The Hamiltonian Property of Generalized de Bruijn Digraphs

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It has been shown that the generalized de Bruijn digraphs $G_B(n, d)$ and the Imase-Itoh digraphs $G_I(n, d)$ have Hamiltonian circuits if $\gcd(n, d) > 1$. For $\gcd(n, d) = 1$ the problem remains open except for $d = 1$ and 2. In this paper we give a unified proof for all $d \geq 3$ that both $G_B(n, d)$ and $G_I(n, d)$ have Hamiltonian circuits.

1. Introduction

A generalized de Bruijn digraph $G_B(n, d)$, first proposed independently by Imase and Itoh [5] and Reddy, Pradhan, and Kuhl [8], is a digraph with $n$ nodes labeled by the residues of modulo $n$ and the set of $nd$ links $\{i \rightarrow di + r \pmod n: 0 \leq i \leq n - 1, 0 \leq r \leq d - 1\}$. The well-known de Bruijn digraph is a special case of $G_B(n, d)$ when $n$ is a power of $d$. Imase and Itoh also proposed $G_I(n, d)$, known as Imase-Itoh digraph, which has the set of $nd$ links $\{i \rightarrow d(n - 1 - i) + r \pmod n: 0 \leq i \leq n - 1, 0 \leq r \leq d - 1\}$, a reverse

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type of $G_B(n,d)$. Various properties of $G_B(n,d)$ and $G_I(n,d)$ have been studied in [1–8]. One of the properties still unsettled is the Hamiltonian property, i.e., whether the graph contains a Hamiltonian circuit as a subgraph. As the loop or ring is widely used in parallel and distributed computing, it is important to know if the graphs considered contain a Hamiltonian cycle.

Let $gcd(n,d) = \lambda$. It was shown in [3] that if $\lambda > 1$, then both $G_B(n,d)$ and $G_I(n,d)$ are Hamiltonian and if $\lambda = 1$, then $G_B(n,2)$ is not Hamiltonian. In this paper we prove that if $\lambda = 1$ and $d > 2$, then both $G_B(n,d)$ and $G_I(n,d)$ are Hamiltonian. One of us (Hwang) also shows in the Appendix that if $\lambda = 1$ then $G_I(n,2)$ is Hamiltonian if and only if $n = 3^m$ for $m = 1, 2, ...$. Since it is trivial to verify that $G_B(n,1)$ and $G_I(n,1)$ are never Hamiltonian except for $G_I(2,1)$, we have completely settled the question when $G_B(n,d)$ and $G_I(n,d)$ are Hamiltonian.

2. The General Approach

We assume $\lambda = 1$ and $d \geq 3$ throughout this paper. While the argument is valid for both the $G_B(n,d)$ and the $G_I(n,d)$ case, the details are given only for $G_B(n,d)$.

Let $L$ be the set of $n$ links $\{i \rightarrow di + 1 \pmod{n}: i = 0, 1, ..., n - 1\}$. Since $\lambda = 1$, $i \neq j$ implies $di + 1 \neq dj + 1 \pmod{n}$. Therefore every node has one inlink and one outlink in $L$ and $L$ consists of a set of disjoint circuits $C_1, ..., C_m$. If $m = 1$, we are through. If $m > 1$, we propose a method to combine two circuits into one. By iteratively using this method eventually there is only one circuit left.

Suppose that $i \in C_x$ and $i + 1 \in C_y$, $x \neq y$. Let $j$ and $k$ be the two nodes preceding $i$ and $i + 1$, respectively, on $C_x$ and $C_y$. Then we can replace the two links $j \rightarrow i$ and $k \rightarrow i + 1$ by $j \rightarrow i + 1$ and $k \rightarrow i$, thus combining $C_x$ and $C_y$ into one circuit. Granted, the latter two links are not in $L$. But $i + 1 = (dj + 1) + 1 = dj + 2$ and $i = (i + 1) - 1 = dk + 1 - 1 = dk$. Therefore $j \rightarrow i + 1$ and $k \rightarrow i$ are indeed links of $G_B(n,d)$. We call the operation of replacing $j \rightarrow i$ and $k \rightarrow i + 1$ by $j \rightarrow i + 1$ and $k \rightarrow i$ the interchange of $(i,i+1)$.

In iteratively interchanging two adjacent nodes there is one constraint to observe. For example, suppose $d = 3$ and the pair $(i,i+1)$ has been interchanged. Then we cannot interchange $(i+1,i+2)$ any longer. To see this let $h \rightarrow i + 2$ be in $L$. Then the interchange yields the two new links $h \rightarrow i + 1$ and $j \rightarrow i + 2$. While $i + 1 = (i + 2) - 1 = dh$ implies that $h \rightarrow i + 1$ is in $G_B(n,d)$, $j \rightarrow i + 2$ is not a link in $G_B(n,d)$ since $i + 2 = dj + 3$. Hence the interchange is illegitimate. For the same reason we cannot have the interchange $(i - 1,i)$ after the interchange $(i,i+1)$ has taken place. Of course, if $d = 4$, then $j \rightarrow i + 2$ is a link in $G_B(n,d)$ and the exchange $(i+1,i+2)$ is
legitimate, but we cannot have the interchange \((i + 2, i + 3)\), or \((i - 1, i)\), after \((i, i + 1)\), \((i + 1, i + 2)\) have been interchanged. In general, let \(P = \{(i_1, i_1 + 1), \ldots, (i_p, i_p + 1)\}\) be the set of interchanged pairs. If we draw each pair as an edge (undirected) on the \(n\) nodes \(0, 1, \ldots, n-1\) arranged in a cycle, then no chain can contain more than \(d - 1\) nodes. Any \(P\) satisfying this constraint will be called a \textit{legitimate interchange set}. Let \(L^*\) denote the undirected version of \(L\).

**Lemma 1.** If \(P\) is a legitimate interchange set and \(L^* \cup P\) is a connected graph, then \(G_B(n, d)\) is Hamiltonian.

**Proof.** We give a procedure which merges the \(m\) circuits in \(L\) into one circuit in \(m - 1\) iterations. Set \(Q = P\) and \(C = L\) to initialize. At each iteration choose \((i, i + 1) \in Q\) such that \((i - 1, i) \notin Q\) (or \(i - 1\) and \(i\) are on the same circuit) and that \(i\) and \(i + 1\) are on different circuits \(C_x\) and \(C_y\) of \(C\). The connectivity of \(L^* \cup P\) guarantees the existence of such \((i, i + 1)\). Make the \((i, i + 1)\) interchange and merge \(C_x\) and \(C_y\) into one circuit (updating \(C\)). Update \(Q\) by setting \(Q = Q - \{i, i + 1\}\).

**Remark.** This procedure avoids the possibility of interchanging \((i - 1, i)\) after \((i, i + 1)\) has been interchanged (which may cause problems).

### 3. The Main Results

We first consider \(G_B(n, d)\). Let \(P^*\) be an interchange set which consists of all adjacent pairs \((i, i + 1)\) (including \((n - 1, 0)\)) except for \(i = w(d - 1)\) for \(w = 0, 1, \ldots, \lfloor n/(d - 1) \rfloor - 1\). Clearly, \(P^*\) is legitimate. Define \(P^* = P^* \cup \{(0, 1)\}\). Since \(0 \to 1\) is a link in \(L\), \(L^* \cup P^*\) is connected if and only if \(L^* \cup P^*\) is connected. Let \(M\) be a graph with \(n\) nodes and the edge set \(\{(di + 1, di + 1 + d) : (i, i + 1) \in P^*\}\). \((i, i + 1) \in P^*\) implies that \(di + 1\) is connected to \(di + 1 + d\) through the edges \((i, di + 1), (i, i + 1), (i + 1, d(i + 1) + 1)\) in \(L^* \cup P^*\). Therefore if \(M \cup P^*\) is connected, then \(L^* \cup P^*\) must be.

**Lemma 2.** Each node in \(M\) is incident to either one or two edges.

**Proof.** Since \(\gcd(n, d) = 1\), \(i \to id + 1 \pmod{n}\) is a one to one mapping from \(\{0, 1, \ldots, n - 1\}\) onto itself. By the definition of \(M\), \((di + 1, di + 1 + 1) \in M\) if and only if \((i, i + 1) \in P^*\). Therefore \(P^*\) is isomorphic to \(M\) under the mapping \(i \to di + 1\). Since each node in \(P^*\) is incident to either one or two edges, Lemma 2 follows.

Let \(S\) denote the set of nodes \(i\) in \(M\) which have no edge of the type \((i, i + d)\).
LEMMA 3. No consecutive \(d + 1\) nodes \(i, i + 1, \ldots, i + d\) can all be in \(S\).

Proof. From Lemma 2, \(i \in S\) implies \((i, i - d)\) is an edge in \(M\). Hence \(i - d \notin S\) by the definition of \(S\).

LEMMA 4. Let \(i, i + 1, \ldots, i + d - 1\) be a run of \(d\) (consecutive) \(S\)-nodes. Then \(i \neq w(d - 1)\) for \(w = 1, \ldots, \lceil n/(d - 1) \rceil - 1\).

Proof. Note that \(S = \{d[w(d - 1)] + 1: w = 1, \ldots, \lceil n/(d - 1) \rceil - 1\}\), i.e., the nodes in \(S\) form an arithmetic progression (mod \(n\)) with the first node \(d(d - 1) + 1\) and the difference \(d(d - 1)\). Since the arithmetic progression covers a total distance of
\[
\left(\left\lfloor \frac{n}{d-1}\right\rfloor - 2\right) d(d-1) < \left(\frac{n}{d-1} - 1\right) d(d-1) = nd - d(d-1),
\]

it covers the \(n\)-cycle at most \(d\) rounds (the last round may be incomplete). Clearly, if there exists a run of \(d\) \(S\)-nodes, the progression must send a node to this run in every round (or there wouldn't be \(d\) nodes in the run). This rules out the case that \(d(d - 1) > n\). Furthermore, let \(v_i\) denote the node in the run contributed by the \(i\)th round. Then \(v_i - v_{i+1}\) must be a constant for \(i - 1, \ldots, d - 1\). Clearly, the only constants to allow \((v_1, \ldots, v_d)\) to be a run are \(\pm 1\). Therefore, \(v_1\) must be an endpoint of the run. However, \(v_1 = ud(d-1) + 1\) for some \(u = 1, \ldots, \lfloor (n-1)/(d(d-1)) \rfloor\). Therefore \(v_1 \neq w(d - 1)\) for some \(w = 1, \ldots, \lceil n/(d - 1) \rceil - 1\). So Lemma 4 follows if \(v_1\) is the starting node of the run. If \(v_1\) is the ending node of the run, then the starting node is \(v_1 - d + 1 = (ud - 1)(d-1) + 1\), again not equal to \(w(d - 1)\) for some \(w = 1, \ldots, \lceil n/(d - 1) \rceil - 1\). Therefore, regardless of which endpoint is \(v_1\), Lemma 4 is true.

Note that \(P^*\) consists of a set of chains all of length \(d - 1\) except the chain containing nodes 0 and 1 is of length at least \(d\). Denote this long chain by \(W_0\) and let \(W_j\) denote the \(j\)th chain succeeding \(W_0\) in the counter-clockwise order. Note that the largest element in each \(W_j\) is of the form \(w(d - 1)\) for some \(w \in \{1, \ldots, \lceil n/(d - 1) \rceil - 1\}\).

THEOREM 1. \(G_\lambda(n, d)\) is Hamiltonian for \(\lambda = 1\) and \(d \geq 3\).

Proof. It suffices to prove that \(M \cup P^*\) is connected. We prove this by showing that \(W_j\) is connected to \(W_0\) through links of \(M\). Our proof is an induction proof on \(j\).

Let \(W_j = (i + 1, \ldots, i + d - 1)\). For \(j = 1\), if \(W_j\) contains a node \(i + k\) not in \(S\), then the edge \((i + k, i + k + d) \in M\), where \(i + k + d \in W_0\). Hence \(W_1\) is connected to \(W_0\). If all nodes of \(W_1\) are in \(S\), then \(W_1\) is connected to \(W_2\) through the edge \((i + 2, i + 2 - d)\). Furthermore, since \(i\) is the largest
element in $W_2$, $i = w(d - 1)$ for some $w \in \{1, ..., \lceil n/(d - 1) \rceil - 1$. By Lemma 4, $i$ is not in $S$, i.e., the edge $(i, i + d) \in M$ where $i + d \in W_0$. Hence $W_1$ is connected to $W_0$ through $W_2$.

For general $j > 1$ the argument is similar. If $W_j$ contains a node $i + k$ not in $S$, then the edge $(i + k, i + k + d) \in M$ where $i + k + d$ is either in $W_{j-1}$ or $W_{j-2}$. By the induction assumption both $W_{j-1}$ and $W_{j-2}$ are connected to $W_0$. If all nodes of $W_j$ are in $S$, then $W_j$ is connected to $W_{j+1}$ through the edge $(i + 2, i + 2 - d)$. Furthermore, by Lemma 4, $i$ is not in $S$, i.e., the edge $(i, i + d) \in M$ where $i + d \in W_{j-1}$. Hence $W_j$ is connected to $W_{j-1}$, and by induction, to $W_0$.

**Theorem 2.** $G_\lambda(n, d)$ is Hamiltonian for $\lambda = 1$ and $d \geq 3$.

**Proof.** Let $L^* = \{(i, d(n - 1 - i) + 1): i = 0, 1, ..., n - 1\}$ and let $P$ be an interchange set consisting of all adjacent pairs $(i, i + 1)$ except for $i = \lfloor n - w(d - 1) \rfloor - 1$. Clearly, $P$ is legitimate. Denote $P_1^* = P \cup \{(0, 1)\}$. Since $(n-1, 0) \in P$ and $(n-1, 1) \in L^*$, $L^* \cup P$ is connected if $L^* \cup P_1^*$ is connected. Let $M_1$ be a graph with $n$ nodes and the edge set $\{(d(n - 1 - i) + 1, d(n - 1 - i) + 1 - d): (i, i + 1) \in P_1^*\}$. $(i, i + 1) \in P_1^*$ implies that $d(n - 1 - i) + 1$ is connected to $d(n - 1 - i) + 1 - d$ through edges in $L^* \cup P_1^*$. Therefore, if $M_1 \cup P_1^*$ is connected, then $L^* \cup P_1^*$ must be. It suffices to consider the case that $M_1 \cup P_1^*$ is not connected.

Let $x$ be the smallest number such that $W_x$ is not connected to $W_0$. Since $W_0$ has at least $d$ nodes, $M_1$ does not contain the edge $[i - d, i]$ for any $i \in W_x$. Consequently, $M_1$ must contain the edge $[i, i + d]$, i.e., $W_x$ is connected to $W_{x+1}$. Now let $y$ be the smallest number such that $W_y$ is not connected to $W_x$. Then we can show that $W_y$ is connected to $W_{y+1}$ similarly. Since an edge of $M_1$ skips only $d - 1$ nodes, the above argument implies that each connected component of $M_1 \cup P_1^*$ contains consecutive nodes.

Let $C = (i, i + 1, ..., i + j)$ be the nodes in a connected component of $M_1 \cup P_1^*$ and let $k$ be the largest $k$ such that $i \leq i + kd \leq i + j$. Then $C$ contains neither the edge $(i - d, i)$ nor the edge $(i + kd, i + (k + 1)d)$. Consider the mapping from $M_1$ to $P_1^*$. This implies the existence of a component in $P_1^*$ with at most $k$ edges. But we know each component in $P_1^*$ has at least $d - 2$ edges. Hence $k \geq d - 2$ and $C$ contains at least $(d - 2)d + 1$ edges (a more careful analysis yields $(d - 1)d$ edges).

Define $P_2^* = \{(i, i + 1): (i, i + 1)$ is connected in $M_1 \cup P_1^*\}$. Define $M_2 = \{(d(n - 1 - i) + 1, d(n - 1 - i) + 1 - d): (i, i + 1) \in P_2^*\}$. Since we have just proved that a component in $P_2^*$ has at least $(d - 2)d + 1$ edges, an analogous argument shows that a component of $P_2^* \cup M_2$ contains at least $[(d - 2)d + 1]d + 1$ edges. Since the minimum component size is growing
from $M_1 \cup P_1^*$ to $M_2 \cup P_2^*$, a recursive argument shows that there exists a $j, j < \log_d n$, such that $M_j \cup P_j^*$ is connected. This in turn implies that $L^* \cup P_1^*$ is connected. The proof is complete.

One referee gave the interesting example $n = 17$ and $d = 3$ such that $M_1 \cup P_1^*$ as defined in Theorem 2 is disconnected. This example forced us to look into $M_j \cup P_j^*$ for $j > 1$.

APPENDIX: THE CASE $d = 2$

**Theorem 1.** $G_1(n, 2)$ is Hamiltonian for odd $n$ if and only if $n = 3^m$, $m = 1, 2, \ldots$.

**Proof.** Suppose that we proceed to construct a Hamiltonian circuit $H$ of $G_1(n, 2)$. Because of symmetry we may assume that $H$ contains the arc $(n - 1)/2 \to 0$ (as versus the arc $(n - 1)/2 \to n - 1$). Then the arc $(n - 1)/2 \to n - 1$ cannot be in $H$, hence the arc $0 \to n - 1$ must be in $H$. Consequently, the arc $n - 1 \to 0$ cannot be in $H$; hence, the arc $n - 1 \to 1$ must be in $H$. Repeating this argument it is easily seen that all arcs $i \to -2i - 2$ can be in $H$.

Define $f^0(x) = x, f(x) = 2x \cdot 1 \pmod{n}, f^k(x) = f(f^{k-1}(x))$. Then $H$ is indeed a Hamiltonian circuit if $x, f(x), f^2(x), \ldots, f^{n-1}(x)$ are all distinct modulo $n$. Set $x = 0$. Then it is easily solved that

$$f^k(0) = \frac{2^k - 1}{3}, \quad k \text{ even}$$

$$= -\frac{2^k + 1}{3}, \quad k \text{ odd}.$$ 

**Lemma.** $f^{c3^m}(0)$ is divisible by $3^m$ but not by $3^m + 1$ for $c$ an integer not a multiple of 3 and $m = 0, 1, \ldots$.

**Proof.** Write $c = 6a + b$ where $0 \leq b \leq 5$. Then

$$2^c = 2^{6a} \cdot b = 64^a 2^b = 2^b \pmod{9}.$$ 

Therefore it is easily verified that $f^c(0)$ is not divisible by 3 if $c$ is not a multiple of 3. This verifies the lemma for $m = 0$. We now prove the general case by induction on $m$. For $c$ even and not a multiple of 3

$$f^{c3^m}(0) = \frac{2^{c3^m} - 1}{3} = \frac{(2^{c3^{m-1}})^3 - 1}{3}$$

$$= \frac{(2^{3^{m-1}} - 1)(2^{3^{m-1}} + 2^{3^{m-1}} + 1)}{3}$$

$$= f^{c3^{m-1}}(0) \cdot (2^{2c3^{m-1}} + 2^{3^{m-1}} + 1).$$
It is easily verified that

$$2^{c \cdot 3^{m-1}} + 2^{c_3^{m-1}} + 1 \equiv 3 \pmod{9}$$

and by induction $f^{c \cdot 3^{m-1}}(0)$ is divisible by $3^{m-1}$ but not by $3^m$, hence the lemma. The odd $c$ case can be similarly proved.

Note that the Lemma implies that the smallest $k$ such that $f^k(0)$ is divisible by $3^m$ is $k = 3^m$, which is the "if" part of the theorem. We next prove the "only if" part. Let $n$ be an odd number and we write $n = 3^m n'$, where $n' > 1$ is not divisible by 3. By the Euler theorem

$$2^{\phi(n')} \equiv 1 \pmod{n'},$$

where $\phi$ is the Euler function.

Therefore

$$2^{3^m \phi(n')} - 1$$

is divisible by $n'$. Furthermore, since $\phi(n')$ is even,

$$f^{3^m \phi(n')}(0) = \frac{2^{3^m \phi(n')} - 1}{3}$$

is divisible by $n'$. On the other hand, $f^{3^m \phi(n')}(0)$ is divisible by $3^m$ by the lemma, hence it is divisible by $n = 3^m n'$. Since

$$3^m \phi(n') < 3^m n' = n,$$

$n$ is not the smallest $k$ such that $f^k(0)$ is divisible by $n$.

Corollary. $G_1(3^m, 2)$ has exactly two Hamiltonian circuits and they are arc-disjoint.

Proof. By symmetry, the set of arcs $i \rightarrow -2i - 2$, $i = 0, 1, ..., n - 1$, also constitutes a Hamiltonian circuit.

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