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Discrete Mathematics 308 (2008) 367–374

DISCRETE MATHEMATICS

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Directed complete bipartite graph decompositions: Indirect constructions

Peter J. Dukes[a], Charles J. Colbourn[b], Violet R. Syrotiuk[b]

[a]*Mathematics and Statistics, University of Victoria, Victoria, BC, Canada V8W 3P4*
[b]*Computer Science and Engineering, Arizona State University, Tempe, AZ 85287-8809, USA*

Received 24 November 2004; accepted 27 November 2006
Available online 2 June 2007

Abstract

Edge-decompositions of the complete λ-fold directed graph $\vec{K}_n$ into (uniform) directed complete bipartite subgraphs $\vec{K}_{a,b}$ form a model for wireless communication in sensor networks. Each node can be in one of three states: asleep (powered down), listening, or transmitting. Communication requires that the sender be transmitting, the destination listening, and no other node near the receiver transmitting. We represent nodes of the network as the vertices of $\vec{K}_n$, and time slots for communication as blocks of the graph decomposition. A block with out-vertices A and in-vertices B corresponds to a slot in which the nodes in A are transmitting, those in B are receiving, and all others are asleep. Thus, such a decomposition of $\lambda\vec{K}_n$ guarantees that every ordered pair of nodes in the associated network can communicate in λ time slots. Additional constraints are needed to minimize interference by a third node. Some recursive constructions for these graph decompositions are established, with particular emphasis on properties minimizing interference.

Keywords: Graph decompositions; Graph designs; Packings; Coverings

1. Introduction

Let $\lambda\vec{K}_n$ denote the graph with vertex set V, with $|V|=n$, having every possible directed edge λ times. Let $\vec{K}_{a,b}$ be the complete bipartite directed graph in which the vertex set is a disjoint union $A\cup B$ with $|A|=a$ and $|B|=b$, and an arc is directed from each vertex of A to each vertex of B. The vertices in A and B are *out-vertices* and *in-vertices*, respectively. A particular subgraph $\vec{K}_{a,b}$ of $\lambda\vec{K}_n$ can be described by the ordered pair $[A, B]$ of out-vertices and in-vertices.

Let G be any graph. A *G-design* of *order* n and *index* λ is a decomposition of the edges of $\lambda\vec{K}_n$ into copies of G, called *blocks*. In this paper, we consider $\vec{K}_{a,b}$-designs. Some divisibility conditions are immediate.

Proposition 1.1. *If there exists a $\vec{K}_{a,b}$-design of order n and index λ, then*

$$a \mid \lambda(n-1), \quad b \mid \lambda(n-1) \quad \text{and} \quad ab \mid \lambda n(n-1).$$

E-mail addresses: dukes@math.uvic.ca (P.J. Dukes), colbourn@asu.edu (C.J. Colbourn), syrotiuk@asu.edu (V.R. Syrotiuk).

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doi:10.1016/j.disc.2006.11.050

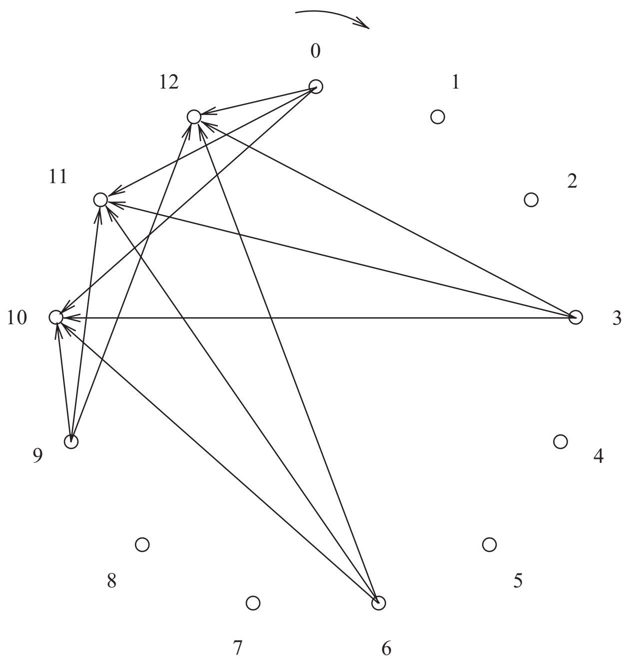

Fig. 1. A cyclic $\vec{K}_{4,3}$-design of order 13.

By the main result in [6], the conditions of Proposition 1.1 are asymptotically sufficient in n for fixed a, b, and λ. In light of the motivation to follow, we are particularly interested in existence of $\vec{K}_{a,b}$-designs for certain values of n. While this question does not appear to be directly addressed in the literature, much related work has been done.

The existence question for G-designs of order n and index 1, where G is a given "directed star" (some orientation of the edges of $K_{1,b}$) was solved in [3]. In [4], some results are given concerning the minimum number of $\vec{K}_{a,b}$s (perhaps of different sizes) required to decompose the edges of a directed graph, and of $\vec{K}_n$ in particular. Adapting a matrix product characterization from [4], one has that a necessary and sufficient condition for the existence of a $\vec{K}_{a,b}$-design of order n and index λ is that there are 0, 1-matrices C and D such that C is $n \times s$ with row-sum a, D is $s \times n$ with column-sum b, and $CD = \lambda(J_n - I_n)$. (Here, I_n and J_n are the $n \times n$ identity and all-ones matrices, respectively.)

When λ is relatively prime to both a and b (in particular when $\lambda = 1$), the necessary conditions in Proposition 1.1 are equivalent to $n \equiv 1 \pmod{ab}$. Indeed if either $a = 1$ or $b = 1$, the third condition becomes redundant. On the other hand, if $a, b > 1$, the first two conditions tell us that no factor of $d > 1$ of a (or b) can divide n, for otherwise $n \equiv 0, 1 \pmod{d}$. So the third condition implies $ab \mid (n - 1)$.

Theorem 1.2. *Suppose $(\lambda, a) = (\lambda, b) = 1$. There exists a $\vec{K}_{a,b}$-design of order n and index λ if and only if $n \equiv 1 \pmod{ab}$.*

Proof. It suffices to exhibit, for any positive integers a, b, m, a $\vec{K}_{a,b}$-design of order $n = mab + 1$ and index 1. Let $a' = ma$. In [5], it is shown that there exists a $\vec{K}_{a',b}$-design of order $a'b + 1$ and index 1 whose blocks are the cyclic shifts in $\mathbb{Z}_{a'b+1}$ of

$$[\{0, b, 2b, \ldots, (a' - 1)b\}, \{(a' - 1)b + 1, (a' - 1)b + 2, \ldots, a'b\}].$$

See Fig. 1 for a cyclic $\vec{K}_{4,3}$-design of order 13. Replace each such block $[A', B]$ with m blocks $[A_1, B], \ldots, [A_m, B]$, where $\{A_1, \ldots, A_m\}$ is a partition of A' into a-sets. The result is a $\vec{K}_{a,b}$-design. $\square$

"Breaking up" the out-vertices in each block is considered in more generality in Theorem 3.1.

We now provide some motivation for the study of $\vec{K}_{a,b}$-designs; see [5] for a more detailed discussion. Consider a sensor network with n nodes, represented by the set V, and m time slots $\{1, 2, \ldots, m\}$ for communication. In each slot j,

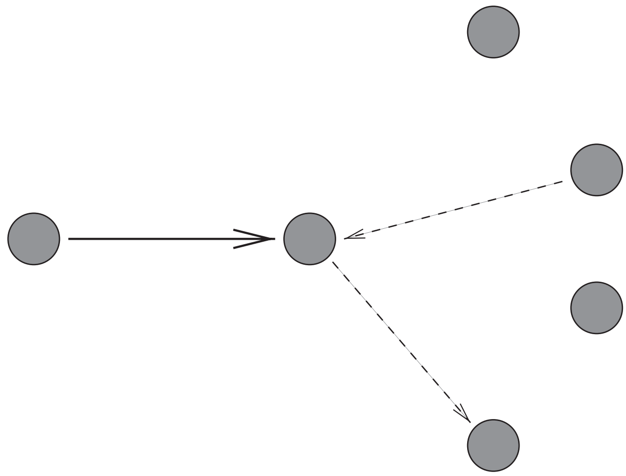

Fig. 2. Interference at the receiver.

we partition the nodes into three states $[A_j, B_j, S_j]$: transmitting, receiving, and sleeping. Such an assignment for all slots is a *frame schedule*. Assume in every time slot that there are fixed numbers $a, b, n-a-b$ of nodes in each state. We associate bipartite subgraphs $\vec{K}_{a,b}$ with time slots so that slot j has vertex bipartition $[A_j, B_j]$. Now suppose time slots are associated with the blocks of a $\vec{K}_{a,b}$-design of order n and index λ. This guarantees that every ordered pair (x, y) of distinct nodes in the corresponding network has exactly λ time slots in which x is transmitting and y is receiving.

While this model appears to provide ample transmission opportunities, in practice a further condition is required. A transmission from node x to y *fails* if there is a node $x' \neq x$ transmitting in proximity to y. Fig. 2 depicts nodes that can interfere with a transmission; the solid arrow depicts the desired transmission, while the dashed ones depict ones that cause collisions. For a pair (x, y) of distinct vertices, define $\sigma(x, y) = \{j : x \in A_j \text{ and } y \in B_j\}$. For $y \in V$, let $\Sigma(y)$ denote the family of sets $\{\sigma(x, y) : x \in V \backslash \{y\}\}$. For a given node y, $\Sigma(y)$ contains, for each node $x \neq y$, the set consisting of slots in which x can transmit and y can receive.

There are two meaningful ways to bound interference at the receiver y. First, we require that for some parameter d, it is never the case that the union of some d of the sets in $\Sigma(y)$ contains any other set in $\Sigma(y)$. A family of subsets satisfying this property is called *d-cover-free*. This condition on $\Sigma(y)$ guarantees that, for any nearby node $x \neq y$, there is some time slot for transmission from x to y provided that y has no more than d neighbours. Alternatively, we could stipulate that, for all $x, x' \neq y$, the intersection of $\sigma(x, y)$ with $\sigma(x', y)$ is bounded by some parameter r. This condition also guarantees an available slot from x to y provided there are fewer than λ/r other neighbours of y, but it does so by bounding the interference from every possible neighbour. We summarize the two extra conditions on $\vec{K}_{a,b}$-designs of index λ:

- Global (Cover-Free) Condition: for all y, $\Sigma(y)$ is a d-cover-free family;
- Local (Intersection) Condition: for all y and $x \neq x'$, $|\sigma(x, y) \cap \sigma(x', y)| \leqslant r$.

The local condition implies the global condition for an appropriate choice of r. When transmitter x and receiver y are to communicate in slot j, as indicated by the presence of arc (x, y) in the $\vec{K}_{a,b}$ for slot j, they are prevented by doing so exactly when another active transmitter, x', appears among the a transmitters in that $\vec{K}_{a,b}$. Now the number of slots in which x' can interfere with the $x \to y$ communication is equal to the number of times the $\vec{K}_{2,1}$ with partition $[\{x, x'\}, \{y\}]$ appears in a block of the $\vec{K}_{a,b}$-design. In meeting the local condition, then, our goal is to minimize the number of occurrences of any specific $\vec{K}_{2,1}$.

The $\vec{K}_{2,1}$-*replication number* of a $\vec{K}_{a,b}$-design, denoted by r_{21}, is the maximum number of blocks containing a given $\vec{K}_{2,1}$ as a subgraph. We have an elementary lower bound:

Proposition 1.3. *In a $\vec{K}_{a,b}$-design of order n and index λ,*

$$r_{21} \geqslant \frac{\lambda(a-1)}{n-2}.$$

Proof. Count in two ways the number of ordered pairs $(H, [A, B])$, where H is a $\vec{K}_{2,1}$ contained in the block $[A, B]$. Each of $\lambda n(n-1)/ab$ blocks contains $\binom{a}{2} b$ copies of $\vec{K}_{2,1}$, while each of $3\binom{n}{3}$ candidates for H is contained in at most $r_{2,1}$ blocks. □

At this time, we comment on the practical size of a and b relative to n. Suppose in our application that every node has D neighbours on average. It is optimal to maximize the number of $(D+1)$-sets of nodes with exactly one transmitter, one receiver, and the rest sleeping. (More precisely, it is not important whether more neighbours are receiving; however, for energy conservation the sleep state is preferred for these neighbours.) This quantity is

$$f(a, b) = ab\binom{n-a-b}{D-1},$$

a function of two variables a and b. With some elementary calculations, $f(a, b)$ is maximized for $a = b = n/(D+1)$. This justifies making the simplifying assumption that the number of nodes transmitting and receiving per slot are equal $(a = b)$, and that this common number is chosen depending on the expected neighbourhood size. For this reason, we focus on the existence of $\vec{K}_{a,a}$-designs.

In [5], numerous direct constructions are developed; here, various indirect design-theoretic constructions are given. We conclude with a summary and some open questions.

2. Packcovers

As seen in [5] and in Theorem 1.2, direct techniques are successful at producing $\vec{K}_{a,b}$-designs of index 1. For our application to sensor networks, it is desirable to have schedules with $\lambda > 1$, since the additional question of minimizing interference becomes more meaningful for higher λ. In the next three sections, we address this issue by presenting constructions that increase λ. Here, we adapt a standard design-theoretic construction to $\vec{K}_{a,b}$-designs. The rough idea is to place $\vec{K}_{a,b}$-designs of small order on the blocks of a large pairwise-balanced set system. We first require a definition.

A t-(v, k, λ) *covering* (*packing*) is a collection of k-subsets (again called *blocks*) of a v-set such that every t-subset is contained in at least (at most) λ blocks. A t-(v, k, λ) *design* is both a covering and a packing. See [2] for further information.

Theorem 2.1. *Suppose there exists a* 2-(v, n, λ_1) *design which is also a* 3-(v, n, λ_2) *packing. Further suppose there is a $\vec{K}_{a,b}$-design of order n and index λ with $\vec{K}_{2,1}$-replication number r. Then there exists a $\vec{K}_{a,b}$-design of order v and index $\lambda\lambda_1$ with $\vec{K}_{2,1}$-replication number at most $r\lambda_2$.*

Proof. Suppose the 2-design has points V and blocks $\mathscr{X}$. Replace each block $X \in \mathscr{X}$ with a copy of the blocks of the hypothesized $\vec{K}_{a,b}$-design on the points X. Every pair of points in V is contained in λ_1 blocks in $\mathscr{X}$, and thus in $\lambda\lambda_1$ copies of $\vec{K}_{a,b}$ in the result. By the packing condition, the distinct points $x, x', y \in V$ are contained in at most λ_2 members of $\mathscr{X}$. So the $\vec{K}_{2,1}$ $[\{x, x'\}, \{y\}]$ appears in at most $r\lambda_2$ blocks of the resulting $\vec{K}_{a,b}$-design. □

In the sensor network application, the 2-design in this theorem may be replaced by a 2-covering if we are willing to tolerate more time slots. The main ingredient in this construction motivates further study.

Definition 2.2. A t-$(v, k, \lambda_1, \lambda_2)$ *packcover* is a collection of sets that forms the blocks of both a t-(v, k, λ_1) covering and a $(t+1)$-(v, k, λ_2) packing.

Two sources of packcovers are now given, together with examples of how the construction in Theorem 2.1 can be applied to produce frame schedules. Consider any 3-(v, n, λ_2) design (see [2] for existence results). Its blocks also form a 2-(v, n, λ_1) design with $\lambda_1 = \lambda_2(v-2)/(n-2)$; hence it is a 2-$(v, n, \lambda_1, \lambda_2)$ packcover.

Example 2.3. Start with the packcover that is a 3-(257, 17, 1) design (this is a "spherical" geometry [2]). This design has 4112 blocks of size 17; every 2-subset occurs in 17 blocks and every 3-subset is in a unique block. From an addition set construction, there is a $\vec{K}_{4,4}$ design of order 17 having index 1 and 17 blocks. Placing a copy of these 17 $\vec{K}_{4,4}$s on each block of the packcover, to obtain $4112 \cdot 17$ $\vec{K}_{4,4}$s forming a $\vec{K}_{4,4}$-design of index 17, supporting $D = 17$ active neighbours as guaranteed by the local condition. Indeed every pair $\{x, y\}$ appears in 17 of the 4112 blocks, but not more than once together with another node x'. Moreover, such a node x' can collide with the transmission at most once in the $\vec{K}_{4,4}$ design of order 17. Only eight of the 257 nodes are awake in each slot, but of course the price that is paid is a very long schedule! Nevertheless, this example demonstrates that large sizes of active neighbourhoods can be tolerated even when most nodes sleep, at the expense of a long schedule.

More generally, we can require in a 2-design that at most λ' blocks contain any given triple of points. A classical inequality on 2-designs guarantees a nontrivial bound on λ' when v/k is small.

Proposition 2.4. *In a* 2-(v, k, λ) *design, any* 3-*subset of points is contained in at most*

$$\lambda' = \left\lceil \lambda \frac{(v-1)(v-2)}{(2v-k-2)(k-1)} \right\rceil - 1$$

blocks. Thus, any 2-(v, k, λ) *design is a* 2-$(v, k, \lambda, \lambda')$ *packcover.*

Proof. According to Raghavarao's inequality [7], m blocks have intersection at most $k(b-m)/(r+m(v-k-1))$, where $b = \lambda v(v-1)/k(k-1)$ and $r = \lambda(v-1)/(k-1)$. Setting this to be at most 2 and solving for m, we see that $\lambda' + 1$ blocks have intersection of size at most 2. □

A 2-(v, k, λ) design is *supersimple* if any two blocks intersect in at most 2 points, and hence every such supersimple design is a 2-$(v, k, \lambda, 1)$ packcover. See [1] for a survey of the known results on supersimple designs. Other than supersimple designs that arise from known 3-designs, knowledge is limited primarily to designs with "small" block sizes. The application here suggests a potential value in finding supersimple designs with large block size that do not arise in this way.

3. Breaking up out-vertices

Although the packcover construction increases the order of $\vec{K}_{a,b}$-designs, one drawback is that it fails to change the block size. Indeed the associated $\vec{K}_{a,b}$-designs may exist, but certain choices of a and b may not be suitable for frame schedules. For instance, the analysis at the end of Section 1 supports taking $a \approx b$.

In this section, we present a construction which converts $\vec{K}_{g,b}$-designs of order n and index 1 for $g \geqslant b$ to $\vec{K}_{a,b}$-designs of order n and index λ for $a \approx b$. The idea is to replace a single block with g out-vertices by several blocks with the same in-vertices, but some family of a-subsets of the out-vertices. Of course, a similar construction works by interchanging the roles of in-vertices and out-vertices. But from the viewpoint of sensor networks, a judicious choice of subsets of out-vertices in each block ensures that interference with all slots in which one transmitter is active requires many other transmitters to also be active. In order to characterize such sets of out-vertices, we give a definition. Given an indexed collection of subsets $\{A_1, \ldots, A_h\}$ of some set X, we define its *dual* as the set system $\{\{i : A_i \ni x\} : x \in X\}$.

Theorem 3.1. *Suppose there exists a $\vec{K}_{g,b}$-design of order n and index* 1 *and a* 1-(g, a, λ) *design with blocks $\mathscr{A}$. Then there exists a $\vec{K}_{a,b}$-design of order n and index λ. If $\mathscr{A}$ is also a* 2-(g, a, λ_2) *packing, then the resulting $\vec{K}_{a,b}$-design has $\vec{K}_{2,1}$-replication number at most λ_2. Moreover, if the dual of $\mathscr{A}$ is a d-cover-free family, then the resulting $\vec{K}_{a,b}$-design satisfies the global condition.*

Proof. Replace each block $[X, B]$ of the $\vec{K}_{g,b}$-design by $[A_1, B], \ldots, [A_h, B]$, where $A_1, \ldots, A_h$ are blocks of a 1-(g, a, λ) design on the point set X. (Here, $h = \lambda g/a$.) Since the blocks of this 1-design have size a, the resulting blocks are all $\vec{K}_{a,b}$. They form the required $\vec{K}_{a,b}$-design since every $x \in X$ is contained in exactly λ of the A_i. Since the ingredient design has index 1, a given $\vec{K}_{2,1}$ $[\{x, x'\}, \{y\}]$ appears in the result in as many blocks as $\{x, x'\}$ appears in $\mathscr{A}$, and this is at most λ_2. For the global condition, we are concerned with the block indices associated with a particular in-vertex, and require these sets to be d-cover-free; hence when stated for the sets $\mathscr{A}$ of out-vertices, we require the dual to be d-cover-free. □

By using a $\vec{K}_{g,b}$-design as a starting point, in each block we essentially fix a selection of b receivers, and restrict our attention to g possible transmitters. For a set of h slots, we then choose a subset of a of the g potential transmitters in each of the h slots. In this way, for a given transmitter–receiver pair we need only be concerned with opportunities in these h slots (of which there are λ), and among these we need only address collisions from the g transmitters. It is possible to extend the initial $\vec{K}_{g,b}$ design by permitting it to have index greater than one. Each $\vec{K}_{g,b}$ could be "broken up" in the same manner; however, intersections among the g transmitters from two $\vec{K}_{g,b}$s that share an arc complicate the analysis. We do not pursue this more general construction here; rather in the next section, we examine a related method for keeping these undesired intersections among transmitter sets small.

We conclude this section with an example, again with particular emphasis on our motivating application.

Example 3.2. Assume $m=13$, $a=4$. Consider the set $S=\{1,2,5,7\}$ in $\mathbb{Z}_{13}$. The 12 ordered pairs of distinct elements from S exhaust the nonzero elements of $\mathbb{Z}_{13}$. (In the terminology of the previous section, this is a difference set.) It follows that the collection of 13 sets $S+x$, $x \in \mathbb{Z}_{13}$, forms a 3-cover-free family, and every point belongs to exactly four sets. If x is used to index this family, its dual is easily seen to be $-S+x$, $x \in \mathbb{Z}_{13}$. Now, suppose we have a $\vec{K}_{13,4}$-design of order n and index 1 (we see later how to construct a $\vec{K}_{13,4}$-design of order 53, for example). Given some block, say $[\{0,1,2,\ldots,12\},\{\alpha,\beta,\gamma,\delta\}]$, replace it with 13 copies of $\vec{K}_{4,4}$ on the bipartitions $[-S+x,\{\alpha,\beta,\gamma,\delta\}]$, $x \in \mathbb{Z}_{13}$. The result is a $\vec{K}_{4,4}$-design of order n and index 4 satisfying the global condition with $d=3$. Using the $\vec{K}_{13,4}$-design of order 53, we obtain $53 \cdot 13 = 689$ slot schedules for 53 nodes with 4 transmitting, 4 receiving, and 45 sleeping in each slot. This frame schedule supports $d+1=4$ active neighbours.

4. Overlaying designs of index 1

Of course, an easy construction to increase the index of $\vec{K}_{a,b}$-designs is to simply repeat the collection of blocks. For scheduling sensor networks, this is undesirable. Rather than repeating *the same* frame schedule, say λ times, we apply permutations to the vertex labels in each copy so as to minimize the largest possible intersection of λ sets of out-vertices, one chosen from each permuted copy of the design. Here, we investigate this idea of "overlaying" designs to increase the index while enforcing an interference criterion. For simplicity, we assume $a=b$.

More precisely, let $\pi_0, \pi_1, \ldots, \pi_{\lambda-1}$ be permutations of $\{1,\ldots,n\}$ with π_0 the identity. For fixed a-subsets $A_0, A_1, \ldots, A_{\lambda-1}$, and a given s-subset $S \subset A_0$, there are $N_s = \binom{a}{s}^{\lambda-1}(s!)^{\lambda-1}(n-s)!^{\lambda-1}$ choices for $(\pi_0,\ldots,\pi_{\lambda-1})$ such that $\bigcap_{i=0}^{\lambda-1}\pi_i(A_i) \supset S$. By the principle of inclusion–exclusion, the number of choices for the π_i such that $\bigcap_{i=0}^{\lambda-1}\pi_i(A_i)=S$ is

$$\begin{aligned} M_s &= \sum_{i=s}^{a}(-1)^{i-s}\binom{a-s}{i-s}N_i \\ &= \sum_{i=s}^{a}(-1)^{i-s}\binom{a-s}{i-s}\binom{a}{i}^{\lambda-1}(i!)^{\lambda-1}((n-i)!)^{\lambda-1}. \end{aligned}$$

There are $(n!)^{\lambda-1}$ total choices for the π_i. So given a $\vec{K}_{a,a}$-design of order $n=ma^2+1$ and index 1, with blocks $\{B_1,\ldots,B_{mn}\}$, there exists a choice of permutations with the intersection of *any* λ blocks of the form $\pi_i(B_j)$ having size at most 1 provided that

$$(m(ma^2+1))^{\lambda}\sum_{i=s}^{a}(-1)^i(i-1)\binom{a}{i}^{\lambda}\binom{ma^2+1}{i}^{1-\lambda} < 1.$$

Define $\Lambda(a,m)$ to be the minimum λ such that this inequality holds for a given a and m. Table 1 shows values of $\Lambda(a,m)$ we have determined by computer.

In practice, by choosing λ large enough, there is a high probability that *any* vector of permutations has the desired property. Computer search for permutations of the nodes of some frame may also yield some $\lambda < \Lambda(a,m)$ for which the same property holds.

Table 1
$\Lambda(a, m)$

$a\backslash m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	10	13	15	17	18	19	19	20	20	21	21	22	22	23	23	23
4	18	22	25	27	29	30	31	32	33	33	34	35	35	36	36	37
5	26	33	36	39	41	42	44	45	46	47	48	49	49	50	51	51
6	36	44	48	51	54	56	57	59	60	61	62	63	64	65	66	66
7	47	56	61	65	67	70	72	73	75	76	78	79	80	81	82	82
8	58	68	74	78	82	84	87	89	91	92	94	95	96	97	98	99

5. A sum construction

We have seen certain constructions that increase the order of $\vec{K}_{a,b}$-designs, such as with packcovers. However, it may be desirable for applications to "fine-tune" such an increase. This issue is addressed here by presenting a construction that can add the orders of two $\vec{K}_{a,b}$-designs. In practice, one may want to choose one of these orders small relative to the other. For interference analysis, we focus on the local condition.

Theorem 5.1. *Suppose there exist $\vec{K}_{a,b}$-designs of orders m and n, each of index λ. Suppose further that a, b divide both λm and λn. Then there exists a $\vec{K}_{a,b}$-design of order $m+n$. Moreover, if the designs of orders m and n each have $\vec{K}_{2,1}$-replication number at most r, and if there exist* 2-(m, a, λ_1) *and* 2-(n, a, λ_2) *packings, then the resulting design has $\vec{K}_{2,1}$-replication number at most* $\max\{r, \lambda_1, \lambda_2\}$.

Proof. Take the designs of orders m and n on disjoint sets of points U and V. Since $a, b \mid \lambda m, \lambda n$, there exist 1-$(m, a, \lambda)$ and 1-(m, b, λ) designs, say with blocks $\mathscr{A}$, $\mathscr{B}'$, and similarly 1-(n, a, λ) and 1-(n, b, λ) designs, say with blocks $\mathscr{A}'$, $\mathscr{B}$. Together with the blocks of the $\vec{K}_{a,b}$-designs on U and V, we add all the blocks $[A, B]$ with $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $[A', B']$ with $A \in \mathscr{A}'$, $B \in \mathscr{B}'$. All arcs crossing between U and V are now covered exactly λ times. If, in addition, the 1-designs are also 2-packings with the given indices, then we have bounds on the number of "crossing" $\vec{K}_{a,b}$s containing a given $\vec{K}_{2,1}$. $\square$

Example 5.2. There is a $\vec{K}_{4,4}$-design of order 16 and index 4 with $\vec{K}_{2,1}$ replication number 1 [5]. Now, applying Theorem 5.1 with $m=n=16$ yields a $\vec{K}_{4,4}$-design of order 32 with index 4. Furthermore, there is a 1-(16, 4, 4) design having pair replication $\leqslant 1$: take four resolution classes of a 2-(16, 4, 1) design. So we can obtain $r_{21}=1$ in the resultant design.

6. Conclusions

In conjunction with the direct constructions in [5], the indirect methods provided here yield a large variety of $\vec{K}_{a,a}$-designs and decompositions in which the global (cover-free) and local (intersection) conditions are met. In turn, these provide schedules for transmission, reception, and idleness in sensor or ad hoc networks. This interpretation of directed graph decompositions as energy-efficient schedules motivates further study of such decompositions. In particular, while designs, packings, and coverings have all been extensively studied, the packcovers employed here suggest a strengthening of packing and covering, but a weakening of design requirements. At the same time, while supersimple designs provide examples of packcovers, the relaxation of the requirements for packcovers in general provide new existence questions.

Acknowledgements

The research of V.R. Syrotiuk is supported in part by the National Science Foundation (NSF) under Grant ANI-0105985. Any opinions, findings, conclusions, or recommendations expressed are those of the authors and do not necessarily reflect the views of NSF.

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