\overline{\text{FBD}}$	FBD	FBD	–

Table 17
The observed belief dominances between alternatives on criterion g_4.

	a_1	a_2	a_3	a_4	a_5
a_1	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_2	FBD	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	FBD
a_3	FBD	FBD	–	$\overline{\text{FBD}}$	FBD
a_4	FBD	FBD	$\overline{\text{FBD}}$	–	FBD
a_5	FBD	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–

Table 18
The concordance indexes.

	a_1	a_2	a_3	a_4	a_5
a_1	–	0.6	0.6	0.2	0.6
a_2	0.4	–	0.3	0.3	0.4
a_3	0.1	0.7	–	0.3	0.3
a_4	0.1	0.7	0.2	–	0.1
a_5	0.6	0.6	0.9	0.5	–

Table 19
The incomparability indexes.

	a_1	a_2	a_3	a_4	a_5
a_1	–	0	0.3	0.5	0
a_2	0	–	0	0	0
a_3	0.3	0	–	0.3	0
a_4	0.5	0	0.3	–	0.2
a_5	0	0	0	0.2	–

Table 20
The observed belief dominances between alternatives on criterion g_1.

	a_1	a_2	a_3	a_4	a_5
a_1	–	FBD	FBD	FBD	FBD
a_2	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$	FBD	$\overline{\text{FBD}}$
a_3	$\overline{\text{FBD}}$	FBD	–	FBD	FBD
a_4	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$
a_5	$\overline{\text{FBD}}$	FBD	$\overline{\text{FBD}}$	FBD	–

Table 21
The observed belief dominances between alternatives on criterion g_2.

	a_1	a_2	a_3	a_4	a_5
a_1	–	FBD	FBD	FBD	FBD
a_2	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_3	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_4	FBD	FBD	FBD	–	FBD
a_5	FBD	FBD	FBD	FBD	–

Table 22
The observed belief dominances between alternatives on criterion g_3.

	a_1	a_2	a_3	a_4	a_5
a_1	–	$\overline{\text{FBD}}$	FBD	FBD	FBD
a_2	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_3	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_4	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	FBD	–	$\overline{\text{FBD}}$
a_5	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	FBD	FBD	–

Table 23
The observed belief dominances between alternatives on criterion g_4.

	a_1	a_2	a_3	a_4	a_5
a_1	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_2	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	FBD
a_3	$\overline{\text{FBD}}$	FBD	–	FBD	FBD
a_4	$\overline{\text{FBD}}$	FBD	$\overline{\text{FBD}}$	–	FBD
a_5	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–

Table 24
The concordance indexes.

	a_1	a_2	a_3	a_4	a_5
a_1	–	0.6	0.9	0.9	0.9
a_2	0	–	0	0.2	0.1
a_3	0	0.3	–	0.3	0.3
a_4	0.2	0.3	0.5	–	0.3
a_5	0.2	0.6	0.5	0.9	–

Table 25
The incomparability indexes.

	a_1	a_2	a_3	a_4	a_5
a_1	–	0.4	0.1	0.1	0.1
a_2	0.4	–	0.5	0.3	0.3
a_3	0.1	0.5	–	0	0
a_4	0.1	0.3	0	–	0
a_5	0.1	0.3	0	0	–

Table 26
The observed belief dominances between alternatives on criterion g_1.

	a_1	a_2	a_3	a_4	a_5
a_1	–	FBD	FBD	$\overline{\text{FBD}}$	FBD
a_2	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_3	$\overline{\text{FBD}}$	FBD	–	$\overline{\text{FBD}}$	FBD
a_4	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$
a_5	$\overline{\text{FBD}}$	FBD	FBD	$\overline{\text{FBD}}$	–

Table 27
The observed belief dominances between alternatives on criterion g_2.

	a_1	a_2	a_3	a_4	a_5
a_1	–	FBD	FBD	FBD	FBD
a_2	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_3	$\overline{\text{FBD}}$	FBD	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_4	$\overline{\text{FBD}}$	FBD	FBD	–	$\overline{\text{FBD}}$
a_5	FBD	FBD	FBD	FBD	–

Table 28
The observed belief dominances between alternatives on criterion g_3.

	a_1	a_2	a_3	a_4	a_5
a_1	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_2	FBD	–	FBD	FBD	FBD
a_3	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–	FBD	$\overline{\text{FBD}}$
a_4	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–	$\overline{\text{FBD}}$
a_5	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	FBD	FBD	–

Table 29
The observed belief dominances between alternatives on criterion g_4.

	a_1	a_2	a_3	a_4	a_5
a_1	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$
a_2	FBD	–	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	FBD
a_3	FBD	FBD	–	$\overline{\text{FBD}}$	FBD
a_4	FBD	FBD	$\overline{\text{FBD}}$	–	FBD
a_5	FBD	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	$\overline{\text{FBD}}$	–

Table 30
The concordance indexes.

	a_1	a_2	a_3	a_4	a_5
a_1	–	0.6	0.6	0.2	0.6
a_2	0.4	–	0.3	0.3	0.4
a_3	0.1	0.7	–	0.3	0.3
a_4	0.1	0.3	0.2	–	0.1
a_5	0.3	0.6	0.9	0.5	–

Table 31
The incomparability indexes.

	a_1	a_2	a_3	a_4	a_5
a_1	–	0	0.3	0.5	0.3
a_2	0	–	0	0.2	0
a_3	0.3	0	–	0.3	0
a_4	0.5	0.2	0.3	–	0.2
a_5	0.3	0	0	0.2	–

Appendix C. Expert 3: The partial belief dominances between alternatives/The concordance and incomparability indexes.

Tables 26–31.

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