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Functional limit theorems for random quadratic forms

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We prove a functional central limit theorem and a functional law of the iterated logarithm for quadratic forms in independent random variables.

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random quadratic forms * independent random variables * functional central limit theorem * functional law of the iterated logarithm * strong invariance principle

Introduction

We consider a sequence (X_n) of independent random variables (r.v.'s) defined on a common probability space with $EX_n=0$, $EX_n^2=1$ for all n. Let (a_{ij}) be a symmetric double array of real numbers (i.e. $a_{ij}=a_{ji}$ for all i and j), $A_n=(a_{ij})_{i,j=1,\dots,n}$. Put

$$Q_0\equiv 0, \qquad Q_n=\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(X_iX_j-EX_iX_j),$$

$$Y_n=Q_n-Q_{n-1}, \qquad B_n=\operatorname{var} Q_n.$$

We assume throughout that $B_n\to\infty$.

The quadratic forms (q.f.'s) Q_n form a martingale with respect to the σ-fields generated by $X_1,\dots,X_n$.

Results concerning the almost sure (a.s.) convergence (e.g. Varberg, 1966, 1968; Sjögren, 1982; Cambanis et al., 1985; Krakowiak and Szulga, 1986; Kwapien and Woyczynski, 1987) and the strong law of large numbers (SLLN) (e.g. Wilmesmeyer and Wright, 1979, 1982; Fernholz and Teicher, 1980; Szulga and Woyczynski, 1981) show that q.f.'s in independent r.v.'s behave very much like weighted sums of independent r.v.'s. Q.f.'s satisfy a fairly general law of the iterated logarithm (LIL) (e.g. Mikosch, 1988a,b, 1989, 1990) and central limit theorem (CLT) (e.g. Rotar', 1973, 1975a,b; de Jong 1987; Guttorp and Lockhart, 1988; and the references cited therein) with nonrandom normalization.

In the present paper we consider the weak and strong limit behaviour of the q.f.'s Q_n. More exactly, we prove functional CLT's and LIL's (FCLT's and FLIL's) provided that $B_n^{-1/2}Q_n$ converges weakly to a standard Gaussian law (in short, $B_n^{-1/2}Q_n \to^{d} \mathrm{N}(0,1)$). We use exponential estimates for tail probabilities, an estimate of the distance between the distribution functions of q.f.'s in Gaussian and non-Gaussian r.v.'s, certain moment estimates as well as a strong invariance principle basing on an SLLN for random times. It is natural that we shall need some algebra for matrices and their eigenvalues, but also some elementary theory of random q.f.'s. We refer to standard books (e.g. Gantmacher, 1971; Lancaster, 1982; Johnson and Kotz, 1980).

1. The central limit theorem

1.1. Preliminaries

Beginning with Sevastyanov (1961) (who determined the class of possible limit distributions for q.f.'s in Gaussian r.v.'s) there have appeared quite a few papers concerning the CLT for q.f.'s.

De Jong (1987) and Guttorp and Lockhart (1988) gave surveys of results on this topic. From the papers cited therein Rotar's (1975a,b) work is of special interest. He proved under quite general assumptions that the distribution of a q.f. in independent mean-zero, square-integrable r.v.'s is close to the distribution of the q.f. in independent identically distributed (i.i.d.) Gaussian r.v.'s with the same coefficient matrix and covariance structure. Rotar's result suggests that the class of q.f.'s in Gaussian r.v.'s is of crucial importance.

It is well known that Q_n permits the representation

$$Q_n = \sum_{i=1}^{n} \lambda_i^{(n)}((Z_i^{(n)})^2 - 1) \tag{1.1}$$

where $Z_i^{(n)}$, $i=1,\dots,n$, are orthogonal r.v.'s with $EZ_i^{(n)}=0$, $E(Z_i^{(n)})^2=1$, and $\lambda_i^{(n)}$, $i=1,\dots,n$, are the eigenvalues of A_n. We suppress the dependence on n in $Z_i^{(n)}$, $\lambda_i^{(n)}$, in the sequel.

The spectral norm of A_n is given by

$$\mu_n = \max_{i=1,\dots,n} |\lambda_i^{(n)}|.$$

For any matrix $B=(b_{ij})_{i,j=1,\dots,n}$ the Frobenius norm $\|B\|$ is defined by

$$\|B\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}^2 .$$

Suppose for the moment that the X_n are i.i.d. N(0, 1) r.v.'s. Then the Z_i in (1.1) are i.i.d. N(0, 1) r.v.'s for every fixed n. It is not difficult to see that

$$B_n^{-1/2} Q_n \xrightarrow{\mathrm{d}} \mathrm{N}(0,1)$$

if and only if

$$B_n^{-1}\mu_n^2 \to 0. \tag{1.2}$$

((1.2) is equivalent to the Lindeberg condition; e.g. Petrov, 1972.)

From Rotar' (1975) (cf. also Guttorp and Lockhart, 1988) or from de Jong (1987, Theorem 5.2), we obtain a more general result:

Theorem 1.1. *Assume that (X_n) is a sequence of independent uniformly square integrable r.v.'s with $EX_n=0$, $EX_n^2=1$ for all n and that $a_{nn}=0$ for all n. If* (1.2) *is satisfied then $B_n^{-1/2}Q_n \to^{\mathrm{d}} \mathrm{N}(0,1)$.* □

For an interpretation of condition (1.2) the following lemma is useful. To formulate it we need some further notations:

$$\tilde{A}_n = (\tilde{a}_{ij})_{i,j=1,\dots,n}, \quad \tilde{a}_{ij}=a_{ij} \text{ if } i>j,\ \tilde{a}_{ij}=0 \text{ otherwise.} \tag{1.3}$$

Lemma 1.2. *The following relations are equivalent*:

(i) $\|A_n\|^{-1}\mu_n \to 0.$

(ii) $\|A_n\|^{-v}\sum_{i=1}^{n}|\lambda_i|^v \to 0$ *for some* $v>2$.

(iii) $\|A_n\|^{-k}\|A_n^k\| \to 0$ *for some integer* $k \geq 2$.

Moreover, if $1<\liminf EX_n^4 \leq \limsup EX_n^4 < \infty$ *or if* $a_{nn}=0$ *for all n, then each of the conditions* (i)–(iii) *is equivalent to* (1.2).

If $a_{nn}=0$ *for all n, the condition*

(iv) $\|\tilde{A}_n\|^{-2}\|\tilde{A}_n^{\mathrm{T}}\tilde{A}_n\| \to 0,$

implies (1)–(3).

If $a_{ij}\geq 0$ *for all i and j and if* $a_{nn}=0$ *for all n, then* (i)–(iv) *are equivalent.*

Each of the conditions (i)–(iii) *implies that*

$$\|A_n\|^{-2}\max_{i=1,\dots,n}\sum_{i=1}^{n}a_{ij}^2 \to 0. \tag{1.4}$$

If $a_{nn}=0$ *for all n then* (iv) *implies* (1.4).

Remarks. (1) If (ii) (if (iii)) is satisfied for some $v>2$ (some integer $k\geq 2$), then (ii) (then (iii)) is true for every $v>2$ (every integer $k \geq 2$).

(2) Condition (1.2) means that the array of rowwise independent r.v.'s $(B_n^{-1/2}\lambda_i^{(n)}((Z_i^{(n)})^2-1))$ satisfies the assumption of infinitesimality (cf. Petrov, 1972).

Condition (1.4) is equivalent to the relation

$$B_n^{-1}\max_{i=1,\dots,n}\operatorname{var}(Q_i-Q_{i-1}) \to 0,$$

provided that $a_{nn}=0$ for all n or that $1<\liminf EX_n^4 \leq \limsup EX_n^4 < \infty$.

Condition (1.4) is more general than (1.2).

(3) It follows from Theorem 1.1 and Lemma 1.2 that (1.2) and the conditions (i)–(iv) are of crucial importance for the CLT-behaviour of q.f.'s. The ratios $B_n^{-1}\mu_n^2$, $\|A_n\|^{-2}\|A_n^2\|$, $\|\tilde{A}_n\|^{-2}\|\tilde{A}_n^{\mathrm{T}}\tilde{A}_n\|$ figure explicitly or implicitly in most of the results given below.

Proof of Lemma 1.2. It is an immediate consequence of Schur's theorem (e.g. Lancaster, 1982) that

$$\|A_n^k\|^2 = \sum_{i=1}^n \lambda_i^{2k}, \quad k \geq 1. \tag{1.5}$$

For $v > 2$ we have

$$\|A_n\|^{-v}\mu_n^v \leq \|A_n\|^{-v} \sum_{i=1}^n |\lambda_i|^v \leq \|A_n\|^{-v+2}\mu_n^{v-2}.$$

Then the equivalence of (i)–(iii) is obvious. Since

$$\|A_n^2\|^2 = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}^2\right)^2 + 2\sum_{i=2}^n \sum_{j=1}^{i-1}\left(\sum_{k=1}^n a_{ik}a_{kj}\right)^2$$

the relation (1.4) follows from (iii) with $k=2$. The equivalence of (i) and (1.2) under the assumptions of the lemma (i.e. $a_{nn}=0$ for all n or $1 < \liminf EX_n^4 \leq \limsup EX_n^4 < \infty$) is a consequence of the identity

$$B_n = \sum_{i=1}^n a_{ii}^2(EX_1^4 - 1) + 4\sum_{i=2}^n \sum_{j=1}^{i-1} a_{ij}^2. \tag{1.6}$$

Now assume that $a_{nn}=0$ for all n and that (iv) is satisfied. Let $\nu_1 \leq \cdots \leq \nu_n$ be the eigenvalues of $\tilde{A}_n^{\mathrm{T}}\tilde{A}_n$. We have

$$\nu_n^2 \leq \sum_{i=1}^n \nu_i^2 = \|\tilde{A}_n^{\mathrm{T}}\tilde{A}_n\|^2. \tag{1.7}$$

Since $\nu_n^{1/2}$ is the spectral norm of $\tilde{A}_n$ and since $A_n = \tilde{A}_n + \tilde{A}_n^{\mathrm{T}}$ we get by the triangular inequality for spectral norms that $\mu_n \leq 2\nu_n^{1/2}$. By (1.7), the relation (iv) implies that $\|A_n\|^{-1}\mu_n \to 0$, i.e. (i) is satisfied.

If $a_{ij} \geq 0$ for all i and j, then $\|\tilde{A}_n^{\mathrm{T}}\tilde{A}_n\|^2 \leq \|A_n^2\|^2$, so that (iii) implies (iv). This concludes the proof of the lemma. □

Frequently we shall make use of the following estimates.

Lemma 1.3. *Assume that $c_1 = \sup_n E|X_n|^p < \infty$ for some $p > 1$. Then*

$$E\left|\sum_{i=1}^n a_i X_i\right|^p \leq c\left(\sum_{i=1}^n a_i^2\right)^{p/2} \tag{1.8}$$

for any reals a_i, some constant c depending on p and c_1. If $a_{nn}=0$ for all n or if the X_n are i.i.d. N(0, 1) *r.v.'s, then*

$$E|Q_n\|^p \leqslant c\|A_n\|^p = c2^{-p/2}B_n^{p/2} \tag{1.9}$$

for $p \geqslant 2$, some constant c depending on p and c_1.

Proof. (1.8) is a consequence of the Marcinkiewicz-Zygmund and Minkowski inequalities as well as of the uniform boundedness of $E|X_i|^p$. By Burkholder's and Minkowski's inequalities and by (1.8) we obtain

$$\begin{aligned} E|Q_n|^p &\leqslant cE\left(\sum_{i=1}^n Y_i^2\right)^{p/2} \\ &\leqslant c\left(\sum_{i=1}^n (E|Y_i|^p)^{2/p}\right)^{p/2} \\ &\leqslant c\left(\sum_{i=2}^n \left(E\left|\sum_{j=1}^{i-1} a_{ij}X_j\right|^p\right)^{2/p}\right)^{p/2} \\ &\leqslant c\left(\sum_{i=2}^n \sum_{j=1}^{i-1} a_{ij}^2\right)^{p/2} \end{aligned}$$

provided that $a_{nn}=0$ for all n.

(Here and in the sequel c stands for constants which may be different from line to line and even from formula to formula and whose value is not of interest.)

If the X_n are i.i.d. N(0, 1) r.v.'s then (1.9) is a consequence of (1.1), (1.8) and (1.5) with $k=1$. □

1.2. A Berry-Esseen estimate

Assume that (ξ_n) is a sequence of i.i.d. N(0, 1) r.v.'s. Put

$$Q(X_1,\ldots,X_n) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}X_iX_j,$$

$$Q_n(\xi) = Q(\xi_1,\ldots,\xi_n), \qquad Q_n(X) = Q(X_1,\ldots,X_n),$$

$$F_{\xi,n}(x) = P(Q_n(\xi) < x), \qquad F_{X,n}(x) = P(Q_n(X) < x).$$

Using the representation (1.1) and a non-uniform version of the Berry-Esseen inequality in the CLT (e.g. Petrov, 1972) we get

$$\sup_x (1+|x|^3)|P(B_n^{-1/2}Q_n < x) - \Phi(x)| \leqslant B_n^{-3/2}\sum_{i=1}^n |\lambda_i|^3 \leqslant cB_n^{-1/2}\mu_n, \tag{1.10}$$

where $\Phi(x)$ denotes the distribution function of ξ_1. Gamkrelidze, Rotar' (1977) and Rotar' and Shervashidze (1985) estimated the distance

$$\Delta_n = \sup_x |F_{\xi,n}(x) - F_{X,n}(x)|.$$

We extend these results slightly. The proof is essentially due to Gamkrelidze, Rotar' (1977). Put

$$L_{p,n} = B_n^{-p/2} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}^2 \right)^{p/2}, \quad p>0.$$

Proposition 1.4. *Assume that* $c_1 = \sup E|X_n|^p < \infty$ *for some integer* $p>2$. *Suppose that* $E\xi_i^k = EX_i^k$, $k=2,\ldots,p-1$, *and* $a_{ii}=0$, *for all i. If* $B_n^{-1}\mu_n^2 \to 0$ *then*

$$\Delta_n = \mathrm{O}(L_{p,n}^{1/(p+1)}), \quad n\to\infty. \tag{1.11}$$

Remarks. (1) Relation (1.11) can be rewritten in the form $\Delta_n \leqslant cL_{p,n}^{1/(p+1)}$ for large n and for a constant c depending on c_1 and p.

(2) By Lemma 1.2, the condition $B_n^{-1}\mu_n^2 \to 0$ implies (1.4). Hence $L_{p,n}\to 0$ as $n\to\infty$. Moreover,

$$L_{p,n} \leqslant \left(B_n^{-1} \max_{i=1,\ldots,n} \sum_{i=1}^{n} a_{ij}^2 \right)^{(p-2)/2}.$$

(3) The relation (1.11) remains true in more general situations, i.e. if the ratio $B_n^{-1}\mu_n^2$ is bounded above by a 'small' constant (cf. Gamkrelidze and Rotar', 1977; Rotar' and Shervashidze, 1985).

(4) According to Theorem 1.1, $F_{X,n}(B_n^{1/2}x + EQ_n)$ and $F_{\xi,n}(B_n^{1/2}x+EQ_n)$ both tend to $\Phi(x)$ whenever $B_n^{-1}\mu_n^2 \to 0$.

Proof of Proposition 1.4. Without loss of generality assume that $\|A_n\|=1$. By Lemma 1.2, $\sum_{i=1}^n |\lambda_i|^3 \to 0$. According to Gamkrelidze and Rotar' (1977) this enables us to apply Esseen's lemma (e.g. Petrov, 1972) for large n,

$$|F_{\xi,n}(x) - F_{X,n}(x)| \leqslant c\left\{ \int_0^T t^{-1}|f_X(t)-f_\xi(t)|\,\mathrm{d}t + qT^{-1} \right\}, \quad T>0, \tag{1.12}$$

with constants c, $q>0$. Here f_ξ, f_X are the characteristic functions of $F_{\xi,n}(x)$, $F_{X,n}(x)$, respectively (we suppress the index n in f_X and f_ξ). Put $f_0 = f_X$, $f_n = f_\xi$ and let f_i denote the characteristic functions of $\tilde{Q}_i = Q(\xi_1,\ldots,\xi_i, X_{i+1},\ldots,X_n)$. Then

$$|f_X(t) - f_\xi(t)| \leqslant \sum_{i=1}^{n} |f_i(t) - f_{i-1}(t)|.$$

We have $\tilde{Q}_{j-1} = X_jV_j + W_j$ where V_j and W_j do not depend on X_j, $\tilde{Q}_j = \xi_jV_j + W_j$.

$$\begin{aligned} |f_j(t) - f_{j-1}(t)| &\leqslant |E[\exp\{\mathrm{i}tW_j\}E(\exp\{\mathrm{i}tX_jV_j\} - \exp\{\mathrm{i}t\xi_jV_j\} \mid \mathcal{F}_j)]| \\ &\leqslant E|E(\exp\{\mathrm{i}tX_jV_j\} - \exp\{\mathrm{i}t\xi_jV_j\} \mid \mathcal{F}_j)| \\ &\leqslant E\left| E\left(\left[1 + \mathrm{i}tX_jV_j + \cdots + \frac{(\mathrm{i}tX_jV_j)^p}{p!}\,\Theta(tX_jV_j) \right] \right.\right. \\ &\qquad \left.\left. - \left[1 + \mathrm{i}t\xi_jV_j + \cdots + \frac{(\mathrm{i}t\xi_jV_j)^p}{p!}\,\tilde{\Theta}(t\xi_jV_j) \right] \,\middle|\, \mathcal{F}_j \right) \right| \end{aligned} \tag{1.13}$$

where $|\Theta(x)|\leq 1$, $|\tilde{\Theta}(x)|\leq 1$, and $\mathscr{F}_j$ is the σ-algebra generated by X_k, ξ_k, $k\neq j$. The moments of X_j and ξ_j coincide up to order $(p-1)$. Hence

$$|f_j(t)-f_{j-1}(t)|\leq c|t|^p E|V_j|^p, \tag{1.14}$$

and by Lemma 1.3,

$$E|V_j|^p\leq c\left(\sum_{i=1}^n a_{ij}^2\right)^{p/2}. \tag{1.15}$$

Now (1.12)-(1.15) yield

$$|F_{\xi,n}(x)-F_{X,n}(x)|\leq c\{L_{p,n}T^p+T^{-1}\}\leq cL_{p,n}^{1/(p+1)}.$$

This concludes the proof. □

1.3. The FCLT

Assume for the moment that (X_n) is a sequence of i.i.d. N(0, 1) r.v.'s. Define the polygonal functions

$$g_n(t)=\begin{cases} B_n^{-1/2}\sum_{i=1}^k \lambda_i(Z_i^2-1), & \text{for } t=2B_n^{-1}(\lambda_1^2+\cdots+\lambda_k^2),\\ 0, & \text{for } t=0,\\ \text{linearly interpolated elsewhere}, & 0\leq t\leq 1.\end{cases}$$

Let W denote Brownian motion on [0, 1] and the symbol $\to^{\mathrm{d}}$ stands for weak convergence in $C(0,1)$.

Proposition 1.5. *$g_n\to^{\mathrm{d}} W$ if and only if $B_n^{-1}\mu_n^2\to 0$.* □

The necessity follows from the fact that $g_n\to^{\mathrm{d}} W$ implies $B_n^{-1/2}Q_n\to^{\mathrm{d}} \mathrm{N}(0,1)$. The sufficiency is a consequence of an FCLT for double arrays or rowwise independent r.v.'s due to Prokhorov (1956) (see Billingsley, 1977, Exercise 1, Section 10). The polygonal functions g_n are not convenient from a practical point of view: They require the knowledge of all eigenvalues of A_n. Therefore we define another sort of piecewise linear function which depends only on the B_i.

Put $t_{kn}=B_n^{-1}B_k$, $k=0,\ldots,n$, and define

$$f_n(t)=\begin{cases} B_n^{-1/2}Q_k, & \text{for } t=t_{kn},\ k=0,\ldots,n,\\ \text{linearly interpolated elsewhere}, & 0\leq t\leq 1.\end{cases}$$

Theorem 1.6. *Let (X_n) be a sequence of independent r.v.'s, $EX_n=0$, $EX_n^2=1$ for all n. Assume that $\|\tilde{A}_n\|^{-2}\|\tilde{A}_n^{\mathrm{T}}\tilde{A}_n\|\to 0$. If $a_{nn}=0$ for all n and $\sup E|X_n|^p<\infty$ for some $p>2$ or if the X_n are i.i.d. N(0, 1) r.v.'s and $B_n^{-1}\max_{i=1,\ldots,n} a_{ii}^2\to 0$, then $f_n\to^{\mathrm{d}} W$.*

Remarks. (1) The matrices $\tilde{A}_n$ were defined by (1.3).

(2) The condition $\|\tilde{A}_n\|^{-2}\|\tilde{A}_n^{\mathrm{T}}\tilde{A}_n\| \to 0$ implies that $B_n^{-1}\mu_n^2 \to 0$ (cf. Lemma 1.2).

Proof of Theorem 1.6. We show that the conditions of Brown's (1971) FCLT for martingales are satisfied, i.e.

$$B_n^{-1} \sum_{i=1}^{n} E(Y_i^2 | \mathscr{F}_{i-1}) \xrightarrow{\mathrm{P}} 1, \tag{1.16}$$

$$S_{n1} = B_n^{-1} \sum_{i=1}^{n} E(Y_i^2 I(|Y_i| > \varepsilon B_n^{1/2}) | \mathscr{F}_{i-1}) \xrightarrow{\mathrm{P}} 0 \quad \forall \varepsilon > 0. \tag{1.17}$$

Here $\to^{\mathrm{P}}$ denotes convergence in probability, $\mathscr{F}_i$ is the σ-field generated by $X_1, \ldots, X_i$, $i \geqslant 1$, $\mathscr{F}_0$ is the trivial σ-field.

First assume that $a_{nn} = 0$ for all n. We have

$$S_{n1} \leqslant c B_n^{-1} \sum_{i=1}^{n} B_n^{-(p-2)/2} E(|Y_i|^p | \mathscr{F}_{i-1})$$

$$\leqslant c B_n^{-p/2} \sum_{i=2}^{n} \left| \sum_{j=1}^{i-1} a_{ij} X_j \right|^p \quad \text{a.s.}$$

By Lemma 1.3,

$$E S_{n1} \leqslant c B_n^{-p/2} \sum_{i=2}^{n} \left(\sum_{j=1}^{i-1} a_{ij}^2 \right)^{p/2}$$

$$\leqslant c \left(B_n^{-1} \max_{i=2,\ldots,n} \sum_{j=1}^{i-1} a_{ij}^2 \right)^{(p-2)/2}.$$

By Lemma 1.2, the right-hand side of this inequality tends to zero. This implies (1.17).

Next we show (1.16). It suffices to prove that

$$4 B_n^{-1} \sum_{i=2}^{n} \left(\sum_{j=1}^{i-1} a_{ij} X_j \right)^2 - 1 \xrightarrow{\mathrm{P}} 0,$$

or, equivalently, that

$$4 B_n^{-1} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left(\sum_{k=i \vee j+1}^{n} a_{ik} a_{kj} \right) (X_i X_j - E X_i X_j)$$

$$= 4 B_n^{-1} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (\tilde{A}_n^{\mathrm{T}} \tilde{A}_n)_{ij} (X_i X_j - E X_i X_j) \xrightarrow{\mathrm{P}} 0.$$

Thus it suffices to show that

$$S_{n2} = B_n^{-1} \sum_{i=1}^{n-1} \left(\sum_{j=i+1}^{n} a_{ij}^2 \right) (X_i^2 - 1) \xrightarrow{\mathrm{P}} 0$$

and

$$S_{n3} = B_n^{-1} \sum_{i=2}^{n} \sum_{j=1}^{i-1} (\tilde{A}_n^{\mathrm{T}} \tilde{A}_n)_{ij} X_i X_j \xrightarrow{\mathrm{P}} 0.$$

We have, by Lemma 1.3,

$$ES_{n3} \leq B_n^{-1} \|\tilde{A}_n^{\mathrm{T}} \tilde{A}_n\|^2,$$

$$E|S_{n2}|^{p/2} \leq c\left(B_n^{-2} \sum_{i=1}^{n-1} \left(\sum_{j=i+1}^{n} a_{ij}^2 \right)^2 \right)^{p/4}$$

$$\leq c\left(B_n^{-1} \max_{i=1,\dots,n-1} \sum_{j=i+1}^{n} a_{ij}^2 \right)^{p/4},$$

and the right-hand sides of both inequalities tend to zero because of Lemma 1.2. This concludes the proof for the case $a_{nn} = 0$.

Now assume that the X_n are i.i.d. N(0, 1) r.v.'s. Then (1.16) can be rewritten in the form

$$4B_n^{-1}\left(\sum_{i=2}^{n} \left(\sum_{j=1}^{i-1} a_{ij} X_j \right)^2 - \sum_{i=2}^{n} \sum_{j=1}^{i-1} a_{ij}^2 \right) \xrightarrow{\mathrm{P}} 0.$$

The proof of (1.16) is then analogous to the case $a_{nn} = 0$.

It suffices for (1.17) to show that

$$B_n^{-2} \sum_{i=1}^{n} EY_i^4 \to 0.$$

Now,

$$EY_i^4 \leq c\left\{ a_{ii}^4 + E\left(\sum_{j=1}^{i-1} a_{ij} X_j \right)^4 \right\}$$

$$\leq c\left\{ a_{ii}^4 + \left(\sum_{j=1}^{i-1} a_{ij}^2 \right)^2 \right\}$$

$$\leq cE\left(\sum_{j=1}^{i} a_{ij}^2 \right)^2.$$

Hence

$$B_n^{-2} \sum_{i=1}^{n} EY_i^4 \leq cB_n^{-1} \max_{i=1,\dots,n} \sum_{j=1}^{i} a_{ij}^2$$

and the right-hand side of this inequality tends to zero, by (1.4) and the assumption that $B_n^{-1} \max_{i=1,\dots,n} a_{ii}^2 \to 0$. This concludes the proof of the theorem. □

Remark. I conjecture that Theorem 1.6 remains valid if $a_{nn} = 0$ for all n and if the r.v.'s X_n are uniformly square integrable. I can prove this only under the assumption that $a_{ij} \geq 0$ for all i and j. The proof of this fact is analogous to the proof of Theorem 5.1 in de Jong (1987). He used Brown's (1971) FCLT and a truncation argument.

2. The law of the iterated logarithm

2.1. Preliminaries

Results of LIL-type depend heavily on estimates for tail probabilities and on maximal inequalities. In the case of q.f.'s in Gaussian r.v.'s these estimates depend on the asymptotic behaviour of the ratio $B_n^{-1}\mu_n^2$(cf. Mikosch, 1988b, 1990).

We cite some LIL-results for q.f.'s in i.i.d. N(0, 1) r.v.'s. Define $\log x = \max(1, \ln x)$, $\log_2 x = \log\log x$, $x > 0$.

For $d > 1$ and $\alpha > 1$ let $n_k = n_k(\alpha, d)$ be an integer sequence satisfying the condition

$$B_{n_k}^{-1} d^{k^\alpha} \to 1. \tag{2.1}$$

Such sequences are well defined for every $d > 1$, $\alpha > 1$, e.g. if $B_n^{-1}B_{n+1} \to 1$. The latter condition is satisfied if $B_n^{-1}\mu_n^2 \to 0$. Indeed, $B_n^{-1}\mu_n^2 \to 0$ implies (1.4), by Lemma 1.2, so that $B_n^{-1}B_{n+1} \to 1$ is an immediate consequence.

Theorem 2.1 (Mikosch, 1988b, 1990). *Assume that (X_n) is a sequence of i.i.d.* N(0, 1) *r.v.'s. Then the following statements are true*:

(A) $\limsup (2B_n \log_2^2 B_n)^{-1/2}|Q_n| \leq 1$ *a.s.*

(B) *If*

$$\mu_n^2 = \mathrm{o}(B_n/\log B_n) \tag{2.2}$$

then

$$\limsup\,(2B_n \log_2 B_n)^{-1/2}|Q_n| \leq 1 \quad a.s.$$

(C) *Assume that* (2.2) *is satisfied and define sequences* (n_k) *with* (2.1) *for some* $d > 1$ *and every* $\alpha > 1$. *If*

$$\sum_{i=1}^{n_{k-1}} \sum_{j=n_{k-1}+1}^{n_k} a_{ij}^2 = \mathrm{o}(B_{n_k}/\log_2 B_{n_k})$$

for every $\alpha > 1$, *then*

$$\limsup \pm(2B_n \log_2 B_n)^{-1/2} Q_n = 1 \quad a.s.$$

(D) *Assume that there exist sequences* (n_k) *with* (2.1) *for some* $d > 1$ *and every* $\alpha > 1$. *If*

$$\sum_{i=1}^{n_{k-1}} \sum_{j=n_{k-1}+1}^{n_k} a_{ij}^2 = \mathrm{o}(B_{n_k})$$

for every $\alpha > 1$, $\mu_n \uparrow \infty$ *and* $\liminf \mu_n/\|A_n\| > 0$, *then*

$$\limsup\,(2\mu_n \log_2^2 \mu_n)^{-1/2}|Q_n| = 1 \quad a.s. \qquad \square$$

Remark. The condition (2.2) is of crucial importance for the LIL-behaviour of Gaussian q.f.'s. This condition is very much like Kolmogorov's condition on the r.v.'s in Kolmogorov's LIL for sums of independent r.v.'s; the exponential estimates are also very similar to the Kolmogorov case (see Mikosch, 1988b, 1990; cf. Petrov, 1972). Condition (2.2) implies that $B_n^{-1}\mu_n^2 \to 0$, i.e. the CLT $B_n^{-1/2}Q_n \to^{\mathrm{d}} \mathrm{N}(0,1)$ holds. The cases (A) and (D) give some information about the LIL-behaviour if the condition $B_n^{-1}\mu_n^2 \to 0$ is not satisfied. Then $B_n^{-1/2}Q_n$ shows 'noncentral' limit behaviour.

We apply an approach to the FLIL basing on an embedding technique for the martingale Q_n. This method was used by Strassen (1964) in order to prove his well-known invariance principle and the FLIL for i.i.d. sum processes.

Theorem 2.2 (e.g. Hall and Heyde, 1980, Theorem A.1). *Let* $\{S_n = \sum_{i=1}^n Y_i, \mathscr{F}_n, n \geq 1\}$ *be a zero-mean, square-integrable martingale. Then there exists a probability space supporting a (standard) Brownian motion* W *and a sequence of nonnegative r.v.'s* $\tau_1, \ldots, \tau_n$ *with the following properties. If* $T_n = \sum_{i=1}^n \tau_i$, $S'_n = W(T_n)$, $Y'_1 = S'_1$, $Y'_n = S'_n - S'_{n-1}$, *for* $n \geq 2$, *and* $\mathscr{S}_n$ *is the σ-field generated by* $S'_1, \ldots, S'_n$ *and* $W(t)$, $0 \leq t \leq T_n$, *then*:

(i) $\{S_n, n \geq 1\} =^{\mathrm{d}} \{S'_n, n \geq 1\}$.

(ii) T_n *is* $\mathscr{S}_n$*-measurable.*

(iii) *For each real number* $r \geq 1$,

$$E(\tau_n^r \mid \mathscr{S}_{n-1}) \leq C_r E(|Y'_n|^{2r} \mid \mathscr{S}_{n-1}) = C_r E(|Y'_n|^{2r} \mid Y'_1, \ldots, Y'_{n-1}) \quad a.s.$$

for some constants C_r.

(iv) $\quad E(\tau_n \mid \mathscr{S}_{n-1}) = E((Y'_n)^2 \mid \mathscr{S}_{n-1}) \quad a.s.$ □

2.2. A strong law for random times

Throughout it is assumed that

$$a_{nn} = 0 \quad \text{for all } n.$$

It is possible to avoid this condition but the assumptions on (a_{ij}) will become more complicated then and we will also need higher moment conditions. According to Theorem 2.2 we may and do suppose that Brownian motion W and nonnegative r.v.'s $\tau_1, \tau_2, \ldots$ are given on a common probability space such that $Q_n = W(T_n)$, $T_n = \sum_{i=1}^n \tau_i$ and the properties (i)–(iv) of Theorem 2.2 are satisfied.

We use the notations

$$Q_n = \sum_{i=1}^n Y_i, \quad Y_i = X_i V_i, \quad V_1 = 0, \quad V_i = 2\sum_{j=1}^{i-1} a_{ij} X_j, \quad R_n = \sum_{i=1}^n V_i^2.$$

Lemma 2.3. *Assume that* $\sup E|X_n|^{2p}<\infty$ *for some* $p\in(1,2]$. *If*

$$\sum_{i=2}^{\infty} B_i^{-p}\left(\sum_{j=1}^{i-1} a_{ij}^2\right)^p<\infty, \tag{2.3}$$

then

$$B_n^{-1}(T_n-R_n)\to 0 \quad a.s. \tag{2.4}$$

Proof. Let $\mathcal{H}_n^i$, $i=1$, 2, 3, be the σ-fields generated by Q_i, $i\leq n$, and $W(t)$, $0\leq t\leq T_n$; by Y_i, $i\leq n$; by X_i, $i\leq n$; respectively. Put

$$M_n^1=T_n-\sum_{i=1}^{n} E(\tau_i\,|\,\mathcal{H}_{i-1}^1),$$

$$M_n^2=\sum_{i=2}^{n}(E(\tau_i\,|\,\mathcal{H}_{i-1}^1)-Y_i^2),$$

$$M_n^3=\sum_{i=2}^{n}(Y_i^2-V_i^2).$$

Note that $(M_n^i, \mathcal{H}_n^i)$, $i=1,2,3$, are martingales. It suffices to show that $\lim B_n^{-1}M_n^i=0$ a.s., $i=1$, 2, 3. According to a result of Chow (see Stout, 1974, Corollary 2.8.5) it suffices to show that

$$\sum B_n^{-p}E(|M_n^i-M_{n-1}^i|^p\,|\,\mathcal{H}_{n-1}^i)<\infty \quad \text{a.s.}, \quad i=1,2,3, \tag{2.5}$$

for some $p\in(1,2]$. Using Theorem 2.2 and Lemma 1.3 we get

$$E|M_n^i-M_{n-1}^i|^p\leq CE|Y_n|^{2p}\leq C\left(\sum_{j=1}^{i-1} a_{ij}^2\right)^p. \tag{2.6}$$

Now relation (2.4) is immediate from (2.5), (2.6) and (2.3). □

Lemma 2.4. *Assume that* $B_n^{-1}B_{n+1}\to 1$ *and define for* $\tau>0$ *the integers* $n_k=n_k(\tau)$ *by* $B_{n_k-1}\leq(1+\tau)^k<B_{n_k}$. *Suppose that for each* $\varepsilon>0$ *there exists a* $\tau_0>0$ *such that for* $\tau\in(0,\tau_0)$,

$$\sum P(|R_{n_k}-B_{n_k}|>\varepsilon B_{n_k})<\infty.$$

Then

$$B_n^{-1}R_n\to 1 \quad a.s.$$

Proof. Note that $ER_n=B_n$. It suffices to show that

$$P\Big(\max_{n_{k-1}<n\leq n_k} B_n^{-1}|R_n-ER_n|>\varepsilon \text{ i.o.}\Big)=0 \tag{2.7}$$

(i.o. — infinitely often) for each $\varepsilon > 0$, some $\tau > 0$. We have

$$P\left(\max_{n_{k-1} < n \le n_k} B_n^{-1} |R_n - ER_n| > \varepsilon \right)$$

$$\le P\left(\max_{n_{k-1} < n \le n_k} (B_n^{-1} R_n - 1) > \varepsilon \right) + P\left(\max_{n_{k-1} < n \le n_k} (1 - B_n^{-1} R_n) > \varepsilon \right)$$

$$\le P(R_{n_k} > (1+\varepsilon) B_{n_{k-1}}) + P(R_{n_{k-1}} < -(1-\varepsilon) B_{n_k})$$

$$\le P(R_{n_k} > (1+\tfrac{1}{2}\varepsilon) B_{n_k}) + P(R_{n_{k-1}} < -(1-\tfrac{1}{2}\varepsilon) B_{n_{k-1}}) \tag{2.8}$$

for $\tau > 0$ close to zero and k sufficiently large. Now the statement follows from (2.8), (2.7) and the Borel-Cantelli lemma. □

Proposition 2.5. *Assume that* (X_n) *is a sequence of i.i.d.* N(0, 1) *r.v.'s. If*

$$\sum_{i=2}^{\infty} B_i^{-2} \left(\sum_{j=1}^{i-1} a_{ij}^2 \right)^2 < \infty, \tag{2.9}$$

$$\| \tilde{A}_n^{\mathrm{T}} \tilde{A}_n \| = \mathrm{o}(B_n / \log_2 B_n), \tag{2.10}$$

then

$$B_n^{-1} T_n \to 1 \quad a.s.$$

Remarks. (1) The matrices $\tilde{A}_n$ are defined by (1.3).

(2) By (1.7), (2.10) implies that $\mu_n^2 = \mathrm{o}(B_n / \log_2 B_n)$, i.e. condition (2.2) is satisfied (cf. Theorem 2.1).

Proof of Proposition 2.5. The conditions of Lemma 2.3 are obviously satisfied. It remains to show that Lemma 2.4 is applicable. From (2.9) we have

$$B_{n-1}^{-1} B_n = 1 - 2 B_n^{-1} \sum_{j=1}^{n-1} a_{nj}^2 \to 1.$$

The r.v. R_n is a q.f. in i.i.d. N(0, 1) r.v.'s with $ER_n = B_n$ and

$$\operatorname{var} R_n = 2 \| \tilde{A}_n^{\mathrm{T}} \tilde{A}_n \|^2. \tag{2.11}$$

Indeed,

$$R_n = \sum_{i=2}^{n} \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \tilde{a}_{ij} \tilde{a}_{ik} X_j X_k$$

$$= \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} X_j X_k \sum_{i=2}^{n} \tilde{a}_{ij} \tilde{a}_{ik}$$

$$= \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} X_j X_k \sum_{i=j \vee k+1}^{n} a_{ij} a_{ik}$$

$$= \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} X_j X_k (\tilde{A}_n^{\mathrm{T}} \tilde{A}_n)_{jk}. \tag{2.12}$$

From Mikosch (1988b) we get

$$P(B_n^{-1}|R_n - ER_n| > \varepsilon) \leq C \exp\left\{-\frac{(1-\delta)\varepsilon B_n}{(2 \operatorname{var} R_n)^{1/2}}\right\}$$

for $\delta \in (0, 1)$, some $C = C(\delta)$. Recalling (2.10), (2.11) and the definition of (n_k) we get

$$P(B_{n_k}^{-1}|R_{n_k} - ER_{n_k}| > \varepsilon) \leq k^{-2}$$

for each $\varepsilon > 0$, $\tau > 0$, k sufficiently large. An application of Lemma 2.4 yields the statement of the proposition. □

Proposition 2.6. *Assume that* $\sup EX_n^6 < \infty$. *Suppose that* (2.9) *and* (2.10) *are satisfied. If the conditions*

$$\sum_k B_{n_k}^{-3} \sum_{i=1}^{n_k} \left(\sum_{j=i+1}^{n_k} a_{ij}^2 \right)^3 < \infty, \tag{2.13}$$

$$\sum_k \left\{ \frac{\sum_{i=1}^{n_k-1} \left(\sum_{j=1, i\neq j}^{n_k-1} (\tilde{A}_{n_k}^{\mathrm{T}} \tilde{A}_{n_k})_{ij}^2 \right)^{3/2}}{\left(\sum_{i=1}^{n_k-1} \sum_{j=1, i\neq j}^{n_k-1} (\tilde{A}_{n_k}^{\mathrm{T}} \tilde{A}_{n_k})_{ij}^2 \right)^{3/2}} \right\}^{1/4} < \infty \tag{2.14}$$

hold for $\tau > 0$ *in a neighbourhood of the origin then*

$$B_n^{-1} T_n \to 1 \quad a.s.$$

Remarks. (1) Each of the following conditions implies (2.13):

$$\sum B_{n_k}^{-2} \max_{i=1,\dots,n_k-1} \left(\sum_{j=i+1}^{n_k} a_{ij}^2 \right)^2 < \infty,$$

$$\sum B_l^{-3} \sum_{i=1}^{l} \left(\sum_{j=i+1}^{l} a_{ij}^2 \right)^3 < \infty.$$

(2) (2.14) is satisfied if

$$\sum_k \left\{ \frac{\max_{i=1,\dots,n_k-1} \left(\sum_{j=1, i\neq j}^{n_k-1} (\tilde{A}_{n_k}^{\mathrm{T}} \tilde{A}_{n_k})_{ij}^2 \right)}{\sum_{i=1}^{n_k-1} \sum_{j=1, i\neq j}^{n_k-1} (\tilde{A}_{n_k}^{\mathrm{T}} \tilde{A}_{n_k})_{ij}^2} \right\}^{1/8} < \infty.$$

Proof of Proposition 2.6. The assumptions of Lemma 2.3 are satisfied. It remains to show that the conditions of Lemma 2.4 are fulfilled. Clearly, $B_{n-1}^{-1} B_n \to 1$. Recalling (2.12) we define

$$R_{n1} = \sum_{i=1}^{n-1} (\tilde{A}_n^{\mathrm{T}} \tilde{A}_n)_{ii} (X_i^2 - 1), \qquad R_{n2} = (R_n - ER_n) - R_{n1}.$$

By a non-uniform version of the Berry–Esseen estimate for sums of independent r.v.'s (cf. (1.10), see Petrov, 1972) and by the uniform boundedness of EX_n^6 we get

$$P(|R_{n1}|>\varepsilon B_n)=2\left(1-\Phi\left(\frac{\varepsilon B_n}{(\operatorname{var} R_{n1})^{1/2}}\right)\right)+\mathrm{O}\left(\frac{\sum_{i=1}^{n}(\tilde{A}_n^{\mathrm{T}}\tilde{A}_n)_{ii}^3}{B_n^3}\right). \tag{2.15}$$

We have

$$\operatorname{var} R_{n1}\leqslant C\|\tilde{A}_n^{\mathrm{T}}\tilde{A}_n\|^2$$

which together with (2.10) implies that

$$\sum\left(1-\Phi\left(\frac{\varepsilon B_{n_k}}{(\operatorname{var} R_{n_k,1})^{1/2}}\right)\right)<\infty. \tag{2.16}$$

Now the relation

$$\sum P(|R_{n_k,1}|>\varepsilon B_n)<\infty$$

follows from (2.15), (2.16) and (2.13). It remains to show that

$$\sum P(|R_{n_k,2}|>\varepsilon B_{n_k})<\infty. \tag{2.17}$$

An application of Proposition 1.4 with $p=3$ yields

$$P(|R_{n2}|>\varepsilon B_n)=P(|R_{n2}(\xi)|>\varepsilon B_n)$$
$$+\mathrm{O}\left(\left\{\frac{\sum_{i=1}^{n-1}\left(\sum_{j=1,i\neq j}^{n-1}(\tilde{A}_n^{\mathrm{T}}\tilde{A}_n)_{ij}^2\right)^{3/2}}{\left(\sum_{i=1}^{n-1}\sum_{j=1,i\neq j}^{n-1}(\tilde{A}_n^{\mathrm{T}}\tilde{A}_n)_{ij}^2\right)^{3/2}}\right\}^{1/4}\right), \tag{2.18}$$

where $R_{n2}(\xi)$ is obtained from R_{n2} by replacing all X_i by i.i.d. N(0, 1) r.v.'s ξ_i. The convergence of the series

$$\sum P(|R_{n_k,2}(\xi)|>\varepsilon B_{n_k})$$

can be proved analogously to the proof of Proposition 2.5. Thus (2.17) is a consequence of (2.18) and (2.14). This concludes the proof. □

Remarks. (1) The assumptions of the Propositions 2.5 and 2.6 are satisfied, for instance, for q.f.'s of the form

$$\sum_{k=1}^{s}\sum_{i=1}^{n-l_k}X_iX_{i+l_k},\quad l_k>0,\ s\geqslant 1.$$

Generally speaking, the conditions of these propositions mean that 'the main coefficient mass' of the matrices A_n is concentrated 'along the diagonal'.

(2) The moment conditions of Proposition 2.6 can be weakened. Considering the proof we can see that the probability $P(|R_{n1}|>\varepsilon B_n)$, can be estimated by a non-uniform Berry–Esseen inequality that requires less than a third moment of the r.v.'s X_i^2 (see e.g. Petrov, 1972). The simplest way to get an estimate for $P(|R_{n2}|>\varepsilon B_n)$ is to use Cebyshev's inequality, i.e.

$$P(|R_{n2}|>\varepsilon B_n)\leqslant \text{const } B_n^{-2}\|\tilde{A}_n^{\mathrm{T}}\tilde{A}_n\|^2,$$

which requires only a second moment condition on the X_i. Better estimates of $P(|R_{n2}|>\varepsilon B_n)$ can be obtained from non-uniform Berry–Esseen inequalities for martingales which were proved by Häusler and Joos (1988).

Clearly, if one uses other probability estimates, one has to modify the conditions (2.13), (2.14), i.e. one has to modify the conditions on the matrices A_n.

The conditions (2.13) and (2.14) are essentially conditions on the growth of $B_n^{-1}\max_{i=1,\dots,n}\sum_{j=1}^n a_{ij}^2$ (cf. Lemma 1.2).

2.3. A strong invariance principle with applications

Define the process $Q(t)$, $t\geqslant 0$, by

$$Q(t)=\begin{cases} Q_n & \text{for } t=B_n,\\ \text{linear interpolation} & \text{elsewhere.}\end{cases}$$

Theorem 2.7. *Assume that the conditions of Proposition* 2.5 *or* 2.6 *are satisfied. There exists a probability space with $Q(t)$ and Brownian motion $W(t)$ defined on it such that*

$$\max_{t\leqslant s}|Q(t)-W(t)|=o((s\log_2 s)^{1/2})\quad a.s.,\quad s\to\infty. \qquad \square$$

The proof is an immediate consequence of the SLLN's for T_n given by the Propositions 2.5 and 2.6. It can be handled analogously as the i.i.d.-sum case (e.g. Stout, 1974, Theorem 5.3.2).

We can apply Theorem 2.7 straightforwardly to get an FLIL for q.f.'s. For $0\leqslant t\leqslant 1$ put

$$h_n(t)=(2B_n\log_2 B_n)^{-1/2}Q(B_nt).$$

Define

$$\mathscr{R}=\left\{h\in\mathbb{C}(0,1)\colon h(0)=0,\ h \text{ absolutely continuous}, \int_0^1 (h'(t))^2\,dt\leqslant 1\right\},$$

the Strassen compact (cf. Stout, 1974).

Theorem 2.8. *Assume that the conditions of Propositions* 2.5 *or* 2.6 *are satisfied. Then, with probability* 1, (h_n) *is relatively compact in* $\mathbb{C}(0,1)$ (*with respect to the sup-norm*) *and its set of limit points coincides with* $\mathscr{R}$. $\square$

For the proof we may restrict ourselves to the consideration of $(2B_n\log_2 B_n)^{-1/2}W(B_nt)$. Then the proof follows as in the Strassen case (e.g. Stout, 1974).

Corollary 2.9. *Assume that the conditions of Proposition* 2.5 *or* 2.6 *are satisfied. Then the limit points of* $((2B_n\log_2 B_n)^{-1/2}Q_n)$ *coincide with the interval* $[-1,+1]$ *with probability* 1. $\square$

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