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Letter to the Editor

In [4, 5] it has been shown that the numbers $\cot^{(r)} \pi k/n$, $0<k<n/2$, $(k, n)=1$, are $\mathbb{Q}$-linearly independent for $r=0, 1, 2, \ldots$. The proofs are based on the nonvanishing of $L(r+1, \chi)$ (χ a nonprincipal Dirichlet character) and group determinant considerations. For $r=0$ this result had been known previously ([1–3], see also references in [4]). Our aim is to point out that this proposition is quite elementary for $r \geqslant 1$. Indeed, suppose there is a relation

$$\sum_{\substack{0<k<n/2\\(k,n)=1}} a_k \cot^{(r)} \frac{\pi k}{n} = 0, \qquad a_k \in \mathbb{Q}, \qquad \text{not all } a_k = 0. \tag{$*$}$$

The numbers $i^{r+1} \cot^{(r)} \pi k/n$, k as in $(*)$, are all $\mathbb{Q}$-conjugate to one another. Hence, on applying a suitable Galois automorphism, we can assume that a_1 has the largest absolute value of all a_k in $(*)$. But then $(*)$ is impossible, for the partial fraction expansion of $\cot x$ yields

$$\left|\sum a_k \cot^{(r)} \frac{\pi k}{n}\right| = \left|\sum a_k\left((-1)^r r! \frac{n^{r+1}}{\pi^{r+1}} \sum_{\substack{j \in \mathbb{Z}\\ j \equiv k \bmod n}} j^{-r-1}\right)\right|$$

$$\geqslant \frac{r!\, n^{r+1}}{\pi^{r+1}} |a_1| \left(1 - \sum_{j=2}^{\infty} j^{-r-1}\right) > 0.$$

The last step is similar to a standard proof of the simple fact $L(r+1, \chi) \neq 0$, $r \geqslant 1$. If $r=0$, the proposition implies $L(1, \chi) \neq 0$ in a number of cases (cf. [1]), so this case is essentially deeper.

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