JOURNAL OF DIFFERENTIAL EQUATIONS 60, 201–217 (1985)

Circulant Matrices and Differential-Delay Equations

ROGER D. NUSSBAUM*

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

Received January 10, 1984; revised September 27, 1984

INTRODUCTION

The general motivation for this paper is the study of nontrivial periodic solutions of parametrized families of differential-delay equations of the form

$$\dot{y}(t) = h_\lambda(y(t-N_1), y(t-N_2),..., y(t-N_k)), \tag{0.1}$$

where $h_\lambda: \mathbb{R}^k \to \mathbb{R}$ is a continuous map and $N_1, N_2,..., N_k$ are nonnegative integers. By a periodic solution $y(t)$ of period p of Eq. (0.1) we mean a nonconstant solution $y(t)$ such that $y(t+p) = y(t)$ for all t (so p need not be the minimal period). We shall be interested in finding integers $p \geqslant 3$ and corresponding conditions on h_λ which insure that Eq. (0.1) has no periodic solution of period p for any $\lambda > 0$. As discussed, for example, in [5], any such result can, when used in conjunction with the global Hopf bifurcation theorem or Fuller's index or the theory of nonlinear cone mappings, provide a great deal of information about the existence and qualitative properties of periodic solutions of Eq. (0.1).For the most part we shall leave such applications to the reader: the results in [5] serve as a model.

The general approach in this paper is as follows: If one assumes that $y(t)$ is a periodic solution of Eq. (0.1) of integral period p and one defines $u_j(t) = y(t-j+1)$ for $1 \leqslant j \leqslant p$, one discovers that $u(t) = (u_1(t),..., u_p(t))$ satisfies a "cyclic system of ordinary differential equations" (see [5]), say

$$\dot{u}(t) = g(u(t)). \tag{0.2}$$

We shall prove that Eq. (0.1) has no periodic solutions of period p basically by proving that every solution of Eq. (0.2) satisfies $\lim_{t\to\infty} |u(t)| = 0$ or

* Partially supported by N.S.F. Grant MCS 82-01316 and as a visiting member at the Courant Institute, 1983–1984.

0022-0396/85 $3.00

$\lim_{t \to \infty} |u(t)| = \infty$, and the key step in proving the latter statement will be the construction of appropriate Lyapunov functions for Eq. (0.2). If one linearizes Eq. (0.2) about 0, one is led to the equation

$$\dot{v}(t) = A(v(t)),$$

where A is a real, circulant matrix (defined in Sect. 1). Thus the fact that circulant matrices enter our work is, in a general sense, unsurprising.

If $p \geqslant 3$, it is shown in [5] that Eq. (0.1) presents too general a class (although see Remark 1.4), and in fact the immediate motivation for this paper comes from the equation

$$\dot{x}(t) = -\lambda x(t) + \lambda f(x(t-1)) \tag{0.3}$$

(studied in Sect. 1) and the equation

$$\dot{x}(t) = -\lambda f(x(t-1)) - \mu f(x(t-2)), \tag{0.4}$$

which is studied in Section 2.

Indeed, perhaps the most striking feature of Theorem 1.1 is that, when it is applied to Eq. (0.3) it gives, in a very sharp sense, best possible results. Specifically, we shall discuss when Eq. (0.3) has no periodic solution of period $2m+1$ (and hence none of period $2+(1/m)$). If our assumptions are weakened even slightly, there exists $\lambda > 0$ such that Eq. (0.3) has a periodic solution of period $2+(1/m)$ (see Remark 1.2). Such theorems, when combined with results in [4], provide a variety of information about Eq. (0.3) (see Remark 1.3).

1. Circulant Matrices and Periodic Solutions of Differential-Delay Equations

We begin by recalling some known results about circulant matrices; further information and details can be found in [1]. A square matrix A is called "circulant" if

$$A = \begin{pmatrix} a_1 & a_2 & & \cdots & a_n \\ a_n & a_1 & a_2 & & a_{n-1} \\ a_{n-1} & a_n & a_1 & & a_{n-2} \\ & & & \ddots & \\ a_2 & a_3 & & \cdots & a_1 \end{pmatrix}$$

and one writes $A = \text{circ}(a_1, a_2, \ldots, a_n)$. More formally, if b_{ij} is the entry in row i and column j, then $b_{ij} = a_{j-i+1}$, where the subscripts of a_k are written modulo n, so $a_0 \equiv a_n$, $a_{-1} = a_{n-1}$, etc.

If z is an n^{th} root of unity (so $z^n = 1$) and one defines $v_z \in C^n$ by

$$v_z = \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{pmatrix}, \tag{1.1}$$

it is easy to check that v_z is an eigenvector of A with corresponding eigenvalue λ_z given by.

$$\lambda_z = \sum_{j=1}^{n} a_j z^{j-1}. \tag{1.2}$$

If $\langle v, w \rangle$ denotes the standard inner product of vectors v and w in $\mathbb{C}^n$, so

$$\langle v, w \rangle = \sum_{j=1}^{n} v_j \bar{w}_j,$$

then $\langle v_{z_1}, v_{z_2} \rangle = 0$ whenever z_1 and z_2 are unequal nth roots of unity and v_z is as in Eq. (1.1). This is because

$$\langle v_{z_1}, v_{z_2} \rangle = \sum_{j=1}^{n} (z_1 \bar{z}_2)^{j-1} = \frac{(z_1 z_2)^n - 1}{z_1 z_2 - 1} = 0. \tag{1.3}$$

If N is an $n \times n$ circulant matrix (real or complex) and N^* denotes its conjugate transpose, one can easily prove that N^* is a circulant matrix and hence has the same eigenvectors $v_z \in \mathbb{C}^n$ as N. It follows that $NN^* = N^*N$, i.e., N is normal.

We shall only use our next lemma for the case of circulant matrices, and in that case we could easily prove it directly without reference to the general theory of normal operators. However, we prefer to place the result within the framework of normal operators.

LEMMA 1.1. *Let H be a real Hilbert space and let $\mathscr{H} = H + iH$ denote its complexification with the standard inner product derived from $\langle \cdot, \cdot \rangle$, the inner product on H. Suppose that $N: H \to H$ is a bounded, normal linear operator and denote by the same letter N the standard extension $N: \mathscr{H} \to \mathscr{H}$ and by $\sigma(N)$ the spectrum of the extension. Then one has*

$$\inf\{\mathrm{Re}(\langle Nu, u \rangle): u \in \mathscr{H}, \|u\| = 1\} = \inf\{\langle Nu, u \rangle: u \in H, \|u\| = 1\}$$
$$= \inf\{\mathrm{Re}(\lambda): \lambda \in \sigma(N)\}. \tag{1.4}$$

In particular, if $N = \text{circ}(a_1, a_2,..., a_n)$ *is an* $n \times n$, *real circulant matrix, then*

$$\inf\left\{\langle Ny, y\rangle : y \in R^n \text{ and } \sum_{j=1}^{n} y_j^2 = 1\right\} = \min\left\{\text{Re}\left(\sum_{j=1}^{n} a_j z^{j-1}\right) : z^n = 1\right\}. \tag{1.5}$$

Proof. Write $N = A + iB$, where $A = ((N + N^*)/2)$ and $B = ((N - N^*)/2i)$ are commuting, self-adjoint operators on $\mathscr{H}$. If $u \in \mathscr{H}$ and $\|u\| = 1$, so $u = x + iy$ with $\|x\|^2 + \|y\|^2 = 1$,

$$\text{Re}(\langle Nu, u\rangle) = \text{Re}(\langle Au, u\rangle) = \langle Ax, x\rangle + \langle Ay, y\rangle. \tag{1.6}$$

If $\alpha = \inf\{\langle Nv, v\rangle : v \in H, \|v\| = 1\}$, it follows that

$$\text{Re}\langle Nu, u\rangle = \langle Ax, x\rangle + \langle Ay, y\rangle \geqslant \alpha\|x\|^2 + \alpha\|y\|^2 = \alpha, \tag{1.7}$$

and this implies

$$\inf\{\text{Re}(\langle Nu, u\rangle) : u \in \mathscr{H}, \|u\| = 1\} \geqslant \inf\{\langle Nu, u\rangle : u \in H, \|u\| = 1\}. \tag{1.8}$$

The opposite inequality is obvious.

Now we need to recall a classical result from the theory of normal operators (see [2, p. 112, Problem 171]): if N is any bounded, normal operator on a complex Hilbert space and $W(N) = \{\langle Nz, z\rangle : \|z\| = 1\}$, then $W(N)$ is the closed, convex hull of $\sigma(N)$. Equation (1.4) follows easily from this fact.

The last statement of the lemma follows immediately from Eq. (1.4) and the fact that if $N = \text{circ}(_1, a_2,..., a_n)$, N is normal with eigenvalues λ_z ($z^n = 1$) given by Eq. (1.2).

Before using Lemma 1.1 we need to recall a special case of a theorem in [3] or [7]; an argument in Section 1 of [5], although applied to a specific example, works in greater generality and also gives the result we shall state. Suppose that $F: \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitzian function and that $V: \mathbb{R}^n \to \mathbb{R}$ is a C^1 map. Define $\dot{V}: R^n \to R$ by

$$\dot{V}(x) \equiv \langle \nabla V(x), F(x)\rangle \tag{1.9}$$

(where $\nabla V(x)$ is the gradient of V and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$) and assume that $\dot{V}(x) \neq 0$ for all $x \neq 0$. If $x(t) = x(t; x_0)$ is the unique solution of

$$\dot{x}(t) = F(x(t)),$$

$$x(0) = x_0,$$

and $x(t)$ is defined on a maximal interval $(\alpha_{x_0}, \beta_{x_0})$, then

$$\lim_{t \to \alpha} \|x(t)\| = \infty$$

or

$$\lim_{t \to \alpha} \|x(t)\| = 0,$$

where α denotes α_{x_0} or β_{x_0}. Note that in [7] it is assumed that V is C^2, but the second derivative is needed only if one does not assume that $\dot{V}(x) \neq 0$ for all $x \neq 0$. Note also that if there exist constants A and B such that

$$\|F(x)\| \leqslant A\,\|x\| + B$$

for all x (as will be true in almost all of our applications), then standard theory of ODEs implies $\alpha_{x_0} = -\infty$ and $\beta_{x_0} = +\infty$.

Our next lemma treats a class of ordinary differential equations which may seem unnatural but which we shall show arise naturally in connection with differential-delay equations.

LEMMA 1.2. *Let $B = (b_{ij})$ be an $n \times n$ real matrix and define $c = c(B)$ by*

$$c(B) = \inf\{\langle By, y\rangle : y \in \mathbb{R}^n,\ \|y\| = 1\} \tag{1.10}$$

Assume that β_i, $1 \leqslant i \leqslant n$, are positive reals and that $g_i \colon \mathbb{R} \to \mathbb{R}$, $1 \leqslant i \leqslant n$, are locally Lipschitzian functions such that $ug_i(u) < 0$ for all $u \neq 0$. If $c(B) < 0$, assume that

$$|g_i(u)| < -\left(\frac{\beta_i}{c(B)}\right)|u| \tag{1.11}$$

for all $u \neq 0$. Then if $x(t) = (x_1(t), x_2(t),..., x_n(t))$ is any solution of

$$\dot{x}_i(t) = -\beta_i x_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)), \qquad 1 \leqslant i \leqslant n, \tag{1.12}$$

$\lim_{t \to \gamma} \|x(t)\| = +\infty$ or $\lim_{t \to \gamma} \|x(t)\| = 0$, where (α, δ) is the maximal interval of definition of $x(t)$ and $\gamma = \alpha$ or $\gamma = \delta$.

Proof. Suppose $x(0) = x_0$, $x(t)$ satisfies (1.12) and is defined on a maximal interval $(\alpha_{x_0}, \delta_{x_0})$. Define, for $x \in \mathbb{R}^n$,

$$V(x) = \sum_{i=1} G_i(x_i), \tag{1.13}$$

where $G_i(u) = \int_0^u g_i(s)\,ds$. By the remarks preceding Lemma 1.2, it suffices to prove $\dot{V}(x) > 0$ for $x \neq 0$, and a calculation gives

$$\dot{V}(x) = -\sum_{i=1}^{n} \beta_i x_i g_i(x_i) + \langle Bg, g \rangle, \tag{1.14}$$

where g in Eq. (1.14) is given by

$$g = \begin{pmatrix} g_1(x_1) \\ g_2(x_2) \\ \vdots \\ g_n(x_n) \end{pmatrix}. \tag{1.15}$$

The assumptions of the lemma imply

$$\dot{V}(x) \geqslant \sum_{i=1}^{n} \beta_i |x_i|\, |g_i(x_i)| + c(B) \sum_{i=1}^{n} |g_i(x_i)|^2. \tag{1.16}$$

If $c(B) \geqslant 0$, Eq. (1.16) implies $\dot{V}(x) > 0$ for $x \neq 0$. If $c(B) < 0$, Eq. (1.11) implies (for $x \neq 0$),

$$\dot{V}(x) \geqslant \sum_{i=1}^{n} |g_i(x_i)|\, [\beta_i |x_i| + c(B)\, |g_i(x_i)|] > 0, \tag{1.17}$$

which completes the proof. ▮

Remark 1.1 Suppose that

$$c(B) = \inf\{\mathrm{Re}(\lambda) \colon \lambda \text{ is an eigenvalue of } B\}, \tag{1.18}$$

as will be true for real, normal matrices (see Lemma 1.1). Assume also that $\beta_1 = \beta_2 = \cdots = \beta_n$ and that $c(B) < 0$. If λ is an eigenvalue of B such that $\mathrm{Re}(\lambda) = c(B)$ and $\mathrm{Im}(\lambda) = v$ and u is a corresponding eigenvector and if

$$g_i(t) = \left(\frac{\beta_i}{c(B)}\right) t$$

for all t, a calculation shows that, for $k = \beta_1 v / c(B)$,

$$x(t) = \mathrm{Re}(e^{ikt} u)$$

is a periodic or constant solution of Eq. (1.12). In this crude sense Lemma 1.2 is a best-possible result.

We shall now apply Lemmas 1.1 and 1.2 to the question of existence of certain periodic solutions of differential-delay equations. In this paper, a map $x: \mathbb{R} \to \mathbb{R}^n$ is said to be periodic of period p if $x(t+p) = x(t)$ for all t and $x(t)$ is not constant; thus p need *not* be the minimal period.

THEOREM 1.1. *Consider the differential-delay equation*

$$\dot{y}(t) = -\alpha y(t) + \sum_{j=1}^{n} a_j f(y(t-j)). \tag{1.19}$$

Assume that $\alpha > 0$, *that* $a_j \in \mathbb{R}$ *for* $1 \leqslant j \leqslant n$, *and that* $f: \mathbb{R} \to \mathbb{R}$ *is a locally Lipschitzian map such that* $yf(y) < 0$ *for all nonzero* y. *Define* $A = \operatorname{circ}(a_1, a_2, ..., a_n)$ *and*

$$c(A) = \min \left\{ \operatorname{Re} \left(\sum_{i=j}^{n} a_j z^{j-1} \right) : z \in \mathbb{C} \text{ and } z^n = 1 \right\}, \tag{1.20}$$

and if $c(A) < 0$, *assume that*

$$|f(y)| < -\left(\frac{\alpha}{c(A)} \right) |y| \tag{1.21}$$

for all $y \neq 0$. *Then Eq.* (1.19) *has no periodic solution* $y(t)$ *of period* n/m *for any positive integer* m.

Proof. If $y(t)$ were a periodic solution of period n/m, $y(t)$ would be of period n. For $1 \leqslant j \leqslant n$ define $x_j(t) = y(t-j+1)$, $B = A = \operatorname{circ}(a_1, a_2, ..., a_n)$ $\beta_j = \alpha$ and $g_j(u) = f(u)$. A calculation shows that in this notation $x(t) = (x_1(t), x_2(t), ..., x_n(t))$ is a periodic solution of Eq. (1.12). However, Lemmas 1.1 and 1.2 imply that

$$\lim_{t \to \infty} \|x(t)\| = \infty \text{ or } \lim_{t \to \infty} \|x(t)\| = 0,$$

which gives a contradiction. ∎

Our immediate motivation for proving Theorem 1.1 comes from studying qualitative properties of periodic solutions of

$$\dot{x}(t) = -\lambda x(t) + \lambda f(x(t-1)), \qquad \lambda > 0. \tag{1.22}$$

If $f: \mathbb{R} \to \mathbb{R}$ is a continuous map such that $|f(x)| \leqslant b\,|x|$ for all x and $b < 1$, the results of Section 1 in [4] imply that any continuous map x: $[t_0, \infty]$ which satisfies Eq. (1.22) for $t \geqslant t_0 + 1$ must satisfy $\lim_{t \to +\infty} x(t) = 0$. On the other hand, suppose that $xf(x) < 0$ for all nonzero x, that f is bounded and f is C^1 on a neighborhood of 0, and that $f'(0) = -k < -1$. If $\pi/2 < \nu_0 < \pi$, $\cos(\nu_0) = -1/k$ and $\lambda_0 = \nu_0/\sqrt{k^2-1}$, it is proved in [4] that Eq. (1.22) has a "slowly oscillating" periodic solution for each $\lambda > \lambda_0$. (A periodic solution $x(t)$ of Eq. (1.22) is called slowly oscillating if $x(0)=0$, $x(t)>0$ for $0<t<z_1$, where $z_1>1$, $x(t)<0$ for $z_1<t<z_2$, where $z_2-z_1>1$, and $x(t+z_2)=x(t)$ for all t). If f is also odd, the slowly oscillating periodic solution $x(t)$ can be chosen so $x(0)=0$, $x(t)>0$ on $(0, z_1)$, $z_1>1$, and $x(t+z_1)=-x(t)$ for all t.

If $f\colon \mathbb{R}\to\mathbb{R}$ is a locally Lipschitzian map such that $xf(x)<0$ for all nonzero x and n is a positive integer one obtains directly from Theorem 1.1 that if

$$|f(x)|<\left|\frac{1}{c_n}\right| |x| \qquad \text{for } x\neq 0, \tag{1.23}$$

where

$$c_n=\inf\{\operatorname{Re}(z)\colon z^n=1\}, \tag{1.24}$$

Eq. (1.27) has no periodic solution of period n/m, m a positive integer. If n is even, $c_n=-1$, and this result is uninteresting: a slight refinement of the previous remarks shows that Eq. (1.22) has no periodic solutions at all if Eq. (1.23) holds. If $n=2m+1$, $c_n=-\cos(\pi/(2m+1))$, and one obtains

COROLLARY 1.1. *Assume that $f\colon \mathbb{R}\to\mathbb{R}$ is a locally Lipschitzian map such that $xf(x)<0$ for all $x\neq 0$ and*

$$|f(x)|<\frac{|x|}{\cos\left(\dfrac{\pi}{2m+1}\right)}, \qquad x\neq 0. \tag{1.25}$$

Then for every $\lambda>0$, Eq. (1.22) has no periodic solution of period $2m+1$ and hence no periodic solution of period $2+(1/m)$.

Remark 1.2. Corollary 1.1 is, in a sense which we shall make precise, a best-possible result. Assume that $f\colon \mathbb{R}\to\mathbb{R}$ is a locally Lipschitzian map such that $xf(x)<0$ for all $x\neq 0$. In addition, assume that f is C^1 on a neighborhood of 0, that $k=-f'(0)>1/\cos(\pi/(2m+1))$, and that there are positive constants A and B such that $-B\leqslant f(x)\leqslant A$ whenever $-B\leqslant x\leqslant A$. Define $v_0\in(\pi/2,\pi)$ by

$$\cos(v_0)=-\frac{1}{k}.$$

It is proved in Corollary 3.2 of [4] that for every number p such that $2<p<2\pi/v_0$ there exists $\lambda>0$ and a slowly oscillating periodic solution

$x_\lambda(t)$ of Eq. (1.27) such that the minimal period of $x_\lambda(t)$ is p and $-B < x_\lambda(t) < A$ for all t. Our assumptions imply that

$$\cos(v_0) = -\frac{1}{k} > \cos\left(\frac{2m\pi}{2m+1}\right),$$

so $v_0 < 2m\pi/(2m+1)$ and $2\pi/v_0 > 2 + (1/m)$, and there exists *a* $\lambda > 0$ such that the minimal period of $x_\lambda(t)$ is $2 + (1/m)$. Note that if $-f'(0) > 1/\cos(\pi/(2m+1))$, Eq. (1.25) cannot hold for all x but might well hold except on a small neighborhood of 0.

Remark 1.3. If we define $S = \{(\lambda, x): \lambda > 0$ and $x(t)$ is a slowly oscillating periodic solution of Eq. (1.22)$\}$ and topologize S by identifying it with its image under the one–one map J defined by $J: S \to (0, \infty) \times C[0, 1]$ and $J(\lambda, x) = (\lambda, \phi)$, where $\phi = x \mid [0, 1]$, then it is proved in [4] that $\mathscr{S}$, the closure of S, equals $S \cup \{(\lambda_0, 0)\}$, where $\lambda_0 = v_0/\sqrt{k^2 - 1}$. Furthermore, if $(\lambda, x) \in S$, then $-B < x(t) < A$ for all t. Let $\mathscr{S}_0$ be the connected component of $\mathscr{S}$ which contains $(\lambda_0, 0)$. It is proved in [4] that if $p(\lambda, x)$ is the minimal period of $x(t)$ and $p(\lambda_0, 0) \equiv 2\pi/v_0$, then the map $p: \mathscr{S} \to R$ is continuous. Thus if f satisfies the assumptions of Corollary 1.1 and $2\pi/v_0 < 2 + (1/m)$ (m a positive integer), then a simple connectivity argument implies that $p(\lambda, x) < 2 + (1/m)$ for all $(\lambda, x) \in \mathscr{S}_0$. In particular, for each $\lambda > \lambda_0$ Eq. (1.22) has a slowly oscillating periodic solution of period less than $2 + (1/m)$. If f is odd and one uses Theorem 1.2 below similar reasoning shows that for each $\lambda > \lambda_0$, Eq. (1.22) has a slowly oscillating periodic solution of period $p < 4$. Note that by using arguments like those in [6] one can actually prove that for certain odd functions $f(x)$ and a range of λ values Eq. (1.22) also has slowly oscillating periodic solutions of period $p > 4$.

We have already remarked that if f is odd, Eq. (1.22) will, for $\lambda > \lambda_0$, have a periodic solution $x(t)$ such that $x(t + z_1) = -x(t)$ for all t. Our next theorem shows that, under mild assumptions on f, $z_1 \neq 2$.

THEOREM 1.2. *Assume that $f: \mathbb{R} \to \mathbb{R}$ is locally Lipschitzian, $xf(x) < 0$ for all $x \neq 0$ and f is odd. Then, for every $\lambda > 0$, Eq.* (1.22) *has no solution $x(t)$ (other than the zero solution) such that $x(t+2) = -x(t)$ for all t.*

Proof. If $x(t)$ satisfies Eq. (1.22) and $x(t+2) = -x(t)$, define $y(t) = x(t-1)$ and observe that

$$\begin{aligned} \dot{x}(t) &= -\lambda x(t) + \lambda f(y(t)), \\ \dot{y}(t) &= -\lambda y(t) - \lambda f(x(t)). \end{aligned} \tag{1.26}$$

The uniqueness theorem for ODEs implies that $(x(t), y(t)) \neq (0, 0)$.

If $F(u)=\int_0^u f(s)\,ds$, observe that

$$\frac{d}{dt}F(x(t))+F(y(t))=-\lambda f(x)x-\lambda f(y)\,y>0, \tag{1.27}$$

so our previous remarks imply that

$$\lim_{t\to\infty}\|(x(t),y(t))\|=\infty$$

or

$$\lim_{t\to\infty}\|(x(t),y(t))\|=0,$$

which contradicts the periodicity of $x(t)$. ∎

Remark 1.4. All of the previous results have somewhat stronger consequences than one might think. For example, if $g\colon \mathbb{R}^{m+1}\to\mathbb{R}$ is a map and $f(y)\equiv g(y,y,\dots,y)$ satisfies (1) $yf(y)<0$ for all $y\neq 0$, (2) f is locally Lipschitzian, and (3) $|f(y)|<2\,|y|$ for all y, then Corollary 1.1 implies that

$$\dot{y}(t)=-\lambda y(t)+\lambda g(y(t-1),y(t-4),y(t-7),\dots,y(t-1-3m))$$

has, for $\lambda>0$, no solution of period 3.

2. Some Criteria for Nonexistence of Solutions of Period 3

The central technical point of the arguments of Section 1 was explicitly to minimize a certain real-valued function, namely $\phi(y)=\langle Ay,y\rangle$, where A is a circulant matrix and $|y|=1$. We want to sketch how the same basic idea, namely the explicit minimization of a certain real-valued function, leads to a substantial improvement in results in Section 1 of [5] about the equation

$$\dot{y}(t)=-\lambda f(y(t-1)-\mu f(y(t-2)). \tag{2.1}$$

We begin with the basic technical lemma.

Lemma 2.1. *Assume that* $0<a<A$ *and* $\beta>-1$ *and consider the map* $\phi\colon\mathbb{R}^6\to\mathbb{R}$ defined by

$$\sum_{i\in Z^3}\left[(\beta+1)\,\alpha_i y_i^2+(\beta-1)\,\alpha_{i+1}y_iy_{i+1}-(\beta-1)\,\alpha_i y_i y_{i+1}\right]=\phi(y,\alpha), \tag{2.2}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $y = (y_1, y_2, y_3)$ and the indices in Eq. (2.2) are taken mod 3 (so $y_4 = y_1$, etc.). If T is given by

$$T = \left\{ (y, \alpha) \in \mathbb{R}^6 \colon \sum y_i^2 = 1 \text{ and } a \leqslant \alpha_j \leqslant A \text{ for } 1 \leqslant j \leqslant 3 \right\}$$

and

$$m = \min\{\phi(y, \alpha) \colon (y, \alpha) \in T\},$$

then

$$m = (\beta + 1)a - \left(\frac{A - a}{2}\right)(\sqrt{D} - \beta - 1), \tag{2.3}$$

where

$$D = (\beta + 1)^2 + 2(\beta - 1)^2. \tag{2.4}$$

Proof. Because $\phi(y, \alpha)$ is linear in α, ϕ achieves its minimum (subject to constraints) when two of the α_j equal a and the other A, or when two of the α_j equal A and the other a, when all α_j equal a or when all α_j equal A. One computes that $\phi(y, a, a, a) = (\beta + 1)a$ and $\phi(y, A, A, A) = (\beta + 1)A$ for $(y, \alpha) \in T$, and we leave it to the reader to verify that

$$\begin{aligned} &\min\{\phi(y, \alpha) \colon \|y\| = 1, \alpha_1 = \alpha_2 = a, \alpha_3 = A\} \\ &\quad = \min\{\phi(y, \alpha) \colon \|y\| = 1, \alpha_1 = \alpha_3 = a, \alpha_2 = A\}, \end{aligned}$$

and so forth. Thus we find that

$$m = \min\{\phi(y, \alpha) \colon \|y\| = 1, \alpha_1 = \alpha_2 = a, \alpha_3 = A\}$$

or

$$m = \min\{\phi(y, \alpha) \colon \|y\| = 1, \alpha_1 = \alpha_2 = A, \alpha_3 = a\}$$

or $m = (\beta + 1)a$. For convenience, write $\alpha_* = (a, a, A)$ and $\alpha^* = (A, A, a)$. A calculation gives

$$\begin{aligned} &\min_{|y| = 1} \phi(y, \alpha_*) \\ &\quad = (\beta + 1)a + (A - a) \min_{|y| = 1} [(\beta + 1)y_2^2 + (\beta - 1)y_1 y_2 - (\beta - 1)y_0 y_2] \\ &\quad = (\beta + 1)a + (A - a)\min\{(\beta + 1)u^2 - |\beta - 1|\, u(v + w) \colon \\ &\qquad u^2 + v^2 + w^2 = 1, u, v, w, \geqslant 0\}. \end{aligned} \tag{2.5}$$

One can verify that, for fixed u with $0 \leqslant u \leqslant 1$, the minimum of the expression in Eq. (2.5) occurs for $v = w$, so

$$\min_{|y|=1} \phi(y, \alpha_*) = (\beta+1)a + (A-a)\min\{(\beta+1)u^2 - |\beta-1|(\sqrt{2})u(\sqrt{1-u^2}) \colon 0 \leqslant u \leqslant 1\}. \tag{2.6}$$

A similar argument shows

$$\min_{|y|=1} \phi(y, \alpha^*) = (\beta+1)a + (A-a)\min\{(\beta+1)(1-u^2) - |\beta-1|(\sqrt{2})u(\sqrt{1-u^2}) \colon 0 \leqslant u \leqslant 1\}. \tag{2.7}$$

If one makes the substitution $r^2 = 1 - u^2$ in Eq. (2.7), one sees that

$$\min_{|y|=1} \phi(y, \alpha^*) = \min_{|y|=1} \phi(y, \alpha_*). \tag{2.8}$$

If one defines $\Theta(u, v, w)$ by

$$\Theta(u, v, w) = (\beta+1)u^2 - |\beta+1|u(v+w), \tag{2.9}$$

the previous remarks show that it suffices to compute

$$\min\{\Theta(u, v, w) \colon u^2 + v^2 + w^2 = 1,\ u, v, w, \geqslant 0\} \equiv m_1. \tag{2.10}$$

One can check that the minimum in Eq. (2.6) occurs for some u with $0 < u < 1$, so our previous remarks imply that the minimum in Eq. (2.10) occurs for $u > 0$ and $v = w > 0$. If the minimum occurs at (u, v, w), Lagrange multiplier theory implies that there exists $\lambda \in \mathbb{R}$ such that

$$2(\beta+1)u - |\beta-1|(v+w) = 2\lambda u \tag{2.11}$$

and

$$-|\beta-1|\, u = 2\lambda u \tag{2.12}$$

and

$$-|\beta-1|u = 2\lambda w. \tag{2.13}$$

If $\lambda = 0$, one deduces that $u = 0$, $v + w = 0$ and finally (because v and w are nonnegative) that $v = w = 0$, a contradiction. Thus $\lambda \neq 0$ and

$$v = w = -\frac{|\beta-1|u}{2\lambda}.$$

Substituting the previous equation into Eq. (2.11) and factoring out u gives

$$2(\beta+1) + \frac{(\beta-1)^2}{\lambda} = 2\lambda. \tag{2.14}$$

Equation (2.12) implies that $\lambda<0$, so we obtain from the quadratic Eq. (2.14) that

$$\lambda=\frac{(\beta+1)}{2}-\frac{\sqrt{D}}{2}. \tag{2.15}$$

Equation (2.15) determines u, because

$$1=u^2+v^2+w^2=u^2\left[1+\frac{(\beta-1)^2}{2\lambda^2}\right], \tag{2.16}$$

and hence $v=w=\sqrt{1-u^2}/(\sqrt{2})$. Substituting in Eq. (2.9), we have

$$m_1=(\beta+1)u^2-(\beta-1)u(\sqrt{2})(\sqrt{1-u^2}), \tag{2.17}$$

where u and λ are given in Eqs. (2.15) and (2.16). Finally, we have

$$\min_{|y|=1} \phi(y, \alpha_*)=(\beta+1)a+(A-a)m_1,$$

and an unpleasant calculation shows that the right hand side of Eq. (2.17) equals the constant in Eq. (2.3). ∎

Remark 2.1. If $\phi(y, \alpha)$ is defined by Eq. (2.2) and $\beta<-1$, an analogous argument (which we leave to the reader) proves that

$$\min\{-\phi(y, \alpha): y_1^2+y_2^2+y_3^2=1, a\leqslant\alpha_j\leqslant A \text{ for } 1\leqslant j\leqslant 3\}$$
$$=|\beta+1|a-\left(\frac{A-a}{2}\right)(\sqrt{D}-|\beta+1|),$$

where D is given in Eq. (2.4).

We shall prove our main theorems with the aid of Lemma 2.1 and Remark 2.1. Define $P\colon \mathbb{R}^3\to\mathbb{R}^3$ by

$$P(y)=\tfrac{1}{3}(2y_1-y_2-y_3, 2y_2-y_1-y_3, 2y_3-y_1-y_2), \tag{2.18}$$

and if $|u|$ denotes the Euclidean norm of $u\in\mathbb{R}^3$, define (as in [5]) $V\colon \mathbb{R}^3\to\mathbb{R}$ by

$$V(y)=\tfrac{1}{2}(3|P(y)|^2-|y|^2). \tag{2.19}$$

If Q is the orthogonal projection of $\mathbb{R}^3$ onto $\{(t, t, t)\colon t\in\mathbb{R}\}$, then $P=I-Q$, so V has a natural geometrical interpretation.

LEMMA 2.2. *Consider the system of three differential equations*

$$\dot{y}_i(t)=-f_{i+1}(y_{i+1}(t))-\beta f_{i+2}(y_{i+2}(t)), \tag{2.20}$$

where $1 \leqslant i \leqslant 3$ *and the subscripts are written* mod 3. *Assume that* $\beta \neq -1$ *and that, for* $1 \leqslant i \leqslant 3$, $f_i: \mathbb{R} \to \mathbb{R}$ *is locally Lipschitzian and* $uf_i(u) > 0$ *for all* $u \neq 0$. *If* $\beta \neq 1$, *assume that there exist positive constants a and A such that*

$$a|u| \leqslant |f_i(u)| \leqslant A|u| \tag{2.21}$$

for all u, where

$$\left[\frac{|\beta+1|}{(\beta-1)^2}\right][|\beta+1| + \sqrt{(\beta+1)^2 + 2(\beta-1)^2}] + 1 > \frac{A}{a}. \tag{2.22}$$

If $y(t)$ *satisfies Eq.* (2.20) *on* (c, d), *where* (c, d) *is a maximal interval of definition, then* $\lim_{t \to \alpha} |y(t)| = \infty$ *or* $\lim_{t \to \alpha} |y(t)| = 0$ *for* $\alpha = c$ *or* $\alpha = d$.

Proof. If $g: \mathbb{R}^3 \to \mathbb{R}^3$ is the map given by the right-hand side of Eq. (2.20) and $V(y)$ is given by Eq. (2.19), a calculation shows (see [5, Sect. 1])

$$\begin{aligned}\dot{V}(y) &= \langle \nabla V(y), g(y) \rangle \\ &= \sum_{i \in Z_3} [(\beta+1) y_i f_i(y_i) + (\beta-1) y_i f_{i+1}(y_{i+1}) - (\beta-1) y_{i+1} f_i(y_i)].\end{aligned} \tag{2.23}$$

If we write $f_i(y_i) = \alpha_i y_i$, then $a \leqslant \alpha_i \leqslant A$ and

$$\dot{V}(y) = \phi(y, \alpha),$$

where ϕ is as in Lemma 2.1. The theorem will follow if we prove that $\dot{V}(y) \neq 0$ for all $y \neq 0$, and this will be true if

$$\min\{\phi(y, \alpha): y_1^2 + y_2^2 + y_3^2 = 1, a \leqslant \alpha_i \leqslant A \text{ for } 1 \leqslant i \leqslant 3\} > 0 \tag{2.24}$$

or

$$\min\{-\phi(y, \alpha): |y| = 1, a \leqslant \alpha_i \leqslant A \text{ for } 1 \leqslant i \leqslant 3\} > 0 \tag{2.25}$$

(If $\beta = 1$, Eq. (2.23) implies $\dot{V}(y) > 0$ without Eq. (2.24) or (2.25)). Lemma 2.1 and Remark 2.1 imply that Eq. (2.24) or Eq. (2.25) will hold if

$$|\beta+1|a - \left(\frac{A-a}{2}\right)(\sqrt{D} - |\beta+1|) > 0,$$

and the latter inequality is equivalent to Eq. (2.22). ∎

Our next theorem is essentially a more symmetric statement of Lemma 2.2.

THEOREM 2.1. *Consider the system of three differential equations*

$$\dot{x}_i(t) = -\lambda g_{i+1}(x_{i+1}(t)) - \mu g_{i+2}(x_{i+2}(t)), \tag{2.26}$$

where $1 \leqslant i \leqslant 3$ *and subscripts are written* mod 3. *Assume that* λ *and* μ *are real numbers with* $\lambda \neq -\mu$ *and that (for* $1 \leqslant i \leqslant 3$) $g_i\colon \mathbb{R} \to \mathbb{R}$ *is locally Lipschitzian and* $ug_i(u) > 0$ *for all* $u \neq 0$. *If* $\lambda \neq \mu$, *assume that there exist positive constants a and A such that*

$$a|u| \leqslant |g_i(u)| \leqslant A|u|$$

for all u, where

$$1 + \frac{|\lambda+\mu|\,(\sqrt{(\lambda+\mu)^2 + 2(\lambda-\mu)^2} + |\lambda+\mu|)}{|\lambda-\mu|^2} > \frac{A}{a}. \tag{2.27}$$

If $x(t) = (x_1(t), x_2(t), x_3(t))$ *satisfies Eq.* (2.26) *and is defined on maximal interval* (c, d), *then*

$$\lim_{t \to \alpha} |x(t)| = \infty$$

or

$$\lim_{t \to \alpha} |x(t)| = 0,$$

where $\alpha = c$ *or* $\alpha = d$.

Proof. If $\lambda > 0$, define $y_i(t) = x_i(t), f_i(v) = \lambda g_i(v)$ and $\beta = \mu/\lambda$ and note that $y(t)$ satisfies Eq. (2.20) and β satisfies Eq. (2.22) if $\beta \neq 1$, so the theorem follows from Lemma 2.2 in this case. If $\lambda < 0$, define $\tilde{x}_i(t) = x_i(-t)$, $\tilde{\lambda} = -\lambda$ and $\tilde{\mu} = -\mu$ and note that

$$\tilde{x}'_i(t) = -\tilde{\lambda} g_{i+1}(\tilde{x}_{i+1}(t)) - \tilde{\mu} g_{i+2}(\tilde{x}_{i+2}(t)),$$

which is the case previously considered (note that Eq. (2.27) is unchanged if $\tilde{\lambda}$ is substituted for λ and $\tilde{\mu}$ for μ).

If $\mu > 0$, defing $\tilde{x}_1(t) = x_1(t)$, $\tilde{x}_2(t) = x_3(t)$, $\tilde{x}_3(t) = x_2(t)$, $\tilde{g}_1(v) = g_1(v)$, $\tilde{g}_2(v) = g_3(v)$, and $\tilde{g}_3(v) = g_2(v)$ and note that

$$\tilde{x}'_i(t) = -\tilde{\lambda} \tilde{g}_{i+1}(t)) - \tilde{\mu} \tilde{g}_{i+2}(\tilde{x}_{i+2}(t)),$$

where $\tilde{\lambda} \equiv \mu$ and $\tilde{\mu} \equiv \lambda$. This is the first case considered (note that Eq. (2.27) is unchanged if λ is substitued for μ and μ for λ).

Finally, if $\mu < 0$, we can reduce to the case $\mu > 0$ by the same trick used to reduce the case $\lambda < 0$ to the $\lambda > 0$. Because we assume $\lambda \neq -\mu$, either λ or μ is nonzero, and the theorem is proved. ∎

Our main interest in Theorem 2.1 is as a tool to obtain information about differential-delay equations.

THEOREM 2.2. *Consider the differential-delay equation*

$$\dot{x}(t) = -\lambda g(x(t-1)) - \mu g(x(t-2), \tag{2.28}$$

where λ *and* μ *are real numbers such that* $\lambda \neq -\mu$ *and* $g: \mathbb{R} \to \mathbb{R}$ *is a locally Lipschitzian map such that* $xg(x) > 0$ *for* $x \neq 0$. *If* $\lambda \neq \mu$, *assume that there exist positive constants* a *and* A *such that*

$$a|x| \leqslant |g(x)| \leqslant A|x|$$

and such that Eq. (2.27) *is satisfied. Then Eq.* (2.28) *has no periodic solution* $x(t)$ *such that* $x(t+3) = x(t)$ *for all* t (*other than* $x(t) \equiv 0$).

Proof. Assume $x(t)$ satisfies Eq. (2.28) and is periodic of period 3. Define $x_i(t) = x(t-i+1)$ and $g_i(v) = g(v)$ for $1 \leqslant i \leqslant 3$ and note that $x_i(t)$ satisfies Eq. (2.26). It follows from Theorem 2.1 that

$$\lim_{t \to \infty} |x(t)| = \infty$$

or

$$\lim_{t \to \infty} |x(t)| = 0,$$

and in either case we have contradicted periodicity. ∎

Remark 2.2. The results of [5] imply that Eq. (2.28) has no periodic solution of period 3 if

$$1 + \frac{|\lambda + \mu|}{|\lambda - \mu|} > \frac{A}{a}, \tag{2.29}$$

and one can easily see that Eq. (2.27) always gives a better result. For example, if $\mu = 0$, Eq. (2.29) would imply $2 > A/a$, while Eq. (2.27) would require $2 + \sqrt{3} > A/a$.

REFERENCES

1. P. J. DAVIS, "Circulant Matrices", Wiley, New York, 1979.
2. P. R. HALMOS, "A Hilbert Space Problem Book", Van Nostrand, Princeton, N. J., 1967.
3. M. B. KUDAEV, "Classification of multi-dimensional systems of ordinary differential equations by the method of Lyapunov functions," *Differentsial'nye* **1** (1965), 346–356 J. Kravneniya [Russian]; *Differential Equations* **1** (1965).

4. J. MALLET-PARET AND R. D. NUSSBAUM, "Global continuation and asymptotic behavior for periodic solutions of a differential-delay equation," submitted for publication.
5. R. D. NUSSBAUM AND A. J. B. POTTER, Cyclic differential equations and period three solutions of differential-delay equations, *J. Differential Equations* **46** (1982), 379–408.
6. R. D. NUSSBAUM, Asymptotic analysis of functional differential equations and solutions of long period, *Arch. Rational. Mech. anal.* **81** (1983), 373–397.
7. J. A. YORKE, A theorem on Liapunow functions using $\dot{V}$, *Math. Systems Theory* **4** (1970), 40–45.