The self-intersections of a Gaussian random field

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Abstract

Let $X(t)$ be an $(N, d, \alpha)$ non-deterministic Gaussian field. In this paper, the sufficient conditions for existence of the $k$-multiple points, the Hausdorff measure and the Hausdorff dimension for the $k$-multiple times set $\{(t_1, t_2, \ldots, t_k) : X(t_1) = X(t_2) = \cdots = X(t_k)\}$ for distinct $t_1, t_2, \ldots, t_k$ and the local times of the process $Y(T) = \{X(t_2) - X(t_1), \ldots, X(t_k) - X(t_{k-1})\}$ are evaluated.

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1. Introduction and main results

For our purposes we first introduce some notation to be used in this paper. We write

$$\langle x, y \rangle = \sum_{i=1}^{d} x_i y_i \quad \text{for any } x, y \in R^d;$$

$$T = (t_1, \ldots, t_k), \quad t_i \in R_+^N, i = 1, \ldots, k; \quad S = (s_1, \ldots, s_k), \quad s_i \in R_+^N, i = 1, \ldots, k;$$

$$B(S, r) = \{T : |t_i - s_i| < r, i = 1, \ldots, k\};$$

$$R^{Nk}_\eta = \left\{ T : \min_{i \neq j} |t_i - t_j| > \eta \right\}; \quad I^{Nk}_\eta = \{ T : T \in B(S, \eta), S \in R^{Nk}_3 \}.$$
Let \( Y(T) = \{X(t_2) - X(t_1), \ldots, X(t_k) - X(t_{k-1})\} \). Then \( Y(T) \) is called the self-intersection process of \( X(t) \). And the local time \( L(x, T) \) of \( Y(T) \) is called the self-intersection local time of \( X \). Furthermore, a point \( x \) is called the \( k \)-multiple point of \( X \) if there exist \( k \) distinct time points \( t_1, \ldots, t_k \) such that \( X(t_1) = X(t_2) = \cdots = X(t_k) = x \). Let \( I = \{(t_1, t_2, \ldots, t_k) : X(t_1) = X(t_2) = \cdots = X(t_k), \text{ for distinct } t_1, t_2, \ldots, t_k\} \). Then we say \( I \) is the \( k \)-multiple times set of \( X \). The existence of multiple points and the Hausdorff dimension of the \( k \)-multiple times set for some processes have been investigated in depth. For example, Taylor [19] studied the multiple points for the sample paths of the symmetric stable process, Hendricks [9] studied the multiple points for a process in \( R^2 \) with stable components, Kôno [10] investigated the double points of Gaussian sample paths, Cuzick [4,6] studied the local properties and the existence of multiple points for the Gaussian vector field, Geman et al. [8] obtained the intersections of Brownian paths in the plane, Rosen [14] studied the self-intersections for plan Brownian motion and fractional Brownian motion, Talagrand [18] evaluated the Hausdorff measure of trajectories for multi-parameter fractional Brownian motion and so on. And there are different tools for studying these problems, such as potential theory, capacity and local times. In this paper, we will establish the Hausdorff measure and Hausdorff dimension of the multiple times sets for a locally non-deterministic Gaussian field \( X(t) \) via studying the local times of \( Y(T) \). For our purposes, we shall introduce the following definition.

**Definition.** Let \( X(t) = (X_1(t), \ldots, X_d(t)), t \in R^N_+ \), be a stationary Gaussian field with mean zero and whose components \( X_i, i = 1, \ldots, d \), are independent. Suppose that for each \( i = 1, \ldots, d \), there exist a non-decreasing continuous function \( \sigma_i(x) \) and constants \( C, \delta > 0 \) such that

\[
E(X_i(t) - X_i(s))^2 \leq \sigma_i^2(|t - s|) \tag{1.1}
\]

and

\[
\text{Var}(X_i(t_n) - X_i(t_{n-1})|X_i(t_j) - X_i(t_{j-1}), 1 \leq j \leq n - 1) \geq C\sigma_i^2(|t_n - t_{n-1}|), \tag{1.2}
\]

where \( t_0, t_1, \ldots, t_n \) are distinct points lying in a cube of edge length at most \( \delta \) and satisfying

\[
|t_j - t_{j-1}| \leq |t_j - t_i|
\]

for all \( i < j \leq n \), \( \sigma_i(x) \) is a function with index \( \alpha_i \), that is,

\[
\alpha_i = \sup \left\{ \alpha > 0, \limsup_{|t| \to 0} |t|^{-\alpha} \sigma_i(|t|) = 0 \right\} = \inf \left\{ \alpha > 0, \liminf_{|t| \to 0} |t|^{-\alpha} \sigma_i(|t|) = \infty \right\}
\]

and satisfying that there exist constants \( M_1, M_2 \) such that for any \( x \in (0, 1] \)

\[
M_1 \leq \liminf_{r \to 0} \frac{\sigma_i(rx)}{\sigma_i(r)x^{\alpha_i}} \leq \limsup_{r \to 0} \frac{\sigma_i(rx)}{\sigma_i(r)x^{\alpha_i}} \leq M_2. \tag{1.3}
\]

Then we shall say that \( X(t) \) is a locally non-deterministic (LND) Gaussian field with index \( \alpha = (\alpha_1, \ldots, \alpha_d) \). A typical example of a LND Gaussian field is the so-called fractional Brownian motion taking values in \( R^d \) with index \( \alpha \), that is \( X(t) = (X_1(t), X_2(t), \ldots, X_d(t)) \), where \( X_1, \ldots, X_d \) are independent copies of the centered real valued Gaussian random field \( Y(t) \) with covariance function

\[
EY(t)Y(s) = \frac{1}{2}(|t|^{2\alpha} + |s|^{2\alpha} - |t - s|^{2\alpha}).
\]
There is a long history of the study of LND Gaussian processes. For more information, we refer the reader to [2,12,1] and so on. And there are some different definitions of the LND Gaussian field. For example the definition we introduce here is slightly different from that of [6] and [21]. Cuzick [6] introduced a different LND Gaussian field by replacing (1.3) with a conditional covariance and studied the sufficient conditions for the existence of $k$-multiple points. Rosen [14] showed that as $X$ is a planar Brownian motion, then the $k$-multiple self-intersection local time $L(x, B)$ of $X$ has the following property

$$L(x, B) \leq C_2 |B|^{1/k} \left( \log \log |B| \right)^{k-1} \text{ a.s.}$$

(1.4)

Moreover, if $X$ is a fractional Brownian motion taking values in $R^d$ with index $\alpha$, then

$$P(\dim(I) = Nk - d(k-1)\alpha > 0),$$

(1.5)

where $\dim(I)$ denotes the Hausdorff dimension of set $I$. Recently, Xiao [21] introduced a strongly locally non-deterministic Gaussian field $X$ with index $\alpha$, that is, $X(t) = (X_1(t), X_2(t), \ldots, X_d(t))$ is a LND Gaussian field with $\sigma_1(x) = \sigma_2(x) = \cdots = \sigma_d(x) = x^\alpha L(x)$, where $L(x)$ is a slowly varying function and satisfies

$$L(x) = \exp \left( \int_x^\infty \frac{\varepsilon(t)}{t} \, dt \right),$$

$\varepsilon(t) : [0, a] \to R$ is a bounded measurable function and $\lim_{x \to 0} \varepsilon(x) = 0$. He obtained the Hölder condition for the local time and the Hausdorff measure for the level set of $X$ in the case of $N > d/d$. It is clear that the definition of Xiao for the strongly LND Gaussian field is contained in ours. Talagrand [18] established the Hausdorff measure for the multiple points for the fractional Brownian motion taking values in $R^d$ with index $\alpha$.

There are two objects of this paper. First, we shall generalize the result (1.4) for planar Brownian motion to that of the LND Gaussian field $X$, that is, we shall consider the self-intersection local time of $X$, and use a result on this local time to obtain a sufficient condition for the existence of $k$-multiple points. Second, we shall use a method similar to that of Talagrand [18] to establish the Hausdorff measure for the $k$-multiple times sets for this LND Gaussian field. The results that we obtain are sharper and more general than those of (1.5).

In the sequel, $K_0, K_1, \ldots$ denote positive absolute constants. The following is our main result.

**Theorem 1.1.** Let $X(t), t \in R^N_+$, be a LND Gaussian field with index $\overline{\alpha}$ and $Y(T) = (X(t_2) - X(t_1), \ldots, X(t_k) - X(t_{k-1}))$ for any fixed $k$. Assume that $Nk > (k-1) \sum_{i=1}^d \alpha_i$, then for any Borel set $B \in \mathcal{B}$, $Y(T)$ has square integrable local times $L(x, \cdot)$ on $B$.

**Theorem 1.2.** Let $Y(T)$ be defined as above with $Nk > (k-1) \sum_{i=1}^d \alpha_i$, and let $L(x, \cdot)$ be its local times, $L^*(B(S, r)) = \sup_{x \in R^{(k-1)d}} L(x, B(S, r))$. Then for any $x \in R^{(k-1)d}$, $S \in R^N_+$ with $s_i \neq s_j$ as $i \neq j$,

$$\limsup_{r \to 0} \frac{L(x, B(S, r))}{\phi_1(r)} \leq K_0 \text{ a.s.}$$

(1.6)

and for any Borel set $E \in \mathcal{B}$,
\[ \limsup_{r \to 0} \sup_{S \in E} \frac{L(x, B(S, r))}{\phi_2(r)} \leq K_1 \quad \text{a.s.} \] (1.7)

In particular, as \( \sigma_1(x) = \sigma_2(x) = \cdots = \sigma_d(x) \), we have
\[ \limsup_{r \to 0} \frac{L^*(B(S, r))}{\phi_1(r)} \leq K'_0 \quad \text{a.s.} \] (1.6′)

and
\[ \limsup_{r \to 0} \sup_{S \in E} \frac{L^*(B(S, r))}{\phi_2(r)} \leq K'_1 \quad \text{a.s.} \] (1.7′)

where \( \phi_1(r) = \frac{r^{NK}(\log \log 1/r)^{\beta}}{(d \prod_{i=1}^{d} \sigma_i(r))^{k-1}} \), \( \phi_2(r) = \frac{r^{NK}(\log 1/r)^{\beta}}{(d \prod_{i=1}^{d} \sigma_i(r))^{k-1}} \) and \( \beta = (k - 1) \sum_{i=1}^{d} \alpha_i / N \).

**Theorem 1.3.** Let \( X(t), t \in R^N_+ \), be a LND Gaussian field with index \( \alpha \). If \( Nk > (k - 1) \sum_{i=1}^{d} \alpha_i \), then for any open set \( B \in R^N_+ \), \( X(t) \) has a point of multiplicity on \( B \) with positive probability. Otherwise, if \( Nk < (k - 1) \sum_{i=1}^{d} \alpha_i \), there are no points of multiplicity with probability one.

**Theorem 1.4.** Let \( I = \{T = (t_1, \ldots, t_k) \in R^{NK}_+ : X(t_1) = \cdots = X(t_k), t_i \neq t_j, i \neq j \} \). Suppose that \( Nk > (k - 1) \sum_{i=1}^{d} \alpha_i \) and on \( R - \{0\} \) there exists a continuous function \( \varphi(x) = (\varphi_1(x), \ldots, \varphi_d(x)) \) such that \( \frac{d(\sigma_i(x))}{dx} = \varphi_i(x), 1 \leq i \leq d \), then for any set \( Q \in \mathcal{R} \),
\[ K_2 L(0, Q) < \phi_1 - m(I \cap Q) < \infty \quad \text{a.s.} \] (1.8)

**Corollary 1.1.**
\[ P \left\{ \dim(I \cap Q) = Nk - (k - 1) \sum_{i=1}^{d} \alpha_i \right\} > 0. \] (1.9)

**Remark.** It is easy to see that Theorem 1.2 is more general than conclusion (1.4). The result of Theorem 1.3 is similar to that of [6]. The result of Theorem 1.4 is sharper than that of [14]. Cuzick mentioned that by [5], with positive probability the Hausdorff dimension of the \( k \)-multiple time set for the Gaussian field that he discussed is \( Nk - (k - 1) \sum_{i=1}^{d} \alpha_i \). It is well known that the Hausdorff measure is always more difficult to deal with and sharper than that of the Hausdorff dimension. So, the conclusion of Theorem 1.4 is sharper than Cuzick’s claim on the Hausdorff dimension result. Furthermore, the corollary of Theorem 1.4 is true for all \( I \cap B, B \in \mathcal{R} \), instead of just \( I \), so it is more general than that of Cuzick and (1.5). The idea of the proof is completely different from that of [5]. The method we used can also be applied to determine the Hausdorff properties of the multiple times sets and level sets for other processes.

2. The existence and uniform Hölder condition of the local times

In this section, we will use arguments similar to those of [14] and [8] to prove Theorems 1.1 and 1.2.

Before starting to prove the theorems, we give the following lemmas that we need.
Lemma 2.1. Let \( u_0 = 0 \) and \( C_k \) be a constant. Then for any fixed integer \( 1 \leq p \leq k \),
\[
\int_{R^{(k-1)d}} \exp \left\{ -\frac{kC_k}{2(k-1)} \sum_{i=1, i \neq p}^{k} |(u_{i-1} - u_i, \sigma(|t_i - s_i|)|^2 \right\} du_1 \cdots du_{k-1}
\leq K_3 \prod_{i=1, i \neq p}^{k} \prod_{j=1}^{d} (\sigma_j(|t_i - s_i|))^{-1}, \tag{2.1}
\]
where \( \sigma(x) = (\sigma_1(x), \ldots, \sigma_d(x)) \).

Proof. Let \( v_i = u_{i-1} - u_i \). Then the left side of (2.1) is equal to
\[
\int_{R^{(k-1)d}} \exp \left\{ -\frac{kC_k}{2(k-1)} \sum_{i=1, i \neq p}^{k} |(v_i, \sigma(|t_i - s_i|)|^2 \right\} dv_1 \cdots dv_{k-1}
= \prod_{i=1, i \neq p}^{k} \int_{R^d} \exp \left\{ -\frac{kC_k}{2(k-1)} \sum_{j=1}^{d} |v_i^j \sigma_j(|t_i - s_i|)|^2 \right\} dv_i^1 \cdots dv_i^d
\leq K_3 \prod_{i=1, i \neq p}^{k} \prod_{j=1}^{d} (\sigma_j(|t_i - s_i|))^{-1}.
\]
This completes our proof. \( \square \)

The following lemma is an extension of the Abel transform.

Lemma 2.2. Let \( t_i^j \in R^N, u_i^0 = u_k^l = t_0^0 = 0 \) for all \( i = 1, \ldots, k, l = 1, \ldots, n \). Then we have
\[
\sum_{j=1}^{n} \sum_{i=1}^{k-1} u_i^j (X(t_{i+1}^j) - X(t_i^j)) = \sum_{j=1}^{n} \sum_{i=1}^{k} \omega_i^j (X(t_i^j) - X(t_{i+1}^j)),
\]
where \( \omega_i^j = \sum_{l=j}^{n} (u_{i-1}^l - u_i^l), 1 \leq i \leq k, 1 \leq j \leq n \).

Lemma 2.3. Let \( \pi \) be a permutation of \( \{1, 2, \ldots, k\} \), then if \( \sum_{i=1}^{k-1} u_i(a_{i+1} - a_i) = \sum_{i=1}^{k-1} v_i (a_{\pi(i+1)} - a_{\pi(i)}) \), we have
\[
u_i = \sum_{i:=[\pi(i), \pi(i+1)] \geq [l, l+1]} \text{sgn}(\pi(i+1) - \pi(i)) v_i
\]
and
\[
u_{\pi(l)} - \nu_{\pi(l-1)} = v_l - v_{l-1}.
\]

The proof can be found in [14].

The following lemma can be deduced by an argument similar to that of Seneta [15].

Lemma 2.4. Assume that \( \sigma_i(x) \) satisfies the condition (1.3) and \( N > \alpha_i \theta \), then for \( r > 0 \) small enough
\[
\int_0^1 \frac{x^{N-1}}{(\sigma_i(r x))^\theta} dx \leq K_4(\sigma_i(r))^{-\theta} \int_0^1 \frac{x^{N-1}}{x^{\alpha_i \theta}} dx
\]
and
\[ \int_{B(0,1)} \frac{1}{(\sigma_i(r|t|))^\theta} dr \leq K_5(\sigma_i(r))^{-\theta} \int_{B(0,1)} \frac{1}{|t|^\alpha \theta} dt \]
where \( B(0,1) \) is the ball in \( \mathbb{R}^N \) with radius 1 and center 0.

**Lemma 2.5.** Let \( \theta > 0 \) with \( \theta \sum_{i=1}^{d} \alpha_i < N, 0 < r < \delta \) and \( s \in \mathbb{R}^N \). Then
\[ \int_{(B(s,r))^n} \left( \prod_{i=1}^{n} \prod_{j=1}^{d} \sigma_j(|t_{\pi(i)} - t_{\pi(i-1)}|) \right)^{-\theta} dt_1 \ldots dt_n \]
\[ \leq K_5^n (n!)^{\alpha/N} \cdot r^{Nn} \cdot \left( \prod_{i=1}^{d} \sigma_i(r) \right)^{-n\theta} \]
where \( \pi \) is a permutation of \( \{1, \ldots, n\} \) such that \( t_{\pi(0)} = 0, t_{\pi(1)} = t_1, |t_{\pi(i)} - t_{\pi(i-1)}| = \min\{|t_j - t_{\pi(i-1)}|, j \in \{1, \ldots, n\} \cap (\pi(1), \ldots, \pi(i-1))\} \).

**Proof.** By Lemma 2.4 for \( r \) small enough we have
\[ \int_{(B(s,r))^n} \left( \prod_{i=1}^{n} \prod_{j=1}^{d} \sigma_j(|t_{\pi(i)} - t_{\pi(i-1)}|) \right)^{-\theta} dt_1 \ldots dt_n \]
\[ = r^{Nn} \int_{(B(s,1))^n} \left( \prod_{i=1}^{n} \prod_{j=1}^{d} \sigma_j(r|t_{\pi(i)} - t_{\pi(i-1)}|) \right)^{-\theta} dt_1 \ldots dt_n \]
\[ \leq K_5^n r^{Nn} \left( \prod_{j=1}^{d} \sigma_j(r) \right)^{-n\theta} \int_{(B(s,1))^n} \prod_{i=1}^{n} \left( \min_{l \leq i-1} |t_i - t_l| \right)^{-\theta} \sum_{j=1}^{d} \alpha_j \] \[ \leq K_6^n r^{Nn} \left( \prod_{j=1}^{d} \sigma_j(r) \right)^{-n\theta} \prod_{i=1}^{n} \sum_{j=1}^{d} \alpha_j / N \]
(2.2)

Here, we have used the fact from [21, Lemma 2.3] that for any integer \( n \geq 1 \) and any distinct \( t_1, \ldots, t_n \in B(s,r), \)
\[ \int_{B(s,r)} \left( \min_{1 \leq j \leq n} |t - t_j|^\alpha \right)^{-\theta} dt \leq A_0 r^{N} \left( |rn^{-1/N}|^\alpha \right)^{-\theta}. \] \( \square \)

**Proof of Theorem 1.1.** By using Fourier analysis (see [2]), in order to prove the existence of the local time of \( Y(T) \), it is enough to show that
\[ \int_B \int_B \int_{R^{k-1}d} E \exp\{i \langle u, Y(T) - Y(S) \rangle\} du \, dS \, dT < \infty. \]  (2.3)

By the definition of \( \mathfrak{R} \), for any \( B \in \mathfrak{R} \) there exist \( \eta > 0, S' \in R_{3\eta}^{N_k} \) such that \( B = B(S', \eta) = \prod_{i=1}^{k} B(s_i', \eta) \). It follows that for any \( S, T \in B \) and all \( 1 \leq i \leq k, s_i \) is the point closest to \( t_i \) among \( \{s_1, \ldots, s_k, t_1, \ldots, t_k\} \), i.e., \( |s_i - t_i| = \min(|s_i - t_i|, |s_i - s_j|, |s_i - t_j|, 1 \leq j \neq i \leq k) \).
By (1.2) and a well known result of [12], we see that there exists a constant $C_k$ such that for any $t_i, 0 \leq i \leq k$, defining as in (1.2),

$$\text{Var} \left\{ \sum_{j=1}^{k} u_i^{j}(X_i(t_j) - X_i(t_{j-1})) \right\} \geq C_k \sum_{j=1}^{k} |u_i^{j} \sigma_i(|t_j - t_{j-1}|)|^2.$$

Put $u_k = u_0 = 0, v_i = u_{i-1} - u_i, i = 1, \ldots, k$. Then by Lemma 2.1 and the generalized Hölder inequality, the left side of (2.3) is equal to

$$\int_B \int_B \int_{\mathbb{R}^d} \exp \left\{ -\frac{1}{2} \text{Var} \left( \sum_{i=1}^{k} \langle v_i, X(t_i) - X(s_i) \rangle \right) \right\} \text{d}v \text{d}s \text{d}T$$

$$\leq \int_B \int_B \int_{\mathbb{R}^d} \exp \left\{ -\frac{1}{2} C_k \sum_{i=1}^{k} \sum_{j=1}^{d} |v_i^j \sigma_j(|t_i - s_i|)|^2 \right\} \text{d}v \text{d}s \text{d}T$$

$$= \int_B \int_B \int_{\mathbb{R}^d} \prod_{p=1}^{k} \exp \left\{ -\frac{1}{2(k-1)} C_k \sum_{i=1, i \neq p}^{k} \sum_{j=1}^{d} |v_i^j \sigma_j(|t_i - s_i|)|^2 \right\} \text{d}v \text{d}s \text{d}T$$

$$\leq K_3 \int_B \int_B \prod_{p=1}^{k} \int_{B(x, \epsilon)} \int_{B(x, \epsilon)} \left( \prod_{j=1}^{d} \sigma_j(|t_i - s_i|) \right)^{-\frac{k-1}{k}} \text{d}s \text{d}t.$$  

(2.4)

Therefore on taking $n = 2$ in Lemma 2.5, the right side of (2.4) is bounded. It follows that $Y(T)$ has local times on $B$, and the proof of Theorem 1.1 is completed. \hfill \Box

**Lemma 2.6.** Assume that $X(t), t \in \mathbb{R}_+^N$, satisfies the conditions of Theorem 1.1 and $L(x, \cdot)$ is the local time of $Y(t)$, then there exists $\delta > 0$ such that for any $0 < r < \delta, B = B(s, r) \in \mathcal{B}, x, y \in \mathbb{R}^{(k-1)d},$ even integer $n \geq 2$ and $0 < \gamma < \min\{1, \frac{Nk-(k-1)\sum_{i=1}^{d} \alpha_i}{2 \alpha_0}\}$

$$E[\text{L}(x, B)]^n \leq K_7^n (n!)^\beta r^{Nnk} \left( \prod_{i=1}^{d} \sigma_i(r) \right)^{n(k-1)},$$

(2.5)

$$E[\text{L}(x + y, B) - \text{L}(x, B)]^n \leq K_8^n |y|^n (n!)^\gamma r^{Nnk} \left( \prod_{i=1}^{d} \sigma_i(r) \right)^{n(k-1)} (\sigma_0(r))^{n\gamma},$$

(2.6)

where $\zeta = \frac{(k-1)\sum_{i=1}^{d} \alpha_i + \gamma (\alpha_0 + N)}{N}$ and $\alpha_0$ denotes the $\alpha_i$ corresponding to $\sigma_0(x) := \min\{\sigma_1(x), \ldots, \sigma_d(x)\}$. 


Proof. It follows from (25.5) and (25.7) in [7] that for any \( x, y \in \mathbb{R}^{d(k-1)} \) and any integer \( n \geq 1 \)

\[
E[L(x, B)]^n \\
= (2\pi)^{-nd(k-1)} \int_{B^n} \int_{\mathbb{R}^{nd(k-1)}} \exp \left\{ -i \sum_{j=1}^{n} \langle u^j, x \rangle \right\} \\
\cdot E \exp \left\{ i \sum_{j=1}^{n} \langle u^j, Y(T^j) \rangle \right\} \, du^* dT^* \tag{2.7}
\]

and for any even integer \( n \geq 2 \),

\[
E[L(x + y, B) - L(x, B)]^n \\
= (2\pi)^{-nd(k-1)} \int_{B^n} \int_{\mathbb{R}^{nd(k-1)}} \prod_{j=1}^{n} (\exp\{-i \langle u^j, x + y \rangle\} - \exp\{-i \langle u^j, x \rangle\}) \\
\cdot E \exp \left\{ i \sum_{j=1}^{n} \langle u^j, Y(T^j) \rangle \right\} \, du^* dT^* \tag{2.8}
\]

where \( u^* = (u^1, \ldots, u^n), T^* = (T^1, \ldots, T^n), u^i \in \mathbb{R}^{(k-1)d} \) and \( T^i \in \mathbb{R}^{Nk}_+ \).

By (2.7), it follows that

\[
E[L(x, B)]^n \\
\leq (2\pi)^{-nd(k-1)} \int_{B^n} \int_{\mathbb{R}^{nd(k-1)}} \exp \left\{ -\frac{1}{2} \text{Var} \left( \sum_{j=1}^{n} \langle u^j, Y(T^j) \rangle \right) \right\} \, du^* dT^* =: H.
\]

Since \( T^j = (t^j_1, \ldots, t^j_k) \in B, j = 1, \ldots, n \), it follows that for any fixed \( i, j, l \)

\[
|t^j_i - t^j_l| < \min\{|t^p_i - t^q_l|, i' \neq i \text{ for all } 1 \leq p, q \leq n\}.
\]

Therefore, for any fixed \( i = \{1, 2, \ldots, k\} \), we can find a permutation \( \pi^i \) of \( \{t^1_i, \ldots, t^n_i\} \) such that \( t^i_{\pi^i(0)} = 0, t^i_{\pi^i(1)} = t^1_i \) and

\[
|t^i_{\pi^i(j)} - t^i_{\pi^i(j-1)}| = \min\{|t^i_l - t^i_{\pi^i(j-1)}|, l \in \{1, \ldots, n\} \cap (\pi(1), \ldots, \pi(j-1))\}
\]

Then by Lemmas 2.2 and 2.3, we have

\[
H = (2\pi)^{-nd(k-1)} \int_{B^n} \int_{\mathbb{R}^{nd(k-1)}} \exp \left\{ -\frac{1}{2} \text{Var} \left( \sum_{j=1}^{k-1} \sum_{i=1}^{k} \langle u^j_i, X(t^j_{i+1}) - X(t^j_i) \rangle \right) \right\} \, du^* dT^* \\
= (2\pi)^{-nd(k-1)} \int_{B^n} \int_{\mathbb{R}^{nd(k-1)}} \exp \left\{ -\frac{1}{2} \text{Var} \left( \sum_{j=1}^{k-1} \sum_{i=1}^{k} \langle u^j_i, X(t^j_i) - X(t^j_{i-1}) \rangle \right) \right\} \, du^* dT^* \\
= (2\pi)^{-nd(k-1)} \\
\times \int_{B^n} \int_{\mathbb{R}^{nd(k-1)}} \exp \left\{ -\frac{1}{2} \text{Var} \left( \sum_{j=1}^{k} \sum_{i=1}^{k} \langle a^j_i, X(t^i_{\pi^i(j)}) - X(t^i_{\pi^i(j-1)}) \rangle \right) \right\} \, du^* dT^*
\]
\[
\begin{align*}
= (2\pi)^{-nd(k-1)} \int_{B^n} \int_{R^{nd(k-1)}} \exp \left\{ -\frac{1}{2} \mathrm{Var} \left( \sum_{j=1}^{n} \sum_{i=1}^{k} \sum_{l=1}^{d} a_{il}^j, X_i(t_i^{\pi^i(j)}) \right) \right. \\
&\left. - X_i(t_i^{\pi^i(j-1)}) \right\} \, du^* \, dT^* \\
= (2\pi)^{-nd(k-1)} \int_{B^n} \int_{R^{nd(k-1)}} \prod_{p=1}^{k} \left\{ \exp \left( -\frac{k}{2(k-1)} \right) \right. \\
&\left. \mathrm{Var} \left( \sum_{j=1}^{n} \sum_{i=1}^{k} \sum_{l=1}^{d} (a_{il}^j, X_i(t_i^{\pi^i(j)}) - X_i(t_i^{\pi^i(j-1)}) \right) \right\} \frac{1}{k} \, du^* \, dT^* \\
&\text{where}
\end{align*}
\]

\begin{align*}
v_i^j &= \sum_{m=j}^{n} (u_{m-1}^m - u_{i}^m), \quad a_i^{j-1} = v_i^{\pi^i(j)} - v_i^{\pi^i(j-1)}, \quad i = 1, \ldots, k. & (2.10)
\end{align*}

By (1.2), we have

\[
\det \mathrm{Cov} \left( X_i(t_i^{\pi^i(j)}) - X_i(t_i^{\pi^i(j-1)}), 1 \leq i \leq k, i \neq p, 1 \leq j \leq n \right)
\]

\[
= \prod_{j=1}^{n} \prod_{i=1,i\neq p}^{k} \det \mathrm{Cov} \left( X_i(t_i^{\pi^i(j)}) - X_i(t_i^{\pi^i(j-1)}), X_i(t_m^{\pi^i(q)}), 1 \leq m < i, m \neq p, 1 \leq q \leq n \right)
\]

\[
\geq C^{n(k-1)} \prod_{j=1}^{n} \prod_{i=1,i\neq p}^{k} \left( \sigma_i(|t_i^{\pi^i(j)} - t_i^{\pi^i(j-1)}) \right)^2 .
\]

Therefore by the generalized Hölder inequality, the right side of (2.9) is less than

\[
(2\pi)^{-nd(k-1)} \int_{B^n} \prod_{p=1}^{k} \left\{ \int_{R^{nd(k-1)}} \exp \left( -\frac{k}{2(k-1)} \right) \mathrm{Var} \left( \sum_{j=1}^{n} \sum_{i=1,i\neq p}^{k} \sum_{l=1}^{d} (a_{il}^j, X_i(t_i^{\pi^i(j)}) \right) \right. \\
&\left. - X_i(t_i^{\pi^i(j-1)}) \right\} \frac{1}{k} \, du^* \, dT^* \\
\leq (2\pi K'_0)^{-nd(k-1)/2} \int_{B^n} \prod_{p=1}^{k} \left\{ \prod_{l=1}^{d} \det \mathrm{Cov} \left( X_i(t_i^{\pi^i(j)}), X_i(t_i^{\pi^i(j-1)}) \right), \\
\left. 1 \leq i \leq k, i \neq p, 1 \leq j \leq n \right\} \frac{1}{k} \, dT^* \\
\leq (2\pi K'_0 C^2)^{-nd(k-1)/2} \int_{B^n} \prod_{p=1}^{k} \prod_{i=1,i\neq p}^{k} \prod_{j=1}^{n} \prod_{l=1}^{d} \left\{ \sigma_i(|t_i^{\pi^i(j)} - t_i^{\pi^i(j-1)}) \right\}^{-1/k} \, dT^*
\]
\[
(2\pi K_6^2)^{-nd(k-1)/2} \int_{B^n} \prod_{i=1}^{k} \prod_{j=1}^{n} \prod_{l=1}^{d} \left| \sigma_l (t_i^{\pi^t(j)} - t_i^{\pi^t(j-1)}) \right|^{-\frac{k-1}{k}} dt^* \\
\]
\[
\leq K_7^2 (n!)^\beta \prod_{i=1}^{k} \prod_{j=1}^{n} \left\{ N \left( \prod_{l=1}^{d} \sigma_l (r) \right)^{-\frac{1}{k}} \right\} = K_7 (n!)^\beta R_k n^k \left( \prod_{l=1}^{d} \sigma_l (r) \right)^{-n(k-1)}
\]
and the proof of (2.5) is complete.

Now we turn to showing (2.6). By (2.8) and the elementary inequality \(|e^{ix} - 1| \leq 2^{1-\gamma} |x|^\gamma\) for any \(x \in \mathbb{R}\) and \(0 < \gamma < 1\), and by the argument above, we have
\[
E[L(x + y, B) - L(x, B)]^n \leq (2\pi)^{-nd(k-1)} 2^{n(1-\gamma)} |y|^{n\gamma} \int_{B^n} \int_{R^{nd(k-1)}} \left( \prod_{j=1}^{n} |u^j|^\gamma \right) \\
\cdot \exp \left\{ -\frac{1}{2} \text{Var} \left( \sum_{j=1}^{n} \langle u^j, Y(T^j) \rangle \right) \right\} du^* dt^* \\
= (2\pi)^{-nd(k-1)} 2^{n(1-\gamma)} |y|^{n\gamma} \int_{B^n} \int_{R^{nd(k-1)}} \left( \prod_{j=1}^{n} |u^j|^\gamma \right) \\
\times \exp \left\{ -\frac{1}{2} \text{Var} \left( \sum_{j=1}^{n} \sum_{i=1}^{k} \langle a_i^j, X(t_i^{\pi^t(j)}) - X(t_i^{\pi^t(j-1)}) \rangle \right) \right\} du^* dt^* \\
=: (2\pi)^{-nd(k-1)} 2^{n(1-\gamma)} |y|^{n\gamma} J.
\]

Let \(u_k^j = u_0^j = 0\) for all \(j \leq n\), then \(\sum_{k=1}^{k} \sum_{i=1, i \neq p}^{k} (u_i^j - u_{i-1}^j) + (u_p^j - u_{p-1}^j) = 0\). Therefore for any \(p \leq k\)
\[
|u^j| = K_9 \left( \sum_{i=1, i \neq p}^{k} |u_i^j - u_{i-1}^j| + |u_p^j - u_{p-1}^j| \right) \leq 2K_9 \sum_{i=1, i \neq p}^{k} |u_i^j - u_{i-1}^j|.
\]

On the other hand, it follows from (2.10) that for all \(1 \leq i \leq k\) and \(1 \leq j \leq n\).
\[
u_i^{\pi^t(n)} - u_{i-1}^{\pi^t(n)} = a_i^n \tag{2.11}
\]
and
\[
a_i^j - a_i^{j-1} = u_i^{\pi^t(j)} - v_i^{\pi^t(j)-1} = u_i^{\pi^t(j)-1} - u_{i-1}^{\pi^t(j)-1}. \tag{2.12}
\]

Hence, we have
\[
\prod_{j=1}^{n} |u^j| \leq (2K_9)^n \prod_{j=1}^{n} \sum_{i=1, i \neq p}^{k} (|a_i^j| + |a_i^{j-1}|)
\]
which on combining with the generalized Hölder theorem yields that
\[
J \leq (2K_9)^{n\gamma} \int_{B^n} \int_{R^{nd(k-1)}} \prod_{p=1}^{k} \left( \prod_{j=1}^{n} \sum_{i=1, i \neq p}^{k} (|a_i^j| + |a_i^{j-1}|) \right)^{\gamma}
\]

\begin{align*}
&\times \exp \left( -\frac{k}{2(k-1)} \text{Var} \left( \sum_{j=1}^{n} \sum_{i=1, i \neq p}^{k} \sum_{l=1}^{d} (a_{ij}^{j}, X_i(t_i^{\pi^i(j)})) \right) \\
&\quad - X_i(t_i^{\pi^i(j-1)}) \right) \right)^{\frac{1}{k}} \text{d}a^* \text{d}T^* \\
&\leq (2K_9)^{n^\gamma} \int_{B^n} \left\{ \prod_{p=1}^{k} \int_{\mathbb{R}^{nd(k-1)}} \left[ \left( \sum_{j=1}^{n} \sum_{i=1, i \neq p}^{k} \sum_{l=1}^{d} (a_{ij}^{j} + |a_{il}^{j-1}|) \right)^{\gamma} \\
&\quad \times \exp \left( -\frac{k}{2(k-1)} \text{Var} \left( \sum_{j=1}^{n} \sum_{i=1, i \neq p}^{k} \sum_{l=1}^{d} (a_{ij}^{j}, X_i(t_i^{\pi^i(j)})) \right) \\
&\quad - X_i(t_i^{\pi^i(j-1)}) \right) \right] \right\} \text{d}a^* \text{d}T^*. \\
&\quad \frac{1}{k}
\end{align*}

Without loss of generality, we may assume that \( \sigma_0(x) = \sigma_d(x) \). Then by elementary calculations and an argument similar to that of [21], we can get that the right side of (2.13) is no more than

\begin{align*}
&\left( K_9 \right)^{n^\gamma} (n!)^\gamma \int_{B^n} \prod_{j=1}^{n} \prod_{p=1}^{k} \left\{ \sigma_d(t_i^{\pi^i(j)} - t_i^{\pi^i(j-1)}) \right\}^{-\gamma/k} \\
&\times \left( \prod_{l=1}^{d} \sigma_l(t_i^{\pi^i(j)} - t_i^{\pi^i(j-1)}) \right)^{-\frac{k-1}{k}} \text{d}T^* \\
&\leq \left( K_6 K_9 \right)^{n^\gamma} (n!)^\gamma \left( \prod_{i=1}^{d} \sigma_i(r) \right)^{-\frac{n(k-1)}{n^\gamma}} \\
&\quad \cdot \left( \prod_{i=1}^{N(nk)} \sigma_i(r) \right)^{-\frac{n^\gamma}{n}} \\
&\quad \cdot \frac{1}{n} (n+\sigma_d)^{\gamma+(k-1)} \sum_{i=1}^{d} a_i \\
&\quad \cdot \frac{1}{n} (n+\sigma_d)^{\gamma+(k-1)} \sum_{i=1}^{d} a_i \\
&\quad \cdot \left( \prod_{i=1}^{N} \sigma_i(r) \right)^{n(k-1)} \\
&\quad \cdot \left( \sigma_d(r) \right)^{n^\gamma}
\end{align*}

So,

\begin{align*}
E[L(x+y,B) - L(x,B)]^n \\
&\leq (2\pi)^{-nd(k-1)} 2^{n(1-\gamma)} |y|^{n^\gamma} \left( K_6 K_9 \right)^{n^\gamma} (n!)^\gamma \\
&\quad \cdot \frac{1}{n} \left( \prod_{i=1}^{d} \sigma_i(r) \right)^{n(k-1)} \\
&\quad \cdot \left( \sigma_d(r) \right)^{n^\gamma}
\end{align*}
\[ K_n^\gamma = \frac{N^{(N+\alpha)\gamma + (k-1)\sum_{i=1}^d \alpha_i}}{(n!)^{n(k-1)} \left( \prod_{i=1}^d \sigma_i(r) \right)^{n\gamma}}. \]

And the proof of (2.6) is complete. \(\square\)

**Lemma 2.7.** For any \(B = B(S, r) \in \mathcal{B} \), there exist finite positive constants \(b_1, b_2 \) depending only on \(N, k, \alpha_i, i = 1, \ldots, d, \) such that for any \(a > 0 \), we have

\[
\begin{align*}
\mathbb{P} \left\{ L(x + X(s), B) \geq \frac{K_7 ar^{Nk}}{\left( \prod_{i=1}^d \sigma_i(r) \right)^{k-1}} \right\} &\leq b_1 \exp \left\{ -\frac{a^{1/\beta}}{2} \right\}, \quad (2.14) \\
\mathbb{P} \left\{ L(x + y + X(s), B) - L(x + X(s), B) \geq \frac{K_8 ar^{Nk}|y|^{\gamma}}{\left( \prod_{i=1}^d \sigma_i(r) \right)^{k-1} (\sigma_0(r))^{\gamma}} \right\} &\leq b_2 \exp \left\{ -\frac{a^{1/\zeta}}{2} \right\}.
\end{align*}
\]

**Proof.** Let \(\Lambda = \frac{L(x+X(s), B)}{K_7 r^{Nk} \left( \prod_{i=1}^d \sigma_i(r) \right)^{k-1}}\), then by Chebyshev’s inequality and Lemma 2.6, we get that for any integer \(n\),

\[ E\Lambda^n \leq (n!)^{-\beta}. \]

Then along the lines of the proof of Lemma 3.8 in [3] (see also Lemma 2.7.9 in [11]), we can draw the conclusion (2.14). The proof of (2.15) is similar. \(\square\)

**Lemma 2.8.** Let \(Y(T)\) be defined like that in Theorem 1.1 and then for any \(r > 0\) small enough, \(S \in \mathbb{R}^{Nk}\),

\[ \mathbb{P} \left\{ \sup_{T \in B(S, r)} |Y(T) - Y(S)| \geq u\sigma'_0(r) \right\} \leq \exp\{-u^2/K_{10}\}, \]

where \(\sigma'_0(x) := \max\{\sigma_1(x), \ldots, \sigma_d(x)\}\).

**Proof.** By a consequence of [17], we have

\[
\begin{align*}
&\mathbb{P} \left\{ \sup_{T \in B(S, r)} |Y(T) - Y(S)| \geq u\sigma'_0(r) \right\} \\
\leq &\mathbb{P} \left\{ \sup_{T \in B(S, r)} \sum_{i=1}^{k-1} (|X(t_{i+1}) - X(s_{i+1})| + |X(t_i) - X(s_i)|) \geq u\sigma'_0(r) \right\} \\
\leq &\mathbb{P} \left\{ \sup_{T \in B(S, r)} \sum_{i=1}^{k} \sum_{j=1}^{d} |X_j(t_i) - X_j(s_i)| \geq u\sigma'_0(r)/2 \right\}
\end{align*}
\]
We now turn to proving Theorem 1.2.

**Proof of Theorem 1.2.** By Lemma 2.7 and a standard argument, it is easy to derive (1.6) and (1.7). Here we shall only give the proof of (1.6′) and (1.7′). For any fixed $S = (s_1, \ldots, s_k) \in R^N$ with $s_i \neq s_j$ as $i \neq j$, let $B_m = B(S, 2^{-m})$, $m \in N$. It follows from the condition $\sigma_1(x) = \sigma_2(x) = \cdots = \sigma_d(x)$ that $\sigma_0(x) = \sigma'_0(x) = \cdots = \sigma_1(x)$. So, by Lemma 2.8, we have

$$P \left\{ \sup_{T \in B_m} |Y(T) - Y(S)| \geq \sigma_0(2^{-m}) \sqrt{2K_{10} \log m} \right\} \leq m^{-2}.$$  

By the Borel–Cantelli lemma,

$$\limsup_{m \to \infty} \sup_{T \in B_m} \frac{|Y(T) - Y(S)|}{\sigma_0(2^{-m}) \sqrt{2K_{10} \log m}} \leq 1 \ a.s. \quad (2.16)$$

Let $\theta_m = \sigma_0(2^{-m})(\log m)^{-2}$.

$$G_m = \{ x : x \in R^{(k-1)d}, |x| \leq \sigma_0(2^{-m}) \sqrt{2K_{10} \log m}, x = p\theta_m, \text{for some } p \in Z^{(k-1)d} \}.$$  

Then for $m$ large enough, the cardinality of $G_m$ is no more than $\sqrt{2K_{10} \log m}^{3kd}$. It follows from Lemma 2.7 that

$$P \{ L(x + Y(S), B_m) \geq K_7a_1^\beta \phi_1(2^{-m}) \text{ for some } x \in G_m \}$$

$$\leq \sqrt{2K_{10} \log m}^{3kd} b_1 \exp \left\{ -\frac{1}{2} a_1 \log \log 2^m \right\}$$

$$= b_1 \sqrt{2K_{10} \log m}^{3kd} (m \log 2)^{-a_1/2} =: A_m.$$  

Take $a_1 \geq 3$, then $\sum_{m=1}^\infty A_m < \infty$. Therefore, by the Borel–Cantelli lemma

$$\limsup_{m \to \infty} \sup_{x \in G_m} \frac{L(x + Y(S), B_m)}{K_7a_1^\beta \phi_1(2^{-m})} \leq 1 \ a.s. \quad (2.17)$$

For any fixed integers $m, h \geq 1$, and any $x \in G_m$, define

$$F(m, h, x) = \left\{ y \in R^{(k-1)d} : y = x + \theta_m \sum_{j=1}^h \epsilon_j 2^{-j}, \epsilon_j \in \{0, 1\}^{(k-1)d}, 1 \leq j \leq h \right\}.$$  

Then by Lemma 2.7, we have

$$P \left( \bigcup_{h=1}^\infty \{ L(y_1 + Y(S), B_m) - L(y_2 + Y(S), B_m) \} \right)$$

$$\geq K_8 2^{-mNk} |y_1 - y_2|^\gamma (a_2h \log \log 2^m)^k \left( \prod_{i=1}^d (2^{-m}) \right)^{k-1} (\sigma_0(2^{-m}))^\gamma$$

for some $x \in G_m$ and
\[ y_1, y_2 \in F(m, h, x) \text{ with } |y_1 - y_2| = \theta_m \varepsilon 2^{-h}, \varepsilon \in \{0, 1\}^{(k-1)d} \]

\[ \leq |G_m| \sum_{h=1}^{\infty} 2^{kd} b_2 \exp \left\{ -\frac{1}{2} a_2 h \log \log 2^m \right\} \]

\[ \leq b_2 \sqrt{2K_{10}(\log m)}^{3kd} \sum_{h=1}^{\infty} 2^{kd} (m \log 2)^{-a_2 h/2}. \]

Select \(a_2\) so large that

\[ \sum_{m=1}^{\infty} b_2 \sqrt{2K_{10}(\log m)}^{3kd} \sum_{h=1}^{\infty} 2^{kd} (m \log 2)^{-a_2 h/2} < \infty, \]

then by the Borel–Cantelli lemma again we have that except for finitely many \(m\),

\[ |L(y_1 + Y(S), B_m) - L(y_2 + Y(S), B_m)| \leq \frac{K_8 2^{-mNk} |y_1 - y_2|^\gamma (a_2 h \log \log 2^m)^\zeta}{\left( \prod_{i=1}^{d} \sigma_i(2^{-m}) \right)^{k-1} (\sigma_0(2^{-m}))^\gamma} \quad (2.18) \]

holds almost surely for all \(x \in G_m, h \geq 1\) and any \(y_1, y_2 \in F(m, h, x)\) with \(|y_1 - y_2| = \theta_m \varepsilon 2^{-h}, \varepsilon \in \{0, 1\}^{(k-1)d}\).

For fixed \(m\) and \(y \in R^{(k-1)d}\) with \(|y| \leq \sigma_0(2^{-m}) \sqrt{2K_{10} \log m}\), we can represent \(y\) in the form \(y = \lim_{h \to \infty} y_h\) with \(y_h = x + \theta_m \sum_{j=1}^{h-1} \varepsilon_j 2^{-j}\), \(y_0 = x \in G_m\) and \(\varepsilon_j \in \{0, 1\}^{(k-1)d}\). Then by (2.18) and continuity of the local times \(L(\cdot, B_m)\), it follows that

\[ |L(y + Y(S), B_m) - L(x + Y(S), B_m)| \leq \sum_{h=1}^{\infty} K_8 2^{-mNk} |y_h - y_{h-1}|^\gamma (a_2 h \log \log 2^m)^\zeta \]

\[ \leq K_{11} \phi_1(2^{-m}) \quad a.s. \quad (2.19) \]

Combining (2.17) with (2.19) yields that

\[ L(y + Y(S), B_m) \leq K_{12} \phi_1(2^{-m}) \quad a.s. \]

for any \(y \in R^{(k-1)d}\) with \(|y| < \sigma_0(2^{-m}) \sqrt{2K_{10} \log m}\). Therefore

\[ \sup_{x \in R^{(k-1)d}} L(x, B_m) = \sup_{x \in Y(B_m)^*} L(x, B_m) \leq K_{12} \phi_1(2^{-m}) \quad a.s. \quad (2.20) \]

where \(Y(B_m)^*\) denotes the closure of \(Y(B_m)\).

For given small \(r > 0\), we can find some \(m\) such that \(2^{-m} < r \leq 2^{-m+1}\). Then (2.20) implies that

\[ \sup_{x \in R^{(k-1)d}} L(x, B(S, r)) \leq K_{12} \phi_1(r). \]

This completes the proof of (1.6').
Now we turn to proving (1.7'). The proof is very similar to that of (1.6'). For any Borel set $E \in \mathcal{R}$, without loss of generality, it is enough to consider the case of $E = \bigotimes_{i=1}^{N_k} [1, 2] \cap R_{\eta}^{N_k}$ for some $\eta > 0$. Let $D$ be the family of $2^{mN_k}$ dyadic cubes

$$Q_{lm} = \bigotimes_{i=1}^{N_k} [1 + (l_i - 1)/2^m, 1 + l_i/2^m] \cap R_{\eta}^{N_k},$$

$$l = (l_1, \ldots, l_{N_k}) \in \{1, \ldots, 2^m\}^{N_k} =: J_m.$$

Let $\phi'_m = \sigma_0(2^{-m})m^{-2}$ and

$$G'_m = \{ x \in R^{(k-1)d} : |x| \leq m\sigma_0(2^{-m})2^{mN_k}, x = p\phi'_m, p \in \mathbb{Z}^{(k-1)d} \}.$$ 

Then by Lemma 2.7, we have

$$P \{ L(x, Q_{lm}) \leq 2^{mN_k} \cdot \#G'_m b_1 \exp \left\{ -\frac{1}{2} a_1 \log 2^m \right\} \leq K_{13} 2^{m(N_k+(k-1)dN_k-a_1)/2} \cdot m^{3(k-1)d} =: E_m'. $$

Select $a_1$ so large that $\sum_{m=1}^{\infty} E_m' < \infty$, then the Borel–Cantelli lemma implies that

$$\limsup_{m \to \infty} \sup_{l \in J_m, x \in G'_m} \frac{L(x, Q_{lm})}{K_7 a_1^2 \phi_2(2^{-m})} \leq 1 \text{ a.s.} \tag{2.21}$$

For any fixed integers $m, h \geq 1$ and $x \in G'_m$, we set

$$F'(m, h, x) = \left\{ y \in R^{(k-1)d} : y = x + \phi'_m \sum_{j=1}^{h} \epsilon_j 2^{-j}, \epsilon_j \in \{0, 1\}^{(k-1)d}, 1 \leq j \leq h \right\}.$$

Then

$$P \left( \bigcup_{l \in J_m} \bigcup_{h=1}^{\infty} |L(y_1, Q_{lm}) - L(y_2, Q_{lm})| \geq \frac{K_8 2^{-mN_k}|y_1 - y_2|^\gamma (a_2 h \log 2^m)\zeta}{\left( \prod_{i=1}^{d} \sigma_i (2^{-m}) \right) (\sigma_0(2^{-m}))^\nu} \right)$$

for some $x \in G'_m$ and $y_1, y_2 \in F'(m, h, x)$ with $|y_1 - y_2| = \phi'_m \epsilon 2^{-h}, \epsilon \in \{0, 1\}^{(k-1)d}$,

$$\leq 2^{mN_k} \cdot \#G'_m \sum_{h=1}^{\infty} 2^{kdh} b_2 \exp \left\{ -a_2 h \log 2^m / 2 \right\} \leq K_{14} 2^{mN_k} (m^3 2^{mN_k})^{(k-1)d} \sum_{h=1}^{\infty} 2^{kdh-a_2 mh/2} =: F_m.$$
We can choose \(a_2\) large enough such that \(\sum_{m=1}^{\infty} F_m < \infty\), which on combining with the Borel–Cantelli lemma yields that except for finitely many \(m\),

\[
|L(y_1, Q_{lm}) - L(y_2, Q_{lm})| \leq \frac{K_8 2^{-mNk} |y_1 - y_2|^{\gamma} (a_2 h \log 2^m)^{\xi}}{\left(\prod_{i=1}^{d} \sigma_i(2^{-m})\right)^{k-1} (\sigma_0(2^{-m}))^{\gamma}}
\]  
(2.22)

holds almost surely for all \(x \in G'_m\), \(h \geq 1\), \(l \in J_m\) and any \(y_1, y_2 \in F'(m, h, x)\) with \(|y_1 - y_2| = \theta_m \epsilon 2^{-h}, \epsilon \in \{0, 1\}^{(k-1)d}\).

Note that \(Y(T)\) is almost surely continuous on \(E\). By Lemma 2.8, it is easy to show that except for finitely many \(m\),

\[
\sup_{T \in E} |Y(T)| \leq m \sigma_0(2^{-m}) 2^{mNk} \quad a.s.
\]  
(2.23)

Therefore if \(m\) large enough and \(y \in Y(E) := \{y \in R^{(k-1)d} : y = Y(T), T \in E\}\), we can represent \(y\) with \(y = \lim_{h \to \infty} y_h\), where \(y_h = x + \theta_m \sum_{j=1}^{h} \epsilon_j 2^{-j}, y_0 = x \in G'_m\). By (2.22), we have

\[
|L(y, Q_{lm}) - L(x, Q_{lm})| \leq \sum_{h=1}^{\infty} \frac{K_8 2^{-mNk} |y_h - y_{h-1}|^{\gamma} (a_2 h \log 2^m)^{\xi}}{\left(\prod_{i=1}^{d} \sigma_i(2^{-m})\right)^{k-1} (\sigma_0(2^{-m}))^{\gamma}}
\]

\[
\leq \sum_{h=1}^{\infty} \frac{K_8 2^{-mNk} (\sigma_0(2^{-m})m^{-2} 2^{-h})^{\gamma} (a_2 h \log 2^m)^{\xi}}{\left(\prod_{i=1}^{d} \sigma_i(2^{-m})\right)^{k-1} (\sigma_0(2^{-m}))^{\gamma}}
\]

\[
= \sum_{h=1}^{\infty} \frac{K_8 (\log 2^{m})^{\gamma-\beta} (a_2 h)^{\xi}}{m^{2\gamma} 2^{bh\gamma}} \phi_2(2^{-m}) < K_1 \phi_2(2^{-m}) \quad a.s.
\]  
(2.24)

Combining (2.21)–(2.24) and a monotonicity argument we can draw the conclusion as desired. This finishes the proof of Theorem 1.2. \(\square\)

3. The existence of the multiple points

In this section, we will show the existence of the \(k\)-multiple points.

Proof of Theorem 1.3. Given any Borel set \(B \in R^N_k\), we can choose set \(A \in \mathcal{R}\) such that \(A \subseteq B\). From the proof of Theorem 1.1, it is easy to see that if \(Nk > (k-1) \sum_{i=1}^{d} \alpha_i\), then

\[
\int_A \int_A [\text{det cov}(Y(T) - Y(S))]^{-1/2} dSdT = (2\pi)^{-(k-1)d/2} \int_A \int_A \int_{R^{(k-1)d}} E \exp[i \langle u, Y(T) - Y(S) \rangle] dudSdT < \infty.
\]

Hence, it follows from Theorem 1.3 of [4] (see also [6]) that \(Y(T)\) hits zero with positive probability for some \(T \in A\). Therefore, \(X(t)\) has multiple points with positive probability.

In the case of \(Nk < (k-1) \sum_{i=1}^{d} \alpha_i\), it follows directly from Theorem 3 of [6] that \(X(t)\) has no multiple point with probability 1.
4. The lower bound of the Hausdorff measure

With the self-intersection local time of $X$ and the upper density theorem, we will obtain the lower bound of the Hausdorff measure of the $k$-multiple times sets in this section.

**Lemma 4.1.** For any $x \in R^{(k-1)d}$, let $L(x, \cdot)$ be the local time of $Y$ at $x$. Then we have that for almost all $S \in Q, Q \in H$,

$$\limsup_{r \to 0} \frac{L(x, B(S, r))}{\phi_1(r)} \leq K_{16} \quad \text{a.s.} \quad (4.1)$$

that is

$$L \left( x, \left\{ S \in Q : \limsup_{r \to 0} \frac{L(x, B(S, r))}{\phi_1(r)} > K_{16} \right\} \right) = 0 \quad \text{a.s.}$$

**Proof.** Let $f_m(S) = L(x, B(S, 2^{-m})), A_m = \{ S \in Q : \frac{f_m(S)}{\phi_1(2^{-m})} \geq K_{17} \}$, where $K_{17}$ will be determined later. Then

$$EL(x, A_m) \leq \frac{E \int_Q |f_m(S)|^n L(x, dt)}{(K_{17} \phi_1(2^{-m}))^n}. \quad (4.2)$$

Let $T^i \in R_{nk}^N, T^1 = S, u^i \in R^{d(k-1)}, i = 1, \ldots, n + 1$, and $u^* = (u^1, \ldots, u^{n+1}), T^* = (T^1, \ldots, T^{n+1})$. By an argument similar to that of proposition 4.1 in [21], we have that for any positive integer $n \geq 1$,

$$E \int_Q |f_m(S)|^n L(x, dt)$$

$$= (2\pi)^{-(n+1)d(k-1)} \int_Q \int_{B^n(S, 2^{-m})} \int_{R^{(n+1)d(k-1)}} \exp \left\{ -i \sum_{j=1}^{n+1} < x, u^j > \right\}$$

$$\cdot E \exp \left( i \sum_{j=1}^{n+1} < u^j, Y(T^j) > \right) du^* dT^*$$

$$\leq (2\pi)^{-(n+1)d(k-1)} \int_Q \int_{B^n(S, 2^{-m})} \exp \left\{ -\frac{1}{2} \text{Var} \left( \sum_{j=1}^{n+1} < u^j, Y(T^j) > \right) \right\} du^* dT^*$$

$$\leq K_{18}^n (n!)^\beta (2^{-m})^{Nnk} \left( \prod_{l=1}^{d} \sigma_l(2^{-m}) \right)^{-n(k-1)}$$

which on combining with (4.2) implies that

$$EL(x, A_m) \leq \frac{K_{18}^n (n!)^\beta (2^{-m})^{Nnk} \left( \prod_{l=1}^{d} \sigma_l(2^{-m}) \right)^{-n(k-1)}}{K_{17} 2^{-mNk} (\log \log 2^m) \beta \left( \prod_{l=1}^{d} \sigma_l(2^{-m}) \right)^{-(k-1)n-n^\beta}}$$

$$\leq \left( \frac{K_{18}}{K_{17}} \right)^n \left( \frac{n}{\log \log 2^m} \right)^{n\beta}.$$
Take \( n = \lfloor \log m \rfloor \) and \( K_{17} \) to be large enough such that \( E L(x, A_m) \leq m^{-2} \). This implies that

\[
E \left( \sum_{m=1}^{\infty} L(x, A_m) \right) < \infty.
\]

Therefore for \( L(x, \cdot) \), we have that for almost all \( S \in Q \),

\[
\limsup_{m \to \infty} \frac{f_m(S)}{\phi_1(2^{-m})} \leq K_{17} \quad a.s.
\]

(4.3)

By (4.3) and a monotonicity argument, it follows that (4.1) is true. \( \Box \)

**Proposition 4.1.** Assume that \( N_k > (k - 1) \sum_{i=1}^{d} \alpha_i \), then for any \( Q \in \mathcal{R} \)

\[
\phi_1 - m(I \cap Q) \geq K_1^{-1} L(0, Q).
\]

**Proof.** Obviously, we can consider \( Y^{-1}(0) \cap Q \) instead of \( I \cap Q \). Let \( A = \{ S \in Q : \limsup_{r \to 0} L(0, B(S, r))/\phi_1(r) \geq K_{16} \} \). Then by Lemma 4.1,

\[
L(0, A) = 0 \quad a.s.
\]

Using the upper density theorem of Rogers and Taylor [13], we have

\[
\phi_1 - m(I \cap Q) = \phi_1 - m(Y^{-1}(0) \cap Q) \geq \phi_1 - m(Y^{-1}(0) \cap (QA^c)) \geq K_1^{-1} L(0, QA^c)
\]

\[
= K_1^{-1} L(0, Q) \quad a.s.
\]

This completes the proof of Proposition 4.1. \( \Box \)

5. The upper bound of the Hausdorff measure

In this section, we are going to establish the upper bound for the Hausdorff measure of the \( k \)-multiple times sets. For this purpose, first we give some lemmas that we need.

Let \( \eta(s), s \in E \), be a Gaussian process and for any \( s, t \in E \), we define the distance apart of \( s \) and \( t \) by

\[
d(s, t) = (E(\eta(s) - \eta(t))^2)^{1/2}.
\]

We denote by \( N(E, \varepsilon) \) the smallest number of open \( d \)-balls of radius \( \varepsilon \) needed to cover \( E \). The following two lemmas are basic facts for \( \eta(s) \).

**Lemma 5.1.** For any \( u > 0 \), we have

\[
P \left\{ \sup_{s, t \in E} |\eta(s) - \eta(t)| \geq K_{19} \left( u + \int_{0}^{a} \sqrt{\log N(E, \varepsilon)} d\varepsilon \right) \right\} \leq \exp(-u^2/a^2).
\]

This is just Lemma 2.1 of Talagrand [17].

**Lemma 5.2.** Let \( \psi \) be a function with \( N(E, \varepsilon) \leq \psi(\varepsilon) \) for all \( \varepsilon > 0 \). If there exists some constant \( C \) such that for all \( \varepsilon > 0 \),

\[
\psi(\varepsilon)/C \leq \psi(\varepsilon/2) \leq C \psi(\varepsilon).
\]

Then

\[
P \left\{ \sup_{s, t \in E} |\eta(s) - \eta(t)| \leq u \right\} \geq \exp(-\psi(u)/K_{20}).
\]

This has been proved in Talagrand [16].
Given $r$ small enough and Lemma 5.1 is true. Let $X_i(t), i = 1, \ldots, d,$ be the components of $X(t)$ and $R_i(s, t) = E X_i(s) X_i(t).$ Then there exist non-negative symmetric measures $\Delta_i(d\lambda), 1 \leq i \leq d$ on $R^N - \{0\}$ satisfying

$$\int_{R^N} \frac{\lambda^2}{1 + \lambda^2} \Delta_i(d\lambda) < \infty$$

such that

$$R_i(s, t) = \int_{R^N} (e^{i <t, \lambda>} - 1)(e^{-i <s, \lambda>} - 1) \Delta_i(d\lambda).$$

Furthermore, there is a centered complex valued Gaussian random measure $W(d\lambda) = (W_1(d\lambda), \ldots, W_d(d\lambda))$ such that

$$X_i(t) = \int_{R^N} (e^{i <t, \lambda>} - 1) W_i(d\lambda)$$

and for any $A, B \subseteq R^N_+,$

$$E W_i(A) W_i(B) = \Delta_i(A \cap B), \quad W_i(-A) = \overline{W_i(A)}.$$

In order to obtain the small ball probability for $X,$ we will use the spectral representation of $X$ to create the independence. Given $0 < a < b < \infty,$ we consider the process

$$X_i(a, b, t) = \int_{a \leq |\lambda| \leq b} (e^{i <t, \lambda>} - 1) W_i(d\lambda).$$

Clearly for $a < b < a' < b',$ $X_i(a, b, t)$ is independent to $X_i(a', b', t).$ Next we show how well $X_i(a, b, t)$ approximates $X_i(t).$

Lemma 5.3. Let $X_i, i = 1, \ldots, d,$ be the components of $X.$ For some constant $C' > 0$ and any $C' < a < b,$

$$E (X_i(a, b, t) - X_i(t))^2 \leq K_{21}[r^2 a^2 \sigma_i^2(a^{-1}) + \sigma_i^2(b^{-1})].$$

The proof is similar to that of Lemma 3.4 in [20], and here we omit the details.

Lemma 5.4. Let $\psi_i(s) = \inf\{t \geq 0 : \sigma_i(t) \geq s\}, A = r^2 a^2 \sigma_i^2(a^{-1}) + \sigma_i^2(b^{-1}).$ Suppose that there exists $C' > 0$ such that for any $C' < a < b,$ $0 < r < 1/C',$ it follows that $\psi_i(\sqrt{A}) \leq \frac{1}{2} r,$ then for any $u \geq K_{22} A \log(K_7 r/\psi_i(\sqrt{A})))^{1/2},$ we have

$$P \left\{ \sup_{|t| \leq r} |X_i(t) - X_i(a, b, t)| \geq u \right\} \leq \exp(-K_{23} u^2 / A). \quad (5.1)$$

Proof. Let $\xi_i(t) = X_i(t) - X_i(a, b, t).$ Then $\xi_i(t)$ is a Gaussian process with $E \xi_i(t) = 0, E \xi_i^2(t) \leq K_{21} A$ for $|t| \leq r.$ By Lemma 5.1, it follows that (5.1) is true. \hfill \Box

Lemma 5.5. Given $r$ small enough and $0 < \varepsilon < 1,$ we have that for any $0 < a < b,$

$$P \left\{ \sup_{|t| \leq r} |X_i(a, b, t)| \leq \varepsilon \sigma_i(r) \right\} \geq \exp(-K_{24} e^{-N/|a|}). \quad (5.2)$$

Proof. Let $E = \{t : |t| \leq r\}$ and the distance on $E$ be

$$d_i(s, t) = \left( E(X_i(a, b, s) - X_i(a, b, t))^2 \right)^{1/2}.$$
Then \(d_i(s, t) \leq \sigma_i(|s-t|)\) and \(N(E, \varepsilon) \leq K_{25}(r/\psi_i(\varepsilon))^N\), where \(\psi_i(\varepsilon)\) is defined in Lemma 5.4. By condition (1.3) and Lemma 5.2,

\[
P \left\{ \sup_{|t| \leq r} |X_i(a, b, t)| \leq \varepsilon \sigma_i(r) \right\} \geq P \left\{ \sup_{|t| \leq r} |X_i(a, b, t)| \leq 2M_2^{-1} \sigma_i(\varepsilon^{1/\alpha_i} r) \right\} \\
\geq \exp \left\{ -K_{25} \frac{r}{K_{20}} \left( \frac{1}{\psi(2M_2^{-1} \sigma_i(\varepsilon^{1/\alpha_i} r))} \right)^N \right\} =: \exp \left\{ -K_{24} e^{-N/\alpha_i} \right\}.
\]

This completes the proof of Lemma 5.5. \(\Box\)

**Lemma 5.6.** There exists a \(\delta > 0\) such that for any \(0 < r_0 < \delta\) and \(U = (u_1, \ldots, u_k)\),

\[
P \left\{ \exists r, r_0^2 \leq r \leq r_0, \sup_{l \leq k} \sup_{|t-u_l| \leq 2\sqrt{N}r} |X_i(t) - X_i(u_l)| \leq K_{26} \sigma_i(r) \times (\log \log 1/r)^{-\alpha_i/N}, i \leq d \right\} \\
\geq 1 - d \exp\{-\log 1/r_0)^{1/2}\}.
\]

**Proof.** The proof is similar to that of [18]. First we show that for any given \(a, b, r\),

\[
P \left\{ \sup_{l \leq k} \sup_{|t-u_l| \leq 2\sqrt{N}r} |X_i(a, b, t) - X_i(a, b, u_l)| \leq \varepsilon \sigma_i(r) \right\} \geq \exp\{-K_{24} e^{-N/\alpha_i}\}. \tag{5.3}
\]

To see this, we simply apply Lemma 5.2 to the \(R^k\) valued process 

\[Z_i(t_1, \ldots, t_k) = (X_i(a, b, t_1), \ldots, X_i(a, b, t_k)).\]

Note that

\[E|Z_i(t_1, \ldots, t_k) - Z_i(t'_1, \ldots, t'_k)|^2 \leq \sum_{i \leq k} \sigma_i^2(|t_i - t'_i|).
\]

Then, by an argument similar to that of Lemma 5.5, we have the conclusion (5.4). The rest of the proof for this lemma is very similar to that of Proposition 3.1 in [20]. We only need to replace Xiao’s Corollary 3.1 by our Lemma 5.4. Here we will not give the details. \(\Box\)

Similar to the argument of (2.16), we have

**Lemma 5.7.** If \(X_i, i \leq d\), are the components of \(X(t)\). Then

\[
\lim \sup_{n \to \infty} \sup_{s, t, |s-t| \leq 2^{-n}} \frac{|X_i(t) - X_i(s)|}{\sigma_i(2^{-n}) \sqrt{n}} \leq K_{27} \quad a.s.
\]

The following proposition concerns the upper bound of the Hausdorff measure.

**Proposition 5.1.** Suppose \(X\) is given by Theorem 1.4, then

\[
\phi_1 - m(I \cap Q) < \infty \quad a.s.
\]
Without loss of generality, we assume
\[ Q = \bigotimes_{i=1}^{k} Q_i = \left\{ T : T = \bigotimes_{i=1}^{k} B(s_i, \eta), S = (s_1, \ldots, s_k) \in \mathbb{R}_{2\eta}^{N_k} \right\}. \]

In order to obtain Proposition 5.1, the key is to construct a random covering of \( Q \). To this end, we will slip the process \( X \) into two independent processes \( X^{(1)} \) and \( X^{(2)} \) such that \( X^{(1)} \) is a very small perturbation of \( X \) and then construct the random covering depending only on \( X^{(1)} \). Let \( S' = (s'_1, \ldots, s'_k) \) be a point such that \( |s_i - s'_i| = 2\eta, 1 \leq k \) and define \( \Sigma'_2 = \sigma \{ X(s'_i), 1 \leq k \} \), \( X^{(2)}(t) = E(X(t)|\Sigma'_2) \), \( X^{(1)}(t) = X(t) - X^{(2)}(t) \). It is easy to see that \( X^{(1)}, X^{(2)} \) are independent. Furthermore, we have

**Lemma 5.8.** If the variance function \( \sigma^2(x) \) of \( X \) satisfies the condition of Theorem 1.4, then for any \( l \leq k, i \leq d, u_1, u_2 \in Q_l \), we have
\[ |E(X_i(u_1) - X_i(u_2))X(s'_i)| \leq K_{28}|u_1 - u_2|. \]

**Proof.** By the fact that \( EX_i(t)X_i(s) = \frac{1}{2}(\sigma^2(|t|) + \sigma^2(|s|) - \sigma^2(|t-s|)) \), we have that the left side of (5.5) is equal to
\[
\begin{align*}
\frac{1}{2} & \left| \sigma^2(|u_1|) - \sigma^2(|u_2|) + \sigma^2(|u_2 - s'_i|) - \sigma^2(|u_1 - s'_i|) \right| \\
& \leq \frac{1}{2} \left( |\sigma^2(|u_1|) - \sigma^2(|u_2|)| + |\sigma^2(|u_2 - s'_i|) - \sigma^2(|u_1 - s'_i|)| \right) \\
& = \frac{1}{2} \left( \left| \int_{|u_1|}^{|u_2|} \varphi_i(x)dx \right| + \left| \int_{|u_1 - s'_i|}^{|u_2 - s'_i|} \varphi_i(x)dx \right| \right) \leq K_{28}|u_1 - u_2| 
\end{align*}
\]
where the last inequality follows from the fact that for any \( u \in Q_l, |s'_i - u| \geq |s'_i - s_i| - |s_i - u| \geq \eta > 0. \]

**Lemma 5.9.** For any \( u_1, u_2 \in Q_l, i \leq d, \)
\[ |X_i^{(2)}(u_1) - X_i^{(2)}(u_2)| \leq K_{29}|u_1 - u_2| \max_{1 \leq l \leq k} |X_i(s'_i)|. \]

**Proof.** Since \( X^{(2)}(t) = E(X(t)|\Sigma'_2) \) and \( X_1, \ldots, X_d \) are independent, it follows that
\[ X_i^{(2)}(t) = \sum_{l,j \leq k} a_{ij}^{(l)} E(X_i(t)X_j(s'_i))X_j(s'_j) \]
where \( a_{ij}^{(l)} \) depends only on \( X_i(s'_i), \ldots, X_j(s'_j) \). Hence by Lemma 5.8, we have the conclusion of Lemma 5.9. \( \square \)

**Proof of Proposition 5.1.** We say a cube \( C \) is a dyadic cube with order \( n \) if there exists a non-negative integer vector \( \mathbf{m}_i = (m_{i1}, \ldots, m_{Ni}) \), such that \( C \) can be represented as
\[
C := C^n = \bigotimes_{l=1}^{k} C_l^n = \bigotimes_{l=1}^{k} \left[ \frac{m_l}{2^n}, \frac{m_l + 1}{2^n} \right] \\
= \bigotimes_{l=1}^{k} \left[ \frac{m_{i1}}{2^n}, \frac{m_{i1} + 1}{2^n} \right] \times \cdots \times \left[ \frac{m_{Ni}}{2^n}, \frac{m_{Ni} + 1}{2^n} \right].
\]
Let

\[ H_n = \left\{ T \in Q : \exists r \in [2^{-2n}, 2^{-n}] \text{ such that } \sup_{l \leq k} \sup_{|t-s| \leq 2\sqrt{Nr}} |X_i(t_l) - X_i(s_l)| \leq K_{26} \sigma_i(r)(\log \log 1/r)^{-\alpha_i/N}, i \leq d \right\}; \]

\[ H'_n = \left\{ T \in Q : \exists r \in [2^{-2n}, 2^{-n}] \text{ such that } \sup_{l \leq k} \sup_{|t-s| \leq 2\sqrt{Nr}} |X_i^{(1)}(t_l) - X_i^{(1)}(s_l)| \leq K'_{26} \sigma_i(r)(\log \log 1/r)^{-\alpha_i/N}, i \leq d \right\}; \]

\[ \Omega_{n1} = \left\{ \omega : \lambda(H_n) \geq \lambda(Q)(1 - \exp(-\sqrt{n}/4)) \right\}; \]

\[ \Omega_{n2} = \left\{ \omega : \lambda(H'_n) \geq \lambda(Q)(1 - \exp(-\sqrt{n}/4)) \right\}; \]

\[ \Omega_{n3} = \left\{ \omega : \max_{l \leq k} |X_i(s_l)| \leq 2^{\beta_i n}, i \leq d \right\}, \text{ where } 0 < \beta_i < 1 - \alpha_i; \]

\[ \Omega_{n4} = \left\{ \omega : \text{ for each dyadic cube } C = \bigotimes_{l=1}^k C_l \text{ with order } n, C \cap Q \neq \emptyset \right\} \]

and

\[ \sup_{l \leq k} \sup_{s_l, t_l \in C_l} |X_i(s_l) - X_i(t_l)| \leq k_{27} \sigma_i(2^{-n})\sqrt{n}, i \leq d \}\]
$C^n(U)$ has the property that for all $i \leq d$,
\[ \sup_{i \leq k} \sup_{s,t \in C^n(u_i) \cap Q_i} |X_i^{(1)}(t) - X_i^{(1)}(s)| \leq 8K_{26}' \sigma_i (2^{-n})(\log \log 2^n)^{\alpha_i/N}, \]
then we say it is a good dyadic cube. Otherwise, we say it is a bad dyadic cube.

It follows from (5.8) that each point $U = (u_1, \ldots, u_k) \in H'_n$ is contained in a good dyadic cube with order $p$, $n \leq p \leq 2n$. Therefore we can find a disjoint family of good dyadic cubes $H_1(n)$ to cover the set $H'_n$. That is,
\[ H'_n \subseteq \bigcup_{p=n}^{2n} V^p =: V \]
where $V^p$ is a disjoint union of good dyadic cubes $C^p$ with order $p$. Moreover, the family $H_1(n)$ depends only on $X_i^{(1)}(t)$. Let $H_2(n)$ be a family of bad dyadic cubes of order $2n$ on $R_{+}^{Nk}$, none of which meet $H_1(n)$. Then $Q - V$ can be covered by the dyadic cubes in $H_2(n)$. If $\Omega_{n2}$ occurs, the number of such bad dyadic cubes is at most
\[ K_{29} \lambda(Q) 2^{2Nkn} \exp(-\sqrt{n}/4). \]

Let $H(n) = H_1(n) \cup H_2(n)$. For any $A \in H(n)$, we pick a distinguished point $U_A = (u_{A1}, \ldots, u_{Ak})$ of $A$ and define
\[ \Omega_A = \{ \omega : |X_i^{(u_{Al})} - X_i(u_{A1})| \leq r_i(A), 2 \leq l \leq k, i \leq d \}, \]
where
\[ r_i(A) = \begin{cases} 32K_{26}' \sigma_i(|A|)(\log \log |A|)^{-\alpha_i/N} & \text{if } A \in H_1(k) \\ 8K_{27} \sigma_i(|A|)(\log |A|)^{-1/2} & \text{if } A \in H_2(k) \end{cases} \]
and $|A|$ denotes the side length of set $A$. Furthermore, we set $\mathcal{F}(n) = \{ A \in H(n) : \Omega_A \text{ occurs} \}$. Next we show that for $n$ large enough on $\Omega_n$, $\mathcal{F}(n)$ covers $I \cap Q$.

For any $T \in I \cap Q$, we have that
1. $X(t_1) = X(t_2) = \cdots = X(t_k)$,
2. there exists a dyadic cube $A$ such that $T \in A$.

Assume that $A \in H_1(n)$ with order $p$, then $n \leq p \leq 2n$ and by Lemma 5.9, we have that for $n$ large enough, on $\Omega_n$, for $i \leq d, l \leq k$,
\[ |X_i^{(u_{Al})} - X_i(u_{A1})| \leq |X_i^{(1)}(u_{Al}) - X_i^{(1)}(t_l)| + |X_i^{(1)}(u_{A1}) - X_i^{(1)}(t_1)| + |X_i^{(2)}(u_{Al}) - X_i^{(2)}(t_l)| + |X_i^{(2)}(u_{A1}) - X_i^{(2)}(t_1)| \leq 16K_{26}' \sigma_i(2^{-p})(\log \log 2^p)^{\alpha_i/N} + 2K_{29}2^{-p+\beta_i} \leq 32K_{26}' \sigma_i(2^{-p})(\log \log 2^p)^{\alpha_i/N} = r_i(A). \]
Similarly, as $A \in H_2(n)$ with order $p$, we also have that
\[ |X_i^{(u_{Al})} - X_i(u_{A1})| \leq r_i(A). \]
So, for $n$ large enough, on $\Omega_n$, the event $\Omega_A$ happens, which implies that $\mathcal{F}(n)$ covers $I \cap Q$.

Let $\Sigma_1 = \sigma(X^{(1)}(t) : t \in R_{+}^{Nk})$. Noting that $X^{(1)}, X^{(2)}$ are independent and
\[ P \left\{ |X_i^{(2)}(u_{Al}) - X_i^{(2)}(u_{A1}) - x| < r, 2 \leq l \leq k \right\} \leq K_{30} r^{k-1} \quad (5.10) \]
(this inequality can be obtained by an argument similar to one in [18]), we have
\[ P(\Omega_A | \Sigma_1) \leq \prod_{i=1}^{d} K_{30} r_i^{k-1}(A). \]  
(5.11)
By (5.11) and Fatou’s lemma, it follows that
\[ E(\phi_1 - m(I \cap Q)) \leq \liminf_{n \to \infty} E \left( I_{\Omega_n} \sum_{A \in \mathcal{F}(n)} \phi_1(|A|) \right) \]
\[ \leq \liminf_{n \to \infty} E \left( I_{\Omega_n} \sum_{A \in \mathcal{H}(n)} E(I_{\Omega_A | \Sigma_1}) \phi_1(|A|) \right) \]
\[ \leq \liminf_{n \to \infty} K_{31} E \left( I_{\Omega_n} \sum_{A \in \mathcal{H}(n)} |A|^{Nk} \right) \]
\[ \leq K_{31} \lambda(Q). \]
This yields the conclusion of Proposition 5.1. □

6. The proofs of Theorem 1.4 and Corollary 1.1

Proof of Theorem 1.4. Combining Propositions 4.1 and 5.1, we have the conclusion of Theorem 1.4. □

Proof of Corollary 1.1. Let cov(T) be the covariance matrix of Y(T), then by (2.7) we have that for any \( B \in \mathcal{H} \),
\[ E L(0, B) = (2\pi)^{-d(k-1)} \int_B \int_{\mathbb{R}^{d(k-1)}} E \exp\{i \langle u, Y(T) \rangle\} du dT \]
\[ = (2\pi)^{-d(k-1)} \int_B \int_{\mathbb{R}^{d(k-1)}} \exp \left\{ -\frac{1}{2} \text{Var}(\langle u, Y(T) \rangle) \right\} du dT \]
\[ = (2\pi)^{-d(k-1)/2} \int_B [\text{det(cov}(T))]^{-1} dT > 0, \]
which implies that \( P\{L(0, B) > 0\} > 0 \), which on combining with Theorem 1.4 yields the conclusion of Corollary 1.1. □

References