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Characterizations of hemirings by their h-ideals

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ABSTRACT

In this paper we characterize hemirings in which all h-ideals or all fuzzy h-ideals are idempotent. It is proved, among other results, that every h-ideal of a hemiring R is idempotent if and only if the lattice of fuzzy h-ideals of R is distributive under the sum and h-intrinsic product of fuzzy h-ideals or, equivalently, if and only if each fuzzy h-ideal of R is intersection of those prime fuzzy h-ideals of R which contain it. We also define two types of prime fuzzy h-ideals of R and prove that, a non-constant h-ideal of R is prime in the second sense if and only if each of its proper level set is a prime h-ideal of R.

1. Introduction

The notion of semiring was introduced by Vandiver in 1934 [1]. Semirings which provide a common generalization of rings and distributive lattices appear in a natural manner in some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics (see for example [2–6]). Hemirings, as semirings with commutative addition and zero element, have also proved to be an important algebraic tool in theoretical computer science (see for instance [7,8]). Some other applications of semirings with references can be found in [9,8,5]. On the other hand, the notions of automata and formal languages have been generalized and extensively studied in a fuzzy frame work (cf. [10–12]).

Ideals play an important role in the structure theory of hemirings and are useful for many purposes. But they do not coincide with usual ring ideals. For this reason many results in ring theory have no analogues in semirings using only ideals. Henriksen defined in [13] a more restricted class of ideals in semirings, which is called the class of k-ideals. A more restricted class of ideals has been given by Iizuka [14]. However, in an additively commutative semiring R, ideals of a semiring coincide with ideals of a ring, provided that a semiring is a hemiring. Now we call this ideal an h-ideal of a hemiring.

Investigations of fuzzy semirings were initiated in [15]. Fuzzy h-ideals of a hemiring are studied by many authors, for example [16–18]. The notion of fuzzy sets was introduced by Zadeh [19]. Later it was applied to many branches of mathematics. Investigations of fuzzy semirings were initiated in [15] and [20]. Fuzzy k-ideals are studied in [21–23]. Fuzzy h-ideals of a hemiring are studied by many authors, for example [16–18,24–27]. In this paper we characterize hemirings in which each h-ideal is idempotent. We also characterize hemirings for which each fuzzy h-ideal is idempotent.

2. Preliminaries

Recall that a *semiring* is an algebraic system $(R, +, \cdot)$ consisting of a non-empty set R together with two binary operations on R called addition and multiplication (denoted in the usual manner) such that $(R, +)$ and $(R, \cdot)$ are semigroups and the following distributive laws:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad \text{and} \quad (b + c) \cdot a = b \cdot a + c \cdot a$$

are satisfied for all $a, b, c \in R$.

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A semiring $(R, +, \cdot)$ is called a *hemiring* if $(R, +)$ is a commutative semigroup with a *zero*, i.e., with an element $0 \in R$ such that $a + 0 = 0 + a = a$ and $a \cdot 0 = 0 \cdot a = 0$ for all $a \in R$. By the *identity* of a hemiring $(R, +, \cdot)$ we mean an element $1 \in R$ (if it exists) such that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

A hemiring $(R, +, \cdot)$ with a commutative semigroup $(R, \cdot)$ is called *commutative*.

A non-empty subset I of a hemiring R is called a *left (right) ideal* of R if (i) $a + b \in I$ for all $a, b \in I$ and (ii) $ra \in I$ ($ar \in I$) for all $a \in I$, $r \in R$. Obviously $0 \in I$ for any left (right) ideal I of R.

A non-empty subset A of a hemiring R is called an *ideal* of R if it is both a left and a right ideal of R. A left (right) ideal A of a hemiring R is called a *left (right) k-ideal* of R if for any $a, b \in A$ and $x \in R$ from $x + a = b$ it follows $x \in A$. A left (right) ideal I of a hemiring R is called a *left (right) h-ideal* of R if for any $a, b \in I$ and $x, y \in R$ from $x + a + y = b + y$ it follows $x \in I$. Every left (right) h-ideal is a left (respectively, right) k-ideal. The converse is not true [22].

Lemma 2.1. *The intersection of any collection of left (right) h-ideals in a hemiring R also is a left (right) h-ideal of R.*

By h-closure of a non-empty subset A of a hemiring R we mean the set

$$\overline{A} = \{x \in R \mid x + a + y = b + y \text{ for some } a, b \in A, y \in R\}.$$

It is clear that if A is a left (right) ideal of R, then $\overline{A}$ is the smallest left (right) h-ideal of R containing A. So, $\overline{A} = A$ for all left (right) h-ideals of R. Obviously $\overline{\overline{A}} = \overline{A}$ for each non-empty $A \subseteq R$. Also $\overline{A} \subseteq \overline{B}$ for all $A \subseteq B \subseteq R$.

Lemma 2.2 ([18]). $\overline{AB} = \overline{\overline{A}\,\overline{B}}$ *for any subsets A, B of a hemiring R.*

Lemma 2.3 ([18]). *If A and B are, respectively, right and left h-ideals of a hemiring R, then*

$$\overline{AB} \subseteq A \cap B.$$

Definition 2.4 ([18]). A hemiring R is said to be *h-hemiregular* if for each $a \in R$, there exist $x, y, z \in R$ such that $a + axa + z = aya + z$.

Lemma 2.5 ([18]). *A hemiring R is h-hemiregular if and only if for any right h-ideal A and any left h-ideal B, we have*

$$\overline{AB} = A \cap B.$$

Let X be a non-empty set. By a *fuzzy subset* μ of X we mean a membership function $\mu : X \to [0, 1]$. Im μ denotes the set of all values of μ. A fuzzy subset $\mu : X \to [0, 1]$ is non-empty if there exist at least one $x \in X$ such that $\mu(x) > 0$. For any fuzzy subsets λ and μ of X we define

$$\lambda \le \mu \Longleftrightarrow \lambda(x) \le \mu(x),$$
$$(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x) = \min\{\lambda(x), \mu(x)\},$$
$$(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x) = \max\{\lambda(x), \mu(x)\}$$

for all $x \in X$.

More generally, if $\{\lambda_i : i \in I\}$ is a collection of fuzzy subsets of X, then by the *intersection* and the *union* of this collection we mean fuzzy subsets

$$\Big(\bigwedge_{i\in I} \lambda_i\Big)(x) = \bigwedge_{i\in I} \lambda_i(x) = \inf_{i\in I}\{\lambda_i(x)\},$$
$$\Big(\bigvee_{i\in I} \lambda_i\Big)(x) = \bigvee_{i\in I} \lambda_i(x) = \sup_{i\in I}\{\lambda_i(x)\},$$

respectively.

A fuzzy subset λ of a semiring R is called a *fuzzy left (right) ideal* of R if for all $a, b \in R$ we have

(1) $\lambda(a + b) \ge \lambda(a) \wedge \lambda(b)$,
(2) $\lambda(ab) \ge \lambda(b)$, $(\lambda(ab) \ge \lambda(a))$.

Note that $\lambda(0) \ge \lambda(x)$ for all $x \in R$.

Definition 2.6. A fuzzy left (right) ideal λ of a hemiring R is called a *fuzzy left (right)*

- *k-ideal* if $x + y = z \longrightarrow \lambda(x) \ge \lambda(y) \wedge \lambda(z)$,
- *h-ideal* if $x + a + y = b + y \longrightarrow \lambda(x) \ge \lambda(a) \wedge \lambda(b)$

holds for all $a, b, x, y \in R$.

Properties of fuzzy sets defined on an algebraic system $\mathfrak{A} = (X, \mathbb{F})$, where $\mathbb{F}$ is a family of operations (also partial) defined on X, can be characterized by the corresponding properties of some subsets of X. Namely, as it is proved in [28] the following Transfer Principle holds.

Lemma 2.7. *A fuzzy set λ defined on $\mathfrak{A}$ has the property $\mathcal{P}$ if and only if all non-empty subsets $U(\lambda; t) = \{x \in X \mid \lambda(x) \geq t\}$ have the property $\mathcal{P}$.*

For example, a fuzzy set λ of a hemiring R is a fuzzy left ideal if and only if each non-empty subset $U(\lambda; t)$ is a left ideal of R. Similarly, a fuzzy set λ in a hemiring R is a fuzzy left h-ideal of R if and only if each non-empty subset $U(\lambda; t)$ is a left h-ideal of R.

As a simple consequence of the above property, we obtain the following proposition, which was first proved in [16].

Proposition 2.8. *Let A be a non-empty subset of a hemiring R. Then a fuzzy set λ_A defined by*

$$\lambda_A(x) = \begin{cases} t & \text{if } x \in A \\ s & \text{otherwise} \end{cases}$$

where $0 \leq s < t \leq 1$, is a fuzzy left h-ideal of R if and only if A is a left h-ideal of R.

Proposition 2.9. *If* $\operatorname{Im}\lambda_A = \operatorname{Im}\lambda_B$ *then*

(1) $A \subseteq B \longleftrightarrow \lambda_A \leq \lambda_B$,
(2) $\lambda_A \wedge \lambda_B = \lambda_{A\cap B}$.

Proof. Let $A \subseteq B$. For $x \in A$ we have $\lambda_A(x) = t = \lambda_B(x)$. If $x \notin A$, then $\lambda_A(x) = s \leq \lambda_B(x)$. So, $\lambda_A \leq \lambda_B$. Conversely, if $\lambda_A \leq \lambda_B$, then for all $x \in A$ we obtain $t = \lambda_A(x) \leq \lambda_B(x)$. Thus $\lambda_B(x) = t$, i.e., $x \in B$. Consequently, $A \subseteq B$. This proves (1).

To prove (2) let $x \in A \cap B$. Then $x \in A$, $x \in B$ and $\lambda_A(x) \wedge \lambda_B(x) = t = \lambda_{A\cap B}$. If $x \notin A \cap B$, then $\lambda_A(x) = s$ or $\lambda_B(x) = s$. So, $\lambda_A(x) \wedge \lambda_B(x) = s = \lambda_{A\cap B}(x)$, which completes the proof. □

Definition 2.10 (*[16]*). Let λ and μ be fuzzy subsets of a hemiring R. Then the h-product of λ and μ is defined by

$$(\lambda \circ_h \mu)(x) = \begin{cases} \sup\limits_{x+a_1b_1+y=a_2b_2+y} \left(\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)\right) \\ 0 \quad \text{if } x \text{ is not expressed as } x + a_1b_1 + y = a_2b_2 + y. \end{cases}$$

One can prove that if λ and μ are fuzzy left (right) h-ideals in a hemiring R, then so is $\lambda \wedge \mu$. Moreover, if λ is a fuzzy right h-ideal and μ is a fuzzy left h-ideal of R, then $\lambda \circ_h \mu \leq \lambda \wedge \mu$.

Theorem 2.11 (*[18]*). *A hemiring R is h-hemiregular if and only if $\lambda \circ_h \mu = \lambda \wedge \mu$ for any fuzzy right h-ideal λ and fuzzy left h-ideal μ.*

3. *h*-intrinsic product of fuzzy subsets

To avoid repetitions from now R will always mean a hemiring $(R, +, \cdot)$.

Generalizing the concept of h-product of two fuzzy subsets of R, in [29] the following h-intrinsic product of fuzzy subsets is defined:

Definition 3.1. The *h-intrinsic product* of two fuzzy subsets μ and ν on R is defined by

$$(\mu \odot_h \nu)(x) = \sup_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} \left(\bigwedge_{i=1}^{m}\left(\mu(a_i) \wedge \nu(b_i)\right) \wedge \bigwedge_{j=1}^{n}\left(\mu(a'_j) \wedge \nu(b'_j)\right)\right)$$

and $(\mu \odot_h \nu)(x) = 0$ if x cannot be expressed as $x + \sum_{i=1}^{m} a_ib_i + z = \sum_{j=1}^{n} a'_jb'_j + z$.

The following properties of the h-intrinsic product of fuzzy sets proved in [29] will be used in this paper.

Proposition 3.2. *Let $\mu, \nu, \omega, \lambda$ be fuzzy subsets on R. Then*

(1) $\mu \circ_h \nu \leq \mu \odot_h \nu$,
(2) $\mu \leq \omega$ *and* $\nu \leq \lambda \longrightarrow \mu \odot_h \nu \leq \omega \odot_h \lambda$.
(3) $\chi_A \odot_h \chi_B = \chi_{\overline{AB}}$ *for characteristic functions of any subsets of R.*

Theorem 3.3. *If λ and μ are fuzzy h-ideals of R, then so is $\lambda \odot_h \mu$. Moreover, $\lambda \odot_h \mu \leq \lambda \wedge \mu$.*

Proof. Let λ and μ be fuzzy h-ideals of R. Let $x, y \in R$, then

$$(\lambda \odot_h \mu)(x) = \sup_{x+\sum_{i=1}^{m} a_ib_i+z=\sum_{j=1}^{n} a'_jb'_j+z} \left(\bigwedge_{i=1}^{m}\left(\lambda(a_i) \wedge \mu(b_i)\right) \wedge \bigwedge_{j=1}^{n}\left(\lambda(a'_j) \wedge \mu(b'_j)\right)\right)$$

and

$$(\lambda \odot_h \mu)(y) = \sup_{y+\sum_{k=1}^{p} c_k d_k + z' = \sum_{l=1}^{q} c'_l d'_l + z'} \left(\bigwedge_{k=1}^{p} \big(\lambda(c_k) \wedge \mu(d_k)\big) \wedge \bigwedge_{l=1}^{q} \big(\lambda(c'_l) \wedge \mu(d'_l)\big) \right).$$

Thus

$$\begin{aligned}
(\lambda \odot_h \mu)(x+y) &= \sup_{x+y+\sum_{s=1}^{u} e_s f_s + z = \sum_{t=1}^{v} e'_t f'_t + z} \left(\bigwedge_{s=1}^{u} \big(\lambda(e_s) \wedge \mu(f_s)\big) \wedge \bigwedge_{t=1}^{v} \big(\lambda(e'_t) \wedge \mu(f'_t)\big) \right) \\
&\geq \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left(\sup_{y+\sum_{k=1}^{p} c_k d_k + z' = \sum_{l=1}^{q} c'_l d'_l + z'} \left(\begin{array}{l} \bigwedge_{i=1}^{m} \Big(\lambda(a_i) \wedge \mu(b_i)\Big) \wedge \bigwedge_{j=1}^{n} \Big(\lambda(a'_j) \wedge \mu(b'_j)\Big) \\ \wedge \bigwedge_{k=1}^{p} \Big(\lambda(c_k) \wedge \mu(d_k)\Big) \wedge \bigwedge_{l=1}^{q} \Big(\lambda(c'_l) \wedge \mu(d'_l)\Big) \end{array} \right) \right) \\
&= \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left(\bigwedge_{i=1}^{m} \big(\lambda(a_i) \wedge \mu(b_i)\big) \wedge \bigwedge_{j=1}^{n} \big(\lambda(a'_j) \wedge \mu(b'_j)\big) \right) \\
&\quad \wedge \sup_{y+\sum_{k=1}^{p} c_k d_k + z' = \sum_{l=1}^{q} c'_l d'_l + z'} \left(\bigwedge_{k=1}^{p} \big(\lambda(c_k) \wedge \mu(d_k)\big) \wedge \bigwedge_{l=1}^{q} \big(\lambda(c'_l) \wedge \mu(d'_l)\big) \right) \\
&= (\lambda \odot_h \mu)(x) \wedge (\lambda \odot_h \mu)(y).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\lambda \odot_h \mu)(xr) &= \sup_{xr+\sum_{k=1}^{p} g_k h_k + z = \sum_{l=1}^{q} g'_l h'_l + z} \left(\bigwedge_{k=1}^{p} \big(\lambda(g_k) \wedge \mu(h_k)\big) \wedge \bigwedge_{l=1}^{q} \big(\lambda(g'_l) \wedge \mu(h'_l)\big) \right) \\
&\geq \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left(\bigwedge_{i=1}^{m} \big(\lambda(a_i) \wedge \mu(b_i r)\big) \wedge \bigwedge_{j=1}^{n} \big(\lambda(a'_j) \wedge \mu(b'_j r)\big) \right) \\
&\geq \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left(\bigwedge_{i=1}^{m} \big(\lambda(a_i) \wedge \mu(b_i)\big) \wedge \bigwedge_{j=1}^{n} \big(\lambda(a'_j) \wedge \mu(b'_j)\big) \right) = (\lambda \odot_h \mu)(x).
\end{aligned}$$

Analogously we can verify that $(\lambda \odot_h \mu)(rx) \geq (\lambda \odot_h \mu)(x)$ for all $r \in R$. This means that $\lambda \odot_h \mu$ is a fuzzy ideal of R.

To prove that $x + a + y = b + y$ implies $(\lambda \odot_h \mu)(x) \geq (\lambda \odot_h \mu)(a) \wedge (\lambda \odot_h \mu)(b)$ observe that

$$a + \sum_{i=1}^{m} a_i b_i + z_1 = \sum_{j=1}^{n} a'_j b'_j + z_1 \quad \text{and} \quad b + \sum_{k=1}^{l} c_k d_k + z_2 = \sum_{q=1}^{p} c'_q d'_q + z_2, \tag{1}$$

together with $x+a+y = b+y$, gives $x+a+(\sum_{i=1}^{m} a_i b_i + z_1)+y = b+(\sum_{i=1}^{m} a_i b_i + z_1)+y$. Thus $x+\sum_{j=1}^{n} a'_j b'_j + z_1 + y = b+\sum_{i=1}^{m} a_i b_i + z_1 + y$ and, consequently, $x+\sum_{j=1}^{n} a'_j b'_j + (\sum_{k=1}^{l} c_k d_k + z_2) + z_1 + y = b + (\sum_{k=1}^{l} c_k d_k + z_2) + \sum_{i=1}^{m} a_i b_i + z_1 + y = \sum_{q=1}^{p} c'_q d'_q + z_2 + \sum_{i=1}^{m} a_i b_i + z_1 + y = \sum_{i=1}^{m} a_i b_i + \sum_{q=1}^{p} c'_q d'_q + z_2 + z_1 + y$. Therefore

$$x + \sum_{j=1}^{n} a'_j b'_j + \sum_{k=1}^{l} c_k d_k + z_2 + z_1 + y = \sum_{i=1}^{m} a_i b_i + \sum_{q=1}^{p} c'_q d'_q + z_2 + z_1 + y. \tag{2}$$

Now, in view of (1) and (2), we have

$$(\lambda \odot_h \mu)(a) \wedge (\lambda \odot_h \mu)(b) = \sup_{a+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left(\bigwedge_{i=1}^{m} \big(\lambda(a_i) \wedge \mu(b_i)\big) \wedge \bigwedge_{j=1}^{n} \big(\lambda(a'_j) \wedge \mu(b'_j)\big) \right)$$

$$\wedge \sup_{b+\sum_{k=1}^{p} c_k d_k + z' = \sum_{l=1}^{q} c'_l d'_l + z'} \left(\bigwedge_{k=1}^{p} \big(\lambda(c_k) \wedge \mu(d_k)\big) \wedge \bigwedge_{l=1}^{q} \big(\lambda(c'_l) \wedge \mu(d'_l)\big) \right)$$

$$= \sup_{a+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left(\sup_{b+\sum_{k=1}^{p} c_k d_k + z' = \sum_{l=1}^{q} c'_l d'_l + z'} \left(\begin{array}{l} \bigwedge_{i=1}^{m} \big(\lambda(a_i) \wedge \mu(b_i)\big) \wedge \bigwedge_{j=1}^{n} \big(\lambda(a'_j) \wedge \mu(b'_j)\big) \\ \wedge \bigwedge_{k=1}^{p} \big(\lambda(c_k) \wedge \mu(d_k)\big) \wedge \bigwedge_{l=1}^{q} \big(\lambda(c'_l) \wedge \mu(d'_l)\big) \end{array} \right) \right)$$

$$\leq \sup_{x+\sum_{s=1}^{u} g_s h_s + z = \sum_{t=1}^{w} g'_t h'_t + z} \left(\bigwedge_{s=1}^{u} \big(\lambda(g_s) \wedge \mu(h_s)\big) \wedge \bigwedge_{t=1}^{w} \big(\lambda(g'_t) \wedge \mu(h'_t)\big) \right) = (\lambda \odot_h \mu)(x).$$

Thus $(\lambda \odot_h \mu)(a) \wedge (\lambda \odot_h \mu)(b) \leq (\lambda \odot_h \mu)(x)$. This completes the proof that $(\lambda \odot_h \mu)$ is a fuzzy h-ideal of R.
By simple calculations we can prove that $\lambda \odot_h \mu \leq \lambda \wedge \mu$. □

For h-hemiregular hemirings we have stronger result. Namely, as it is proved in [29], the following theorem is valid.

Theorem 3.4. *A hemiring R is h-hemiregular if and only if for any fuzzy right h-ideal λ and any fuzzy left h-ideal μ of R we have $\lambda \odot_h \mu = \lambda \wedge \mu$.*

Comparing this theorem with Theorem 2.11 we obtain

Corollary 3.5. *$\lambda \odot_h \mu = \lambda \circ \mu$ for all fuzzy h-ideals of any h-hemiregular hemiring.*

4. Idempotent *h*-ideals

The concept of h-hemiregularity of a hemiring was introduced in [18] as a generalization of the concept of regularity of a ring. From results proved in [18] (see our Lemma 2.5) it follows that in h-hemiregular hemirings every h-ideal A is *h-idempotent*, that is $\overline{AA} = A$. On the other hand, Theorem 3.4 implies that in such hemirings we have $\lambda \odot_h \lambda = \lambda$ for all fuzzy h-ideals. Fuzzy h-ideals with this property will be called *idempotent*.

Proposition 4.1. *The following statements are equivalent:*

(1) *Each h-ideal of R is h-idempotent.*
(2) *$A \cap B = \overline{AB}$ for each pair of h-ideals of R.*
(3) *$x \in \overline{RxRxR}$ for every $x \in R$.*
(4) *$A \subseteq \overline{RARAR}$ for every non-empty $A \subseteq R$.*
(5) *$A = \overline{RARAR}$ for every h-ideal A of R.*

Proof. Indeed, by Lemma 2.3, $\overline{AB} \subseteq A \cap B$ for all h-ideals of R. Since $A \cap B$ is an h-ideal of R, (1) implies $A \cap B = \overline{(A \cap B)(A \cap B)} \subseteq \overline{AB}$. Thus $A \cap B = \overline{AB}$. So, (1) implies (2). The converse implication is obvious.
It is clear that the smallest h-ideal of R containing $x \in R$ has the form

$$\langle x \rangle = \overline{\langle x \rangle} = \overline{Rx + xR + RxR + Sx},$$

where Sx is a finite sum of x's. If (1) holds, then $\overline{\langle x \rangle} = \overline{\overline{\langle x \rangle}\,\overline{\langle x \rangle}} = \overline{\langle x \rangle \langle x \rangle}$. Consequently,

$$\begin{aligned} x &= 0 + x \in \overline{Rx + xR + RxR + Sx} \\ &= \overline{(Rx + xR + RxR + Sx)(Rx + xR + RxR + Sx)} \subseteq \overline{RxRRxR} \subseteq \overline{RxRxR} \end{aligned}$$

for every $x \in R$. So, (1) implies (3). Clearly (3) implies (4). If (4) holds, then for every h-ideal of R we have $\overline{A} = A \subseteq \overline{RARAR} \subseteq \overline{AA} \subseteq \overline{A} = A$, which proves (5). The implication (5) → (1) is obvious. □

As a consequence of the above result and Lemma 2.5 we obtain the following characterization of h-hemiregularity of commutative hemirings.

Corollary 4.2. *A commutative hemiring is h-hemiregular if and only if all its h-ideals are h-idempotent.*

Proposition 4.3. *The following statements are equivalent:*

(1) *Each fuzzy h-ideal of R is idempotent.*
(2) *$\lambda \odot_h \mu = \lambda \wedge \mu$ for all fuzzy h-ideals of R.*

Proof. Let λ and μ be fuzzy h-ideals of R. Since $\lambda\wedge\mu$ is a fuzzy h-ideal of R such that $\lambda\wedge\mu\le\lambda$ and $\lambda\wedge\mu\le\mu$, Proposition 3.2 implies $(\lambda\wedge\mu)\odot_h(\lambda\wedge\mu)\le\lambda\odot_h\mu$. So, if $\lambda\wedge\mu$ is an idempotent fuzzy h-ideal, then $\lambda\wedge\mu\le\lambda\odot_h\mu$, which together with Theorem 3.3 gives $\lambda\odot_h\mu=\lambda\wedge\mu$. This means that (1) implies (2). The converse implication is obvious. □

Comparing this proposition with Theorem 3.4 we obtain

Corollary 4.4. *A commutative hemiring is h-hemiregular if and only if all its fuzzy h-ideals are idempotent, or equivalently, if and only if $\lambda\odot_h\mu=\lambda\wedge\mu$ holds for all its fuzzy h-ideals.*

Theorem 4.5. *For hemirings with the identity the following statements are equivalent:*

(1) *Each h-ideal of R is h-idempotent.*
(2) *$A\cap B=\overline{AB}$ for each pair of h-ideals of R.*
(3) *Each fuzzy h-ideal of R is idempotent.*
(4) *$\lambda\odot_h\mu=\lambda\wedge\mu$ for all fuzzy h-ideals of R.*

Proof. (1) and (2) are equivalent by Proposition 4.1, (3) and (4) by Proposition 4.3. To prove that (1) and (3) are equivalent observe that the smallest h-ideal containing $x\in R$ has the form RxR. Its closure $\overline{RxR}$ also is an h-ideal. Since, by (1), all h-ideals of R are h-idempotent, we have $\overline{RxR}=\overline{(\overline{RxR})(\overline{RxR})}=\overline{RxRRxR}$ (Lemma 2.2). Thus $x\in\overline{RxR}=\overline{RxRRxR}$ implies

$$x+\sum_{i=1}^{m}r_ixs_iu_ixt_i+z=\sum_{j=1}^{n}r_j'xs_j'u_j'xt_j'+z.$$

But, by Theorem 3.3, for every fuzzy h-ideal of R we have $\lambda\odot_h\lambda\le\lambda$. Hence $\lambda(x)=\lambda(x)\wedge\lambda(x)\le\bigwedge_{i=1}^{m}\Big(\lambda(r_ixs_i)\wedge\lambda(u_ixt_i)\Big)$. Also $\lambda(x)=\lambda(x)\wedge\lambda(x)\le\bigwedge_{j=1}^{n}\big(\lambda(r_j'xs_j')\wedge\lambda(u_j'xt_j')\big)$. Therefore

$$\begin{aligned}\lambda(x)&\le\bigwedge_{i=1}^{m}\Big(\lambda(r_ixs_i)\wedge\lambda(u_ixt_i)\Big)\wedge\bigwedge_{j=1}^{n}\big(\lambda(r_j'xs_j')\wedge\lambda(u_j'xt_j')\big)=M(x,r_i,s_i,r_j',s_j')\\&\le\sup_{x+\sum_{i=1}^{m}r_ixs_iu_ixt_i+z=\sum_{j=1}^{n}r_j'xs_j'u_j't_j'+z}M(x,r_i,s_i,r_j',s_j')=(\lambda\odot_h\lambda)(x).\end{aligned}$$

Hence $\lambda\le\lambda\odot_h\lambda$, which proves $\lambda\odot_h\lambda=\lambda$. So, (1) implies (3).

Conversely, according to Proposition 2.8, the characteristic function χ_A of any h-ideal A of R is a fuzzy h-ideal of R. If it is idempotent, then $\chi_A=\chi_A\odot_h\chi_A=\chi_{\overline{AA}}$ (Proposition 3.2). Thus $A=\overline{AA}$. (3) implies (1). □

Definition 4.6. The *h-sum* $\lambda+_h\mu$ of fuzzy subsets λ and μ of R is defined by

$$(\lambda+_h\mu)(x)=\sup_{x+(a_1+b_1)+z=(a_2+b_2)+z}\Big(\lambda(a_1)\wedge\lambda(a_2)\wedge\mu(b_1)\wedge\mu(b_2)\Big),$$

where $x,a_1,b_1,a_2,b_2,z\in R$.

Theorem 4.7. *The h-sum of fuzzy h-ideals of R also is a fuzzy h-ideal of R.*

Proof. Let λ,μ be fuzzy h-ideals of R. Then for $x,y\in R$ we have

$$\begin{aligned}(\lambda+_h\mu)(x)\wedge(\lambda+_h\mu)(y)&=\sup_{x+(a_1+b_1)+z=(a_2+b_2)+z}\Big(\lambda(a_1)\wedge\lambda(a_2)\wedge\mu(b_1)\wedge\mu(b_2)\Big)\\&\quad\wedge\sup_{y+(a_1'+b_1')+z'=(a_2'+b_2')+z'}\Big(\lambda(a_1')\wedge\lambda(a_2')\wedge\mu(b_1')\wedge\mu(b_2')\Big)\\&=\sup_{\substack{x+(a_1+b_1)+z=(a_2+b_2)+z\\y+(a_1'+b_1')+z'=(a_2'+b_2')+z'}}\begin{pmatrix}\lambda(a_1)\wedge\lambda(a_2)\wedge\mu(b_1)\wedge\mu(b_2)\\\wedge\lambda(a_1')\wedge\lambda(a_2')\wedge\mu(b_1')\wedge\mu(b_2')\end{pmatrix}\\&\le\sup_{\substack{x+(a_1+b_1)+z=(a_2+b_2)+z\\y+(a_1'+b_1')+z'=(a_2'+b_2')+z'}}\begin{pmatrix}\lambda(a_1+a_1')\wedge\lambda(a_2+a_2')\\\wedge\mu(b_1+b_1')\wedge\mu(b_2+b_2')\end{pmatrix}\\&\le\sup_{(x+y)+(c_1+d_1)+z''=(c_2+d_2)+z''}\Big(\lambda(c_1)\wedge\lambda(c_2)\wedge\mu(d_1)\wedge\mu(d_2)\Big)\\&=(\lambda+_h\mu)(x+y).\end{aligned}$$

Similarly,

$$\begin{aligned}(\lambda +_h \mu)(x) &= \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z}\Big(\lambda(a_1)\wedge\lambda(a_2)\wedge\mu(b_1)\wedge\mu(b_2)\Big)\\ &\le \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z}\Big(\lambda(ra_1)\wedge\lambda(ra_2)\wedge\mu(rb_1)\wedge\mu(rb_2)\Big)\\ &\le \sup_{rx+(a_1''+b_1'')+z''=(a_2''+b_2'')+z''}\Big(\lambda(a_1'')\wedge\lambda(a_2'')\wedge\mu(b_1'')\wedge\mu(b_2'')\Big)\\ &= (\lambda +_h \mu)(rx).\end{aligned}$$

Analogously $(\lambda +_h \mu)(x) \le (\lambda +_h \mu)(xr)$. This proves that $(\lambda +_h \mu)$ is a fuzzy ideal of R.

Now we show that $x+a+z = b+z$ implies $(\lambda +_h \mu)(x) \ge (\lambda +_h \mu)(a)\wedge(\lambda +_h \mu)(b)$. For this let $a+(a_1+b_1)+z_1 = (a_2+b_2)+z_1$ and $b+(c_1+d_1)+z_2 = (c_2+d_2)+z_2$. Then,

$$a+(c_2+d_2+z_2)+(a_1+b_1+z_1) = (a_2+b_2+z_1)+(b+c_1+d_1+z_2),$$

whence

$$a+(a_1+c_2)+(b_1+d_2)+(z_1+z_2) = b+(a_2+c_1)+(b_2+d_1)+(z_1+z_2)\,.$$

Consequently

$$a+(a_1+c_2)+(b_1+d_2)+(z_1+z_2+z) = b+z+(a_2+c_1)+(b_2+d_1)+(z_1+z_2)$$

and

$$a+(a_1+c_2)+(b_1+d_2)+(z_1+z_2+z) = x+a+z+(a_2+c_1)+(b_2+d_1)+(z_1+z_2)\,.$$

Thus

$$x+(a_2+c_1)+(b_2+d_1)+(z_1+z_2+z+a) = (a_1+c_2)+(b_1+d_2)+(z_1+z_2+z+a)\,,$$

i.e., $x+(a'+b')+z' = (a''+b'')+z'$ for some $a', b', a'', b'' \in R$.

Therefore

$$\begin{aligned}(\lambda +_h \mu)(a)\wedge(\lambda +_h \mu)(b) &= \sup_{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1}\Big(\lambda(a_1)\wedge\lambda(a_2)\wedge\mu(b_1)\wedge\mu(b_2)\Big)\\ &\quad\wedge \sup_{b+(c_1+d_1)+z_2=(c_2+d_2)+z_2}\Big(\lambda(c_1)\wedge\lambda(c_2)\wedge\mu(d_1)\wedge\mu(d_2)\Big)\\ &= \sup_{\substack{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1\\ b+(c_1+d_1)+z_2=(c_2+d_2)+z_2}}\begin{pmatrix}\lambda(a_1)\wedge\lambda(a_2)\wedge\mu(b_1)\wedge\mu(b_2)\\ \wedge\lambda(c_1)\wedge\lambda(c_2)\wedge\mu(d_1)\wedge\mu(d_2)\end{pmatrix}\\ &\le \sup_{\substack{a+(a_1+b_1)+z_1=(a_2+b_2)+z_1\\ b+(c_1+d_1)+z_2=(c_2+d_2)+z_2}}\begin{pmatrix}\lambda(a_1+c_2)\wedge\lambda(a_2+c_1)\\ \wedge\mu(b_1+d_2)\wedge\mu(b_2+d_1)\end{pmatrix}\\ &\le \sup_{x+(a'+b')+z'=(a''+b'')+z'}\Big(\lambda(a')\wedge\lambda(a'')\wedge\mu(b')\wedge\mu(b'')\Big)\\ &= (\lambda +_h \mu)(x).\end{aligned}$$

Thus $\lambda +_h \mu$ is a fuzzy h-ideal of R. □

Theorem 4.8. *If all h-ideals of R are h-idempotent, then the collection of these h-ideals forms a complete Brouwerian lattice.*

Proof. The collection $\mathcal{L}_R$ of all h-ideals of R is a poset under the inclusion of sets. It is not difficult to see that $\mathcal{L}_R$ is a complete lattice under operations $\sqcup, \sqcap$ defined as $A\sqcup B = \overline{A+B}$ and $A\sqcap B = A\cap B$.

We show that $\mathcal{L}_R$ is a Brouwerian lattice, that is, for any $A, B \in \mathcal{L}_R$, the set $\mathcal{L}_R(A,B) = \{I \in \mathcal{L}_R \mid A\cap I \subseteq B\}$ contains a greatest element.

By Zorn's Lemma the set $\mathcal{L}_R(A,B)$ contains a maximal element M. Since each h-ideal of R is h-idempotent, $\overline{AI} = A\cap I \subseteq B$ and $\overline{AM} = A\cap M \subseteq B$ (Proposition 4.1). Thus $\overline{AI}+\overline{AM} \subseteq B$. Consequently, $\overline{\overline{AI}+\overline{AM}} \subseteq \overline{B} = B$.

Since $\overline{I+M} = I\sqcup M \in \mathcal{L}_R$, for every $x \in \overline{I+M}$ there exist $i_1, i_2 \in I$, $m_1, m_2 \in M$ and $z \in R$ such that $x+i_1+m_1+z = i_2+m_2+z$. Thus

$$dx+di_1+dm_1+dz = di_2+dm_2+dz$$

for any $d \in D \in \mathcal{L}_R$. As $di_1, di_2 \in DI$, $dm_1, dm_2 \in DM$, $dz \in R$, we have $dx \in \overline{DI+DM}$, which implies $D\left(\overline{I+M}\right) \subseteq \overline{DI+DM} \subseteq \overline{\overline{DI}+\overline{DM}} \subseteq B$. Hence $D\left(\overline{I+M}\right) \subseteq B$. This means that $D \cap \left(\overline{I+M}\right) = D\left(\overline{I+M}\right) \subseteq B$, i.e., $\overline{I+M} \in \mathcal{L}_R(A,B)$, whence $\overline{I+M} = M$ because M is maximal in $\mathcal{L}_R(A,B)$. Therefore $I \subseteq \bar{I} \subseteq \overline{I+M} = M$ for every $I \in \mathcal{L}_R(A,B)$. □

Corollary 4.9. *If all h-ideals of R are idempotent, then the lattice $\mathcal{L}_R$ is distributive.*

Proof. Each complete Brouwerian lattice is distributive (cf. [30], 11.11). □

Theorem 4.10. *Each fuzzy h-ideal of R is h-idempotent if and only if the set of all fuzzy h-ideals of R (ordered by $\leq$) forms a distributive lattice under the h-sum and h-intrinsic product of fuzzy h-ideals with $\lambda \odot_h \mu = \lambda \wedge \mu$.*

Proof. Assume that all fuzzy h-ideals of R are idempotent. Then $\lambda \odot_h \mu = \lambda \wedge \mu$ (Proposition 4.3) and, as it is not difficult to see, the set $\mathcal{F}\mathcal{L}_R$ of all fuzzy h-ideals of R (ordered by $\leq$) is a lattice under the h-sum and h-intrinsic product of fuzzy h-ideals.

We show that $(\lambda \odot_h \delta) +_h \mu = (\lambda +_h \mu) \odot_h (\delta +_h \mu)$ for all $\lambda, \mu, \delta \in \mathcal{F}\mathcal{L}_R$. Indeed, for any $x \in R$ we have

$$\begin{aligned}
((\lambda \odot_h \delta) +_h \mu)(x) &= \Big((\lambda \odot_h \delta) +_h \mu\Big)(x) \\
&= \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \Big((\lambda \wedge \delta)(a_1) \wedge (\lambda \wedge \delta)(a_2) \wedge \mu(b_1) \wedge \mu(b_2)\Big) \\
&= \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \Big(\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2) \wedge \delta(a_1) \wedge \delta(a_2)\Big) \\
&= \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \Big(\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)\Big) \\
&\quad \wedge \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \Big(\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)\Big) \\
&= (\lambda +_h \mu)(x) \wedge (\delta +_h \mu)(x) = \Big((\lambda +_h \mu) \wedge (\delta +_h \mu)\Big)(x) \\
&= \Big((\lambda +_h \mu) \odot_h (\delta +_h \mu)\Big)(x).
\end{aligned}$$

So, $\mathcal{F}\mathcal{L}_R$ is a distributive lattice.

The converse statement is a consequence of Proposition 4.3. □

5. Prime ideals

An h-ideal P of R is called *prime* if $P \neq R$ and for any h-ideals A, B of R from $AB \subseteq P$ it follows $A \subseteq P$ or $B \subseteq P$, and *irreducible* if $P \neq R$ and $A \cap B = P$ implies $A = P$ or $B = P$. By analogy a non-constant fuzzy h-ideal δ of R is called *prime* (in the first sense) if for any fuzzy h-ideals λ, μ of R from $\lambda \odot_h \mu \leq \delta$ it follows $\lambda \leq \delta$ or $\mu \leq \delta$, and *irreducible* if $\lambda \wedge \mu = \delta$ implies $\lambda = \delta$ or $\mu = \delta$.

Theorem 5.1. *A left (right) h-ideal P of R is prime if and only if for all $a, b \in R$ from $aRb \subseteq P$ it follows $a \in P$ or $b \in P$.*

Proof. Assume that P is a prime left h-ideal of R and $aRb \subseteq P$ for some $a, b \in R$. Obviously, $A = \overline{Ra}$ and $B = \overline{Rb}$ are left h-ideals of R. So, $AB \subseteq \overline{AB} = \overline{\overline{RaRb}} = \overline{RaRb} \subseteq \overline{RP} \subseteq P$, and consequently $A \subseteq P$ or $B \subseteq P$. Let $\langle x \rangle$ be a left h-ideal generated by $x \in R$. If $A \subseteq P$, then $\langle a \rangle \subseteq \overline{Ra} = A \subseteq P$, whence $a \in P$. If $B \subseteq P$, then $\langle b \rangle \subseteq \overline{Rb} = B \subseteq P$, whence $b \in P$.

The converse is obvious. □

Corollary 5.2. *An h-ideal P of R is prime if and only if for all $a, b \in R$ from $aRb \subseteq P$ it follows $a \in P$ or $b \in P$.*

Corollary 5.3. *An h-ideal P of a commutative hemiring R with identity is prime if and only if for all $a, b \in R$ from $ab \in P$ it follows $a \in P$ or $b \in P$.*

The result expressed by Corollary 5.2 suggests the following definition of prime fuzzy h-ideals.

Definition 5.4. A non-constant fuzzy h-ideal δ of R is called *prime* (in the second sense) if for all $t \in [0, 1]$ and $a, b \in R$ the following condition is satisfied:

if $\delta(axb) \geq t$ for every $x \in R$ then $\delta(a) \geq t$ or $\delta(b) \geq t$.

In other words, a non-constant fuzzy h-ideal δ is prime if from the fact that $axb \in U(\delta; t)$ for every $x \in R$ it follows $a \in U(\delta; t)$ or $b \in U(\delta; t)$. It is clear that any fuzzy h-ideal prime in the first sense is prime in the second sense. The converse is not true.

Example 5.5. In an ordinary hemiring of natural numbers the set of even numbers forms an h-ideal. A fuzzy set

$$\delta(n)=\begin{cases}1 & \text{if } n=0,\\ 0.8 & \text{if } n=2k\neq 0,\\ 0.4 & \text{if } n=2k+1\end{cases}$$

is a fuzzy h-ideal of this hemiring. It is prime in the second sense but it is not prime in the first sense.

Theorem 5.6. *A non-constant fuzzy h-ideal δ of R is prime in the second sense if and only if each its proper level set $U(\delta;t)$ is a prime h-ideal of R.*

Proof. Let a fuzzy h-ideal δ of R be prime in the second sense and let $U(\delta;t)$ be its arbitrary proper level set, i.e., $\emptyset\neq U(\delta;t)\neq R$. If $aRb\subseteq U(\delta;t)$, then $\delta(axb)\geq t$ for every $x\in R$. Hence $\delta(a)\geq t$ or $\delta(b)\geq t$, i.e., $a\in U(\delta;t)$ or $b\in U(\delta;t)$, which, by Corollary 5.2, means that $U(\delta;t)$ is a prime h-ideal of R.

To prove the converse consider a non-constant fuzzy h-ideal δ of R. If it is not prime then there exists $a,b\in R$ such that $\delta(axb)\geq t$ for all $x\in R$, but $\delta(a)<t$ and $\delta(b)<t$. Thus, $aRb\subseteq U(\delta;t)$, but $a\notin U(\delta;t)$ and $b\notin U(\delta;t)$. Therefore $U(\delta;t)$ is not prime. Obtained contradiction proves that δ should be prime. □

Corollary 5.7. *A fuzzy set λ_A defined in Proposition 2.8 is a prime fuzzy h-ideal of R if and only if A is a prime h-ideal of R.*

In view of the Transfer Principle (Lemma 2.7) the second definition of prime fuzzy h-ideals is better. Therefore fuzzy h-ideals which are prime in the first sense will be called *h-prime*.

Proposition 5.8. *A non-constant fuzzy h-ideal δ of a commutative hemiring R with identity is prime if and only if $\delta(ab)=\delta(a)\vee\delta(b)$ for all $a,b\in R$.*

Proof. Let δ be a non-constant fuzzy h-ideal of a commutative hemiring R with identity. If $\delta(ab)=t$, then, for every $x\in R$, we have $\delta(axb)=\delta(xab)\geq\delta(x)\vee\delta(ab)\geq t$. Thus $\delta(axb)\geq t$ for every $x\in R$, which implies $\delta(a)\geq t$ or $\delta(b)\geq t$. If $\delta(a)\geq t$, then $t=\delta(ab)\geq\delta(a)\geq t$, whence $\delta(ab)=\delta(a)$. If $\delta(b)\geq t$, then, as in the previous case, $\delta(ab)=\delta(b)$. So, $\delta(ab)=\delta(a)\vee\delta(b)$.

Conversely, assume that $\delta(ab)=\delta(a)\vee\delta(b)$ for all $a,b\in R$. If $\delta(axb)\geq t$ for every $x\in R$, then, replacing in this inequality x by the identity of R, we obtain $\delta(ab)\geq t$. Thus $\delta(a)\vee\delta(b)\geq t$, i.e., $\delta(a)\geq t$ or $\delta(b)\geq t$, which means that a fuzzy h-ideal δ is prime. □

Theorem 5.9. *Every proper h-ideal is contained in some proper irreducible h-ideal.*

Proof. Let P be a proper h-ideal of R and let $\{P_\alpha \mid \alpha\in\Lambda\}$ be a family of all proper h-ideals of R containing P. By Zorn's Lemma, for any fixed $a\notin P$, the family of h-ideals P_α such that $P\subseteq P_\alpha$ and $a\notin P_\alpha$ contains a maximal element M. This maximal element is an irreducible h-ideal. Indeed, let $M=P_\beta\cap P_\delta$ for some h-ideals of R. If M is a proper subset of P_β and P_δ, then, according to the maximality of M, we have $a\in P_\beta$ and $a\in P_\delta$. Hence $a\in P_\beta\cap P_\delta=M$, which is impossible. Thus, either $M=P_\beta$ or $M=P_\delta$. □

Theorem 5.10. *If all h-ideals of R are h-idempotent, then an h-ideal P of R is irreducible if and only if it is prime.*

Proof. Assume that all h-ideals of R are h-idempotent. Let P be a fixed irreducible h-ideal. If $AB\subseteq P$ for some h-ideals A, B, then $A\cap B=\overline{AB}\subseteq\overline{P}=P$, by Proposition 4.1. Thus $\overline{(A\cap B)+P}=P$. Since $\mathcal{L}_R$ is a distributive lattice, $P=\overline{(A\cap B)+P}=\overline{(A+P)}\cap\overline{(B+P)}$. So either $\overline{A+P}=P$ or $\overline{B+P}=P$, that is, either $A\subseteq P$ or $B\subseteq P$.

Conversely, if an h-ideal P is prime and $A\cap B=P$ for some $A,B\in\mathcal{L}_R$, then $AB\subseteq\overline{AB}=A\cap B=P$. Thus $A\subseteq P$ or $B\subseteq P$. But $P\subseteq A$ and $P\subseteq B$. Hence $A=P$ or $B=P$. □

Corollary 5.11. *In hemirings in which all h-ideals are h-idempotent each proper h-ideal is contained in some proper prime h-ideal.*

Theorem 5.12. *In hemirings in which all fuzzy h-ideals are idempotent a fuzzy h-ideal is irreducible if and only if it is h-prime.*

Proof. Let all fuzzy h-ideals of R will be idempotent and let δ be an arbitrary irreducible fuzzy h-ideal of R. We prove that it is prime. If $\lambda\odot_h\mu\leq\delta$ for some fuzzy h-ideals, then also $\lambda\wedge\mu\leq\delta$. Since the set $\mathcal{FL}_R$ of all fuzzy h-ideals of R is a distributive lattice (Theorem 4.10) we have $\delta=(\lambda\wedge\mu)+_h\delta=(\lambda+_h\delta)\wedge(\mu+_h\delta)$. Thus $\lambda+_h\delta=\delta$ or $\mu+_h\delta=\delta$. But $\leq$ is a lattice order, so $\lambda\leq\delta$ or $\mu\leq\delta$. This proves that a fuzzy h-ideal δ is h-prime.

Conversely, if δ is an h-prime fuzzy h-ideal of R and $\lambda\wedge\mu=\delta$ for some $\lambda,\mu\in\mathcal{FL}_R$, then $\lambda\odot_h\mu=\delta$, which implies $\lambda\leq\delta$ or $\mu\leq\delta$. Since $\leq$ is a lattice order and $\delta=\lambda\wedge\mu$ we have also $\delta\leq\lambda$ and $\delta\leq\mu$. Thus $\lambda=\delta$ or $\mu=\delta$. So, δ is irreducible. □

Theorem 5.13. *The following assertions for a hemiring R are equivalent:*

(1) *Each h-ideal of R is h-idempotent.*
(2) *Each proper h-ideal P of R is the intersection of all prime h-ideals containing P.*

Proof. Let P be a proper h-ideal of R and let $\{P_\alpha \mid \alpha \in \Lambda\}$ be the family of all prime h-ideals of R containing P. Clearly $P \subseteq \cap_{\alpha\in\Lambda} P_\alpha$. By Zorn's Lemma, for any fixed $a \notin P$, the family of h-ideals P_α such that $P \subseteq P_\alpha$ and $a \notin P_\alpha$ contains a maximal element M_a. We will show that this maximal element is an irreducible h-ideal. Let $M_a = K \cap L$. If M_a is a proper subset of K and L, then, according to the maximality of M_a, we have $a \in K$ and $a \in L$. Hence $a \in K \cap L = M_a$, which is impossible. Thus, either $M_a = K$ or $M_a = L$. By Theorem 5.10, M_a is a prime h-ideal. So there exists a prime h-ideal M_a such that $a \notin M_a$ and $P \subseteq M_a$. Hence $\cap P_\alpha \subseteq P$. Thus $P = \cap P_\alpha$.

Assume that each h-ideal of R is the intersection of all prime h-ideals of R which contain it. Let A be an h-ideal of R. If $\overline{A^2} = R$, then, by Lemma 2.3, we have $A = R$, which means such h-ideal is h-idempotent. If $\overline{A^2} \neq R$, then $\overline{A^2}$ is a proper h-ideal of R and so it is the intersection of all prime h-ideals of R containing A. Let $\overline{A^2} = \cap P_\alpha$. Then $A^2 \subseteq P_\alpha$ for each α. Since P_α is prime, we have $A \subseteq P_\alpha$. Thus $A \subseteq \cap P_\alpha = \overline{A^2}$. But $\overline{A^2} \subseteq A$ for every h-ideal. Hence $A = \overline{A^2}$. □

Lemma 5.14. *Let R be a hemiring in which each fuzzy h-ideal is idempotent. If λ is a fuzzy h-ideal of R with $\lambda(a) = \alpha$, where a is any element of R and $\alpha \in [0, 1]$, then there exists an irreducible and h-prime fuzzy h-ideal δ of R such that $\lambda \le \delta$ and $\delta(a) = \alpha$.*

Proof. Let λ be an arbitrary fuzzy h-ideal of R and let $a \in R$ be fixed. Consider the following collection of fuzzy h-ideals of R

$$\mathcal{B} = \{\mu \mid \mu(a) = \lambda(a), \lambda \le \mu\}.$$

$\mathcal{B}$ is non-empty since $\lambda \in \mathcal{B}$. Let $\mathcal{F}$ be a totally ordered subset of $\mathcal{B}$ containing λ, say $\mathcal{F} = \{\lambda_i \mid i \in I\}$. Obviously $\lambda_i \vee \lambda_j \in \mathcal{F}$ for any $\lambda_i, \lambda_j \in \mathcal{F}$. So, for example, $\big(\lambda_i(x) \vee \lambda_j(x)\big) \wedge \big(\lambda_i(y) \vee \lambda_j(y)\big) \le \lambda_i(x+y) \vee \lambda_j(x+y)$ for any $\lambda_i, \lambda_j \in \mathcal{F}$ and $x, y \in R$.

We claim that $\bigvee_{i\in I} \lambda_i$ is a fuzzy h-ideal of R.

For any $x, y \in R$, we have

$$\begin{aligned}
\Big(\bigvee_{i\in I} \lambda_i\Big)(x) \wedge \Big(\bigvee_{i\in I} \lambda_i\Big)(y) &= \Big(\bigvee_{i\in I} \lambda_i(x)\Big) \wedge \Big(\bigvee_{j\in I} \lambda_j(y)\Big) \\
&= \bigvee_{i,j\in I} \big(\lambda_i(x) \wedge \lambda_j(y)\big) \\
&\le \bigvee_{i,j\in I} \Big(\big(\lambda_i(x) \vee \lambda_j(x)\big) \wedge \big(\lambda_i(y) \vee \lambda_j(y)\big)\Big) \\
&\le \bigvee_{i,j\in I} \big(\lambda_i(x+y) \vee \lambda_j(x+y)\big) \\
&\le \bigvee_{i\in I} \lambda_i(x+y) = \Big(\bigvee_{i\in I} \lambda_i\Big)(x+y).
\end{aligned}$$

Similarly

$$\Big(\bigvee_{i\in I} \lambda_i\Big)(x) = \bigvee_{i\in I} \lambda_i(x) \le \bigvee_{i\in I} \lambda_i(xr) = \Big(\bigvee_{i\in I} \lambda_i\Big)(xr)$$

and

$$\Big(\bigvee_{i\in I} \lambda_i\Big)(x) \le \Big(\bigvee_{i\in I} \lambda_i\Big)(rx)$$

for all $x, r \in R$. Thus $\bigvee_{i\in I}$ is a fuzzy ideal.

Now, let $x + a + z = b + z$, where $a, b, z \in R$. Then

$$\begin{aligned}
\Big(\bigvee_{i\in I} \lambda_i\Big)(a) \wedge \Big(\bigvee_{i\in I} \lambda_i\Big)(b) &= \Big(\bigvee_{i\in I} \lambda_i(a)\Big) \wedge \Big(\bigvee_{j\in I} (\lambda_j(b))\Big) \\
&= \bigvee_{i,j\in I} \big(\lambda_i(a) \wedge \lambda_j(b)\big) \\
&\le \bigvee_{i,j\in I} \Big(\big(\lambda_i(a) \vee \lambda_j(a)\big) \wedge \big(\lambda_i(b) \vee \lambda_j(b)\big)\Big) \\
&\le \bigvee_{i,j} \big(\lambda_i(x) \vee \lambda_j(x)\big) \le \bigvee_{i\in i} \lambda_i(x) = \Big(\bigvee_{i\in I} \lambda_i\Big)(x).
\end{aligned}$$

This means that $\bigvee_{i\in I} \lambda_i$ is a fuzzy h-ideal of R. Clearly $\lambda \le \bigvee_{i\in I} \lambda_i$ and $(\bigvee_{i\in I} \lambda_i)(a) = \lambda(a) = \alpha$. Thus $\bigvee_{i\in I} \lambda_i$ is the least upper bound of $\mathcal{F}$. Hence by Zorn's lemma there exists a fuzzy h-ideal δ of R which is maximal with respect to the property that $\lambda \le \delta$ and $\delta(a) = \alpha$.

We will show that δ is an irreducible fuzzy h-ideal of R. Let $\delta = \delta_1 \wedge \delta_2$, where δ_1, δ_2 are fuzzy h-ideals of R. Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$ since $\mathcal{FL}_R$ is a lattice. We claim that either $\delta = \delta_1$ or $\delta = \delta_2$. Suppose $\delta \neq \delta_1$ and $\delta \neq \delta_2$. Since δ is maximal with respect to the property that $\delta(a) = \alpha$ and since $\delta \lneqq \delta_1$ and $\delta \lneqq \delta_2$, so $\delta_1(a) \neq \alpha$ and $\delta_2(a) \neq \alpha$. Hence $\alpha = \delta(a) = (\delta_1 \wedge \delta_2)(a) = \delta_1(a) \wedge \delta_2(a) \neq \alpha$, which is impossible. Hence $\delta = \delta_1$ or $\delta = \delta_2$. Thus δ is an irreducible fuzzy h-ideal of R. By Theorem 5.12, it is also prime. □

Theorem 5.15. *Each fuzzy h-ideal of R is idempotent if and only if each fuzzy h-ideal of R is the intersection of those h-prime fuzzy h-ideals of R which contain it.*

Proof. Suppose each fuzzy h-ideal of R is idempotent. Let λ be a fuzzy h-ideal of R and let $\{\lambda_\alpha \mid \alpha \in \Lambda\}$ be the family of all h-prime fuzzy h-ideals of R which contain λ. Obviously $\lambda \leq \bigwedge_{\alpha\in\Lambda} \lambda_\alpha$. We now show that $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \lambda$. Let a be an arbitrary element of R. Then, according to Lemma 5.14, there exists an irreducible and h-prime fuzzy h-ideal δ such that $\lambda \leq \delta$ and $\lambda(a) = \delta(a)$. Hence $\delta \in \{\lambda_\alpha \mid \alpha \in \Lambda\}$ and $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \delta$. So, $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha(a) \leq \delta(a) = \lambda(a)$. Thus $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha \leq \lambda$. Therefore $\bigwedge_{\alpha\in\Lambda} \lambda_\alpha = \lambda$.

Conversely, assume that each fuzzy h-ideal of R is the intersection of those h-prime fuzzy h-ideals of R which contain it. Let λ be a fuzzy h-ideal of R then $\lambda \odot \lambda$ is also fuzzy h-ideal of R, so $\lambda \odot \lambda = \bigwedge_{\alpha\in\Lambda} \lambda_\alpha$ where λ_α are h-prime fuzzy h-ideals of R. Thus each λ_α contains $\lambda \odot \lambda$, and hence λ. So $\lambda \subseteq \bigwedge_{\alpha\in\Lambda} \lambda_\alpha = \lambda \odot \lambda$, but $\lambda \odot \lambda \subseteq \lambda$ always. Hence $\lambda = \lambda \odot \lambda$. □

6. Semiprime ideals

Definition 6.1. An h-ideal A of R is called *semiprime* if $A \neq R$ and for any h-ideal B of R, $B^2 \subseteq A$ implies $B \subseteq A$. Similarly, a non-constant fuzzy h-ideal λ of R is called *semiprime* if for any fuzzy h-ideal δ of R, $\delta \odot_h \delta \leq \lambda$ implies $\delta \leq \lambda$.

Obviously, each semiprime h-ideal is prime. Each semiprime fuzzy h-ideal is h-prime. The converse is not true (see Example 6.7).

Using the same method as in the proof of Theorem 5.1 we can prove

Theorem 6.2. *A (left, right) h-ideal P of R is semiprime if and only if for every $a \in R$ from $aRa \subseteq P$ it follows $a \in P$.*

Corollary 6.3. *An h-ideal P of a commutative hemiring R with identity is semiprime if and only if for all $a \in R$ from $a^2 \in P$ it follows $a \in P$.*

Theorem 6.4. *The following assertions for a hemiring R are equivalent:*

(1) *Each h-ideal of R is h-idempotent.*
(2) *Each h-ideal of R is semiprime.*

Proof. Suppose that each h-ideal of R is idempotent. Let A, B be h-ideals of R such that $B^2 \subseteq A$. Thus $\overline{B^2} \subseteq \overline{A} = A$. By hypothesis $B = \overline{B^2}$, so $B \subseteq A$. Hence A is semiprime.

Conversely, assume that each h-ideal of R is semiprime. Let A be an h-ideal of R. Then $\overline{A^2}$ is also an h-ideal of R. Also $A^2 \subseteq \overline{A^2}$. Hence by hypothesis $A \subseteq \overline{A^2}$. But $\overline{A^2} \subseteq A$ always. Hence $A = \overline{A^2}$. □

Theorem 6.5. *Each fuzzy h-ideal of R is idempotent if and only if each fuzzy h-ideal of R is semiprime.*

Proof. For any h-ideal of R we have $\lambda \odot_h \lambda \leq \lambda$ (Theorem 3.3). If each h-ideal of R is semiprime, then $\lambda \odot_h \lambda \leq \lambda \odot_h \lambda$ implies $\lambda \leq \lambda \odot_h \lambda$. Hence $\lambda \odot_h \lambda = \lambda$.

The converse is obvious. □

Below we present two examples of hemirings in which all fuzzy h-ideals are semiprime.

Example 6.6. Consider the set $R = \{0, a, 1\}$ with the following two operations:

$+$	0	a	1
0	0	a	1
a	a	a	a
1	1	a	1

$\cdot$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

Then $(R, +, \cdot)$ is a commutative hemiring with identity. It has only one proper ideal $\{0, a\}$. This ideal is not an h-ideal. The only h-ideal of R is $\{0, a, 1\}$, which is clearly h-idempotent.

Since $0 = 0a = a0 = 01 = 10$, for any fuzzy ideal λ of this hemiring we have $\lambda(0) \geq \lambda(a)$ and $\lambda(0) \geq \lambda(1)$ and $\lambda(a) = \lambda(1a) \geq \lambda(1)$. Thus $\lambda(0) \geq \lambda(a) \geq \lambda(1)$. If λ is a fuzzy h-ideal, then $1 + 0 + 1 = 0 + 1$ implies $\lambda(1) \geq \lambda(0) \wedge \lambda(0) = \lambda(0)$, which proves that each fuzzy h-ideal of this hemiring is a constant function. So, $\lambda \odot_h \lambda = \lambda$ for each fuzzy h-ideal λ of R. This, by Theorem 6.5, means that each fuzzy h-ideal of R is semiprime.

Example 6.7. Now, consider the hemiring $R = \{0, a, b, c\}$ defined by the following tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	b	c	a
b	b	c	a	b
c	c	a	b	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	c
c	0	c	b	c

This hemiring has only one h-ideal $A = R$. Obviously this h-ideal is h-idempotent.

For any fuzzy ideal λ of R and any $x \in R$ we have $\lambda(0) \geq \lambda(x) \geq \lambda(a)$. Indeed, $\lambda(0) = \lambda(0x) \geq \lambda(x) = \lambda(xa) \geq \lambda(a)$. This together with $\lambda(a) = \lambda(b+b) \geq \lambda(b) \wedge \lambda(b) = \lambda(b)$ implies $\lambda(a) = \lambda(b)$. Consequently, $\lambda(c) = \lambda(a+b) \geq \lambda(a) \wedge \lambda(b) = \lambda(b)$. Therefore $\lambda(0) \geq \lambda(c) \geq \lambda(b) = \lambda(a)$. Moreover, if λ is a fuzzy h-ideal, then $c + 0 + a = 0 + a$, which implies $\lambda(c) \geq \lambda(0) \wedge \lambda(0) = \lambda(0)$. Thus $\lambda(0) = \lambda(c) \geq \lambda(b) = \lambda(a)$ for every fuzzy h-ideal of this hemiring.

Now we prove that each fuzzy h-ideal of R is idempotent. Since $\lambda \odot_h \lambda \leq \lambda$ always, so we have to show that $\lambda \odot_h \lambda \geq \lambda$. Obviously, for every $x \in R$ we have

$$\begin{aligned}(\lambda \odot_h \lambda)(x) &= \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left(\bigwedge_{i=1}^{m} \big(\lambda(a_i) \wedge \lambda(b_i)\big) \wedge \bigwedge_{j=1}^{n} \big(\lambda(a'_j) \wedge \lambda(b'_j)\big) \right) \\ &\geq \sup_{x+cd+z=c'd'+z} \big(\lambda(c) \wedge \lambda(d) \wedge \lambda(c') \wedge \lambda(d')\big) = \lambda(c) \wedge \lambda(d) \wedge \lambda(c') \wedge \lambda(d').\end{aligned}$$

So, $x + cd + z = c'd' + z$ implies $(\lambda \odot_h \lambda)(x) \geq \lambda(c) \wedge \lambda(d) \wedge \lambda(c') \wedge \lambda(d')$. Hence $0 + 00 + z = 00 + z$ implies $(\lambda \odot_h \lambda)(0) \geq \lambda(0)$. Similarly $a+bb+z = bc+z$ implies $(\lambda \odot_h \lambda)(a) \geq \lambda(b) \wedge \lambda(c) = \lambda(b) = \lambda(a)$, $b+aa+z = bc+z$ implies $(\lambda \odot_h \lambda)(b) \geq \lambda(a) \wedge \lambda(b) \wedge \lambda(c) = \lambda(b)$. Analogously, from $c+00+z = cc+z$ it follows $(\lambda \circ_h \lambda)(c) \geq \lambda(0) \wedge \lambda(c) = \lambda(c)$. This proves that $(\lambda \odot_h \lambda)(x) \geq \lambda(x)$ for every $x \in R$. Therefore $\lambda \odot_h \lambda = \lambda$ for every fuzzy h-ideal of R, which, by Theorem 6.5, means that each fuzzy h-ideal of R is semiprime.

Consider the following three fuzzy sets:

$$\begin{aligned}\lambda(0) = \lambda(c) &= 0.8, \quad \lambda(a) = \lambda(b) = 0.4,\\ \mu(0) = \mu(c) &= 0.6, \quad \mu(a) = \mu(b) = 0.5,\\ \delta(0) = \delta(c) &= 0.7, \quad \delta(a) = \delta(b) = 0.45.\end{aligned}$$

These three fuzzy sets are idempotent fuzzy h-ideals. Since all fuzzy h-ideal of this hemiring are idempotent, by Proposition 4.3, we have $\lambda \odot_h \mu = \lambda \wedge \mu$. Thus $(\lambda \odot_h \mu)(0) = (\lambda \odot_h \mu)(c) = 0.6$ and $(\lambda \odot_h \mu)(a) = (\lambda \odot_h \mu)(b) = 0.4$. So, $\lambda \odot_h \mu \leq \delta$ but neither $\lambda \leq \delta$ nor $\mu \leq \delta$, that is δ is not an h-prime fuzzy h-ideal.

Theorem 6.2 suggests the following definition of semiprime fuzzy h-ideals.

Definition 6.8. A non-constant fuzzy h-ideal δ of R is called *semiprime* (in the second sense) if for all $t \in [0, 1]$ and $a, b \in R$ the following condition is satisfied:

if $\delta(axb) \geq t$ for every $x \in R$ then $\delta(a) \geq t$ or $\delta(b) \geq t$.

In other words, a non-constant fuzzy h-ideal δ is semiprime if from the fact that $axb \in U(\delta; t)$ for every $x \in R$ it follows $a \in U(\delta; t)$ or $b \in U(\delta; t)$. It is clear that any fuzzy h-ideal semiprime in the first sense is semiprime in the second sense. The converse is not true (see Example 5.5).

Theorem 6.9. *A non-constant fuzzy h-ideal δ of R is semiprime in the second sense if and only if each its proper level set $U(\delta; t)$ is a semiprime h-ideal of R.*

Proof. The proof is analogous to the proof of Theorem 5.6. □

Corollary 6.10. *A fuzzy set λ_A defined in Proposition 2.8 is a semiprime fuzzy h-ideal of R if and only if A is a semiprime h-ideal of R.*

In view of the Transfer Principle (Lemma 2.7) the second definition of semiprime fuzzy h-ideals is better. Therefore fuzzy h-ideals which are prime in the first sense should be called *h-semiprime*.

Proposition 6.11. *A non-constant fuzzy h-ideal δ of a commutative hemiring R with identity is semiprime if and only if $\delta(a^2) = \delta(a)$ for every $a \in R$.*

Proof. The proof is similar to the proof of Proposition 5.8. □

7. Conclusion

In the study of fuzzy algebraic system, the fuzzy ideals with special properties always play an important role. In this paper we study those hemirings for which each fuzzy h-ideal is idempotent. We characterize these hemirings in terms of prime and semiprime fuzzy h-ideals. In the future we wanted to study those hemirings for which each fuzzy one sided h-ideal is idempotent and also those hemirings for which each fuzzy h-bi-ideal is idempotent. We also want to establish a fuzzy spectrum of hemirings.

We hope that the research along this direction can be continued, and our results presented in this paper have already constituted a platform for further discussion concerning the future development of hemirings and their applications to study fundamental concepts of the automata theory such as nondeterminism, for example.

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