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A unique factorization theorem for matroids

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For Denis Higgs, who gave us the lift

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Abstract

We study the combinatorial, algebraic and geometric properties of the free product operation on matroids. After giving cryptomorphic definitions of free product in terms of independent sets, bases, circuits, closure, flats and rank function, we show that free product, which is a noncommutative operation, is associative and respects matroid duality. The free product of matroids M and N is maximal with respect to the weak order among matroids having M as a submatroid, with complementary contraction equal to N. Any minor of the free product of M and N is a free product of a repeated truncation of the corresponding minor of M with a repeated Higgs lift of the corresponding minor of N. We characterize, in terms of their cyclic flats, matroids that are irreducible with respect to free product, and prove that the factorization of a matroid into a free product of irreducibles is unique up to isomorphism. We use these results to determine, for K a field of characteristic zero, the structure of the minor coalgebra $K\{\mathcal{M}\}$ of a family of matroids $\mathcal{M}$ that is closed under formation of minors and free products: namely, $K\{\mathcal{M}\}$ is cofree, cogenerated by the set of irreducible matroids belonging to $\mathcal{M}$.

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1. Introduction

We introduced the free product of matroids in a short article [4], in which we used it to settle the conjecture by Welsh [9] that $f_{n+m} \geqslant f_n \cdot f_m$, where f_n is the number of distinct isomorphism classes of matroids on an n-element set. Free product is, in a categorical sense, dual to the direct sum operation, and has properties that are in striking contrast to those of other, better known, binary operations on matroids; most significantly, it is noncommutative. In the present article we initiate a systematic study of the combinatorial, algebraic and geometric properties of this new operation. Our main results include a characterization, in terms of cyclic flats, of matroids that are irreducible with respect to free product, and a unique factorization theorem: every matroid factors uniquely, up to isomorphism, as a free product of irreducible matroids. Hence the set of all isomorphism classes of matroids, equipped with the binary operation induced by free product, is a free monoid, generated by the isomorphism classes of irreducible matroids.

Although we first defined the free product as such in [4], we first became aware of it earlier, while investigating, in [5], the *minor coalgebra* of a minor-closed family of matroids. This coalgebra has as basis the set of all isomorphism classes of matroids in the given family, with coproduct of a matroid $M = M(S)$ given by $\sum_{A\subseteq S} M|A \otimes M/A$, where $M|A$ is the submatroid obtained by restriction to A and M/A is the complementary contraction. If the family is also closed under formation of direct sums then its minor coalgebra is a Hopf algebra, with product determined on the basis of matroids by direct sum. These Hopf algebras, and analogous Hopf algebras based on families of graphs, were introduced in [8], as examples of the more general construction of incidence Hopf algebra. In the dual of the minor coalgebra, the *minor algebra*, the product of matroids M and N (dual basis elements) is a linear combination of those matroids having some restriction isomorphic to M, with complementary contraction isomorphic to N; the coefficient of $L = L(U)$ being the number of subsets $A \subseteq U$ such that $L|A \cong M$ and $L/A \cong N$. In the weak map order, the set of matroids appearing with nonzero coefficient in this product has a minimum element, given by the direct sum $M \oplus N$, and also has a maximum element, which we denote by $M \Box N$; this is the free product of M and N.

After discussing a few preliminaries in the following short section, we begin Section 3 by recalling from [4] the definition, in terms of independent sets, of the free product. As a next step, dictated by the culture of matroid theory, we give cryptomorphic definitions of the free product in terms of bases, circuits, closure, flats and rank function. These various characterizations allow us to demonstrate, in Sections 4 and 5, a number of fundamental properties of free product. In particular: free product satisfies the extremal property mentioned above, that is, $M \Box N$ is maximal in the weak order among matroids having a submatroid equal to M, with complementary contraction equal to N; free product is associative, and commutes with matroid duality; and any minor of a free product $M \Box N$ is itself a free product, namely, the free product of a repeated truncation of a minor of M with a repeated Higgs lift of a minor of N.

We begin Section 6 by giving a characterization of the cyclic flats of a free product, and making the key definition of *free separator* of a matroid $M(S)$ as a subset of S that is comparable by inclusion to all cyclic flats of M. We then prove the theorem that M factors as a free product $P(U) \Box Q(V)$ if and only if the set U is a free separator of M. As a consequence, we find that a nonuniform matroid $M(S)$ is irreducible if and only if the complete sublattice $\mathcal{D}(M)$ of the Boolean algebra 2^S generated by the cyclic flats of M has no *pinchpoint*, that is, single-element crosscut, other than $\emptyset$ and S. (Uniform matroids factor completely, into single-element matroids.) In order to examine free product factorization of matroids in detail, we turn our attention to the set $\mathcal{F}(M)$ of all free separators of a matroid $M(S)$, which, partially ordered by inclusion, is also a sublattice of 2^S. By the theorem mentioned above, there is a one-to-one correspondence between chains from $\emptyset$ to S in $\mathcal{F}(M)$ and factorizations $M = M_1 \square \cdots \square M_k$, according to which M_i is the minor of M determined by the ith interval in the corresponding chain. Factorizations of M into irreducibles thus correspond to maximal chains in $\mathcal{F}(M)$.

We define the *primary flag* $\mathcal{T}_M$ of a matroid M as the chain $T_0 \subset \cdots \subset T_k$ of pinchpoints in the lattice $\mathcal{D}(M)$. We show that $\mathcal{T}_M$ is also the chain of pinchpoints in $\mathcal{F}(M)$ and, furthermore, that the intersection of the lattices $\mathcal{F}(M)$ and $\mathcal{D}(M)$ is precisely $\mathcal{T}_M$. These results, together with a proposition characterizing the intervals $[T_{i-1}, T_i]$ in $\mathcal{F}(M)$, allow us to prove that the free product factorization $M = M_1 \square \cdots \square M_k$ corresponding to the chain $\mathcal{T}_M$ is the unique factorization of M having the property that each M_i is either irreducible, or maximally uniform (in the sense that no free product of consecutive M_i's is uniform). From this fundamental result, our main theorem quickly follows: every matroid factors uniquely up to isomorphism as a free product of irreducible matroids.

In Section 7, we use the unique factorization theorem, together with the extremal property of free product with respect to the weak order, to show that for any class $\mathcal{M}$ of matroids closed under the formation of minors and free products, the minor coalgebra of $\mathcal{M}$ is cofree, cogenerated by the isomorphism classes of irreducible matroids in $\mathcal{M}$. Any minor-closed class of matroids defined by the exclusion of a set of irreducible minors will therefore generate a minor coalgebra that is cofree. This is not the case for certain well-studied classes such as binary or unimodular matroids, because the four point line factors (as the free product of four one-element matroids). But for an infinite field F the class of F-representable matroids is closed under free product and hence its minor coalgebra is cofree.

In conclusion, we sketch in Section 8 a development whereby the minor coalgebra of a free product and minor-closed family of matroids forms a (self-dual) Hopf algebra in an appropriate braided monoidal category.

2. Preliminaries

We denote the disjoint union of sets S and T by $S + T$, the set difference by $S \backslash T$, and the intersection $S \cap T$ by either S_T or T_S. If T is a singleton set $\{a\}$, we write $S + a$ and $S \backslash a$, respectively, for $S + T$ and $S \backslash T$. We write $M = M(S)$ to indicate that M is a matroid with ground set S; in the case that $S = \{a\}$ is a singleton set we write $M(a)$ instead of $M(S)$. We denote the rank and nullity functions of M by ρ_M and ν_M, respectively, and denote by λ_M the *rank-lack* function on M, given by $\lambda_M(A) = \rho(M) - \rho_M(A)$, for all $A \subseteq S$, where $\rho(M) = \rho_M(S)$ is the rank of M.

Given a matroid $M(S)$ and $A \subseteq S$, we write $M|A$ for the restriction of M to A, that is, the matroid on A obtained by deleting $S \backslash A$ from M, and we write M/A for the matroid on $S\backslash A$ obtained by contracting A from M. For all $A \subseteq B \subseteq S$, we denote the minor $(M|B)/A = (M/A)|(B\backslash A)$ by $M(A, B)$.

For any set S, the *free matroid* $I(S)$ and the *zero matroid* $Z(S)$ are, respectively, the unique matroids on S having nullity zero and rank zero. In other words, if $|S| = n$, then $I(S)$ is the uniform matroid $U_{n,n}(S)$ and $Z(S)$ is the uniform matroid $U_{0,n}(S)$. We refer the reader to Oxley [7] and Welsh [10] for any background on matroid theory that might be needed.

3. The free product: cryptomorphic definitions

Definition 3.1 (*Crapo and Schmitt [4]*). The *free product* of matroids $M(S)$ and $N(T)$ is the matroid $M \,\square\, N$ defined on the set $S + T$ whose collection of independent sets is given by

$$\{A \subseteq S + T : A_S \text{ is independent in } M \text{ and } \lambda_M(A_S) \geqslant \nu_N(A_T)\}.$$

The first two propositions of [4] show that $M \,\square\, N$ is indeed a matroid, which contains M and N as complementary minors; specifically, if the ground set of M is S, then

$$(M \,\square\, N)|S = M \qquad \text{and} \qquad (M \,\square\, N)/S = N. \tag{3.2}$$

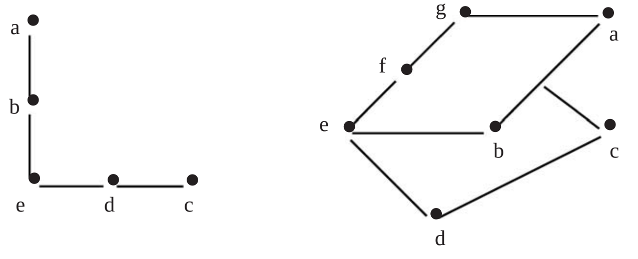
Proposition 3.3. *The collection of bases of* $M(S) \,\square\, N(T)$ *is given by*

$$\{A \subseteq S + T : A_S \textit{ is independent in } M,\ A_T \textit{ spans } N \textit{ and } \lambda_M(A_S) = \nu_N(A_T)\}.$$

Proof. The result follows directly from the definition of the free product. □

Note that it follows immediately from the characterization of the bases of $M \,\square\, N$ that $\rho(M \,\square\, N) = \rho(M) + \rho(N)$, for all M and N.

Example 3.4. Let $S = \{e, f, g\}$ and $T = \{a, b, c, d\}$, and suppose that $M(S)$ is a three-point line, and $N(T)$ consists of two double points *ab* and *cd*. The free products $I(e) \,\square\, N(T)$ and $M(S) \,\square\, N(T)$ are shown below:



According to Proposition 3.3, the matroid $I \,\square\, N$ has as bases all three-element subsets of $\{a, b, c, d\}$, together with all sets of the form $\{e, x, y\}$, where $x \in \{a, b\}$ and $y \in \{c, d\}$; while the bases of $M \,\square\, N$ are the sets of the form $A \cup B$, with $A \subseteq S$, $B \subseteq T$, and either

(i) $A = \emptyset$ and $B = T$,
(ii) $|A| = 1$ and $|B| = 3$, or
(ii) $|A| = 2$ and $|B| = 2$, with B not equal to $\{a, b\}$ or $\{c, d\}$.

Proposition 3.5. *The rank function of* $L = M(S) \,\square\, N(T)$ *is given by*

$$\rho_L(A) \;=\; \rho_M(A_S) + \rho_N(A_T) + \min\{\lambda_M(A_S), \nu_N(A_T)\},$$

for all $A \subseteq S + T$.

Proof. Suppose that $A \subseteq S + T$ and that $\lambda_M(A_S) \geqslant \nu_N(A_T)$. Then for any basis B of $M|A_S$, the set $B \cup A_T$ is a basis for $L|A$, and thus $\rho_L(A) = |B \cup A_T| = |B| + |A_T| = \rho_M(A_S) + \rho_N(A_T) + \nu_N(A_T)$.

If $\lambda_M(A_S) \leqslant \nu_N(A_T)$, choose $C \subseteq A_T$ such that $\rho_N(C) = \rho_N(A_T)$ and $\nu_N(C) = \lambda_M(A_S)$ and note that we then have $|C| = \rho_N(C) + \nu_N(C) = \rho_N(A_T) + \lambda_M(A_S)$. If B is a basis for $M|A_S$, then $B \cup C$ is a basis for $L|A$, and thus $\rho_L(A) = |B \cup C| = \rho_M(A_S) + \rho_N(A_T) + \lambda_M(A_S)$. $\square$

It follows immediately that the nullity function of $L = M(S) \,\square\, N(T)$ is given by

$$\nu_L(A) \;=\; \nu_M(A_S) + \nu_N(A_T) - \min\{\lambda_M(A_S), \nu_N(A_T)\}, \tag{3.6}$$

for all $A \subseteq S + T$, and similarly for the rank-lack function.

Proposition 3.7. *The closure operator on* $L = M(S) \,\square\, N(T)$ *is given by*

$$c\ell_L(A) = \begin{cases} c\ell_M(A_S) \cup A_T & \text{if } \lambda_M(A_S) > \nu_N(A_T), \\ S \cup c\ell_N(A_T) & \text{if } \lambda_M(A_S) \leqslant \nu_N(A_T), \end{cases}$$

for all $A \subseteq S + T$.

Proof. Suppose that $\lambda_M(A_S) > \nu_N(A_T)$. According to Proposition 3.5, the rank of A in L is given by $\rho_L(A) = \rho_M(A_S) + |A_T|$, and if $B = A \cup x$, for any $x \in S + T$, then $\lambda_M(B_S) \geqslant \nu_N(B_T)$, and we have $\rho_L(B) = \rho_M(B_S) + |B_T|$. Hence $x \in c\ell_L(A)$ if and only if $\rho_M(A_S) + |A_T| = \rho_M(B_S) + |B_T|$, that is, if and only if $x \in c\ell_M(A_S) \cup A_T$.

Suppose that $\lambda_M(A_S) \leqslant \nu_N(A_T)$. If $B = A \cup x$, for any $x \in S + T$, then $\lambda_M(B_S) \leqslant \nu_N(B_T)$ and therefore, by Proposition 3.5, $\rho_L(A) = \rho(M) + \rho_N(A_T)$ and $\rho_L(B) = \rho(M) + \rho_N(B_T)$. Hence $x \in c\ell_L(A)$ if and only if $\rho_N(A_T) = \rho_N(B_T)$, that is, if and only if $x \in S \cup c\ell_N(A_T)$. $\square$

As a corollary, we obtain the following description of the flats of a free product in terms of the flats of its factors.

Corollary 3.8. *Suppose that* $L = M(S) \square N(T)$ *and* $A \subseteq S + T$. *If* $\lambda_M(A_S) > \nu_N(A_T)$, *then* A *is a flat of* L *if and only if* A_S *is a flat of* M; *if* $\lambda_M(A_S) \leqslant \nu_N(A_T)$, *then* A *is a flat of* L *if and only if* $A_S = S$ *and* A_T *is a flat of* N.

Proposition 3.9. *A set* $C \subseteq S + T$ *is a circuit in* $L = M(S) \square N(T)$ *if and only if* $C \subseteq S$ *and* $C = C_S$ *is a circuit in* M, *or* C_S *is independent in* M, *the restriction* $N|C_T$ *is isthmusless, and* $\lambda_M(C_S) + 1 = \nu_N(C_T)$.

Proof. By the definition of free product, a subset C of $S + T$ is dependent in L if and only if C_S is dependent in M or $\lambda_M(C_S) < \nu_N(C_T)$. A minimal set with this property is either a circuit in M, or a minimal set with C_S independent in M but with $\lambda_M(C_S) < \nu_N(C_T)$, that is, a set such that $\lambda_M(C_S) + 1 = \nu_N(C_T)$. If such a set C were such that the restriction $N|C_T$ were to have an isthmus d, then C would not be minimal, since we would have $\nu_N(C_T) = \nu_N(C_T \backslash d)$. □

4. Basic properties of the free product

We begin with a lemma showing that the asserted inequality between $\lambda_M(A_S)$ and $\nu_N(A_T)$ in the definition of free product is in fact a property of restrictions and complementary contractions in arbitrary matroids.

Lemma 4.1. *Given a matroid* $L = L(S + T)$, *let* $M = L|S$ *and* $N = L/S$. *Then* $\lambda_M(A_S) \geqslant \nu_N(A_T)$, *for all independent sets* A *in* L.

Proof. The rank function on the contraction $N = L/S$ is determined by the formula $\rho_N(B) = \rho_L(B \cup S) - \rho_L(S) = \rho_L(B \cup S) - \rho(M)$, for all $B \subseteq T$. If $A \subseteq S + T$ is independent in L, then $\rho_L(A_T \cup S) \geqslant |A|$, and so by the above formula, $\rho_N(A_T) \geqslant |A| - \rho(M)$. Thus we have $\nu_N(A_T) = |A_T| - \rho_N(A_T) \leqslant |A_T| - (|A| - \rho(M)) = \lambda_M(A_S)$. □

By definition, the independent sets of the free product $M(S) \square N(T)$ are precisely those subsets of $S + T$ which, according to Lemma 4.1, are necessarily independent in any matroid containing M as a submatroid with complementary contraction N. The following proposition expresses the consequent extremal, or universal, property of the free product.

Proposition 4.2. *For any matroid* $L = L(U)$, *and* $S \subseteq U$, *the identity map on* U *is a rank-preserving weak map* $L|S \square L/S \to L$.

Proof. Let $M = L|S$ and $N = N(T) = L/S$. If A is independent in L, then A_S is independent in M and, by Lemma 4.1, we have $\lambda_M(A_S) \geqslant \nu_N(A_T)$. Hence A is independent in $M \square N$, and so the identity map on $S + T$ is a weak map from $M \square N$ to L, which is clearly rank-preserving. □

Roughly speaking, in a free product $L = M(S) \square N(T)$, the submatroid $L|T$ is the freest matroid, arranged in the most general position possible relative to $M = L|S$ such that the contraction L/S is equal to $N(T)$. In the matroid $M(S)\,\square\, N(T)$ of Example 3.4, as long as $\{a, b\}$ and $\{c, d\}$ are each coplanar with $S = \{e, f, g\}$, and on distinct planes, the contraction by S will be equal to N, as required. In the indicated free product, $\{a, b\}$ and $\{c, d\}$ are simply "in general position" on such planes.

We prove next that free product respects matroid duality and is associative. First, recall that for any matroid $M(S)$, the rank function of the dual matroid M^* satisfies $\rho_{M^*}(B) = |B| - \rho(M) + \rho_M(A)$, or equivalently, $\lambda_M(A) = \nu_{M^*}(B)$, whenever $A + B = S$.

Proposition 4.3 (*Crapo and Schmitt [4]*). *For all matroids M and N,* $(M\,\square\, N)^* = N^*\,\square\, M^*$.

Proof. Suppose that $M = M(S)$, $N = N(T)$, and $A + B = S + T$, so that A is a basis for $M\,\square\, N$ if and only if B is a basis for $(M\,\square\, N)^*$. Then A is a basis for $M\,\square\, N$ if and only if A_S is independent in M, A_T spans N and $\lambda_M(A_S) = \nu_N(A_T)$, which is true if and only if B_S spans M^*, B_T is independent in N^*, and $\nu_{M^*}(B_S) = \lambda_{N^*}(B_T)$, that is, if and only if B is a basis for $N^*\,\square\, M^*$. $\square$

Proposition 4.4. *Free product is an associative operation*.

Proof. Suppose that $M = M(S)$, $N = N(T)$ and $P = P(U)$. Then $A \subseteq S + T + U$ is independent in $(M\,\square\, N)\,\square\, P$ if and only if A_{S+T} is independent in $M\,\square\, N$ and $\lambda_{M\,\square\, N}(A_{S+T}) \geqslant \nu_P(A_U)$. Since A_{S+T} is independent in $M\,\square\, N$, we have

$$\begin{aligned}\lambda_{M\,\square\, N}(A_{S+T}) &= \rho(M\,\square\, N) - |A_{S+T}| \\ &= \rho(M) + \rho(N) - |A_S| - |A_T| \\ &= \lambda_M(A_S) + \rho(N) - |A_T|.\end{aligned}$$

Hence the set A is independent in $(M\,\square\, N)\,\square\, P$ if and only if A_S is independent in M, $\nu_N(A_T) \leqslant \lambda_M(A_S)$ and $\nu_P(A_U) \leqslant \lambda_M(A_S) + \rho(N) - |A_T|$. Adding $\nu_N(A_T)$ to both sides of the last inequality, we may express these three conditions as

$$\nu_M(A_S) \leqslant 0, \qquad \nu_N(A_T) \leqslant \lambda_M(A_S) \quad \text{and} \quad \nu_N(A_T) + \nu_P(A_U) \leqslant \lambda_M(A_S) + \lambda_N(A_T).$$

On the other hand, A is independent in $M\,\square\,(N\,\square\, P)$ if and only if $\nu_M(A_S) \leqslant 0$ and $\nu_{N\,\square\, P}(A_{T+U}) \leqslant \lambda_M(A_S)$. By Eq. (3.6), the latter inequality may be written as

$$\nu_N(A_T) + \nu_P(A_U) \leqslant \lambda_M(A_S) + \min\{\lambda_N(A_T), \nu_P(A_U)\},$$

which holds if and only if $\nu_N(A_T) \leqslant \lambda_M(A_S)$ and $\nu_N(A_T) + \nu_P(A_U) \leqslant \lambda_M(A_S) + \lambda_N(A_T)$. Hence A is independent in $M\,\square\,(N\,\square\, P)$ if and only if it is independent in $(M\,\square\, N)\,\square\, P$. $\square$

The definitions and properties stated above have natural analogs for iterated free products.

Proposition 4.5. *If $L(S) = M_1(S_1) \square \cdots \square M_k(S_k)$, then $A \subseteq S$ is independent in L if and only if*

$$\sum_{i=1}^{j-1} \lambda_{M_i}(A_{S_i}) \geqslant \sum_{i=1}^{j} \nu_{M_i}(A_{S_i}), \tag{4.6}$$

for all j such that $1 \leqslant j \leqslant k$.

Proof. We use induction on k. When $k = 1$, the sum on the left-hand side of the inequality is empty and thus zero; so the result holds. Suppose the result holds for $L' = M_1(S_1) \square \cdots \square M_{k-1}(S_{k-1})$. Then A is independent in $L = L' \square M_k$ if and only if $A'_{S_k} = A_{S_1} + \cdots + A_{S_{k-1}}$ is independent in L' and $\nu_{M_k}(A_{S_k}) \leqslant \lambda_{L'}(A'_{S_k})$, that is, if and only if inequality (4.6) holds for $1 \leqslant j \leqslant k-1$ and, since A'_{S_k} is independent in L',

$$\nu_{M_k}(A_{S_k}) \leqslant \rho(L') - |A'_{S_k}| = \sum_{i=1}^{k-1} \rho(M_i) - |A_{S_i}|.$$

But $\rho(M_i) - |A_{S_i}| = \lambda_{M_i}(A_{S_i}) - \nu_{M_i}(A_{S_i})$, for all i; hence the above inequality is equivalent to inequality (4.6), for $j = k$. □

We will need the following generalization of Proposition 4.2 in Section 7.

Proposition 4.7. *Suppose that $L = L(U)$ and $\emptyset = T_0 \subset \cdots \subset T_k = U$ is a chain of subsets of U, for some $k \geqslant 0$, and let L_i denote the minor $L(T_{i-1}, T_i)$, for $1 \leqslant i \leqslant k$. The identity map on U is a weak map $L_1 \square \cdots \square L_k \to L$.*

Proof. Let $S_i = T_i \backslash T_{i-1}$, for $1 \leqslant i \leqslant k$, so that $L_i = L_i(S_i)$, for all i. By Lemma 4.1 and induction on k, it follows that inequalities (4.6) hold for all independent sets A in L. Hence, by Proposition 4.5, any independent set in L is also independent in $L_1 \square \cdots \square L_k$, that is, the identity map on U is a weak map $L_1 \square \cdots \square L_k \to L$. □

One-element matroids (isthmuses and loops) play a special role in the study of free products.

Example 4.8. Recall that, if $\{a\}$ is any singleton, then $I(a)$ and $Z(a)$ denote the matroids on $\{a\}$ consisting, respectively, of a single point and a single loop. For any set $S = \{s_1, \ldots, s_n\}$, and $k \leqslant n$, the free product $I(s_1) \square \cdots \square I(s_k) \square Z(s_{k+1}) \square \cdots \square Z(s_n)$ is the uniform matroid $U_{k,n}(S)$.

For any matroid M, we write $\mathrm{Loop}(M)$ and $\mathrm{Isth}(M)$, respectively, for the sets of loops and isthmuses of M.

Proposition 4.9. *For all matroids M and N,* $\mathrm{Loop}(M) \subseteq \mathrm{Loop}(M \square N)$, *with* $\mathrm{Loop}(M) = \mathrm{Loop}(M \square N)$, *whenever* $\rho(M) > 0$. *Dually,* $\mathrm{Isth}(N) \subseteq \mathrm{Isth}(M \square N)$, *with equality whenever* $\nu(N) > 0$.

Proof. If x is a loop of M, then x belongs to no independent set of $M \,\Box\, N$; hence x is a loop of $M \,\Box\, N$, and so $\mathrm{Loop}(M) \subseteq \mathrm{Loop}(M \,\Box\, N)$. On the other hand, suppose that $\rho(M) > 0$, and that $N = N(T)$ and $x \in T$. It follows from Proposition 3.5 that $\rho_{M \Box N}(x) = \rho_N(x) + \min\{\rho(M), \nu_N(x)\} = 1$, so x is not a loop in $M \,\Box\, N$, and hence $\mathrm{Loop}(M \,\Box\, N) = \mathrm{Loop}(M)$. The dual statements follow directly from Proposition 4.3. $\Box$

Corollary 4.10. *If $\rho(M) = 0$ or $\nu(N) = 0$, then $M \,\Box\, N = M \oplus N$.*

Example 4.11. For any matroid M, the matroids $M \,\Box\, I$ and $Z \,\Box\, M$ consist of M with, respectively, an isthmus and a loop adjoined, while $M \,\Box\, Z$ and $I \,\Box\, M$ are respectively the free one-point extension and coextension of M (see [7]).

Example 4.12. Because adjoining an isthmus and taking a single-point free extension of a matroid correspond to free multiplication on the right by I and Z, respectively, it follows that the class of matroids introduced in [3], now variously known as *generalized Catalan matroids* [2], *shifted matroids* [1] and *freedom matroids* [5], is the class generated by the single-element matroids under free product.

A *representation* of a matroid $M(S)$ over a field F is a matrix P having entries in F and rows labeled by the elements of S, such that for all $A \subseteq S$, the submatrix P_A of P, consisting of those rows of P whose labels belong to A, has rank $\rho_M(A)$. We can, and shall, always assume that the number of columns in a representation of M is equal to the rank of M. A matroid M is called *F-representable* if there exists a representation of M over F.

Proposition 4.13. *If the matroids $M(S)$ and $N(T)$ are F-representable, and the field F is large enough, then the free product $M \,\Box\, N$ is F-representable.*

Proof. Suppose that P and Q are representations for M and N, respectively. Using the fact that the field F has enough elements, we can construct a $|T| \times \rho(M)$ matrix Z, with rows labelled (arbitrarily) by T, having the following property: given any $A \subseteq S$ which is independent in M, and any $B \subseteq T$ of size $\lambda_M(A) = \rho(M) - |A|$, the matrix

$$\left[\begin{array}{c} P_A \\ \hline Z_B \end{array}\right]$$

is nonsingular. We show that the matrix

$$\left[\begin{array}{c|c} P & 0 \\ \hline Z & Q \end{array}\right]$$

is a representation for the free product $M \,\Box\, N$. Suppose that $A \subseteq S + T$, and let $B \subseteq A_T$ be a basis for A_T in N. Since B is independent in N, the matrix Q_B has independent rows, and hence the matrix R_A has independent rows if and only if the matrix

$$\left[\begin{array}{c} P_{A_S} \\ \hline Z_{A_T \setminus B} \end{array}\right]$$

has independent rows. Since $|A_T \backslash B| = \nu_N(A_T)$, it follows from the construction of Z that this latter matrix has independent rows if and only if A_S is independent in M and $\lambda_M(A_S) \geqslant \nu_N(A_T)$, that is, if and only if A is independent in $M \,\square\, N$. $\square$

Suppose that $\mathcal{A} = \{A_i : i \in I\}$ is an indexed family of subsets of a set S (with repetitions allowed). A set $A \subseteq S$ is a *partial transversal* of $\mathcal{A}$ if there exists an injective map $f : A \to I$ such that $a \in A_{f(a)}$, for all $a \in A$. The set of partial transversals of $\mathcal{A}$ is the collection of independent sets of a matroid, called a *transversal matroid* on S, and denoted by $M(S, \mathcal{A})$. The family $\mathcal{A}$ is a *presentation* of $M(S, \mathcal{A})$. Any transversal matroid M has a presentation with number of sets equal to the rank of M (see [10, p. 244]).

Proposition 4.14. *The free product of transversal matroids is a transversal matroid.*

Proof. Suppose that $M = M(S, \mathcal{A})$ and $N = M(T, \mathcal{B})$ are transversal matroids with respective presentations $\mathcal{A} = \{A_i : i \in I\}$ and $\{B_j : j \in J\}$, where $|I| = \rho(M)$. For all $k \in I + J$, define $U_k \subseteq S + T$ by

$$U_k = \begin{cases} A_k + T & \text{if } k \in I, \\ B_k & \text{if } k \in J. \end{cases}$$

We show that the free product $M \,\square\, N$ is equal to the transversal matroid on $S + T$ having presentation $\mathcal{U} = \{U_k : k \in I + J\}$. Given $A \subseteq S + T$, let $B \subseteq A_T$ be a basis for A_T in N. The set A is independent in $M(S + T, \mathcal{U})$ if and only if there exists injective $f : A \backslash B \to I$ such that $a \in U_{f(a)}$ for all $a \in A \backslash B$, which is the case if and only if A_S is independent in M and $|A_T \backslash B| \leqslant |I| - |A_S|$. Since $|A_T \backslash B| = \nu_N(A_T)$ and $\lambda_M(A_S) = |I| - |A_S|$, for A_S independent in M, it follows that such f exists if and only if A is independent in $M \,\square\, N$. $\square$

5. Minors of free products

The minors of a free product of matroids are perhaps most simply described in terms of the matroid truncation operator and its dual, the Higgs lift operator (see [6]). The *truncation* of a matroid $M(S)$ is the matroid TM whose independent sets are those independent sets A of M satisfying $|A| \leqslant \max\{0, \rho(M) - 1\}$, and the *Higgs lift*, or simply *lift*, of M is the matroid LM whose family of independent sets is $\{A \subseteq S : \nu_M(A) \leqslant 1\}$. Denoting by $T^i M$ and $L^i M$, respectively, the i-fold truncation and lift of $M(S)$, it follows that $T^i M$ has rank equal to $\max\{0, \rho(M) - i\}$, and

$$\rho_{\mathrm{T}^i M}(A) = \min\{\rho_M(A),\ \rho(T^i M)\} \quad \text{and} \quad \lambda_{\mathrm{T}^i M}(A) = \min\{0,\ \lambda_M(A) - i\},$$

for all $A \subseteq S$. The rank of $L^i M$ is $\min\{|S|, \rho(M) + i\}$, and

$$\rho_{\mathrm{L}^i M}(A) = \min\{|A|,\ \rho_M(A) + i\} \quad \text{and} \quad \nu_{\mathrm{L}^i M}(A) = \max\{0,\ \nu_M(A) - i\}$$

for all $A \subseteq S$. The truncation and lift operators are dual to each other, so that $(T^i M)^* = L^i(M^*)$, for all matroids M and $i \geqslant 0$. Truncation commutes with contraction and lift commutes with restriction, so for any matroid $M(S)$ and $i \geqslant 0$,

$$(T^i M)/U = T^i(M/U) \qquad \text{and} \qquad (L^i M)|U = L^i(M|U),$$

for all $U \subseteq S$. We thus shall write expressions such as these without parentheses. The precise manner in which lift and truncation fail to commute with contraction and restriction, respectively, is described by the following proposition.

Proposition 5.1. *For any matroid $M(S)$ and $U \subseteq S$*

$$T^i(M|U) = (T^{i+j}M)|U \qquad \textit{and} \qquad L^i(M/U) = (L^{i+k}M)/U,$$

for all $i \geqslant 0$, where $j = \lambda_M(U)$ and $k = \nu_M(U)$.

Proof. The rank-lack of $A \subseteq U$ in $M|U$ is given by $\lambda_{M|U}(A) = \lambda_M(A) - \lambda_M(U) = \lambda_M(A) - j$, and so $\lambda_{\mathrm{T}^i M|U}(A) = \min\{0,\ \lambda_{M|U}(A) - i\} = \min\{0, \lambda_M(A) - j - i\}$. On the other hand,

$$\begin{aligned}\lambda_{(\mathrm{T}^{i+j}M)|U}(A) &= \lambda_{\mathrm{T}^{i+j}M}(A) - \lambda_{\mathrm{T}^{i+j}M}(U)\\ &= \min\{0,\ \lambda_M(A) - i - j\} - \min\{0,\ \lambda_M(U) - i - j\},\end{aligned}$$

which is equal to $\min\{0, \lambda_M(A) - i - j\}$, since $\lambda_M(U) = j$. The matroids $T^i(M|U)$ and $(T^{i+j}M)|U$ thus have identical rank-lack functions, and are therefore equal. The second equality follows from duality, using the fact that $\lambda_M(U) = \nu_{M^*}(S\backslash U)$, for all $U \subseteq S$. $\square$

In keeping with the notational tradition of performing unary operations before binary operations, in order to avoid a proliferation of parentheses, we adopt the convention that all truncations, lifts, deletions and contractions that may appear in a given expression for a matroid are to be performed before any free products and/or direct sums that appear.

Proposition 5.2. *If $P = M(S) \,\square\, N(T)$ and $U \subseteq S + T$, then*

$$P|U = M|U_S \,\square\, L^i N|U_T \qquad \textit{and} \qquad P/U = T^j M/U_S \,\square\, N/U_T,$$

where $i = \lambda_M(U_S)$ and $j = \nu_N(U_T)$.

Proof. A set $A \subseteq U$ is independent in $P|U$ if and only if A_S is independent in M and $\lambda_M(A_S) \geqslant \nu_N(A_T)$. Using the fact that $\lambda_M(A_S) = \lambda_{M|U_S}(A_S) + \lambda_M(U_S)$ and that $\nu_N(A_T) = \nu_{N|U_T}(A_T)$, we thus have A independent in $P|U$ if and only if A_S is independent in $M|U_S$ and $\lambda_{M|U_S}(A_S) \geqslant \nu_{N|U_T}(A_T) - i$. But $\max\{0, \nu_{N|U_T}(A_T) - i\} = \nu_{L^i N|U_T}(A_T)$, and so A is independent in $P|U$ if and only if A_S is independent in $M|U_S$ and $\lambda_{M|U_S}(A_S) \geqslant \nu_{L^i N|U_T}(A_T)$, that is, if and only if A is independent in $M|U_S \,\square\, L^i N|U_T$.

The second equality follows from the first by duality, that is, by Proposition 4.3, the duality between deletion and contraction, the duality between lift and truncation and the fact that $\lambda_{N^*}(T\backslash U_T) = \nu_N(U_T)$. $\square$

Theorem 5.3. *If* $P = M(S) \Box N(T)$ *and* $U \subseteq V \subseteq S + T$, *then*

$$P(U, V) = (T^j M)(U_S, V_S) \Box (L^i N)(U_T, V_T),$$

where $j = \nu_N(U_T)$ *and* $i = \lambda_M(V_S)$.

Proof. By Proposition 5.2, we have $P|V = M|V_S \Box (L^i N|V_T)$, where $i = \lambda_M(V_S)$, and thus, by the same proposition,

$$\begin{aligned} P(U, V) = (P|V)/U &= (T^k(M|V_S))/U_S \Box (L^i N|V_T)/U_T \\ &= (T^k(M|V_S))/U_S \Box (L^i N)(U_T, V_T), \end{aligned}$$

where $k = \nu_{L^i N|V_T}(U_T) = \max\{0, \nu_N(U_T) - i\} = \max\{0, j - i\}$. If $j \geqslant i$, then by Proposition 5.1,

$$\begin{aligned} (T^k(M|V_S))/U_S &= ((T^{k+i} M)|V_S)/U_S \\ &= (T^j M)(U_S, V_S) \end{aligned}$$

and we thus obtain the desired expression for $P(U, V)$. On the other hand, if $j < i = \lambda_M(V_S)$, then $(T^j M)|V_S = M|V_S$, and $k = 0$, and thus

$$\begin{aligned} (T^k(M|V_S))/U_S &= (M|V_S)/U_S \\ &= ((T^j M)|V_S)/U_S \\ &= (T^j M)(U_S, V_S) \end{aligned}$$

and again we obtain the desired expression for $P(U, V)$. $\Box$

As a special case of Theorem 5.3, we have that the minors of $P = M(S) \Box N(T)$ supported on the sets S and T are obtained by successive truncations of M and Higgs lifts of N, respectively; that is, for all $A \subseteq S$ and $B \subseteq T$,

$$P(A, A \cup T) = L^i N \qquad \text{and} \qquad P(B, B \cup S) = T^j M,$$

where $i = \lambda_M(A)$ and $j = \nu_N(B)$. This is to be compared to the direct sum, where these minors are simply isomorphic to M and N.

The following proposition describes how the lift and truncation operators interact with free product.

Proposition 5.4. *For all matroids M and N, the truncation and lift of the free product* $M \Box N$ *are given by*

$$T(M \Box N) = \begin{cases} M \Box TN & \text{if } \rho(N) > 0, \\ TM \Box N & \text{if } \rho(N) = 0 \end{cases}$$

and

$$L(M \Box N) = \begin{cases} LM \Box N & \text{if } \nu(M) > 0, \\ M \Box LN & \text{if } \nu(M) = 0, \end{cases}$$

for all matroids M and N.

Proof. If $\rho(M) = 0$ then, by Corollary 4.10, we have $M \square N = M \oplus N$, and so $T(M \square N) = T(M \oplus N) = TM \oplus TN = M \oplus TN = M \square TN$. We therefore assume that $\rho(M)$ is nonzero.

Suppose that $M = M(S)$ and $N = N(T)$. Observe that if a set $A \subseteq S+T$ is independent in any of the matroids $T(M \square N)$, $M \square TN$ and $TM \square N$, then A_S is necessarily independent in M. Hence, for the remainder of the proof, we assume that A is some subset of $S+T$ such that A_S is independent in M.

We first consider the case in which $\rho(N) = 0$. The set A is independent in $M \square N$ if and only if $\lambda_M(A_S) \geqslant v_N(A_T)$, which is the case if and only if $|A| \leqslant \rho(M)$, since $\lambda_M(A_S) = \rho(M) - |A_S|$ and $v_N(A_T) = |A_T|$. It follows that A is independent in $T(M \square N)$ if and only if $|A| \leqslant \rho(M) - 1$.

Now A is independent in $TM \square N$ if and only if $\rho_M(A_S) = |A_S| \leqslant \rho(M) - 1$ and $\lambda_{TM}(A_S) \geqslant v_N(A_T)$. Furthermore

$$\begin{aligned}\lambda_{TM}(A_S) &= \max\{\lambda_M(A_S) - 1,\ 0\}\\ &= \max\{\rho(M) - |A_S| - 1,\ 0\},\end{aligned}$$

which is equal to $\rho(M) - |A_S| - 1$, since $|A_S| \leqslant \rho(M) - 1$. Therefore A is independent in $TM \square N$ if and only if $\rho(M) - |A_S| - 1 \geqslant v_N(A_T) = |A_T|$, that is, if and only if $|A| \leqslant \rho(M) - 1$, and hence $T(M \square N) = TM \square N$.

Now suppose that $\rho(N) > 0$. If $\rho_N(A_T) < \rho(N)$ then, by Proposition 3.3, the set A does not span $M \square N$, and so A is independent in $T(M \square N)$ if and only if A is independent in $M \square N$. But since A_T does not span N, and thus $v_{TN}(A_T) = v_N(A_T)$, it follows that A is independent in $M \square N$ if and only if it is also independent in $M \square TN$. If $\rho_N(A_T) = \rho(N)$ then, by Proposition 3.3, we have that A is independent in $T(M \square N)$ if and only if $\lambda_M(A_S) > v_N(A_T)$. But A is independent in $M \square TN$ if and only if $\lambda_M(A_S) \geqslant v_{TN}(A_T) = v_N(A_T) + 1$; hence $T(M \square N) = M \square TN$. The corresponding result for $L(M \square N)$ follows by duality. □

It follows from Proposition 5.4 that, for all matroids M and N, and $i \geqslant 0$,

$$T^i(M \square N) = T^j M \square T^{i-j} N \quad \text{and} \quad L^i(M \square N) = L^{i-k} M \square L^k N, \tag{5.5}$$

where $j = \max\{i - \rho(N), 0\}$ and $k = \max\{i - v(M), 0\}$.

6. Irreducible matroids and unique factorization

A crucial tool for the study of factorization of matroids with respect to free product is the notion of *cyclic flat* of a matroid. Recall that a cyclic flat of M is a flat A which is equal to a union of circuits of M. Alternatively, a flat A is cyclic if and only if the restriction $M|A$ is isthmusless. Observe that in particular, any closure of a circuit in a matroid is a cyclic flat. We begin with the following characterization of the cyclic flats in a free product of matroids.

Proposition 6.1. *A subset $A \neq S$ of $S+T$ is a cyclic flat of $L = M(S) \square N(T)$ if and only if either $A \subseteq S$ and A is a cyclic flat of M, or $A = S \cup B$, where B is a (nonempty) cyclic flat of N. The set S is a cyclic flat of L if and only if M is isthmusless and N is loopless.*

Proof. Suppose that $A \subseteq S + T$ satisfies $\lambda_M(A_S) > v_N(A_T)$ and $A \neq S$. According to Corollary 3.8, A is a flat of L if and only if A_S is a flat of M, in which case any element of A_T is an isthmus of $L|A$. Hence A is a cyclic flat of L if and only if $A_T = \emptyset$ and $A = A_S$ is a cyclic flat of M.

Now suppose that $A \neq S$ and $\lambda_M(A_S) \leqslant v_N(A_T)$. Then by Corollary 3.8, A is a flat of L if and only if $A_S = S$ and A_T is a nonempty flat of N. Given such a flat A, we have $\rho_L(A) = \rho_M(A_S) + \rho_N(A_T) + \min\{\lambda_M(A_S), v_N(A_T)\} = \rho(M) + \rho_N(A_T)$; hence if A is cyclic then A_T must be a cyclic flat of N. On the other hand, if A_T is cyclic in N, then $\rho_L(A \backslash a) = \rho_L(A)$, for all $a \in A_T$, and since $v_N(A_T) > 0$ and $\lambda_M(A_S) = \lambda_M(S) = 0$, it follows that $\rho_L(A \backslash a) = \rho_L(A)$ for all $a \in A_S$ as well. Hence A is cyclic.

Since $\lambda_M(S) = 0$, it follows from Corollary 3.8 that S is a flat of L if and only if N is loopless, in which case the flat S is cyclic if and only if $M = L|S$ is isthmusless. □

Definition 6.2. A set $A \subseteq S$ is a *free separator* of a matroid $M(S)$ if every cyclic flat of M is comparable to A by inclusion.

Note that the empty set and the entire set S are free separators of any matroid $M(S)$; any other free separator is said to be *nontrivial*.

Theorem 6.3. *For any matroid $L(S + T)$, the following are equivalent*:

(i) $L(S + T) = L|S \,\square\, L/S$.
(ii) *S is a free separator of L.*

Proof. The implication (i) ⇒ (ii) is immediate from Proposition 6.1. Conversely, suppose that S is a free separator of L, and let $M = L|S$ and $N = L/S$. We first show that every circuit of L is also a circuit of the free product $M(S) \,\square\, N(T)$. Let C be a circuit of L. If $C \subseteq S$, then C is a circuit of M, and therefore a circuit of $M \,\square\, N$. Suppose that $C \nsubseteq S$. Since C is a circuit, $\rho_L(C \backslash a) = \rho_L(C)$ and thus, by the semimodularity of the rank function, $\rho_L((S \cup C) \backslash a) = \rho_L(S \cup C)$, for all $a \in C$. Hence, for all $a \in C_T$, we have $\rho_N(C_T) = \rho_L(S \cup C) - \rho_L(S) = \rho_L(S \cup C \backslash a) - \rho_L(S) = \rho_N(C_T \backslash a)$, and so $N|C_T$ is isthmus free. Since the closure of a circuit is a cyclic flat, S is a free separator, and $C \nsubseteq S$, we have $S \subseteq c\ell_L(C)$. It follows that $\rho_L(S \cup C) = \rho_L(C) = |C| - 1$, and so $v_L(S \cup C) = |S| - |C_S| + 1$. Therefore

$$\begin{aligned} v_N(C_T) &= v_L(S \cup C) - v_L(S) \\ &= |S| - |C_S| + 1 - (|S| - \rho_L(S)) \\ &= \rho(M) - |C_S| + 1, \end{aligned}$$

which is equal to $\lambda_M(C_S) + 1$, since C_S is independent in L (and thus also in M). By Proposition 3.9, it follows that C is a circuit in $M \,\square\, N$.

We have thus shown that every circuit in L is also a circuit in $L|S \,\square\, L/S$, in other words, the identity map on $S + T$ is a weak map $L \to L|S \,\square\, L/S$. By Proposition 4.2, the identity map on $S + T$ is also a weak map $L|S \,\square\, L/S \to L$; hence $L = L|S \,\square\, L/S$. □

We refer to a nonempty matroid M as *irreducible* if any factorization of M as a free product of matroids contains M as a factor. By convention, the empty matroid is not irreducible. The following restatement of Theorem 6.3 characterizes irreducible matroids.

Theorem 6.4. *For any nonempty matroid $M(S)$, the following are equivalent*:

(i) *M is irreducible with respect to free product*.
(ii) *M has no nontrivial free separator*.

Corollary 6.5. *If M is loopless, isthmusless and disconnected, then M is irreducible*.

Proof. Suppose that $M(S)$ is loopless, isthmusless and disconnected, and write $M(S)$ as the direct sum $P(U) \oplus Q(V)$, with U and V nonempty. Let A be a nonempty proper subset of S. Assume, without loss of generality, that A_U and $V \backslash A$ are nonempty, and let $a \in V \backslash A$. Since Q is loop and isthmus free, a is contained in some circuit C of Q. Now C is also a circuit of M and $a \in c\ell_M(C) = c\ell_Q(C) \subseteq V$; hence $c\ell_M(C)$ neither contains nor is contained in A, and so A is not a free separator of M. □

Corollary 6.6. *If $L = M(S) \square N(T) = P(T) \square Q(S)$, where S and T are nonempty, then L is a uniform matroid*.

Proof. Let C be a circuit of L. By Theorem 6.3, both S and T are free separators of L and hence $c\ell_L(C)$ is comparable to both S and T by inclusion. Since S and T are disjoint and nonempty, the only possibility is that S and T are both contained in $c\ell_L(C)$. Every circuit of L is thus a spanning set for L, and therefore L is uniform. □

We remark that it follows from Proposition 4.3 that a matroid M is irreducible if and only if the dual matroid M^* is irreducible.

Corollary 6.7. *If M is identically self-dual, then M is either uniform or irreducible*.

Proof. Suppose that M is identically self-dual and factors as $P(U) \square Q(V)$, with U and V nonempty. Using Proposition 4.3, we have $P(U) \square Q(V) = M = M^* = Q^*(V) \square P^*(U)$, and hence it follows from Corollary 6.6 that M is uniform. □

Example 6.8. Suppose that $S = \{a, b, c, d\}$ and let $M(S)$ be the matroid in which ab is a double point, collinear with c and d. Then M is self-dual, not uniform, and factors with respect to free product as $I(a) \square Z(b) \square I(c) \square Z(d)$.

For any matroid $M(S)$, we denote by $\mathcal{D}(M)$ the complete sublattice of the Boolean algebra 2^S generated by all cyclic flats of M. Note that $\mathcal{D}(M)$ is a distributive lattice, and contains in particular the empty union and empty intersection of cyclic flats of M, which are equal to Ø and S, respectively.

Proposition 6.9. *A nonempty matroid $M(S)$ is uniform if and only if $|\mathcal{D}(M)| = 2$, that is, if and only if $\mathcal{D}(M) = \{\emptyset, S\}$*.

Proof. Uniform matroids are characterized by the fact that all of their circuits are spanning. Hence $M(S)$ is uniform if and only if it has no cyclic flat that is both nonempty and not equal to S. For nonempty matroids, this is the case if and only if $\mathcal{D}(M) = \{\emptyset, S\}$. $\square$

Definition 6.10. An element x of a partially ordered set P is a *pinchpoint* if the set $\{x\}$ is a crosscut of P, that is, if all elements of P are comparable to x. A pinchpoint of P is *nontrivial* if it is neither minimal nor maximal in P.

A uniform matroid is irreducible with respect to free product if and only if its underlying set is a singleton (see Example 4.8). Irreducibility of nonuniform matroids is characterized in the following theorem.

Theorem 6.11. *For any nonuniform matroid $M(S)$, the following are equivalent*:

(i) *M is irreducible with respect to free product.*
(ii) *The lattice $\mathcal{D}(M)$ contains no nontrivial pinchpoint.*

Proof. If $A \in \mathcal{D}(M)$ is a nontrivial pinchpoint then $A \subseteq S$ is itself a nontrivial free separator, and hence M is not irreducible by Theorem 6.4. Conversely, suppose that $M(S)$ is nonuniform and has a nontrivial free separator $A \subseteq S$. Since M is nonuniform it has a cyclic flat B which is neither empty nor equal to S. If $A \subseteq B$, then the intersection of all cyclic flats of M containing A is a nontrivial pinchpoint of $\mathcal{D}(M)$. If $B \subseteq A$, then the union of all cyclic flats which are contained in A is a nontrivial pinchpoint. $\square$

For any matroid $M(S)$ we denote by $\mathcal{F}(M)$ the set of all free separators of M, ordered by inclusion. We shall see presently that $\mathcal{F}(M)$ is a lattice (in fact distributive). For all $A \subseteq B \subseteq S$, we denote by $[A, B]$ the subinterval $\{U \subseteq S\colon A \subseteq U \subseteq B\}$ of the Boolean algebra 2^S. If A and B are free separators of $M(S)$, then we write $[A, B]_{\mathcal{F}}$ for the subinterval $[A, B] \cap \mathcal{F}(M)$ of $\mathcal{F}(M)$. In the following lemma we show that an interval in the lattice of free separators of a matroid is isomorphic, under the obvious map, to the lattice of free separators of the corresponding minor of the matroid.

Lemma 6.12. *For all free separators $A \subseteq B$ of a matroid $M(S)$, the map from the interval $[A, B]_{\mathcal{F}}$ in $\mathcal{F}(M)$ to the lattice $\mathcal{F}(M(A, B))$ given by $U \mapsto U \backslash A$ is a bijection* (*and thus a lattice isomorphism*).

Proof. If $A \subseteq U \subseteq B$ are free separators of $M(S)$, then it follows from Theorems 5.3 and 6.3 that $M(A, B) = M(A, U) \,\square\, M(U, B)$, and so $U \backslash A$ is a free separator of $M(A, B)$. On the other hand, if $A \subseteq B$ are free separators of M, then M factors as $M = M|A \,\square\, M(A, B) \,\square\, M/B$, and if $V \subseteq B \backslash A$ is a free separator of $M(A, B)$, we have the factorization $M(A, B) = M(A, B)|V \,\square\, M(A, B)/V = M(A, A \cup V) \,\square\, M(A \cup V, B)$. Hence, by associativity of free product, $A \cup V$ is a free separator of M. $\square$

If $U_0 \subset \cdots \subset U_k$ is a chain in $\mathcal{F}(M)$, with $U_0 = \emptyset$ and $U_k = S$, then by Lemma 6.12, we have the factorization $M(S) = M(U_0, U_1) \,\square \cdots \square\, M(U_{k-1}, U_k)$ of M into a free product of nonempty matroids. On the other hand, given any factorization

$M(S) = M_1(S_1) \square \cdots \square M_k(S_k)$, with all M_i nonempty, the sets $U_i = S_0 \cup \cdots \cup S_i$, for $1 \leqslant i \leqslant k$, comprise a chain from $\emptyset$ to S in $\mathcal{F}(M)$. Hence the factorizations of $M(S)$ into free products of nonempty matroids are in one-to-one correspondence with chains from $\emptyset$ to S in the lattice $\mathcal{F}(M)$.

Lemma 6.13. *A matroid $M(S)$ is uniform if and only if $\mathcal{F}(M)$ is equal to the Boolean algebra 2^S.*

Proof. If $M(S)$ is uniform then the only possible cyclic flats of M are $\emptyset$ and S, and so every subset of S is a free separator of M. Conversely, if every subset of S is a free separator of M, then the only possible cyclic flats of M are $\emptyset$ and S, and thus M must be uniform. $\square$

Definition 6.14. The *primary flag* $\mathcal{T}_M$ of a matroid M is the chain $T_0 \subset \cdots \subset T_k$ consisting of all pinchpoints in the lattice $\mathcal{D}(M)$.

Note that the sets belonging to the primary flag of a matroid are, in particular, free separators, and thus the primary flag of M is a chain from $\emptyset$ to S in $\mathcal{F}(M)$.

Proposition 6.15. *If the matroid $M(S)$ has primary flag $T_0 \subset \cdots \subset T_k$, then the lattice $\mathcal{F}(M)$ of free separators of M is equal to the union of intervals $\bigcup_{i=1}^{k}[T_{i-1}, T_i]_{\mathcal{F}}$, where each interval $[T_{i-1}, T_i]_{\mathcal{F}}$ is a Boolean algebra, given by*

$$[T_{i-1}, T_i]_{\mathcal{F}} = \begin{cases} [T_{i-1}, T_i] & \text{if } T_i \text{ covers } T_{i-1} \text{ in } \mathcal{D}(M), \\ \{T_{i-1}, T_i\} & \text{otherwise}, \end{cases}$$

for $1 \leqslant i \leqslant k$.

Proof. By definition, free separators of M are comparable to all cyclic flats of M and hence comparable to all elements of $\mathcal{D}(M)$. Every free separator is thus contained in one of the intervals $[T_{i-1}, T_i]_{\mathcal{F}}$, and so $\mathcal{F}(M) = \bigcup_{i=1}^{k}[T_{i-1}, T_i]_{\mathcal{F}}$.

Suppose that T_i covers T_{i-1} in $\mathcal{D}(M)$. Since T_{i-1} and T_i are consecutive pinchpoints of $\mathcal{D}(M)$, and $\mathcal{D}(M)$ contains all cyclic flats of M, it follows that any $A \subseteq S$ with $T_{i-1} \subseteq A \subseteq T_i$ is a free separator. Hence $[T_{i-1}, T_i]_{\mathcal{F}} = [T_{i-1}, T_i]$.

Now suppose that T_i does not cover T_{i-1} in $\mathcal{D}(M)$. Choose some $D \in \mathcal{D}(M)$ such that $T_{i-1} \subset D \subset T_i$, and let $A \in [T_{i-1}, T_i]_{\mathcal{F}}$. Since A is a free separator, A must be comparable to D. If $A \subseteq D$, then the set $\{E \in \mathcal{D}(M)\colon A \subseteq E \subset T_i\}$ is nonempty, and thus the intersection F of all elements of this set is a pinchpoint of $\mathcal{D}(M)$ satisfying $A \subseteq F \subset T_i$. Since T_{i-1} and T_i are consecutive pinchpoints of $\mathcal{D}(M)$, we therefore have $A = F = T_{i-1}$. Similarly, if $D \subseteq A$, it follows that $A = T_i$. Hence $[T_{i-1}, T_i]_{\mathcal{F}} = \{T_{i-1}, T_i\}$. $\square$

Proposition 6.15 shows, in particular, that $\mathcal{F}(M)$ is a sublattice of the Boolean algebra 2^S, and therefore is a distributive lattice. Observe that the first statement of Proposition 6.15 means that, in addition to being the chain of pinchpoints in $\mathcal{D}(M)$, the primary flag $\mathcal{T}_M$ is also the chain of all pinchpoints in $\mathcal{F}(M)$, and the second statement implies that $\mathcal{D}(M) \cap \mathcal{F}(M) = \mathcal{T}_M$. If a matroid M has primary flag $T_0 \subset \cdots \subset T_k$, we refer to the minors $M(T_{i-1}, T_i)$ as the *primary factors* of M, and refer to the factorization $M = M(T_0, T_1) \,\square\, \cdots \,\square\, M(T_{k-1}, T_k)$ as the *primary factorization* of M.

Theorem 6.16. *The sequence of primary factors of a matroid M is the unique sequence* $M_1, \ldots, M_k$ *of nonempty matroids such that* $M = M_1 \,\square\, \cdots \,\square\, M_k$, *each* M_i *is either irreducible or uniform, and no free product of consecutive* M_i*'s uniform.*

Proof. Suppose that $M(S)$ factors as $M = M_1 \,\square\, \cdots \,\square\, M_\ell$. Let $\mathcal{U} = \{U_0 \subset \cdots \subset U_\ell\}$ be the corresponding chain in $\mathcal{F}(M)$, determined by $M_i = M(U_{i-1}, U_i)$, for $1 \leqslant i \leqslant \ell$, and let $\mathcal{T}_M = \{T_0 \subset \cdots \subset T_k\}$ be the primary flag of M. We show that the sequence $M_1, \ldots, M_\ell$ has the properties described in the theorem if and only if $\mathcal{U} = \mathcal{T}_M$.

Suppose that $\mathcal{U} = \mathcal{T}_M$. By Lemma 6.12 we have $\mathcal{F}(M_i) = \mathcal{F}(M(T_{i-1}, T_i)) \cong [T_{i-1}, T_i]_{\mathcal{F}}$, for $1 \leqslant i \leqslant k$. If T_i covers T_{i-1} in $\mathcal{D}(M)$, it follows from Proposition 6.15 and Lemma 6.13 that M_i is uniform; and if T_i does not cover T_{i-1} in $\mathcal{D}(M)$, then Proposition 6.15 and Theorem 6.4 imply that M_i is irreducible. For $1 \leqslant i \leqslant k-1$, we have $M_i \,\square\, M_{i+1}$ = $M(T_{i-1}, T_i) \,\square\, M(T_i, T_{i+1}) = M(T_{i-1}, T_{i+1})$, and so $\mathcal{F}(M_i \,\square\, M_{i+1}) \cong [T_{i-1}, T_{i+1}]_{\mathcal{F}}$, by Lemma 6.12. This interval has a nontrivial pinchpoint (namely, T_i), and so is not a Boolean algebra; hence by Lemma 6.13, $M_i \,\square\, M_{i+1}$ is not uniform.

For the converse, first note that, since any free separator of M is comparable with all the T_i's, it follows that the union $\mathcal{U} \cup \mathcal{T}_M$ is a chain in $\mathcal{F}(M)$. Hence if $\mathcal{T} \not\subseteq \mathcal{U}$, we can find i and j such that $T_j \in [U_{i-1}, U_i]_{\mathcal{F}}$, with T_j not equal to U_{i-1} or U_i. Then T_j is a nontrivial pinchpoint of $[U_{i-1}, U_i]_{\mathcal{F}} \cong \mathcal{F}(M(U_{i-1}, U_i))$, and hence $M_i = M(U_{i-1}, U_i)$ is neither uniform nor irreducible.

Now suppose that $\mathcal{T}$ is a proper subset of $\mathcal{U}$. We can then find some i and j such that $U_j \in [T_{i-1}, T_i]_{\mathcal{F}}$, with U_j not equal to T_{i-1} or T_i. By Proposition 6.15, we know that T_i covers T_{i-1} in $\mathcal{D}(M)$, from which it follows that $M(T_{i-1}, T_i)$ is uniform. Since $\mathcal{T} \subseteq \mathcal{U}$, we have $T_{i-1} \subseteq U_{j-1}$ and $U_{j+1} \subseteq T_i$; hence the free product $M_j \,\square\, M_{j+1} = M(U_{j-1}, U_j) \,\square\, M(U_j, U_{j+1}) = M(U_{j-1}, U_{j+1})$ is a minor of $M(T_{i-1}, T_i)$ and is thus uniform. □

Theorem 6.16 shows that matroids factor uniquely as free products of minors that are either irreducible or "maximally" uniform. We now wish to consider factorization of matroids into irreducibles. Clearly, given a factorization $M(S) = M(U_0, U_1) \,\square\, \cdots \,\square\, M(U_{k-1}, U_k)$, the factors $M(U_{i-1}, U_i)$ are all irreducible if and only if $U_0 \subset \cdots \subset U_k$ is a maximal chain in the lattice of free separators $\mathcal{F}(M)$. If $M(S) = U_{r,n}$ is uniform of rank r, then any maximal chain in $\mathcal{F}(M) = 2^S$, or equivalently, any ordering $s_1, \ldots, s_n$ of the elements of S, gives a factorization

$$M = I(s_1) \,\square\, \cdots \,\square\, I(s_r) \,\square\, Z(s_{r+1}) \,\square\, \cdots \,\square\, Z(s_n)$$

of M into irreducibles (see Example 4.8). The factorization of a uniform matroid into irreducibles is thus in general far from unique. Up to isomorphism, or course, we do have the unique factorization $U_{r,n} = I^r \,\square\, Z^{n-r}$. In the next theorem we show that, up to isomorphism, arbitrary matroids factor uniquely into irreducibles.

Theorem 6.17. *If* $M \cong M_1 \square \cdots \square M_k \cong N_1 \square \cdots \square N_r$, *where all the* M_i *and* N_j *are irreducible, then* $k = r$ *and* $M_i \cong N_i$, *for* $1 \leqslant i \leqslant k$.

Proof. Since the sets T_i belonging to the primary flag $\mathcal{T}_M$ of M are all pinchpoints of $\mathcal{F}(M)$, it follows that any maximal chain in $\mathcal{F}(M)$ is a refinement of $\mathcal{T}_M$. Hence any factorization of M into irreducibles can be obtained by starting with the primary factorization $M = M(T_0, T_1) \square \cdots \square M(T_{\ell-1}, T_\ell)$, then factoring each $M(T_{i-1}, T_i)$ into irreducibles. Since each $M(T_{i-1}, T_i)$ is either irreducible or uniform, and uniform matroids factor into irreducibles uniquely up to isomorphism, it follows that the factorization of M into irreducibles is unique up to isomorphism. □

The unique factorization theorem (Theorem 6.17) provides a quick proof of the following theorem, which was the main result in [4]:

Theorem 6.18. *Suppose that* $M(S) \square N(T) \cong P(U) \square Q(V)$, *where* $|S| = |U|$. *Then* $M \cong P$ *and* $N \cong Q$.

Proof. Since $M \square N$ and $P \square Q$ have, up to isomorphism, the same factorization into irreducibles, it follows from the fact that $|S| = |U|$ and $|T| = |V|$, that $M \cong P$ and $N \cong Q$. □

For all $n \geqslant 0$, denote by m_n and i_n, respectively, the number of isomorphism classes of matroids and irreducible matroids on n elements, and let $M(t) = \sum_{n \geqslant 0} m_n t^n$ and $I(t) = \sum_{n \geqslant 0} i_n t^n$ be the ordinary generating functions for these numbers. For all $r, k, \geqslant 0$, denote by $m_{r,k}$ and $i_{r,k}$, respectively, the number of isomorphism classes of matroids and irreducible matroids having rank r and nullity k, and let $M(x, y) = \sum_{r,k \geqslant 0} m_{r,k} x^r y^k$ and $I(x, y) = \sum_{r,k \geqslant 0} i_{r,k} x^r y^k$.

Corollary 6.19. *The generating functions* $M(t)$ *and* $I(t)$, *and* $M(x, y)$ *and* $I(x, y)$ *satisfy*

$$M(t) = \frac{1}{1 - I(t)} \qquad \text{and} \qquad M(x, y) = \frac{1}{1 - I(x, y)}.$$

Proof. Unique factorization implies that, for all $n \geqslant 0$,

$$m_n = \sum_{j \geqslant 0} \sum_{n_1 + \cdots + n_j = n} i_{n_1} \cdots i_{n_j},$$

which is the coefficient of t^n in $\sum_{j \geqslant 0} I(t)^j = 1/(1 - I(t))$. The second equation is proved similarly. □

Using Corollary 6.19, we compute the numbers i_n and $i_{r,k}$ in terms of the values of m_n and $m_{r,k}$, for $n, r + k \leqslant 8$. The results are shown in Tables 1 and 2.

The two matroids of size one, namely, the point I and loop Z, are irreducible, and no matroid of size two or three is irreducible. The unique irreducible matroid on four elements is the pair of double points $U_{1,2} \oplus U_{1,2}$. The two irreducible matroids on five elements

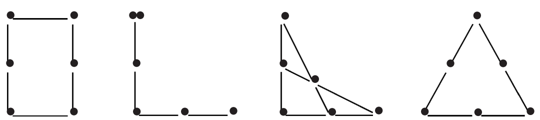
Table 1
The numbers of nonisomorphic matroids, irreducible matroid, of size n, for $0 \leqslant n \leqslant 8$

n	0	1	2	3	4	5	6	7	8
Matroids	1	2	4	8	17	38	98	306	1724
Irreducible matroids	0	2	0	0	1	2	14	66	891

Table 2
The numbers of nonisomorphic matroids (left), irreducible matroids (right), of rank r and nullity k, for $0 \leqslant r+k \leqslant 8$

r	k																	
	0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1	0	1	0	0	0	0	0	0	0
1	1	2	3	4	5	6	7	8		1	0	0	0	0	0	0	0	
2	1	3	7	13	23	37	58			0	0	1	1	3	3	6		
3	1	4	13	38	108	325				0	0	1	8	30	125			
4	1	5	23	108	940					0	0	3	30	629				
5	1	6	37	325						0	0	3	125					
6	1	7	58							0	0	6						
7	1	8								0	0							
8	1									0								

are $U_{1,3} \oplus U_{1,2}$ and its dual $U_{2,3} \oplus U_{1,2}$. On six elements, the irreducibles of rank two are $U_{1,4} \oplus U_{1,2}$, $U_{1,3} \oplus U_{1,3}$ and the truncation $T(U_{1,2} \oplus U_{1,2} \oplus U_{1,2})$, which consists of three collinear double points. The duals of these matroids, $U_{3,4} \oplus U_{1,2}$, $U_{2,3} \oplus U_{2,3}$ and $L(U_{1,2} \oplus U_{1,2} \oplus U_{1,2})$, are the six-element irreducibles of rank four. Finally, on six elements in rank three, the irreducible matroids consist of $U_{2,4} \oplus U_{1,2}$, $U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$, $U_{1,3} \oplus U_{2,3}$, and $U'_{2,3} \oplus U_{1,2}$, where $U'_{2,3}$ is the three-point line $U_{2,3}$, with one point doubled, together with the four matroids shown below:



Since the dual of an irreducible matroid is irreducible, the set of rank-three irreducible matroids on six elements must be closed under duality; in fact, each matroid in this set is self-dual.

7. The minor coalgebra

In this section, and the next, we work over some commutative ring K with unit. All modules, algebras and coalgebras are over K, all maps between such objects are assumed to be K-linear, and all tensor products are taken over K. Given a family of matroids $\mathcal{M}$, we denote by $K\{\mathcal{M}\}$ the free K-module having as basis all isomorphism classes of matroids belonging to $\mathcal{M}$. In what follows, we shall not distinguish notationally between a matroid M and its isomorphism class, or between a family of matroids $\mathcal{M}$ and the set of isomorphism classes of matroids belonging to $\mathcal{M}$; it should always be clear from the context which is meant.

If $\mathcal{M}$ is a minor-closed family, then the *minor coalgebra* [5,8] of $\mathcal{M}$ is the free module $K\{\mathcal{M}\}$, equipped with *restriction–contraction coproduct* δ determined by

$$\delta(M) = \sum_{A \subseteq S} M|A \otimes M/A$$

and counit determined by $\varepsilon(M) = \delta_{\emptyset,S}$, for all $M = M(S)$ in $\mathcal{M}$. If $\mathcal{M}$ is also closed under formation of direct sums, then $K\{\mathcal{M}\}$ is a Hopf algebra, with product determined on the basis $\mathcal{M}$ by direct sum. For any minor-closed family $\mathcal{M}$, the coalgebra $K\{\mathcal{M}\}$ is bigraded, with homogeneous component $K\{\mathcal{M}\}_{r,k}$ spanned by all isomorphism classes of matroids in $\mathcal{M}$ having rank r and nullity k. When $\mathcal{M}$ is also closed under direct sum, this is a Hopf algebra bigrading.

For all matroids N_1, N_2 and $M = M(S)$, the *section coefficient* $\binom{M}{N_1, N_2}$ is the number of subsets A of S such that $M|A \cong N_1$ and $M/A \cong N_2$; hence if $\mathcal{M}$ is a minor-closed family, the restriction–contraction coproduct satisfies

$$\delta(M) = \sum_{N_1, N_2} \binom{M}{N_1, N_2} N_1 \otimes N_2, \tag{7.1}$$

for all $M \in \mathcal{M}$, where the sum is taken over all (isomorphism classes of) matroids N_1 and N_2. More generally, for matroids $N_1, \dots, N_k$ and $M = M(S)$, the *multisection coefficient* $\binom{M}{N_1,\dots,N_k}$ is the number of sequences $(S_0, \dots, S_k)$ such that $\emptyset = S_0 \subseteq \cdots \subseteq S_k = S$ and the minor $M(S_{i-1}, S_i)$ is isomorphic to N_i, for $1 \leqslant i \leqslant k$. The iterated coproduct $\delta^{k-1}\colon K\{\mathcal{M}\} \to K\{\mathcal{M}\} \otimes \cdots \otimes K\{\mathcal{M}\}$ is thus determined by

$$\delta^{k-1}(M) = \sum_{N_1,\dots,N_k} \binom{M}{N_1, \dots, N_k} N_1 \otimes \cdots \otimes N_k,$$

for all $M \in \mathcal{M}$.

For any family of matroids $\mathcal{M}$, we define a pairing $\langle \cdot, \cdot \rangle\colon K\{\mathcal{M}\} \times K\{\mathcal{M}\} \to K$ by setting $\langle M, N \rangle = \delta_{M,N}$, for all $M, N \in \mathcal{M}$, and thus identify the graded dual module $K\{\mathcal{M}\}^*$ with the free module $K\{\mathcal{M}\}$. In the case that $\mathcal{M}$ is minor-closed, we refer to the (graded) dual algebra $K\{\mathcal{M}\}^*$ as the *minor algebra* of $\mathcal{M}$; the product in the minor algebra is thus determined by

$$M \cdot N = \sum_{L \in \mathcal{M}} \binom{L}{M, N} L,$$

for all $M, N \in \mathcal{M}$.

We partially order the set of all isomorphism classes of matroids by setting $M \geqslant N$ if and only if there exists a bijective weak map from M to N. The following result provides us with critical necessary conditions for a matroid to appear in a given product of matroids in $K\{\mathcal{M}\}^*$.

Proposition 7.2. *For all matroids L, M and N,*

$$\binom{L}{M, N} \neq 0 \quad \Longrightarrow \quad M \oplus N \leqslant L \leqslant M \,\square\, N.$$

Proof. Suppose that $M = M(S)$ and $N = N(T)$. Given a matroid L such that $\binom{L}{M,N} \neq 0$ we may assume that $L = L(S + T)$, where $L|S = M$ and $L/S = N$. The semimodularity of ρ_L implies that $\rho_L(A_S) + \rho_L(S \cup A) \leqslant \rho_L(S) + \rho_L(A)$, for all $A \subseteq S + T$, and so $\rho_{M\oplus N}(A) = \rho_M(A_S) + \rho_N(A_T) = \rho_L(A_S) + \rho_L(S \cup A) - \rho_L(S) \leqslant \rho_L(A)$, and hence the identity on $S + T$ is a weak map $L \to M \oplus N$. On the other hand, according to Proposition 4.2, the identity on $S + T$ is a weak map $M \,\square\, N \to L$; hence $M \oplus N \leqslant L \leqslant M \,\square\, N$. $\square$

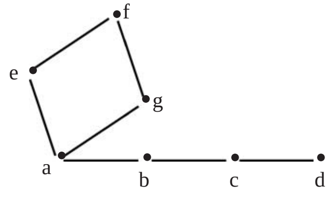
Similarly, using Proposition 4.7 instead of Proposition 4.2, we obtain

$$\binom{L}{M_1, \ldots, M_k} \neq 0 \quad \Longrightarrow \quad M_1 \oplus \cdots \oplus M_k \leqslant L \leqslant M_1 \,\square\, \cdots \,\square\, M_k, \tag{7.3}$$

for all L and $M_1, \ldots, M_k \in \mathcal{M}$.

The following example shows that the converse of Proposition 7.2 does not hold.

Example 7.4. Suppose L is the rank 4 matroid on the set $U = \{a, b, c, d, e, f, g\}$ pictured below.



If M is a three point line on the set $\{a, b, c\}$, and N is a four point line on $\{d, e, f, g\}$, then the free product $M \,\square\, N$ consists of a three point line on $\{a, b, c\}$, together with points d, e, f, g in general position in 3-space, and the identity map on U is thus a weak map $M \,\square\, N \to L$. Now if M' is a three point line on $\{e, f, g\}$ and N' is a four point line on $\{a, b, c, d\}$, then the identity on U is a weak map $L \to M' \oplus N'$. Since $M \cong M'$ and $N \cong N'$, we thus have $M \oplus N \leqslant L \leqslant M \,\square\, N$. But L has no three point line as a restriction with a four point line as complementary contraction, and so $\binom{L}{M,N} = 0$.

If a family $\mathcal{M}$ is closed under formation of free products then $K\{\mathcal{M}\}$, with product determined by the free product on the basis $\mathcal{M}$, is an associative algebra. We denote $K\{\mathcal{M}\}$, equipped with this algebra structure, by $K\{\mathcal{M}\}_\square$.

Proposition 7.5. *If $\mathcal{M}$ is a free product-closed family of matroids, then the algebra $K\{\mathcal{M}\}_{\Box}$ is free, generated by the set of irreducible matroids belonging to $\mathcal{M}$.*

Proof. Because the set $\mathcal{M}$ is a basis for $K\{\mathcal{M}\}_{\Box}$, the result follows directly from unique factorization, Theorem 6.17. □

For all matroids M and N, we denote by $c(N, M)$ the section coefficient $\binom{N}{M_1,\ldots,M_k}$, where $M_1, \ldots, M_k$ is the sequence of irreducible factors of M.

Theorem 7.6. *Suppose that $\mathcal{M}$ is a family of matroids that is closed under formation of minors and free products. If K is a field of characteristic zero, then the minor algebra $K\{\mathcal{M}\}^*$ is free, generated by the set of irreducible matroids belonging to $\mathcal{M}$.*

Proof. For each matroid M belonging to $\mathcal{M}$, let P_M denote the product $M_1 \cdots M_k$ in $K\{\mathcal{M}\}^*$, where $M_1, \ldots, M_k$ is the sequence of irreducible factors of M. We can write

$$P_M = \sum_N c(N, M)\, N,$$

where, by (7.3), the sum is taken over all $N \in \mathcal{M}$ such that $N \leqslant M$ in the weak order. Since $c(M, M) \neq 0$, for all matroids M, and K is a field of characteristic zero, it thus follows from the fact that $\mathcal{M}$ is a basis for $K\{\mathcal{M}\}^*$ that $\{P_M : M \in \mathcal{M}\}$ is also a basis for $K\{\mathcal{M}\}^*$. The map $K\{\mathcal{M}\}_{\Box} \to K\{\mathcal{M}\}^*$ determined by $M \mapsto P_M$, which is clearly an algebra homomorphism, is thus bijective and hence an algebra isomorphism. Since $P_M = M$, whenever $M \in \mathcal{M}$ is irreducible, the result follows from Proposition 7.5. □

Example 7.7. The family $\mathcal{M}$ of all matroids is minor-closed and closed under free product. Hence $K\{\mathcal{M}\}^*$ is the free algebra generated by the set of all (isomorphism classes of) irreducible matroids.

Example 7.8. The family $\mathcal{F}$ of freedom matroids (see Example 4.12) is minor-closed and closed under free product. Since all freedom matroids can be expressed as free products of points and loops, it follows that $K\{\mathcal{F}\}^*$ is the free algebra generated by I and Z.

Example 7.9. For any field F, the class $\mathcal{M}_F$ of all F-representable matroids is minor-closed. It follows from Proposition 4.13 that if F is infinite then $\mathcal{M}_F$ is also closed under formation of free products.

Example 7.10. It follows from Proposition 4.14 that the family $\mathcal{T}$ of all transversal matroids is closed under formation of free products. However, since contractions of transversal matroids are not in general transversal, $\mathcal{T}$ is not minor-closed.

Proposition 7.11. *If a family $\mathcal{M}$ of matroids is minor-closed and closed under formation of free products, then $\mathcal{M}$ is also closed under the lift and truncation operations.*

Proof. Suppose that $\mathcal{M}$ is minor-closed and closed under formation of free products. If $\mathcal{M}$ is the class of all free matroids or the class of all zero matroids, or consists only of the empty matroid, then $\mathcal{M}$ is closed under lift and truncation. If $\mathcal{M}$ is none of the above, then it must contain the matroids I and Z. By Proposition 5.2, we have

$$LM = (I \,\square\, M(S))|S \quad \text{and} \quad TN = (M \,\square\, Z(a))/a,$$

for any matroid $M = M(S)$. Hence, if M belongs to $\mathcal{M}$ then so do LM and TM. □

Suppose that $\mathcal{M}$ and K satisfy the hypotheses of Theorem 7.6, and that $\mathcal{M}$ is partially ordered by the weak order. The fact that $c(M, N) \neq 0$ implies $M \leqslant N$, for all $M, N \in \mathcal{M}$, means that we may regard c as an element of the incidence algebra $I(\mathcal{M})$ of the poset $\mathcal{M}$. Since $c(M, M)$ is invertible in K, for all M, it follows that c is invertible in $I(\mathcal{M})$, the inverse given recursively by $c^{-1}(M, M) = c(M, M)^{-1}$, for $M \in \mathcal{M}$, and

$$c^{-1}(M, N) = -c(N, N)^{-1} \sum_{M \leqslant P < N} c^{-1}(M, P)\, c(P, N),$$

for all $M < N$ in $\mathcal{M}$. The inverse of the change of basis map $M \mapsto P_M$ is thus given by

$$M = \sum_{N \leqslant M} c^{-1}(N, M) P_N,$$

for all $M \in \mathcal{M}$. Let $\{Q_M : M \in \mathcal{M}\}$ be the basis of $K\{\mathcal{M}\}$ determined by $\langle Q_M, P_N \rangle = \delta_{M,N}$, for all $M, N \in \mathcal{M}$. Observe that Q_M satisfies

$$Q_M = \sum_{N \geqslant M} c^{-1}(M, N) N, \tag{7.12}$$

for all $M \in \mathcal{M}$. Before stating the next theorem, which is dual to Theorem 7.6, we note that, for any minor-closed family $\mathcal{M}$, the minor coalgebra $K\{\mathcal{M}\}$ is connected, with the empty matroid as unique group-like element. In particular, it follows that the notion of primitive element of $K\{\mathcal{M}\}$ is unambiguous.

Theorem 7.13. *Suppose that $\mathcal{M}$ is a family of matroids that is closed under formation of minors and free products. If K is a field of characteristic zero, then the minor coalgebra $K\{\mathcal{M}\}$ is cofree. The set $\{Q_M : M \in \mathcal{M}$ is irreducible$\}$ is a basis for the subspace of primitive elements of $K\{\mathcal{M}\}$.*

Proof. The fact that $K\{\mathcal{M}\}$ is cofree is equivalent to the fact that $K\{\mathcal{M}\}^*$ is free, which was shown in Theorem 7.6. Let $\varphi: K\{\mathcal{M}\}_\square \to K\{\mathcal{M}\}^*$ be the algebra isomorphism used in the proof of Theorem 7.6, given by $M \mapsto P_M$, for all $M \in \mathcal{M}$. The transpose $\varphi^*: K\{\mathcal{M}\} \to K\{\mathcal{M}\}^*_\square$ is thus a coalgebra isomorphism. For all $M, N \in \mathcal{M}$, we have $\langle \varphi^*(Q_M), N \rangle = \langle Q_M, \varphi(N) \rangle = \langle Q_M, P_N \rangle = \delta_{M,N}$, and hence $\varphi^*(Q_M) = M$. Since the set of all irreducible $M \in \mathcal{M}$ is a basis for the subspace of primitive elements of $K\{\mathcal{M}\}^*_\square$, it follows that $\{Q_M : M \in \mathcal{M}$ is irreducible$\}$ is a basis for the subspace of primitive elements of $K\{\mathcal{M}\}$. □

Example 7.14. Suppose that $\mathcal{M}$ is closed under formation of minors and free products, and that $\mathcal{M}$ contains the irreducible matroid $D = U_{1,2} \oplus U_{1,2}$, consisting of two double points. Since $\mathcal{M}$ is minor-closed, it contains the (irreducible) single-element matroids I and Z. Since $\mathcal{M}$ is also closed under free product, it follows from Table 1 and unique factorization that $\mathcal{M}$ contains all matroids of size less than or equal to four (all such matroids, except D, being free products of I's and Z's).

It is clear from Eq. (7.12) that the primitive elements Q_I and Q_Z in $K\{\mathcal{M}\}$ are equal to I and Z, respectively. In order to compute Q_D, we first observe that $\{N : N > D \text{ in } \mathcal{M}\}$ consists of the two matroids $U_{2,4} = I \,\square\, I \,\square\, Z \,\square\, Z$ and $P = I \,\square\, Z \,\square\, I \,\square\, Z$. Since P is a three point line, with one point doubled, we have $D \leqslant P \leqslant U_{2,4}$. The multisection coefficients $c(M, N)$, for all $M, N \geqslant D$, are given by the matrix

$$\begin{array}{c c} & \begin{array}{ccc} D & P & U_{2,4} \end{array} \\ \begin{array}{c} D \\ P \\ U_{2,4} \end{array} & \begin{pmatrix} 1 & 8 & 16 \\ 0 & 4 & 20 \\ 0 & 0 & 24 \end{pmatrix} \end{array}$$

and the numbers $c^{-1}(M, N)$, for $M, N \geqslant D$, are thus given by the inverse matrix

$$\frac{1}{24}\begin{pmatrix} 24 & -48 & 24 \\ 0 & 6 & -5 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $Q_D = D - 2P + U_{2,4}$.

8. A new twist

If a family of matroids $\mathcal{M}$ is both minor and free product-closed, then the module $K\{\mathcal{M}\}$ has both the structure of a (free) associative algebra, under free product, and a coassociative coalgebra, with restriction–contraction coproduct. Moreover, according to Theorem 7.6, when the ring of scalars is a field of characteristic zero, these structures are dual to one another. In this section we show that free product and restriction–contraction coproduct are compatible in the sense that $K\{\mathcal{M}\}$ is a Hopf algebra in an appropriate braided monoidal category.

By a *matroid module*, we shall mean a free module $K\{\mathcal{M}\}$, where $\mathcal{M}$ is a family of matroids that is closed under formation of lifts and truncations. Given matroid modules $V = K\{\mathcal{M}\}$ and $W = K\{\mathcal{N}\}$, we define the *twist map* $\tau = \tau_{V,W} : V \otimes W \to W \otimes V$ by

$$\tau(M \otimes N) \;=\; L^{\rho(M)} N \otimes T^{\nu(N)} M, \tag{8.1}$$

for all $M \in \mathcal{M}$ and $N \in \mathcal{N}$. If the families $\mathcal{M}$ and $\mathcal{N}$ are also closed under formation of free products, we use the twist map to extend the definition of the free product to a binary operation on $V \otimes W$:

$$(M \otimes N) \,\square\, (P \otimes Q) \;=\; (M \,\square\, L^{\rho(N)} P) \otimes (T^{\nu(P)} N \,\square\, Q), \tag{8.2}$$

for all $M, P \in \mathcal{M}$ and $N, Q \in \mathcal{N}$.

Proposition 8.3. *For all families $\mathcal{M}$ and $\mathcal{N}$, closed under free product, lift and truncation, the product $\square$ given by Eq.* (8.2) *is an associative operation on $K\{\mathcal{M}\} \otimes K\{\mathcal{N}\}$.*

Proof. Suppose that $M_i \in \mathcal{M}$ and $N_i \in \mathcal{N}$, and let $v_i = v(M_i)$ and $\rho_i = \rho(N_i)$, for $1 \leqslant i \leqslant 3$. Then

$$\begin{aligned}
&[(M_1 \otimes N_1) \square (M_2 \otimes N_2)] \square (M_3 \otimes N_3)\\
&\quad = [(M_1 \square L^{\rho_1} M_2) \otimes (T^{v_2} N_1 \square N_2)] \square (M_3 \otimes N_3)\\
&\quad = (M_1 \square L^{\rho_1} M_2 \square L^i M_3) \otimes (T^{v_3}(T^{v_2} N_1 \square N_2) \square N_3)\\
&\quad = (M_1 \square L^{\rho_1} M_2 \square L^i M_3) \otimes (T^k N_1 \square T^{v_3} N_2 \square N_3),
\end{aligned}$$

where $i = \rho(T^{v_2} N_1 \square N_2) = \rho_2 + \max\{\rho_1 - v_2, 0\}$ and, by Eq. (5.5), we have $k = v_2 + \max\{v_3 - \rho_2, 0\}$. On the other hand,

$$\begin{aligned}
&(M_1 \otimes N_1) \square [(M_2 \otimes N_2) \square (M_3 \otimes N_3)]\\
&\quad = (M_1 \otimes N_1) \square [(M_2 \square L^{\rho_2} M_3) \otimes (T^{v_3} N_2 \square N_3)]\\
&\quad = (M_1 \square L^{\rho_1}(M_2 \square L^{\rho_2} M_3)) \otimes (T^j N_1 \square T^{v_3} N_2 \square N_3)\\
&\quad = (M_1 \square L^{\rho_1} M_2 \square L^s M_3) \otimes (T^j N_1 \square T^{v_3} N_2) \square N_3),
\end{aligned}$$

where $j = v(M_2 \square L^{\rho_2} M_3) = v_2 + \max\{v_3 - \rho_2, 0\}$ and, by Eq. (5.5), we have $s = \rho_2 + \max\{\rho_1 - v_2, 0\}$. Since $s = i$ and $j = k$, the two parenthesizations of $(M_1 \otimes N_1) \square (M_2 \otimes N_2) \square (M_3 \otimes N_3)$ are thus equal. $\square$

Proposition 8.4. *If the family $\mathcal{M}$ is minor and free product-closed* (*and thus also closed under lift and truncation*), *then the restriction–contraction coproduct δ is compatible with the free product on $K\{\mathcal{M}\}$, in the sense that δ: $K\{\mathcal{M}\} \to K\{\mathcal{M}\} \otimes K\{\mathcal{M}\}$ is an algebra map.*

Proof. Suppose that $M(S)$ and $N(T)$ belong to $\mathcal{M}$. Using Proposition 5.2, we compute the coproduct of $M \square N$:

$$\begin{aligned}
\delta(M \square N) &= \sum_{A \subseteq S+T} (M \square N)|A \otimes (M \square N)/A\\
&= \sum_{A \subseteq S+T} (M|A_S \square L^{\lambda_M(A_S)} N|A_T) \otimes (T^{v_N(A_T)} M/A_S \square N/A_T)\\
&= \sum_{A \subseteq S+T} (M|A_S \square L^{\rho(M/A_S)} N|A_T) \otimes (T^{v(N|A_T)} M/A_S \square N/A_T)\\
&= \sum_{A \subseteq S+T} (M|A_S \otimes M/A_S) \square (N|A_T \otimes N/A_T),
\end{aligned}$$

which is equal to $\delta(M) \square \delta(N)$. $\square$

We conclude by outlining a categorical framework for these results. Let **M** be the category whose objects are bigraded K-modules $V = \bigoplus_{r,k \geqslant 0} V_{r,k}$, equipped with linear operators

$L = L_V$ and $T = T_V$ satisfying

(i) $L: V_{r,k} \to V_{r+1,k-1}$, if $k > 0$ and $L|V_{r,0} = id_{V_{r,0}}$,
(ii) $T: V_{r,k} \to V_{r-1,k+1}$, if $r > 0$ and $T|V_{0,k} = id_{V_{0,k}}$,
(iii) $TL = LT$, when restricted to $\bigoplus_{r,k \geqslant 1} V_{r,k}$.

We assume that each homogenous component $V_{r,k}$ is a free K-module of finite rank and that $V_{r,0}$ and $V_{0,k}$ have rank one, for all $r, k \geqslant 0$. For homogeneous $x \in V$, we write $\rho(x) = r$ and $\nu(x) = k$ to indicate that x belongs to $V_{r,k}$. The morphisms of **M** are the K-linear maps which commute with L and T. For all objects V and W in **M**, we suppose that the tensor product $V \otimes W$ is bigraded in the usual manner, with

$$(V \otimes W)_{r,k} = \bigoplus_{\substack{r_1+r_2=r \\ k_1+k_2=k}} (V_{r_1,k_1} \otimes W_{r_2,k_2}),$$

for all $r, k \geqslant 0$, and the operators $L = L_{V \otimes W}$ and $T = T_{V \otimes W}$ satisfy

$$L(x \otimes y) = \begin{cases} (Lx) \otimes y & \text{if } \nu(x) > 0, \\ x \otimes Ly & \text{if } \nu(x) = 0 \end{cases}$$

and

$$T(x \otimes y) = \begin{cases} x \otimes Ty & \text{if } \rho(y) > 0, \\ (Tx) \otimes y & \text{if } \rho(y) = 0, \end{cases}$$

for all homogeneous $x \in V$ and $y \in W$; hence **M** is a monoidal category. For all objects V and W in **M** we define the twist map $\tau = \tau_{V,W}: V \otimes W \to W \otimes V$ as in Eq. (8.1), that is, by $\tau(x \otimes y) = L^{\rho(x)} y \otimes T^{\nu(y)} x$, for homogeneous $x \in V$ and $y \in W$. It is readily verified that the twist maps $\tau_{V,W}$ commute with the operators L and T, and so are morphisms in **M**; furthermore, the maps $\tau_{V,W}$ are the components of a natural transformation $\tau: \otimes \Rightarrow \otimes^{\mathrm{op}}$, that is, $(g \otimes f) \circ \tau_{V,W} = \tau_{V',W'} \circ (f \otimes g)$, for all morphisms $f: V \to V'$ and $g: W \to W'$ in **M**. It is then a simple matter to verify that the natural transformation τ satisfies the braid relations:

$$\tau_{U \otimes V, W} = (\tau_{U,W} \otimes 1_V) \circ (1_U \otimes \tau_{V,W}) \quad \text{and} \quad \tau_{U, V \otimes W} = (1_V \otimes \tau_{U,W}) \circ (\tau_{U,V} \otimes 1_W),$$

for all objects U, V, W. Note that the maps $\tau_{V,W}$ are not necessarily isomorphisms in **M** (because different matroids may have the same lifts or truncations). Hence, as long as we allow a notion of braiding that does not require the component morphisms to be isomorphisms, it follows that **M** is a braided monoidal category.

We regard each matroid module $K\{\mathcal{M}\}$ as an object of **M**, bigraded by rank and nullity, with operators L and T determined by lift and truncation on the basis $\mathcal{M}$. If $V = K\{\mathcal{M}\}$, and the family of matroids $\mathcal{M}$ is closed under free product, as well as lift and truncation, then it follows immediately from Proposition 5.4 and the definition of L and T on $V \otimes V$ that the map $\mu_V: V \otimes V \to V$ given by $M \otimes N \mapsto M \,\square\, N$, for all $M, N \in \mathcal{M}$, is a morphism in **M**, and hence V is a monoid object in **M**.

Suppose that $V = K\{\mathcal{M}\}$ and $W = K\{\mathcal{N}\}$ are matroid modules with $\mathcal{M}$ and $\mathcal{N}$ free product-closed. The operation $\square$ on $V \otimes W$ defined by Eq. (8.2) is the composition $\mu_{V \otimes W} =$

$(\mu_V \otimes \mu_W) \circ (1_V \otimes \tau_{V,W} \otimes 1_W)$, which is the standard monoid structure on the product of monoid objects in a braided monoidal category. Associativity of $\mu_{V \otimes W}$ (our Proposition 8.3) follows immediately from the braid relations and the associativity of μ_V and μ_W.

Finally, we note that if $V = K\{\mathcal{M}\}$ is a matroid module, where $\mathcal{M}$ is minor-closed, then the restriction–contraction coproduct δ: $V \to V \otimes V$ commutes with L and T, and so V is a comonoid object in **M**. If $\mathcal{M}$ is also closed under free product, then Proposition 8.4 says that V is a bialgebra in the braided monoidal category **M**. Since V is a connected bialgebra, it is in fact a Hopf algebra, with antipode given by the usual formula. Furthermore, it follows from the proof of Theorem 7.6 that this Hopf algebra is self-dual.

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