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Automorphisms of Group Extensions and Differentials in the Lyndon–Hochschild–Serre Spectral Sequence

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1. INTRODUCTION

Let

$$1 \to N \xrightarrow{i} G \xrightarrow{p} Q \to 1 \tag{1.1}$$

be a group extension and A a (left) G-module. We note, for clarity, that the extension (1.1) and the module A will always be fixed unless the contrary is admitted explicitly. We consider the Lyndon–Hochschild–Serre (LHS) spectral sequence

$$(E_r^{p,q}, d_r^{p,q}), \qquad \text{with} \qquad E_2^{p,q} = H^p(Q, H^q(N, A))$$

[7; 16, p. 351]. The purpose of this paper is to examine the differentials

$$d_2^{0,2}\colon E_2^{0,2} \to E_2^{2,1}, \qquad d_2^{1,1}\colon E_2^{1,1} \to E_2^{3,0}, \qquad d_3^{0,2}\colon E_3^{0,2} \to E_3^{3,0}$$

(we shall usually drop the superscripts and write d_r instead of $d_r^{p,q}$) and the transgression $\tau\colon E_2^{0,2} \rightharpoonup E_2^{3,0}$ ("$\rightharpoonup$" denotes an additive relation). We shall give explicit descriptions (see Section 2 below) in terms of group extensions, crossed 2-fold extensions (see below) and certain automorphisms groups. Our descriptions also turn out to be natural in a strong sense. We note that similar automorphism groups were studied in [20].

The results of this paper were announced in [11]. The differentials we describe in this paper yield, together with the differential d_2: $H^0(Q, H^1(N, A)) \to H^2(Q, A^N)$, all the information about $H^2(G, A)$ that can be obtained from the spectral sequence. This has been pushed further in [13],

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where we have constructed a certain extension $\mathrm{Xpext}(G, N; A)$ of $E_2^{1,1}$ by $E_3^{0,2}$ which fits into a natural exact sequence

$$H^2(Q, A^N) \to H^2(G, A) \to \mathrm{Xpext}(G, N; A) \xrightarrow{\Delta} H^3(Q, A^N) \to H^3(G, A)$$

such that Δ lifts the differential $d_3^{0,3}$ (this was announced in [12]). We also note that in [13] a conceptual description of $d_2\colon E_2^{0,1} \to E_2^{2,0}$ was obtained.

In another paper [14] we shall extend our methods to obtain conceptual descriptions of all differentials

$$d_2^{0,q}\colon E_2^{0,q} \to E_2^{2,q-1} \qquad \text{and} \qquad d_2^{1,q}\colon E_2^{1,q} \to E_2^{3,q-1}, \qquad q > 1.$$

The paper is organised in the following manner: In Section 2 we present our results (Theorems 1, 2 and 3). Section 3 deals with some differentials in the LHS spectral sequence. In Sections 4–6 we prove our theorems. Section 7 offers an example.

Central roles will be played by the concept of a *crossed module* and that of a *crossed 2-fold extension*, the definitions of which we reproduce here for completeness: *A crossed module* (C, Γ, ∂) (Whitehead [23, p. 453]) consists of groups C and Γ, an action of Γ on the left of C, written $(\gamma, c) \mapsto {}^{\gamma}c$, $\gamma \in \Gamma$, $c \in C$, and a homomorphism $\partial\colon C \to \Gamma$ of Γ-groups, where Γ acts on itself by conjugation. The map ∂ must satisfy the rule

$$bcb^{-1} = {}^{\partial(b)}c, \qquad b, c \in C.$$

A *crossed 2-fold extension* ([9] or [10, Sect. 3]) is an exact sequence of groups

$$e^2\colon 0 \to A \to C \xrightarrow{\partial} \Gamma \to Q \to 1,$$

where (C, Γ, ∂) is a crossed module. The group A is then central in C, whence it is Abelian; furthermore, the Γ-action on C induces a Q-action on A. For Q and A fixed, the classes of crossed 2-fold extensions under the similarity relation generated by morphisms $(1, \cdot, \cdot, 1)\colon e^2 \to \hat{e}^2$ of crossed 2-fold extensions constitute an Abelian group naturally isomorphic to the cohomology group $H^3(Q, A)$; this is a special case of the main Theorem in Section 7 of [10] (see also [9]). We note that such an interpretation of group cohomology was found independently by several other people; see Mac Lane [15]. Here we would like to point out, however, that the interpretation of the third cohomology group in terms of crossed 2-fold extensions, although not explicitly recognised, is hidden in an old paper of Mac Lane and Whitehead [17].

2. Results

2.1. *Automorphisms and Group Extensions*

Let Γ be a group and A a (left) Γ-module. Let $\chi_0: \Gamma \to \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(A)$ be the obvious map given by $\chi_0(\gamma) = (i_\gamma, l_\gamma)$, $\gamma \in \Gamma$, where i_γ is the corresponding inner automorphism and l_γ the action $l_\gamma(a) = {}^\gamma a$ of γ on A; here $\mathrm{Aut}(\Gamma)$ denotes the group of automorphisms of Γ, and $\mathrm{Aut}(A)$ that of A as Abelian group. Denote by $\mathrm{Aut}(\Gamma, A)$ the subgroup of $\mathrm{Aut}(\Gamma) \times \mathrm{Aut}(A)$ that consists of pairs (φ, σ) of automorphisms φ of Γ and σ of A such that

$$\sigma({}^\gamma a) = {}^{\varphi(\gamma)}\sigma(a), \qquad \gamma \in \Gamma, \quad a \in A.$$

We call $\mathrm{Aut}(\Gamma, A)$ the *group of automorphisms of the pair* (Γ, A).

PROPOSITION 2.1. *The group* $\mathrm{Aut}(\Gamma, A)$ *is the normaliser in* $\mathrm{Aut}(\Gamma) \times \mathrm{Aut}(A)$ *of* $\chi_0(\Gamma)$.

Let $\mathrm{Out}(\Gamma, A) = \mathrm{Aut}(\Gamma, A)/\chi_0(\Gamma)$ and call it the *group of outer automorphisms of the pair* (Γ, A). We shall now describe an obvious action of $\mathrm{Out}(\Gamma, A)$ on the cohomology $H^*(\Gamma, A)$ (this may be folklore).

Recall that any group homomorphism $f: \Gamma' \to \Gamma$ induces a unique map $f^*: H^*(\Gamma, A) \to H^*(\Gamma', A')$; here A' is the Γ'-module which has the same underlying Abelian group as A but operators from Γ' via f. Now, if $(\varphi, \sigma) \in \mathrm{Aut}(\Gamma, A)$, let $f = \varphi^{-1}$, and consider $(\varphi^{-1})^*: H^*(\Gamma, A) \to H^*(\Gamma, A')$. Since

$$\sigma({}^{\varphi^{-1}(\gamma)}a) = {}^\gamma(\sigma(a)), \qquad \gamma \in \Gamma, \quad a \in A,$$

σ induces $\sigma_*: H^*(\Gamma, A') \to H^*(\Gamma, A)$. Let

$$a_{(\varphi,\sigma)} = \sigma_*(\varphi^{-1})^*: H^*(\Gamma, A) \to H^*(\Gamma, A).$$

We note that it is convenient to invert the automorphism φ for the formal reason that cohomology is contravariant in the group variable.

PROPOSITION 2.2. *The rule* $(\varphi, \sigma) \mapsto a_{(\varphi,\sigma)}$ *induces an action of* $\mathrm{Out}(\Gamma, A)$ *on (the left of)* $H^*(\Gamma, A)$.

In our situation, we have the group extension (1.1) and the G-module A. Let $\Gamma = N$, and let N act on A in the obvious way. Then (1.1) furnishes an *action* $\chi: G \to \mathrm{Aut}(N, A)$ *of* G *on the pair* (N, A) given by $\chi(g) = (i_g, l_g)$, $g \in G$, where i_g is the conjugation $n \mapsto gng^{-1}$, $n \in N$, and l_g the action $l_g(a) = {}^g a$ of g on A. For later reference, denote by $\mathrm{Aut}_G(N, A)$ the image of χ. Since χ extends the above homomorphism $\chi_0: N \to \mathrm{Aut}(N, A)$, it induces an *outer action* $\omega: Q \to \mathrm{Out}(N, A)$ *of* Q *on the pair* (N, A).

PROPOSITION 2.3. *This outer action, combined with the action of* $\mathrm{Out}(N, A)$ *on* $H^*(N, A)$ *given above, yields the standard action of* Q *on* $H^*(N, A)$.

Consider an extension

$$e: 0 \to A \to E \xrightarrow{\pi} N \to 1 \tag{2.1}$$

with Abelian kernel A, where we assume that conjugation in E induces that action of N on A which is obtained by restricting the operators from G to N. Let $\mathrm{Aut}^A(E)$ denote the group of automorphisms of E which map A to itself. Each $\alpha \in \mathrm{Aut}^A(E)$ induces an automorphism l_α of A (as Abelian group) and an automorphism i_α of N such that (i_α, l_α) is a member of $\mathrm{Aut}(N, A)$. The rule $\alpha \mapsto (i_\alpha, l_\alpha)$ is in fact a homomorphism $\mathrm{Aut}^A(E) \to \mathrm{Aut}(N, A)$. If $\mathrm{Aut}^A_G(E)$ denotes the pre-image (in $\mathrm{Aut}^A(E)$) of $\mathrm{Aut}_G(N, A)$ ($\subset \mathrm{Aut}(N, A)$), we have a homomorphism

$$h = h_e: \mathrm{Aut}^A_G(E) \to \mathrm{Aut}_G(N, A)$$

which is determined by e. The kernel of h_e is isomorphic to the group $\mathrm{Der}(N, A)$ of derivations (=crossed homomorphisms) of N in A [5, p. 12; 6, p. 45]. We fix an embedding of $\mathrm{Der}(N, A)$ in $\mathrm{Aut}^A_G(E)$ as follows: If $d: N \to A$ is a derivation (i.e., $d(nm) = d(n) + {}^n d(m)$, m, $n \in N$) define $\alpha_d: E \to E$ by $\alpha_d(x) = d(\pi(x)) \cdot x$, $x \in E$. We now embed $\mathrm{Der}(N, A)$ in $\mathrm{Aut}^A_G(E)$ by the rule $d \mapsto \alpha_d$.

PROPOSITION 2.4. *The map* h_e *is surjective if and only if the class* $[e] \in H^2(N, A)$ *is a member of* $H^2(N, A)^Q$.

Proof. By virtue of Proposition 2.3, $[e] \in H^2(N, A)^Q$ if and only if for each $g \in G$ the map $(l_g)_* \, (i_g^{-1})^*: H^2(N, A) \to H^2(N, A)$ is the identity. This implies the claim.

2.2. *The Differential* $d_2: H^0(Q, H^2(N, A)) \to H^2(Q, H^1(N, A))$

Let e be a group extension (2.1). Assume now that e represents a member of $H^2(N, A)^Q$. In Section 2.1 we associated to e the extension

$$0 \to \mathrm{Der}(N, A) \to \mathrm{Aut}^A_G(E) \to \mathrm{Aut}_G(N, A) \to 1. \tag{2.2}$$

If we replace $\mathrm{Aut}^A_G(E)$ by the fibre product

$$\mathrm{Aut}^A_G(E) \underset{\mathrm{Aut}_G(N,A)}{\times} G,$$

denoted henceforth by $\mathrm{Aut}_G(e)$, we obtain the extension

$$0 \to \mathrm{Der}(N, A) \to \mathrm{Aut}_G(e) \to G \to 1. \tag{2.3}$$

There is an obvious map of (2.1) into (2.3):

$$\begin{array}{ccccccccc} 0 \longrightarrow & A & \longrightarrow & E & \xrightarrow{\pi} & N & \longrightarrow 1 \\ & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle i} & \\ 0 \longrightarrow & \mathrm{Der}(N, A) & \longrightarrow & \mathrm{Aut}_G(e) & \longrightarrow & G & \longrightarrow 1; \end{array} \tag{2.4}$$

here ζ sends $a \in A$ to the inner derivation $(n \mapsto a - {}^n a, n \in N)$, $\beta(x) = (i_x, \pi(x))$, $x \in E$, and i is the inclusion. Inspection proves the following.

Proposition 2.5. *The obvious action of* $\mathrm{Aut}_G(e)$ *on* E *turns* $(E, \mathrm{Aut}_G(e), \beta)$ *into a crossed module.*

A consequence of this is that $\beta(E)$ is normal in $\mathrm{Aut}_G(e)$. We denote the cokernel of β by $\mathrm{Out}_G(e)$, since there is an obvious map $\eta\colon \mathrm{Out}_G(e) \to \mathrm{Out}(E)$, where $\mathrm{Out}(E)$ denotes the group of outer automorphisms of E. If we pass in (2.4) to cokernels, we obtain the extension

$$\bar{e}\colon 0 \to H^1(N, A) \to \mathrm{Out}_G(e) \to Q \to 1. \tag{2.5}$$

It is straightforward to check that the class $[\bar{e}] \in H^2(Q, H^1(N, A))$ depends only on $[e] \in H^2(N, A)^Q = H^0(Q, H^2(N, A))$.

Theorem 1. *The rule* $e \mapsto \bar{e}$ *describes the differential*

$$d_2\colon E_2^{0,2} \to E_2^{2,1}.$$

A proof will be given in Section 4 below. We shall also show that our description is natural in a very strong sense; see Propositions 4.5 and 4.6 below.

For later reference, we note that the above construction also associates with e the crossed 2-fold extension

$$0 \to A^N \to E \xrightarrow{\beta} \mathrm{Aut}_G(e) \to \mathrm{Out}_G(e) \to 1. \tag{2.6}$$

Remark 1. In a picturesque way one could say that the image $d_2[e]$ extends the well known interpretation of $H^1(N, A)$ as the group of automorphisms of E leaving A and $N = E/A$ elementwise fixed, modulo the inner automorphisms induced by elements of A; see e.g., [5, p. 12] or [6, p. 46].

Remark 2. Theorem 1 generalises Theorem 0.2 of [22]; see also p. 265 of [21]. Extensions (1), (6) and (7) in Section 0 of [22] correspond to our extensions (1.1), (2.1) and (2.5), respectively. Sah assumes extension (1.1) to be split with N Abelian (i.e., N a Q-module) and the N-action on A to be trivial. We managed to get rid of all these hypotheses.

Since $E_3^{0,2}$ is the kernel of d_2, we have the following.

COROLLARY 1. *The subgroup $E_3^{0,2}$ of transgressive elements consists of those classes of extensions e for which $\bar{e}$ splits.*

This suggests that $d_2[e]$ should be the obstruction to lifting the outer action $\omega: Q \to \operatorname{Out}(N, A)$ to somewhat of an outer action on E. In fact, if $\operatorname{Out}^A(E)$ denotes the cokernel of the obvious map $E \to \operatorname{Aut}^A(E)$ which sends a member of E to the corresponding inner automorphism, the map $\operatorname{Aut}^A(E) \to \operatorname{Aut}(N, A)$ in Section 2.1 induces a homomorphism $\operatorname{Out}^A(E) \to \operatorname{Out}(N, A)$ the kernel of which is (isomorphic to) $H^1(N, A)$.

COROLLARY 2. *The class $d_2[e] \in H^2(Q, H^1(N, A))$ is the obstruction to lifting the outer action $\omega: Q \to \operatorname{Out}(N, A)$ of Q on (N, A) to $\operatorname{Out}^A(E)$.*

Proof. The map $\eta: \operatorname{Out}_G(e) \to \operatorname{Out}(E)$ maps $\operatorname{Out}_G(e)$ into $\operatorname{Out}^A(E) \subset \operatorname{Out}(E)$ and induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(N, A) & \longrightarrow & \operatorname{Out}_G(e) & \longrightarrow & Q & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\omega} & & \\ 0 & \longrightarrow & H^1(N, A) & \longrightarrow & \operatorname{Out}^A(E) & \longrightarrow & \operatorname{Out}(N, A) & & \end{array}$$

with exact rows, whose right-hand square is a pullback diagram. The claim follows. Q.E.D.

We take the opportunity to correct a slight error in [22]: It is fairly clear from our construction of (2.5) that, in the special case considered by Sah, the middle group of (7) in Section 0 of [22] should be a fibre product

$$A(\Gamma, [f]) \underset{\operatorname{Aut}_\Gamma(K, M)}{\times} \Gamma;$$

here $\operatorname{Aut}_\Gamma(K, M)$ is the image of the obvious map $\Gamma \to \operatorname{Aut}(K) \times \operatorname{Aut}(M)$. Sah's description is correct only if this map is injective, i.e., if the action of Γ on (K, M) is faithful. We also note that, in view of the above, on p. 21 of [22] the group M in line 18 from below should perhaps be replaced by $E(f)$.

Remark 3. In very special cases the differential d_2 can be described as the cup product with certain characteristic classes [1–3]. We tried to obtain such a description in our situation but could not manage to do so.

2.3. *The Differential* $d_2\colon H^1(Q, H^1(N,A)) \to H^3(Q, H^0(N,A))$

Let e_s denote the split extension

$$e_s\colon 0 \to A \to A \rtimes N \to N \to 1.$$

Since $[e_s] \in H^2(N,A)^Q$, the construction in Section 2.2 above associates the extension

$$\bar{e}_s\colon 0 \to H^1(N,A) \to \mathrm{Out}_G(e_s) \to Q \to 1 \tag{2.7}$$

with e_s. The obvious action of G on $A \rtimes N$ induces a canonical section s_0: $G \to \mathrm{Aut}_G(e_s)$ which, in turn, induces a canonical section $\psi_0\colon Q \to \mathrm{Out}_G(e_s)$. Hence we my identify $\mathrm{Aut}_G(e_s)$ and $\mathrm{Out}_G(e_s)$ with $\mathrm{Der}(N,A) \rtimes G$ and $H^1(N,A) \rtimes Q$, respectively, in a canonical way. Further, the crossed 2-fold extension (2.6) now reads

$$0 \to A^N \to A \rtimes N \xrightarrow{\beta_s} \mathrm{Aut}_G(e_s) \to \mathrm{Out}_G(e_s) \to 1. \tag{2.8}$$

Consider a derivation $\delta\colon Q \to H^1(N,A)$ representing a class $[\delta] \in H^1(Q, H^1(N,A))$. Setting $\psi_\delta(q) = \delta(q)\,\psi_0(q)$, $q \in Q$, we obtain a further section $\psi_\delta\colon Q \to \mathrm{Out}_G(e_s)$ in (2.7). Here we identify $H^1(N,A)$ with its isomorphic image in $\mathrm{Out}_G(e_s)$. Pulling back (2.8) along ψ_δ yields the crossed 2-fold extension

$$\tilde{\delta}\colon 0 \to A^N \to A \rtimes N \xrightarrow{\partial} B^\delta \to Q \to 1. \tag{2.9}$$

Here B^δ is the fibre product

$$B^\delta = \mathrm{Aut}_G(e_s) \underset{\mathrm{Out}_G(e_s)}{\times} Q;$$

it will be convenient to take as B^δ the pre-image in $\mathrm{Aut}_G(e_s)$ of $\psi_\delta(Q) \subset \mathrm{Out}_G(e_s)$. Further, the map $\partial\colon A \rtimes N \to B^\delta$ is induced by β_s, and B^δ acts on $A \rtimes N$ in the obvious way. As pointed out in the Introduction, $\tilde{\delta}$ represents a class $[\tilde{\delta}] \in H^3(Q, A^N)$. It is straightforward to check that this class depends only on $[\delta] \in H^1(Q, H^1(N,A))$.

THEOREM 2. *The rule* $\delta \mapsto \tilde{\delta}$ *describes the differential*

$$d_2\colon E_2^{1,1} \to E_2^{3,0}.$$

A proof will be given in Section 5 below. Again we shall show that our description is natural in a very strong sense; see Proposition 5.4 and 5.5.

2.4. *The Transgression* $\tau: E_2^{0,2} \rightharpoonup E_2^{3,0}$ *and the Differential* $d_3: E_3^{0,2} \to E_3^{3,0}$

Let e be a group extension (2.1) whose class is a transgressive element of $H^2(N, A)^Q$; by Corollary 1, the extension $\bar{e}$ associated with e in Section 2.2 splits. Let $\psi: Q \to \mathrm{Out}_G(e)$ be a section. Pulling back (2.6) along ψ yields the crossed 2-fold extension

$$\tilde{e}_\psi: 0 \to A^N \to E \xrightarrow{\partial} B^\psi \to Q \to 1. \tag{2.10}$$

Here B^ψ is the fibre product

$$B^\psi = \mathrm{Aut}_G(e) \underset{\mathrm{Out}_G(e)}{\times} Q,$$

the map $\partial: E \to B^\psi$ is induced by β, and B^ψ acts on E in the obvious way. As we have already explained, $\tilde{e}_\psi$ represents a class $[\tilde{e}_\psi] \in H^3(Q, A^N)$; this class depends on $[e]$ and ψ.

THEOREM 3. (a) *The pairs* $([e], [\tilde{e}_\psi])$, *where* $\bar{e}$ *splits and where* ψ *is a section of* $\bar{e}$, *constitute an additive relation. This additive relation is the transgression* $\tau: E_2^{0,2} \rightharpoonup E_2^{3,0}$.

(b) *Combining this relation with the projection* $E_2^{3,0} \to E_3^{3,0}$ *yields a homomorphism* $E_3^{0,2} \to E_3^{3,0}$ *which is the corresponding differential* d_3.

This will be proved in Section 6 below. Again the descriptions are natural, in a suitable sense. This is, however, best understood in terms of crossed pairs; see Section 2 of [13].

Remark 4. In the special case that N acts trivially on A, a similar result as Theorem 3(a) was obtained by Ratcliffe [18].

3. On Differentials in the LHS Spectral Sequence

Let $(B_*(\ ), \partial)$ denote the Bar resolution. The LHS spectral sequence $(E_r^{p,q}, d_r)$, associated with the group extension (1.1) and the G-module A, is obtained by suitably filtering the bicomplex

$$K^{p,q} = \mathrm{Hom}_Q(B_p(Q), \mathrm{Hom}_N(B_q(G), A))$$

with differentials

$$(\delta' f)(b')(b'') = (-1)^{p+q+1} f(\partial b')(b''), \qquad b' \in B_{p+1}, \quad b'' \in B_q,$$
$$(\delta'' f)(b')(b'') = (-1)^{q+1} f(b')(\partial b''), \qquad b' \in B_p, \quad b'' \in B_{q+1}$$

(see [16, p. 351], where this spectral sequence is called the *Lyndon spectral sequence*). For later use, we denote the cokernel of $\partial: B_{t+1} \to B_t$ by C_t and the kernel of $\partial: B_t \to B_{t-1}$ by J_t; the corresponding canonical maps will be denoted by $pr: B_t \to C_t$ and $k: J_t \to B_t$ (we set $B_{-1} = \mathbb{Z}$).

We shall utilize a variant of the description of $E_2^{p,q}$ and d_2 introduced on pp. 341, 342 of Mac Lane's book [16] in case of homology:

Define $L_2^{p,q} \subset K^{p,q}$ and $M_2^{p,q} \subset L_2^{p,q}$ by

$$L_2^{p,q} = \{a^{p,q};\ \delta'' a^{p,q} = 0 \text{ and } \delta' a^{p,q} = -\delta'' a^{p+1,q-1} \text{ for some } a^{p+1,q-1}\},$$

$$M_2^{p,q} = \{\delta' b^{p-1,q} + \delta'' b^{p,q-1};\ \delta'' b^{p-1,q} = 0\}.$$

Then $E_2^{p,q} = L_2^{p,q}/M_2^{p,q}$, and the differential

$$d_2: E_2^{p,q} \to E_2^{p+2,q-1}$$

is induced by the additive relation

$$\{(a^{p,q}, \delta' a^{p+1,q-1}) \in K^{p,q} \oplus K^{p+2,q-1};\quad \delta' a^{p,q} + \delta'' a^{p+1,q-1} = 0,\ \delta'' a^{p,q} = 0\}. \tag{3.1}$$

Now $a^{p,q}: B_p(Q) \to \mathrm{Hom}_N(B_q(G), A)$ is Q-linear, and the condition $\delta'' a^{p,q} = 0$ means that the image of $a^{p,q}$ is contained in $\mathrm{Hom}_N(J_{q-1}(G), A)$ ($\subset \mathrm{Hom}_N(B_q(G), A)$ via $pr: B_q(G) \to C_q(G) = J_{q-1}(G)$). Hence $a^{p,q} \in L_2^{p,q}$ if and only if there is a commutative square

$$\begin{array}{ccc} B_{p+1}(Q) & \xrightarrow{\ \partial\ } & B_p(Q) \\ \downarrow{\scriptstyle (-1)^{p+1} a^{p+1,q-1}} & & \downarrow{\scriptstyle a^{p,q}} \\ \mathrm{Hom}_N(B_{q-1}(G), A) & \xrightarrow[k^*]{} & \mathrm{Hom}_N(J_{q-1}(G), A). \end{array} \tag{3.2}$$

The cokernel of the second row of (3.2) is (naturally isomorphic to) the cohomology group $H^q(N, A)$. Hence any $a^{p,q} \in L_2^{p,q}$ induces a Q-linear map

$$\alpha = a^{p,q}: C_p(Q) \to H^q(N, A).$$

Conversely, any such α gives rise to a commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow & J_{p+1}(Q) & \xrightarrow{\ k\ } & B_{p+1}(Q) & \\ & \downarrow{\scriptstyle \sigma} & & \downarrow & \\ 0 \longrightarrow & \mathrm{Hom}_N(C_{q-1}(G), A) & \xrightarrow{\ pr^*\ } & \mathrm{Hom}_N(B_{q-1}(G), A) & \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{\ \partial\ } & B_p(Q) & \xrightarrow{\ pr\ } & C_p(Q) & \longrightarrow 0 \\ & \downarrow & & \downarrow{\scriptstyle \alpha} & \\ \xrightarrow{\ k^*\ } & \mathrm{Hom}_N(J_{q-1}(G), A) & \longrightarrow & H^q(N, A) & \longrightarrow 0 \end{array} \tag{3.3}$$

with exact rows such that the combined map $B_p(Q) \to \mathrm{Hom}_N(J_{q-1}(G), A) \to \mathrm{Hom}_N(B_q(G), A)$ is a member of $L_2^{p,q}$. We shall refer to (3.3) as a *lifting* of α; it is uniquely determined by α up to chain homotopy. Hence the class $[\sigma]$ in the cokernel of

$$\mathrm{Hom}_Q(B_{p+1}(Q), \mathrm{Hom}_N(C_{q-1}(G), A))$$
$$\to \mathrm{Hom}_N(J_{p+1}(Q), \mathrm{Hom}_N(C_{q-1}(G), A)),$$

which is the cohomology group $H^{p+2}(Q, \mathrm{Hom}_N(C_{q-1}(G), A))$, depends only on α. Furthermore, $[\sigma]$ depends only on the cohomology class

$$[\alpha] \in H^p(Q, H^q(N, A))$$

that is represented by α, and the rule $[\alpha] \mapsto [\sigma]$ describes a homomorphism

$$H^p(Q, H^q(N, A)) \to H^{p+2}(Q, \mathrm{Hom}_N(C_{q-1}(G), A)).$$

We denote this homomorphism by y.

Remark 3.1. The map y coincides with the map

$$\mathrm{Ext}_Q^p(\mathbb{Z}, H^q(N, A)) \to \mathrm{Ext}_Q^{p+2}(\mathbb{Z}, \mathrm{Hom}_N(C_{q-1}(G), A))$$

given by Yoneda splicing with the second row of (3.3) (we shall, however, not use this fact).

If r denotes the natural projection

$$\mathrm{Hom}_N(C_{q-1}(G), A) \to H^{q-1}(N, A),$$

we have the following.

PROPOSITION 3.1. *The differential*

$$d_2: H^p(Q, H^q(N, A)) \to H^{p+2}(Q, H^{q-1}(N, A))$$

is given by $(-1)^q r_* y$ *(where*

$$r_*: H^{p+2}(Q, \mathrm{Hom}_N(C_{q-1}(G), A)) \to H^{p+2}(Q, H^{q-1}(N, A))$$

is the induced map). In other words: If the Q-linear map $\alpha: C_p(Q) \to H^q(N, A)$ *represents* $[\alpha] \in H^p(Q, H^q(N, A))$, *construct a lifting* (3.3); *then*$(-1)^q$ *times the composite map* $r\sigma$ *represents* $d_2[\alpha] \in H^{p+2}(Q, H^{q-1}(N, A))$.

Proof. A pair $(a^{p,q}, \delta' a^{p+1,q-1})$ belongs to the additive relation (3.1) if and only if it fits into a diagram (3.2). The assertion follows since $\delta' a^{p+1,q-1}: B_{p+2}(Q) \to \mathrm{Hom}_N(B_{q-1}(G), A)$ (or the induced map $J_{p+1}(Q) \to \mathrm{Hom}_N(C_{p-1}(G), A))$ represents the d_2-image of the class represented by $a^{p,q}$.

Q.E.D.

Remark 3.2. The preceding proposition recovers the following cocycle description of d_2: Let the p-cochain $f\colon Q^p \to \mathrm{Hom}_N(B_q(G), A)$ represent $[f] \in H^p(Q, H^q(N, A))$; this means that f maps Q^p to $\mathrm{Hom}_N(J_{q-1}(G), A)$ $(\subset \mathrm{Hom}_N(B_q(G), A)$ via $pr\colon B_q(G) \to J_{q-1}(G))$ in such a way that $rf\colon Q^p \to H^q(N, A)$ is a p-cocycle. It follows that for each $[\sigma_1 | \cdots | \sigma_{p+1}] \in Q^{p+1}$ there exists $h_{[\sigma_1 | \cdots | \sigma_{p+1}]} \in \mathrm{Hom}_N(B_{q-1}(G), A)$ such that

$$h_{[\sigma_1|\cdots|\sigma_{p+1}]}\partial = \sigma_1(f[\sigma_2 | \cdots | \sigma_{p+1}]) + \sum_{i=1}^{p} (-1)^i f[\sigma_1 | \cdots | \sigma_i\sigma_{i+1} | \cdots | \sigma_{p+1}] + (-1)^{p+1} f[\sigma_1 | \cdots | \sigma_p],$$

where $\partial\colon B_q(G) \to B_{q-1}(G)$ is the corresponding map. Define $g\colon Q^{p+2} \to \mathrm{Hom}_N(B_{q-1}(G), A)$ by

$$g[\sigma_1 | \cdots | \sigma_{p+2}] = \sigma_1 h_{[\sigma_2|\cdots|\sigma_{p+2}]} + \sum_{i=1}^{p+1} (-1)^i h_{[\sigma_1|\cdots|\sigma_i\sigma_{i+1}|\cdots|\sigma_{p+2}]} + (-1)^{p+2} h_{[\sigma_1|\cdots|\sigma_{p+1}]}.$$

Then $(-1)^q g$ represents $d_2[f]$. We note that a similar description of d_2: $H^0(Q, H^q(N, A)) \to H^2(Q, H^{q-1}(N, A))$ can be found on p. 21 of [22] (it is clear that (5) must read "$g(\sigma, \tau) = \sigma h_\tau - h_{\sigma\tau} + h_\sigma$").

There is an even more direct description of

$$d_2\colon H^p(Q, H^2(N, A)) \to H^{p+2}(Q, H^1(N, A)).$$

Let

$$1 \to N^G \to F \to G \to 1 \tag{3.4}$$

be the free standard presentation; here F is free on a set $\{x_g; g \in G^*\}$, where $G^* = G - \{1\}$. Let $N^Q \subset F$ denote the pre-image of $N \subset G$.

LEMMA 3.1. *The cokernel of* $\partial^*\colon \mathrm{Hom}_N(\mathbb{Z}G, A) \to \mathrm{Hom}_N(B_1(G), A)$ *is (naturally isomorphic to) the group* $H^1(N^Q, A)$, *and passing to cokernels in the second row of* (3.3) *yields, in case* $q = 2$, *an exact sequence of N-modules*

$$0 \longrightarrow H^1(N, A) \xrightarrow{\text{inf}} H^1(N^Q, A) \xrightarrow{h} \mathrm{Hom}_N(J_1(G), A) \xrightarrow{r} H^2(N, A) \longrightarrow 0. \tag{3.5}$$

Here h is the obvious map that sends the class of $\varphi\colon B_1(G) \to A$ to its restriction $\varphi | J_1(G)$.

Proof. The projection $F \to G$ induces natural isomorphisms $\mathrm{Hom}_N(\mathbb{Z}G, A) \to \mathrm{Hom}_{NQ}(\mathbb{Z}F, A)$ and $\mathrm{Hom}_N(B_1(G), A) \to \mathrm{Hom}_{NQ}(IF, A)$ (here "$I\Gamma$" denotes the augmentation ideal of a group Γ). Q.E.D.

Remark 3.3. The commutator factor group $(N^G)^{Ab}$ is (naturally isomorphic to) $J_1(G)$, and the exact sequence (3.5) is the exact sequence (10.6) in [16, p. 354] associated with the group extension $1 \to N^G \to N^Q \to N \to 1$ and the N-module A.

The following is immediate from the above:

ADDENDUM TO PROPOSITION 3.1. *If the Q-linear map $\alpha: C_p(Q) \to H^2(N, A)$ represents $[\alpha] \in H^p(Q, H^2(N, A))$, construct a lifting*

$$\begin{array}{ccccccccc} 0 \longrightarrow & J_{p+1}(Q) & \longrightarrow & B_{p+1}(Q) & \longrightarrow & B_p(Q) & \longrightarrow & C_p(Q) & \longrightarrow 0 \\ & \downarrow{\scriptstyle \sigma} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & \\ 0 \longrightarrow & H^1(N, A) & \xrightarrow{\text{inf}} & H^1(N^Q, A) & \xrightarrow{h} & \mathrm{Hom}_N(J_1(G), A) & \xrightarrow{r} & H^2(N, A) & \longrightarrow 0. \end{array} \tag{3.6}$$

Then σ represents $d_2[\alpha] \in H^{p+2}(Q, H^1(N, A))$.

Remark 3.4. Proposition 3.1 may be paraphrased by saying that $d_2^{p,2}$ is the map $\mathrm{Ext}_Q^p(\mathbb{Z}, H^2(N, A)) \to \mathrm{Ext}_Q^{p+2}(\mathbb{Z}, H^1(N, A))$ given by Yoneda splicing with (3.5).

We shall also need a description of the differentials

$$d_3: E_3^{p,q} \to E_3^{p+3,q-2}.$$

We shall proceed as follows (cf. [16, p. 342, Ex. 2]): Define

$$L_3^{p,q} = \{a^{p,q} \in K^{p,q};\ Cc(a^{p,q})\}.$$

Here $Cc(a^{p,q})$ shall mean: $a^{p,q}$ maps $B_p(Q)$ into $\mathrm{Hom}_N(J_{q-1}(G), A)$ ($\subset \mathrm{Hom}_N(B_q(G), A)$ as above), and there is a commutative diagram

$$\begin{array}{ccccc} B_{p+2}(Q) & \xrightarrow{\partial} & B_{p+1}(Q) & \xrightarrow{\partial} & B_p(Q) \\ \downarrow{\scriptstyle a^{p+2,q-2}} & & \downarrow{\scriptstyle (-1)^{p+1}a^{p+1,q-1}} & & \downarrow{\scriptstyle a^{p,q}} \\ \mathrm{Hom}_N(B_{q-2}(G), A) & \xrightarrow{\partial^*} & \mathrm{Hom}_N(B_{q-1}(G), A) & \xrightarrow{k^*} & \mathrm{Hom}_N(J_{q-1}(G), A), \end{array} \tag{3.7}$$

where $a^{p+1,q-1} \in K^{p+1,q-1}$, $a^{p+2,q-2} \in K^{p+2,q-2}$. We also define

$$M_3^{p,q} = \{\delta' b^{p-1,q} + \delta'' b^{p,q-1};\ Cb(b^{p-1,q})\}.$$

Here $Cb(b^{p-1,q})$ shall mean: There is a commutative diagram:

$$\begin{array}{ccc} B_{p-1}(Q) & \longrightarrow & B_{p-2}(Q) \\ \downarrow{\scriptstyle b^{p-1,q}} & & \downarrow{\scriptstyle b^{p-2,q+1}} \\ \operatorname{Hom}_N(B_q(G), A) & \longrightarrow & \operatorname{Hom}_N(J_q(G), A), \end{array}$$

where $b^{p-2,q+1} \in K^{p-2,q+1}$. Now $E_3^{p,q} = L_3^{p,q}/M_3^{p,q}$, and the differential d_3: $E_3^{p,q} \to E_3^{p+3,q-2}$ is induced by the additive relation

$$\{(a^{p,q}, \delta' a^{p+2,q-2}) \in K^{p,q} \oplus K^{p+3,q-2};\, Cc(a^{p,q})\}, \tag{3.8}$$

as a closer examination of the arguments in the proof of Proposition 6.1 on p. 341 of [16] shows. Hence

PROPOSITION 3.2. *The differential d_3: $E_3^{p,q} \to E_3^{p+3,q-2}$ may be described as follows: Represent a class in $E_3^{p,q}$ by a Q-linear map α: $C_p(Q) \to H^q(N, A)$, and lift α to*

$$\begin{array}{ccccccc} B_{p+2}(Q) & \xrightarrow{\partial} & B_{p+1}(Q) & & & & \\ \downarrow{\scriptstyle a^{p+2,q-2}} & & \downarrow{\scriptstyle a^{p+1,q-1}} & & & & \\ \operatorname{Hom}_N(B_{q-2}(G), A) & \xrightarrow{\partial^*} & \operatorname{Hom}_N(B_{q-1}(G), A) & & & & \\ & & \xrightarrow{\partial} & B_p(Q) & \longrightarrow & C_p(Q) & \longrightarrow 0 \\ & & & \downarrow{\scriptstyle a^{p,q}} & & \downarrow{\scriptstyle \alpha} & \\ & & \xrightarrow{k^*} & \operatorname{Hom}_N(J_{q-1}(G), A) & \xrightarrow{r} & H^q(N, A) & \longrightarrow 0. \end{array} \tag{3.9}$$

Then $a^{p+2,q-2}$ induces a map $\sigma: J_{p+2}(Q) \to \operatorname{Hom}_N(C_{q-2}(G), A)$, and $(-1)^{p+q+1}$ times the composite map $r\sigma: J_{p+2}(Q) \to H^{q-1}(N, A)$ represents the d_3-image of $[\alpha]$.

ADDENDUM. *The transgression $\tau: E_2^{0,2} \rightharpoonup E_2^{3,0}$ may be described as follows: Let $\alpha: \mathbb{Z} \to H^2(N, A)$ represent a transgressive class; this is the case if and only if α admits a lifting (3.9). Let $\sigma: J_2(Q) \to \operatorname{Hom}_N(\mathbb{Z}, A)$ be the induced map as above. Then $([\alpha], -[\sigma])$ is a member of the transgression τ, and any member of τ may be obtained in this way.*

4. THE PROOF OF THEOREM 1

Let

$$e: 0 \to A \to E \to N \to 1$$

be a group extension (2.1) that represents a member of $H^2(N, A)^Q$. Let

$$c: 0 \to A \to C \to IG \to 0$$

rpresent the corresponding class $[c] \in \mathrm{Ext}_N(IG, A) \cong H^2(N, A)$ (cf. Proposition 4.1 below). The extension c determines a group extension

$$\hat{e}: 0 \to H^1(N, A) \to \hat{E} \to Q \to 1$$

that represents $d_2[e] \in H^2(Q, H^1(N, A))$ (Corollary 4.1 below). Theorem 1 is then proved by showing that $\hat{e}$ is equivalent to the extension $\bar{e}$ (2.5); see Proposition 4.4 below.

4.1. $\mathrm{Ext}_N(IG, A)$ *and* $\mathrm{Opext}(N, A)$

The purpose of this subsection is to develop a conceptual description of the standard map $\mathrm{Opext}(N, A) \to \mathrm{Ext}_N(IG, A)$ that identifies the two models $\mathrm{Opext}(N, A)$ (operator extensions of A by N) and $\mathrm{Ext}_N(IG, A)$ (N-module extension of A by IG) of the abstract group $H^2(N, A)$.

Let

$$1 \to N^G \to F \to G \to 1 \tag{4.1}$$

be the free standard presentation such that F is free on a set $\{x_g; g \in G^*\}$, where $G^* = G - \{1\}$. Let $N^Q \subset F$ denote the pre-image of $N \subset G$. We identify the commutator factor group $(N^G)^{Ab} = N^G/[N^G, N^G]$ with $J_1(G) = \ker(B_1(G) \to B_0(G))$ by the standard rule $n \mapsto pr(n-1)$, $n \in N^G$, where pr: $IF \to B_1(G)$ denotes the projection $(x_g - 1) \mapsto [g]$ (we could also take $n \mapsto pr(1-n)$). Let $M = N^Q/[N^G, N^G]$. Now, if e represents $[e] \in \mathrm{Opext}(N, A)$, we may lift the identity map of N to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_1(G) & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \| & & \\ e: 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 1, \end{array} \tag{4.2}$$

such that μ is N-linear. In order to map $[e]$ to an element of $\mathrm{Ext}_N(IG, A)$, let C_e denote the pushout of

$$\begin{array}{ccc} J_1(G) & \longrightarrow & B_1(G) \\ \downarrow{\scriptstyle \mu} & & \\ A & & \end{array}$$

in the category of N-modules. It yields a commutative diagram of N-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_1(G) & \longrightarrow & B_1(G) & \longrightarrow & IG & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow & & \| & & \\ c_e\colon 0 & \longrightarrow & A & \longrightarrow & C_e & \longrightarrow & IG & \longrightarrow & 0. \end{array} \tag{4.3}$$

PROPOSITION 4.1. *The rule* $e \mapsto c_e$ *induces the standard isomorphism* $\mathrm{Opext}(N, A) \to \mathrm{Ext}_N(IG, A)$; *this isomorphism is canonical up to a sign depending on how* $(N^G)^{Ab}$ *and* $J_1(G)$ *are identified.*

Proof. Straightforward and left to the reader.

We shall always identify $(N^G)^{Ab}$ and $J_1(G)$ by $n \mapsto pr(n-1)$, $n \in N^G$. Then the isomorphism in Proposition 4.1 is canonical.

4.2. *A Semidirect Fibre Product*

Let K be a group and B a K-module. We shall need a conceptual description of the standard map

$$\mathrm{Ext}_K(IK, B) \to \mathrm{Opext}(K, B)$$

that identifies the two models $\mathrm{Ext}_K(IK, B)$ (K-module extensions of B by IK) and $\mathrm{Opext}(K, B)$ (operator extensions of B by K) of the abstract group $H^2(K, B)$:

Let C and D be K-modules, let $h\colon C \to D$ be a map of K-modules, and let $d\colon K \to D$ be a derivation. We call the subgroup of the semidirect product $C \rtimes K$ consisting of the elements $(x, k) \in C \rtimes K$ such that $h(x) = d(k)$ a *semidirect fibre product* and denote it by

$$C \underset{D}{\rtimes} K.$$

Next, let

$$0 \to B \xrightarrow{i} C \xrightarrow{h} D \to 0$$

be an extension of K-modules. The above construction provides us with the uniquely determined group extension

$$0 \to B \xrightarrow{j} C \underset{D}{\rtimes} K \xrightarrow{q} K \to 1;$$

here $j(b) = (i(b), 1)$ and $q(x, k) = k$, where $b \in B$, $(x, k) \in C \rtimes_D K$.

We may, in particular, apply this construction to an extension

$$c\colon 0 \to B \to C \to IK \to 0$$

in connection with the standard derivation $d: K \to IK$, $d(k) = k - 1$, $k \in K$. This yields the group extension (cf. Section 3 of [4])

$$e_c: 0 \to B \to C \underset{IK}{\rtimes} K \to K \to 1.$$

PROPOSITION 4.2. *The rule* $c \mapsto e_c$ *describes the standard isomorphism* $\mathrm{Ext}_K(IK, B) \to \mathrm{Opext}(K, B)$; *this isomorphism is canonical up to sign.*

The proof is easy and is left to the reader. We note, however, that we could construct $C \rtimes_{IK} K$ with respect to the derivation $d(k) = 1 - k$ also. This explains the ambiguity of sign.

Remark. The inverse to the map $\mathrm{Opext}(N, A) \to \mathrm{Ext}_N(IG, A)$ in Proposition 4.1 is given by sending

$$0 \to A \to C \to IG \to 0$$

to

$$0 \to A \to C \underset{IG}{\rtimes} N \to N \to 1,$$

where $d(n) = n - 1 \in IG$, $n \in N$ (cf. Section 4.1).

4.3. *The Proof of Theorem* 1

Let the group extension e represent $[e] \in H^2(N, A)^Q$. Lift the identity map of N to a diagram (4.2) and construct a diagram (4.3). This yields an extension $c = c_e$ of N-modules that represents the corresponding class $[c] \in \mathrm{Ext}_N(IG, A)^Q$ (Proposition 4.1). Let $\alpha: \mathbb{Z} \to \mathrm{Ext}_N(IG, A)$ send 1 to $[c]$. It is clear that the projection $r: \mathrm{Hom}_N(J_1(G), A) \to \mathrm{Ext}_N(IG, A)$ maps μ (occurring in (4.2) and (4.3)) to $\alpha(1)$.

By the Addendum to Proposition 3.1 we have to consider the lifting problem

$$\begin{array}{ccccccccccc} 0 \longrightarrow & J_1(Q) & \longrightarrow & B_1(Q) & \longrightarrow & \mathbb{Z}Q & \longrightarrow & \mathbb{Z} & \longrightarrow 0 \\ & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & \\ 0 \longrightarrow & H^1(N, A) & \xrightarrow{\text{inf}} & H^1(N^Q, A) & \xrightarrow{h} & \mathrm{Hom}_N(J_1(G), A) & \xrightarrow{r} & \mathrm{Ext}_N(IG, A) & \longrightarrow 0, \end{array} \tag{4.4}$$

where h is the map used in Lemma 3.1. A lifting $\alpha_0: \mathbb{Z}Q \to \mathrm{Hom}_N(J_1(G), A)$ is a given by $\alpha_0(1) = \mu$. Now, for $g \in G$, $\alpha_0(p(g)) = l_g \mu l_g^{-1}$, where $p: G \to Q$ is the projection in (1.1); note that for $n \in N$ we have $l_n \mu l_n^{-1} = \mu$ since μ is N-linear. There is no need to construct α_1; we shall instead construct directly a group extension representing $d_2[\alpha]$.

Let $T = \ker(r\colon \mathrm{Hom}_N(J_1(G), A) \to \mathrm{Ext}_N(IG, A))$. Clearly, α_0 induces a Q-map $IQ \to T$, and we may take the fibre product $H^1(N^Q, A) \times_T IQ$ (note that, by exactness, h maps $H^1(N^Q, A)$ onto T).

Proposition 4.3. *The fibre product $H^1(N^Q, A) \times_T IQ$ fits into an extension*

$$0 \to H^1(N, A) \to H^1(N^Q, A) \times_T IQ \to IQ \to 0 \tag{4.5}$$

of Q-modules that represents

$$d_2[\alpha] \in \mathrm{Ext}(IQ, H^1(N, A)) \cong H^2(Q, H^1(N, A)).$$

Proof. We may complete the construction of (4.4) by setting $\alpha_1[q] = [\varphi]$, where $([\varphi], q-1) \in H^1(N^Q, A) \times_T IQ$, $\varphi\colon B_1(G) \to A$ denoting an N-map that represents $[\varphi] \in H^1(N^Q, A)$ (see proof of Lemma 3.1). The assertion is now a consequence of the Addendum to Proposition 3.1. Q.E.D.

Corollary 4.1. *The group extension*

$$\hat{e}\colon 0 \to H^1(N, A) \to H^1(N^Q, A) \underset{T}{\rtimes} Q \to Q \to 1 \tag{4.6}$$

represents $d_2[\alpha] \in \mathrm{Opext}(Q, H^1(N, A)) \cong H^2(Q, H^1(N, A))$. Here $H^1(N^Q, A) \rtimes_T Q$ is the semidirect fibre product with respect to the derivation $d\colon Q \to T$, $d(q) = \alpha_0(q-1)$ $(= l_g \mu l_g^{-1} - \mu$, where $p(g) = q$, $g \in G$, $q \in Q)$, and the map $h\colon H^1(N^Q, A) \to T$, introduced in Section 3.

Proof. Apply Proposition 4.2 to extension (4.5) and observe that $(H^1(N^Q, A) \times_T IQ) \rtimes_{IQ} Q = H^1(N^Q, A) \rtimes_T Q$. Q.E.D.

In the group extension (4.6) the group $H^1(N, A)$ is the cokernel of $k^*\colon \mathrm{Hom}_N(\mathbb{Z}G, A) \to \mathrm{Hom}_N(IG, A)$ and $H^1(N^Q, A)$ is the cokernel of $\partial^*\colon \mathrm{Hom}_N(\mathbb{Z}G, A) \to \mathrm{Hom}_N(B_1(G), A)$ (cf. Lemma 3.1). It is known that the map

$$\nu\colon \mathrm{Hom}_N(IG, A) \to \mathrm{Der}(N, A), \qquad (\nu(\varphi))(n) = \varphi(n-1),$$
$$\varphi \in \mathrm{Hom}_N(IG, A), \qquad n \in N,$$

induces an isomorphism

$$\nu_{\#}\colon \mathrm{coker}(k^*) \to \mathrm{coker}(A \to \mathrm{Der}(N, A));$$

similarly, the map

$$\rho\colon \mathrm{Hom}_N(B_1(G), A) \to \mathrm{Der}(N^Q, A),$$

given by $(\rho(\psi))(n) = (\psi pr)(n-1)$, $\psi \in \mathrm{Hom}_N(B_1(G), A)$, $n \in N^Q$, where pr: $IF \to B_1(G)$ is the projection $x_g - 1 \mapsto [g]$, induces an isomorphism

$$\rho_{\#}\colon \mathrm{coker}(\partial^*) \to \mathrm{coker}(A \to \mathrm{Der}(N^Q, A)).$$

Corresponding to $h\colon \mathrm{coker}(\partial^*) \to T$,

$$\begin{aligned} h'\colon & \mathrm{coker}(A \to \mathrm{Der}(N^Q, A)) \to T \\ & = \ker(r\colon \mathrm{Hom}_N(J_1(G), A) \to H^2(N, A)) \end{aligned}$$

is defined by $h'[d] = \varphi\colon J_1(G) \to A$ such that $\varphi pr(n-1) = d(n)$, where pr is as above.

LEMMA 4.1. *The diagram*

$$\begin{array}{ccc} \mathrm{coker}(k^*) = H^1(N, A) & \xrightarrow{\mathrm{inf}} & H^1(N^Q, A) = \mathrm{coker}(\partial^*) \\ \downarrow{\scriptstyle v_{\#}} & & \downarrow{\scriptstyle \rho_{\#}} \\ \mathrm{coker}(A \to \mathrm{Der}(N, A)) & \longrightarrow & \mathrm{coker}(A \to \mathrm{Der}(N^Q, A)) \end{array}$$

is commutative, where the second row is induced by the projection. Furthermore, $h = h'\rho_{\#}$.

Proof. The first statement is clear. In order to verify the second, let $x = pr(n-1) \in J_1(G)$, $n \in N^G$. For $\psi \in \mathrm{Hom}_N(B_1(G), A)$ we have

$$(h'\rho_{\#}[\psi])(pr(n-1)) = (\rho\psi)(n) = \psi(pr(n-1)).$$

Hence $h'\rho_{\#}[\psi] = \psi | J_1(G) = h[\psi]$, as h was introduced in Lemma 3.1.

Q.E.D.

In view of Lemma 4.1, we shall now take $\mathrm{coker}(A \to \mathrm{Der}(N, A))$ as $H^1(N, A)$ and $\mathrm{coker}(A \to \mathrm{Der}(N^Q, A))$ as $H^1(N^Q, A)$, and we shall no longer distinguish between h and h'. It will be convenient to describe $h\colon H^1(N^Q, A) \to T$ by the rule

$$(h[d])\, pr = d | N^G, \qquad d \in \mathrm{Der}(N^Q, A), \tag{4.7}$$

where $pr\colon N^G \to (N^G)^{Ab}$ is the projection; here $(N^G)^{Ab}$ is identified with $J_1(G)$ as in Section 4.1, above.

PROPOSITION 4.4. *There is a morphism of extensions*

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Der}(N,A) & \longrightarrow & \mathrm{Aut}_G(e) & \longrightarrow & G & \longrightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} & \\ \hat{e}: 0 \longrightarrow & H^1(N,A) & \longrightarrow & H^1(H^Q,A) \rtimes_T Q & \longrightarrow & Q & \longrightarrow 1 \end{array} \tag{4.8}$$

such that the combined map $E \to^{\beta} \mathrm{Aut}_G(e) \to H^1(N^Q, A) \rtimes_T Q$ *is zero.*

Clearly, this establishes Theorem 1, since (4.8) induces an equivalence of extensions $(2.5) \to \hat{e}$.

Proof of Proposition 4.4. From (4.2) we may construct

$$\begin{array}{ccccccccc} 1 \longrightarrow & N^G & \xrightarrow{j} & N^Q & \longrightarrow & N & \longrightarrow 1 \\ & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \theta} & & \| & \\ e: 0 \longrightarrow & A & \longrightarrow & E & \xrightarrow{\pi} & N & \longrightarrow 1, \end{array} \tag{4.9}$$

where $\kappa = \mu pr$ and $\theta = \nu pr$; here "pr" denotes the corresponding projections.

Let $\alpha \in \mathrm{Aut}_G(e)$, and let $g = g_\alpha \in G$ be the image in G. Define $d_\alpha: N^Q \to A$ by

$$d_\alpha(n) = \alpha(\theta(x_g^{-1} n x_g))\, \theta(n^{-1}), \qquad n \in N^Q, \quad x_g \in F;$$

this yields a derivation from N^Q into A, where N^Q acts upon A via the projection $N^Q \to N$.

LEMMA 4.2. *The rule* $\alpha \mapsto d_\alpha$ *induces a derivation* $\mathrm{Aut}_G(e) \to H^1(N^Q, A)$, *where* $\mathrm{Aut}_G(e)$ *acts on* $H^1(N^Q, A)$ *via the obvious projection* $\mathrm{Aut}_G(e) \to Q$.

Proof. Let α, $\beta \in \mathrm{Aut}_G(e)$, and let $x = x_{g_\alpha} \in F$, $y = x_{g_\beta} \in F$, where g_α, $g_\beta \in G$ are the corresponding images. Using additive notation in $H^1(N^Q, A)$, we have to show that

$$[d_{\alpha\beta}] = [d_\alpha] + {}^{q_\alpha}[d_\beta] \in H^1(N^Q, A),$$

where $q_\alpha \in Q$ is the image of α. Now

$$\begin{aligned} d_{\alpha\beta}(n) &= (\alpha\beta\theta(y^{-1}x^{-1}nxy))\, \theta(n^{-1}), \qquad n \in N^Q, \\ &= (\alpha\beta\theta(y^{-1}x^{-1}nxy))(\alpha\theta(x^{-1}n^{-1}x)) + (\alpha\theta(x^{-1}nx))\, \theta(n^{-1}) \\ &= {}^{g_\alpha}(d_\beta(x^{-1}nx)) + d_\alpha(n), \end{aligned}$$

whence the assertion. Q.E.D.

We can now complete the proof of Proposition 4.4: The rule $\alpha \mapsto ([d_\alpha], q_\alpha)$ describes a homomorphism $\operatorname{Aut}_G(e) \to H^1(N^Q, A) \rtimes Q$, where $q_\alpha \in Q$ denotes the image in Q. Moreover, for $n \in N^G$ we have

$$\begin{aligned} d_\alpha(n) &= \alpha\kappa(x_g^{-1} n x_g) - \kappa(n), \qquad g = g_\alpha \in G, \\ &= (l_g \kappa i_{x_g}^{-1} - \kappa)(n), \end{aligned}$$

i.e.,

$$d_\alpha | N^G = (l_g \mu l_g^{-1} - \mu)\, pr,$$

where $pr: N^G \to (N^G)^{Ab}$ is the projection. By rule (4.7) it follows that

$$h[d_\alpha] = l_g \mu l_g^{-1} - \mu = d(pg) = d(q_\alpha),$$

whence $([d_\alpha], q_\alpha) \in H^1(N^Q, A) \rtimes_T Q$. Thus we have a map

$$\operatorname{Aut}_G(e) \to H^1(N^Q, A) \underset{T}{\rtimes} Q.$$

For an element $\alpha = \alpha_d \in \operatorname{Aut}_G(e)$ such that $\alpha(x) = d(\pi x) \cdot x$, $x \in E$, where $d: N \to A$ is a derivation (cf. Section 2.1), we have, for $n \in N^Q$,

$$\begin{aligned} d_\alpha(n) &= (\alpha\theta(n))\, \theta(n^{-1}) \\ &= (dpr)(n), \end{aligned}$$

where $pr: N^Q \to N$ is the projection; note in particular that $g_\alpha = 1 \in G$. It follows that $\operatorname{Aut}_G(e) \to H^1(N^Q, A) \rtimes_T Q$ induces a diagram (4.8). To see that the combined map $E \to^\beta \operatorname{Aut}_G(e) \to H^1(N^Q, A) \rtimes_T Q$ is zero, let $\alpha = \beta(y)$, $y \in E$. Now, for $n \in N^Q$, we have

$$\begin{aligned} d_\alpha(n) &= y\theta(x^{-1} n x) y^{-1} \theta(n^{-1}), \qquad x = x_g, \quad g = g_y \in N, \\ &= y\theta(x^{-1})\, \theta(n) (y\theta(x^{-1}))^{-1}\, \theta(n^{-1}) \\ &= a - {}^n a, \qquad \text{where} \qquad a = y\theta(x^{-1}) \in A. \end{aligned}$$

Hence d_α is an inner derivation, and we are done. Q.E.D.

Remark. The reader might perhaps believe that in our proof of Theorem 1 there is an argument missing which should establish the independence of the choices of the maps μ and ν in (4.2). There is, however, no need to give such an argument: Diagram (4.8) reverses the choices of μ and ν in the sense that (4.2) and (4.8) together show that the whole proof is independent of μ and ν.

4.4. *Naturalness of the Description*

Our description of $d_2: E_2^{0,2} \to E_2^{2,1}$ is *natural* in a strong sense; what this means will be expressed below in Propositions 4.5 and 4.6 for the module variable and the group extension variable, respectively.

Let $\tau: A \to A'$ be a homomorphism of G-modules. If e is a group extension (2.1) let

$$\tau e: 0 \to A' \to E_\tau \to N \to 1$$

be the induced extension, representing $\tau_*[e] \in H^2(N, A')$; cf., e.g., Section 2.2 of [13]. If $[e] \in H^2(N, A)^Q$ then $[\tau e] \in H^2(N, A')^Q$.

PROPOSITION 4.5. *For any extension e of A by N that represents a member of $H^2(N, A)^Q$, the G-map τ induces, in a canonical way, a morphism*

$$(\tau_*, \omega_\tau, 1): \bar{e} \to (\overline{\tau e})$$

of extensions.

Proof. The map $\omega_\tau: \mathrm{Out}_G(e) \to \mathrm{Out}_G(\tau e)$ given in Proposition 2.1 of [13] yields the desired morphism of extensions.

Next, let there be given two group extensions, (1.1) and (1.1)′, and let $\Phi: G' \to G$ be a homomorphism that maps N' into N. Then Φ induces a morphism of extensions and, by abuse of notation, we simply write Φ: $(1.1)' \to (1.1)$.

Now, if e is a group extension (2.1), let

$$e\Phi: 0 \to A \to E^\Phi \to N' \to 1$$

be the induced extension representing $\Phi^*[e] \in H^2(N', A)$; cf., e.g., Section 2.2 of [13]. If e represents a member of $H^2(N, A)^Q$, writing $\mathrm{Out}_{G'}(e) = \mathrm{Out}_G(e) \times_Q Q'$, let

$$\bar{e}\Phi: 0 \to H^1(N, A) \to \mathrm{Out}_{G'}(e) \to Q' \to 1$$

be the induced extension, representing $\Phi^*[\bar{e}] \in H^2(Q', H^1(N, A))$ (here and below the notation "-*" is abused); notice that $e\Phi$ represents a member of $H^2(N', A)^{Q'}$ in this case.

PROPOSITION 4.6. *For any extension e of A by N that represents a member of $H^2(N, A)^Q$, the morphism Φ induces, in a canonical way, morphisms*

$$(1, \hat{\omega}^\Phi, \Phi): \bar{e}\Phi \to \bar{e}$$

and

$$(\Phi^*, \omega^\Phi, 1)\colon \bar{e}\Phi \to (\overline{e\Phi})$$

of extensions.

Proof. The maps $\hat{\omega}^\Phi\colon \mathrm{Out}_{G'}(e) \to \mathrm{Out}_G(e)$ and $\omega^\Phi\colon \mathrm{Out}_{G'}(e) \to \mathrm{Out}_{G'}(e\Phi)$ in Propositions 2.3 and 2.4 of [13], respectively, yield the desired morphisms of extensions.

Notice that Propositions 4.5 and 4.6 imply the (well known) fact that the differential $d_2\colon E_2^{0,2} \to E_2^{2,1}$ is natural in both variables.

5. The Proof of Theorem 2

Let $\delta\colon Q \to H^1(N, A)$ be a derivation. Let $\alpha = \alpha_\delta$ be the corresponding Q-map $IQ \to H^1(N, A)$ ($\alpha(q-1) = \delta(q)$, $q \in Q$). In view of Proposition 3.1, the image $d_2([\alpha]) \in H^3(Q, A^N)$ is obtained as follows: Let $(B_*(Q), \partial)$ be the (normalised) Bar resolution in inhomogeneous form [16, p. 114]. Construct a lifting of α:

$$\begin{array}{ccccccccc}
0 \longrightarrow & J_2(Q) & \longrightarrow & B_2(Q) & \longrightarrow & B_1(Q) & \longrightarrow & IQ & \longrightarrow 0 \\
& \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \alpha} & \\
0 \longrightarrow & \mathrm{Hom}_N(\mathbb{Z}, A) & \longrightarrow & \mathrm{Hom}_N(\mathbb{Z}G, A) & \underset{k^*}{\longrightarrow} & \mathrm{Hom}_N(IG, A) & \longrightarrow & H^1(N, A) & \longrightarrow 0.
\end{array} \tag{5.1}$$

Then $-\sigma$ represents the image $d_2[\alpha] \in H^3(Q, A^N)$.

In order to prove Theorem 2, let (C, F, ∂) be the free crossed module on the standard presentation $(X; R)$ of Q [10, Sect. 4]; here $X = \{u_q; q \in Q^*\}$ and $R = \{(r, s) = u_r u_s u_{rs}^{-1}; r, s \in Q^*\}$. Now choose a lifting $\lambda\colon F \to G$ of the obvious projection $\pi\colon F \to Q$ such that $\pi = p\lambda$, where $p\colon G \to Q$ is the projection in (1.1). Further, let

$$e_{(X;R)}\colon 0 \to J \to C \to F \xrightarrow{\pi} Q \to 1$$

be the corresponding crossed 2-fold extension [10, Sects. 3, 4]. It is known [10, Sects. 2, 4, 9] that J is a Q-module (the action is induced by the F-action on C) generated by the elements

$$u(r, s, t) = {}^{u_r}(s, t)(r, st)(rs, t)^{-1}(r, s)^{-1} \in C,$$

and that the rule

$$u(r, s, t) \mapsto (r[s|t] + [r|st] - [rs|t] - [r|s]) \in J_2(Q)$$

describes an isomorphism $J \to J_2(Q)$. In view of the main Theorem in [10, Sect. 7], Theorem 2 is implied by the following.

PROPOSITION 5.1. *The above map* $\lambda: F \to G$ *and diagram* (5.1) *determine a lifting*

$$\begin{array}{cccccccccc} e_{(X;R)}: 0 \longrightarrow & J & \longrightarrow & C & \longrightarrow & F & \longrightarrow & Q & \longrightarrow 1 \\ & \downarrow{\scriptstyle -\sigma} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_0} & & \| & \\ \bar{\delta}: 0 \longrightarrow & A^N & \longrightarrow & A \rtimes N & \longrightarrow & B^\delta & \longrightarrow & Q & \longrightarrow 1 \end{array} \tag{5.2}$$

of the identity map of Q *in a canonical way.*

5.1. *The Group* B^δ

We wish to describe the group B^δ (introduced in Section 2.3) as the semidirect fibre product $\mathrm{Der}(N, A) \rtimes_{H^1(N,A)} G$ (see Section 4.1) with respect to the derivation $\delta p: G \to H^1(N, A)$ and the natural projection $\mathrm{Der}(N, A) \to H^1(N, A)$; here G acts on $H^1(N, A)$ via the projection $p: G \to Q$ in (1.1). The requisite action of G on $\mathrm{Der}(N, A)$ is gven by the rule $d \mapsto l_g d i_g^{-1}$; here $d \in \mathrm{Der}(N, A)$, $g \in G$, and $i_g: N \to N$ denotes conjugation $n \mapsto gng^{-1}$. Note that this action coincides with that induced from extension (2.3).

LEMMA 5.1. *The projection* $\mathrm{Der}(N, A) \to H^1(N, A)$ *is a* G*-map.*

Proof. Consider the commutative triangle

$$\begin{array}{ccc} \mathrm{Hom}_N(IG, A) & & \\ \downarrow{\scriptstyle \rho} & \searrow & \\ & & H^1(N, A), \\ & \nearrow & \\ \mathrm{Der}(N, A) & & \end{array}$$

where $(\rho(h))(n) = h(n-1)$, $h \in \mathrm{Hom}_N(IG, A)$, $n \in N$. Let $g \in G$ and $n \in N$. For any $h \in \mathrm{Hom}_N(IG, A)$, the computation

$$\begin{aligned} {}^g h(g^{-1}(n-1) - (g^{-1}ng - 1)) &= {}^g h(g^{-1}ng(g^{-1} - 1) - (g^{-1} - 1)) \\ &= {}^{gg^{-1}ng} h(g^{-1} - 1) - {}^g h(g^{-1} - 1) \\ &= {}^{(n-1)}({}^g h(g^{-1} - 1)) \end{aligned}$$

shows that for $g \in G$ fixed the two derivations $N \to A$, given by

$$n \mapsto {}^g h(g^{-1}(n-1)) \; (= \rho(l_g h l_g^{-1})(n)), \qquad n \in N,$$

and

$$n \mapsto {}^g h(g^{-1}ng - 1), \qquad n \in N,$$

differ by an inner derivation only and thus determine the same class in $H^1(N, A)$. The statement of the lemma follows, since the Q-action on $H^1(N, A)$ is induced by the rule

$$(h: IG \to A) \mapsto (l_g h l_g^{-1}: IG \to A),$$

i.e., by the Q-action on $\mathrm{Hom}_N(IG, A)$. Q.E.D.

It follows that the construction of the semidirect fibre product $\mathrm{Der}(N, A) \rtimes_{H^1(N,A)} G$ makes sense. We can now identify this group with B^δ as follows: As already explained in Section 2.3, the group $\mathrm{Aut}_G(e_s)$ splits canonically into $\mathrm{Der}(N, A) \rtimes G$; in fact, a canonical section $G \to \mathrm{Aut}_G(e_s)$ is induced by the (obvious) action of G on $A \rtimes N$. The action of $\mathrm{Der}(N, A) \rtimes G$ on $A \rtimes N$ is given explicitly by the rule

$$^{(d,g)}(a, n) = ({}^g a + d(gng^{-1}), gng^{-1}). \tag{5.3}$$

Furthermore, the group $\mathrm{Out}_G(e_s)$ speits canonically into $H^1(N, A) \rtimes Q$, and we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Der}(N, A) \rtimes_{H^1(N,A)} G & \longrightarrow & \mathrm{Der}(N, A) \rtimes G \\ \downarrow{\scriptstyle \pi_\delta} & & \downarrow \\ Q & \xrightarrow[\psi_\delta]{} & H^1(N, A) \rtimes Q; \end{array}$$

here $\pi_\delta(d, g) = q_g \in Q$ (the image of g in Q), and the other maps are the obvious ones.

PROPOSITION 5.2. *If we identify* $\mathrm{Aut}_G(e_s)$ *with* $\mathrm{Der}(N, A) \rtimes G$ *as above, then* B^δ *is the subgroup* $\mathrm{Der}(N, A) \rtimes_{H^1(N,A)} G$.

The projection $B^\delta \to Q$ is now the map π_δ, the homomorphism $\partial: A \rtimes N \to B^\delta$ is given by $\partial(a, n) = (-d_a^i, n)$, $a \in A$, $n \in N$, and B^δ acts on $A \rtimes N$ by rule (5.3); here $d_a^i: N \to A$ is the inner derivation $d_a^i(n) = {}^n a - a$, $n \in N$.

5.2. *The Construction of the Lifting* (5.2)

For convenience, we shall replace $\mathfrak{F}$ by the crossed 2-fold extension

$$0 \to A^N \to A \rtimes N \xrightarrow{\partial'} \mathrm{Der}(N, A) \rtimes_{H^1(N,A)} G \to Q \to 1, \tag{5.4}$$

where $\partial'(a, n) = (d_a^i, n)$, and where $\mathrm{Der}(N, A) \rtimes_{H^1(N,A)} G$ acts on $A \rtimes N$ by the rule

$$^{(d,g)}(a, n) = ({}^g a - d(gng^{-1}), gng^{-1}).$$

LEMMA 5.2. *The map* $\varphi: A \rtimes N \to A \rtimes N$, $\varphi(a, n) = (-a, n)$, *induces a morphism* $(-1, \varphi, 1, 1)$: $(5.4) \to \tilde{\delta}$ *of crossed 2-fold extensions.*

Proof. Straightforward.

Instead of directly constructing (5.2), we shall construct a morphism $(\sigma, \beta_1, \beta_0, 1)$: $e_{(X;R)} \to (5.4)$ of crossed 2-fold extensions.

We maintain the notation at the beginning of this section; further, if u_q is a free generator of F, $q \in Q^*$, we shall write $\lambda_q = \lambda(u_q)$.

Now define $\beta_0: F \to B^\delta = \mathrm{Der}(N, A) \rtimes_{H^1(N,A)} G$ by the rule

$$\beta_0(u_q) = (d_{\mu_0[q]}, \lambda_q), \qquad q \in Q^*;$$

here $d_{\mu_0[q]}$ denotes the derivation $N \to A$ given by

$$n \mapsto (\mu_0[q])(n-1), \qquad n \in N.$$

LEMMA 5.3. *The map* β_0 *is well defined, i.e.,*

$$[d_{\mu_0[q]}] = \delta p(\lambda_q) \in H^1(N, A), \qquad q \in Q^*.$$

Proof. Clearly $\delta p(\lambda_q) = \delta(q) = \alpha(q-1)$, where $\alpha = \alpha_\delta: IQ \to H^1(N, A)$. The assertion follows, since μ_0 lifts α in (5.1). Q.E.D.

Next we introduce a function

$$\gamma: Q^* \times Q^* \to A \quad (Q^* = Q - \{1\})$$

by

$$\gamma(r, s) = \mu_1[r|s](1) + \mu_0(r[s])(\lambda_r - 1) - \mu_0[rs](\lambda_r \lambda_s \lambda_{rs}^{-1} - 1),$$
$$r, s \in Q^*.$$

LEMMA 5.4. *Let* $r, s \in Q^*$; *then*

$$\beta_0(u_r u_s u_{rs}^{-1}) = (d^i_{\gamma(r,s)}, \lambda_r \lambda_s \lambda_{rs}^{-1})$$

(*where, for* $a \in A$, d^i_a *denotes the inner derivation* $N \to A$ *given by* $d^i_a(n) = {}^n a - a$).

Since C is the free crossed F-module with basis $\{(r, s); r, s \in Q^*\}$ (cf. Section 4 of [10]), we may define $\beta_1: C \to A \rtimes N$ by

$$\beta_1(r, s) = (\gamma(r, s), \lambda_r \lambda_s \lambda_{rs}^{-1}), \qquad r, s \in Q^*.$$

Proof of Lemma 5.4

Using additive notation in $\mathrm{Der}(N, A)$, the first component of $\beta_0(u_r u_s u_{rs}^{-1})$ is the derivation

$$d_{(r,s)} = d_{\mu_0[r]} + {}^{\lambda_r}(d_{\mu_0[s]}) - {}^{\lambda_0}(d_{\mu_0[rs]})\colon N \to A,$$

where $\lambda_0 = \lambda_r \lambda_s \lambda_{rs}^{-1}$. Now, for $n \in N$, we have

$$\begin{aligned}
d_{\mu_0[r]}(n) &= \mu_0[r](n-1);\\
({}^{\lambda_r}(d_{\mu_0[s]}))(n) &= {}^{\lambda_r}(\mu_0[s](\lambda_r^{-1} n \lambda_r - 1))\\
&= {}^{r}(\mu_0[s])(\lambda_r(\lambda_r^{-1} n \lambda_r - 1))\\
&= \mu_0(r[s])((n-1)\,\lambda_r)\\
&= \mu_0(r[s])(n-1) + \mu_0(r[s])((n-1)(\lambda_r - 1))\\
&= \mu_0(r[s])(n-1) + {}^{(n-1)}(\mu_0(r[s])(\lambda_r - 1));\\
({}^{\lambda_0}(d_{\mu_0[rs]}))(n) &= \mu_0[rs](n-1) + {}^{(n-1)}(\mu_0[rs](\lambda_r \lambda_s \lambda_{rs}^{-1} - 1)).
\end{aligned}$$

Hence

$$\begin{aligned}
d_{(r,s)}(n) &= (\mu_0[r] + \mu_0(r[s]) - \mu_0[rs])(n-1)\\
&\quad + {}^{(n-1)}(\mu_0(r[s])(\lambda_r - 1) - \mu_0[rs](\lambda_r \lambda_s \lambda_{rs}^{-1} - 1))\\
&= (\mu_0 \partial[r|s])(n-1) + {}^{(n-1)}(\cdots)\\
&= {}^{(n-1)}(\mu_1[r|s](1) + (\cdots)) = d^i_{\gamma(r,s)}(n).
\end{aligned}$$

5.3. *The Completion of the Proof*

The group $J = \ker(\partial\colon C \to F)$ is (as a Q-module) generated by the elements (cf. [10, Sect. 9])

$$u(r, s, t) = {}^{u_r}(s, t)(r, st)(rs, t)^{-1}(r, s)^{-1}, \qquad r, s, t \in Q^*.$$

The proof of Theorem 2 is now completed by the following.

PROPOSITION 5.3. *The restriction of β_1 to J is the map σ; in that connection $u(r, s, t)$ is to be identified with $(r[s|t] + [r|st] - [rs|t] - [r|s]) \in J_2(Q)$, as already indicated, and A^N is to be identified with* $\mathrm{Hom}_N(\mathbb{Z}, A)$ *in the standard way.*

Proof (it is fuzzy but straightforward). The value

$$\sigma(r[s|t] + [r|st] - [rs|t] - [r|s])$$

is given by the N-map

$$\xi = \mu_1(r[s|t] + [r|st] - [rs|t] - [r|s]) \colon \mathbb{Z}G \to A$$

which, by construction, is trivial on IG (and hence induces an $N-$ map $\mathbb{Z} \to A$). Thus we have to verify that

$$\beta_1(u(r,s,t)) = (\xi(1), 1) \in A \rtimes N.$$

To this end, we calculate in $A \rtimes N$ the product of the following four terms (i), (ii), (iii) and (iv) (in A we use additive notation):

(i) $\beta_1({}^{u_r}(s,t)) = {}^{\beta_0(u_r)}(\gamma(s,t), \lambda_s\lambda_t\lambda_{st}^{-1})$;

(ii) $\beta_1(r, st) = (\gamma(r,st), \lambda_r\lambda_{st}\lambda_{rst}^{-1})$;

(iii) $\beta_1((rs,t)^{-1}) = (-{}^{\lambda_1}\gamma(rs,t), \lambda_1)$, where $\lambda_1 = \lambda_{rst}\lambda_t^{-1}\lambda_{rs}^{-1}$;

(iv) $\beta_1((r,s)^{-1}) = (-{}^{\lambda_2}\gamma(r,s), \lambda_2)$, where $\lambda_2 = \lambda_{rs}\lambda_s^{-1}\lambda_r^{-1}$.

Now

$$\beta_1({}^{u_r}(s,t)) = {}^{(d,\lambda_r)}(\gamma(s,t), \lambda_s\lambda_t\lambda_{st}^{-1}), \qquad \text{where } d = d_{\mu_0[r]},$$
$$= ({}^{\lambda_r}\gamma(s,t) - \mu_0[r](a-1), a), \qquad \text{where } a = \lambda_r\lambda_s\lambda_t\lambda_{st}^{-1}\lambda_r^{-1}.$$

The second component of the product obviously gives $1 \in N$.
Hence in A we have to work out the sum

$$\Sigma = (\text{i}') + (\text{ii}') + (\text{iii}') + (\text{iv}'),$$

where

(i′) ${}^{\lambda_r}\gamma(s,t) - \mu_0[r](a-1)$;

(ii′) ${}^{a}\gamma(r,st)$;

(iii′) $-{}^{\lambda_3}\gamma(rs,t)$, where

$$\lambda_3 = (\lambda_r\lambda_s\lambda_t\lambda_{st}^{-1}\lambda_r^{-1})(\lambda_r\lambda_{st}\lambda_{rst}^{-1})(\lambda_{rst}\lambda_t^{-1}\lambda_{rs}^{-1});$$

$= -{}^{b}\gamma(rs,t)$, where $b = \lambda_r\lambda_s\lambda_{rs}^{-1}$;

(iv′) $-\gamma(r,s)$.

By routine calculations, we get

(i′) $\mu_1(r[s|t])(\lambda_r) + \mu_0(rs[t])(\lambda_r(\lambda_s - 1)) - \mu_0(r[st])(\lambda_r(c-1)) - \mu_0[r](a-1)$, where $c = \lambda_s\lambda_t\lambda_{st}^{-1}$;

(ii′) $\mu_1[r|st](a) + \mu_0(r[st](a(\lambda_r - 1)) - \mu_0[rst](a(d-1))$, where $d = \lambda_r\lambda_{st}\lambda_{rst}^{-1}$;

(iii′) $-\mu_1[rs|t](b) - \mu_0(rs[t])(b(\lambda_{rs} - 1)) + \mu_0[rst](b(e-1))$,
where $e = \lambda_{rs}\lambda_t\lambda_{rst}^{-1}$;

(iv′) $-\mu_1[r|s](1) - \mu_0(r[s])(\lambda_r - 1) + \mu_0[rs](b-1)$.

The sum of terms with $\mu_0(rs[t])$ is

$$\begin{aligned}
&\mu_0(rs[t])(\lambda_r(\lambda_s - 1) - b(\lambda_{rs} - 1))\\
&= \mu_0(rs[t])(b - \lambda_r)\\
&= \mu_0(rs[t])(b-1) - \mu_0(rs[t])(\lambda_r - 1).
\end{aligned}$$

Likewise, we compute the sum of terms with $\mu_0(r[st])$:

$$\begin{aligned}
&\mu_0(r[st])(a(\lambda_r - 1) - \lambda_r(c-1))\\
&= \mu_0(r[st])(\lambda_r - a)\\
&= \mu_0(r[st])(\lambda_r - 1) - \mu_0(r[st])(a-1).
\end{aligned}$$

Finally, the sum of terms with $\mu_0[rst]$ is

$$\begin{aligned}
&\mu_0[rst](b(e-1) - a(d-1))\\
&= \mu_0[rst](a-b)\\
&= \mu_0[rst](a-1) - \mu_0[rst](b-1).
\end{aligned}$$

If we now sum up suitably, we obtain

$$\begin{aligned}
\Sigma &= \mu_1(r[s|t])(\lambda_r) - (\mu_0(r[s]) + \mu_0(rs[t]) - \mu_0(r[st]))(\lambda_r - 1)\\
&\quad + \mu_1[r|st](a) - (\mu_0[r] + \mu_0(r[st]) - \mu_0[rst])(a-1)\\
&\quad - \mu_1[rs|t](b) + (\mu_0[rs] + \mu_0(rs[t]) - \mu_0[rst])(b-1)\\
&\quad - \mu_1[r|s](1)\\
&= \mu_1(r[s|t])(\lambda_r) + \mu_1(r[s|t])(1-\lambda_r)\\
&\quad + \mu_1[r|st](a) + \mu_1[r|st](1-a)\\
&\quad - \mu_1[rs|t](b) - \mu_1[rs|t](1-b)\\
&\quad - \mu_1[r|s](1)\\
&= (\mu_1(r[s|t]) + \mu_1[r|st] - \mu_1[rs|t] - \mu_1[r|s])(1)\\
&= \xi(1).
\end{aligned}$$

Q.E.D.

5.4. *Naturalness of the Description*

Our description of $d_2: E_2^{1,1} \to E_2^{3,0}$ is again *natural* in a strong sense; what this means will be expressed below in Propositions 5.4 and 5.5 for the module variable and the group extension variable, respectively.

Let $\tau: A \to A'$ be a homomorphism of G-modules and denote $\tau_*: H^1(N, A) \to H^1(N, A')$ the induced map. If $\delta: Q \to H^1(N, A)$ is a derivation it is clear that the combined map $\tau_* \delta: Q \to H^1(N, A')$ is a derivation representing the image $\tau_*[\delta] \in H^1(Q, H^1(N, A'))$ (where we abuse the notation "$-_*$").

PROPOSITION 5.4. *For any derivation $\delta: Q \to H^1(N, A)$ the G-map τ induces, in a canonical way, a morphism*

$$(\tau|, \lambda_\tau, \nu_\tau, 1): \tilde{\delta} \to (\tilde{\tau}_* \delta)$$

of crossed 2-fold extensions.

Proof. Let $\lambda_\tau: A \rtimes N \to A' \rtimes N$ be the obvious map. Further, by Proposition 5.2, $B^\delta = \operatorname{Der}(N, A) \rtimes_{H^1(N,A)} G$ and $B^{\tau_*\delta} = \operatorname{Der}(N, A') \rtimes_{H^1(N,A')} G$; now let ν_τ be the obvious map.

Remark. There is a different way of obtaining the above morphism $(\tau|, \lambda_\tau, \nu_\tau, 1)$ of crossed 2-fold extensions. In fact, if $\omega_\tau: \operatorname{Out}_G(e_s) \to \operatorname{Out}_G(\tau e_s)$ is the map in Proposition 4.5 then $\psi_{\tau_*\delta} = \omega_\tau \psi_\delta$. Hence, if $\mu_\tau: \operatorname{Aut}_G(e_s) \to \operatorname{Aut}_G(\tau e_s)$ is the map given in Proposition 2.1 of [13] then μ_τ induces the desired map ν_τ. It is also worth noting that the map

$$\omega_\tau: H^1(N, A) \rtimes Q = \operatorname{Out}_G(e_s) \to \operatorname{Out}_G(\tau e_s) = H^1(N, A') \rtimes Q$$

is the obvious one, where "=" means the obvious isomorphisms explained in Section 5.1.

Next, let there be given a morphism $\Phi: (1.1)' \to (1.1)$ of extensions (notation as in Section 4.4). Denote $\Phi^*: H^1(N, A) \to H^1(N', A)$ the induced map (the notation "$-^*$" will be abused at several places below). If $\delta: Q \to H^1(N, A)$ is a derivation, it is clear that the combined map

$$\delta' = \Phi^* \delta \Phi: Q' \to H^1(N', A)$$

is a derivation representing $\Phi^*[\delta] \in H^1(Q', H^1(N', A))$; let $\hat{B}^\delta$ denote the semidirect fibre product $\operatorname{Der}(N, A) \rtimes_{H^1(N,A)} G'$ with respect to the derivation $G' \to G \to Q \to^\delta H^1(N, A)$, and let $\hat{B}^\delta$ act on $A \rtimes N'$ by the rule (5.3), where the notation is to be suitably modified. Together with the obvious map $\partial: A \rtimes N' \to \hat{B}^{\delta}$ this yields the crossed 2-fold extension

$$\hat{\delta}: 0 \to A^N \to A \rtimes N' \xrightarrow{\partial} \hat{B}^{\delta} \to Q' \to 1$$

which clearly represents $\Phi^*[\tilde{\delta}] \in H^3(Q', A^N)$.

PROPOSITION 5.5. *For any derivation $\delta: Q \to H^1(N, A)$ the morphism Φ induces, in a canonical way, morphisms*

$$(1, \cdot, \cdot, \Phi): \hat{\delta} \to \tilde{\delta}$$

and

$$(\Phi^*, 1, \cdot, 1): \hat{\delta} \to \tilde{\delta}'$$

of crossed 2-fold extensions.

Proof. By Proposition 5.2, we may identify B^{δ} with $\mathrm{Der}(N, A) \rtimes_{H^1(N,A)} G$ and $B^{\delta'}$ with $\mathrm{Der}(N', A) \rtimes_{H^1(N',A)} G'$. Hence Φ induces morphisms of crossed 2-fold extensions as desired.

Remark. There is also a different way of obtaining the morphisms of crossed 2-fold extensions in Proposition 5.5. In fact, let ω^{Φ}: $\mathrm{Out}_{G'}(e_s) \to \mathrm{Out}_{G'}(e_s\Phi)$ be the map in Proposition 4.6, and let $\psi'_{\delta}: Q' \to \mathrm{Out}_{G'}(e_s)$ be the obvious map which is induced by ψ_{δ}. Then

$$\psi_{\delta'} = \omega^{\Phi}\psi'_{\delta}: Q' \to \mathrm{Out}_{G'}(e_s\Phi),$$

whence $\hat{B}^{\delta}$ may be identified with the fibre product $\mathrm{Aut}_{G'}(e_s) \times_{\mathrm{Out}_{G'}(e_s)} Q'$, where $\mathrm{Aut}_{G'}(e_s) = \mathrm{Aut}_G(e_s) \times_G G'$. Further, the maps $\hat{\mu}^{\Phi}$: $\mathrm{Aut}_{G'}(e_s) \to \mathrm{Aut}_G(e_s)$ and μ^{Φ}: $\mathrm{Aut}_{G'}(e_s) \to \mathrm{Aut}_{G'}(e_s\Phi)$ in Propositions 2.3 and 2.4 of [13], respectively, induce the desired maps $\hat{B}^{\delta} \to B^{\delta}$ and $\hat{B}^{\delta} \to B^{\delta'}$. It is also worth noting that the map

$$\omega^{\Phi}: H^1(N, A) \rtimes G' = \mathrm{Out}_{G'}(e_s) \to \mathrm{Out}_{G'}(e_s\Phi) = H^1(N', A) \rtimes G'$$

is the obvious one, where "=" means the obvious isomorphisms; see Section 5.1 above.

Notice that Propositions 5.4 and 5.5 imply the (well known) fact that the differential $d_2: E_2^{1,1} \to E_2^{3,0}$ is natural in both variables.

6. The Proof of Theorem 3

We shall show that the pairs given in Theorem 3 constitute the corresponding transgression. All the rest is straightforward.

Let e be a group extension (2.1) that represents a member of $H^2(N,A)^Q$. We choose a lifting (4.2) of 1_N and construct a diagram (4.3). We then represent $[e]$ by $\alpha\colon \mathbb{Z}\to \operatorname{Ext}_N(IG,A)^Q$ ($\alpha(1)=[c_e]$, c_e as in Section 4.1), and construct a lifting α_0 in

$$\begin{array}{ccc} \mathbb{Z}Q & \longrightarrow & \mathbb{Z} \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} \\ \operatorname{Hom}_N(J_1(G),A) & \xrightarrow[r]{} & \operatorname{Ext}_N(IG,A) \end{array}$$

by setting $\alpha_0(1)=\mu$ (cf. Section 4.3). This induces a map $\eta\colon IQ\to T$ ($=\ker r$). Let

$$\bar{e}\colon 0\to H^1(N,A)\to \operatorname{Out}_G(e)\to Q\to 1$$

be the extension (2.5) associated with e in Section 2.2. By Proposition 4.4 we may identify $\operatorname{Out}_G(e)$ with $H^1(N^Q,A)\rtimes_T Q$, where $h\colon H^1(N^Q,A)\to T$ is the obvious map given by rule (4.7) above and where the requisite derivation $d\colon Q\to T$ is given by $d(q)=\alpha_0(q-1)$, $q\in Q$.

PROPOSITION 6.1. *Let α_0 as above be fixed. The class $[e]\in H^2(N,A)^Q$ is transgressive if and only if there is a Q-map $\chi\colon IQ\to H^1(N^Q,A)$ such that $\eta=h\chi$. In this case, there is a canonical bijection between Q-maps χ with $\eta=h\chi$ and sections $\psi\colon Q\to \operatorname{Out}_G(e)=H^1(N^Q,A)\rtimes_T Q$.*

Proof. By Corollary 1 in Section 2.2, $[e]$ is transgressive if and only if there is a section $\psi\colon Q\to \operatorname{Out}_G(e)=H^1(N^Q,A)\rtimes_T Q$. Any such section determines a derivation $\delta\colon Q\to H^1(N^Q,A)$, hence a Q-map as desired, and vice versa. Q.E.D.

Now let $[e]\in H^2(N,A)^Q$ be transgressive, and let $\psi\colon Q\to \operatorname{Out}_G(e)$ be a section. In view of the above, ψ determines a map $\chi\colon IQ\to H^1(N^Q,A)$ such that $\eta=h\chi$. It follows that α lifts to

$$\begin{array}{ccccccc} 0\longrightarrow & J_2(Q) & \longrightarrow & B_2(Q) & \longrightarrow & B_1(Q) & \\ & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & \\ 0\longrightarrow & \operatorname{Hom}_N(\mathbb{Z},A) & \longrightarrow & \operatorname{Hom}_N(\mathbb{Z}G,A) & \longrightarrow & \operatorname{Hom}_N(B_1(G),A) & \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & \mathbb{Z}Q & \longrightarrow & \mathbb{Z} & \longrightarrow 0 \\ & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & \\ \longrightarrow & \operatorname{Hom}_N(J_1(G),A) & \longrightarrow & \operatorname{Ext}_N(IG,A) & \longrightarrow 0. \end{array} \tag{6.1}$$

By the Addendum to Proposition 3.2, the pair $(\alpha,-\sigma)$ represents the element $(\alpha,-[\sigma])$ of the transgression. Conversely, if $(\alpha,-\sigma)$ represents an element of the transgression, there is a diagram (6.1) (again by the Addendum to Proposition 3.2). Hence

PROPOSITION 6.2. *Any element of the transgression* $\tau: E_2^{0,2} \rightharpoonup E_2^{3,0}$ *may be obtained as follows: Let* $[e] \in H^2(N, A)^Q$ *be transgressive. Represent* $[e]$ *by* $\alpha: \mathbb{Z} \to \operatorname{Ext}_N(IG, A)$ *and lift* α *to* α_0 *as above. Then using* α_0, *identify* $\operatorname{Out}_G(e)$ *and* $H^1(N^Q, A) \rtimes_T Q$ *as above. Let* $\psi: Q \to \operatorname{Out}_G(e)$ *be a section. It induces a derivation* $\delta: Q \to H^1(N^Q, A)$, *hence a* Q*-map* $\chi: IQ \to H^1(N^Q, A)$ *such that* $\eta = h\chi$ *as above. Finally, construct a lifting*

$$\begin{array}{ccccccccc} 0 \longrightarrow & J_2(Q) & \longrightarrow & B_2(Q) & \longrightarrow & B_1(Q) & \longrightarrow & IQ & \longrightarrow 0 \\ & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \chi} & \\ 0 \longrightarrow & \operatorname{Hom}_N(\mathbb{Z}, A) & \longrightarrow & \operatorname{Hom}_N(\mathbb{Z}G, A) & \longrightarrow & \operatorname{Hom}_N(B_1(G), A) & \longrightarrow & H^1(N^Q, A) & \longrightarrow 0. \end{array} \tag{6.2}$$

Then $([e], -[\sigma])$ *is an element of the transgression.*

In view of the main Theorem in [10, Sect. 7], the crucial step in the proof of Theorem 3 is now provided by the following.

PROPOSITION 6.3. *Let* $e_{(X;R)}$ *be the crossed 2-fold extension, associated in Section 5 to the standard presentation* $(X; R)$ *of* Q. *Let* $[e] \in H^2(N, A)^Q$ *be transgressive. Represent* $[e]$ *by* $\alpha: \mathbb{Z} \to \operatorname{Ext}_N(IG, A)$ *and lift* α *to* α_0 *as above. Let* $\psi: Q \to \operatorname{Out}_G(e)$ *be a section, and construct a diagram* (6.1) (*or* (6.2)). *Then* (6.1) *gives rise to a morphism of crossed 2-fold extensions*

$$\begin{array}{ccccccccccc} e_{(X;R)}: 0 \longrightarrow & J & \longrightarrow & C & \longrightarrow & F & \longrightarrow & Q & \longrightarrow 1 \\ & \downarrow{\scriptstyle -\sigma} & & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_0} & & \| & \\ \tilde{e}_\psi: 0 \longrightarrow & A^N & \longrightarrow & E & \longrightarrow & B^\psi & \longrightarrow & Q & \longrightarrow 1. \end{array} \tag{6.3}$$

Proof. The exact sequence

$$0 \to \operatorname{Hom}_N(\mathbb{Z}, A) \to \operatorname{Hom}_N(\mathbb{Z}G, A) \to \operatorname{Hom}_N(B_1(G), A) \to H^1(N^Q, A) \to 0$$

is naturally isomorphic to

$$0 \to \operatorname{Hom}_{N^Q}(\mathbb{Z}, A) \to \operatorname{Hom}_{N^Q}(\mathbb{Z}\bar{F}, A) \to \operatorname{Hom}_{N^Q}(I\bar{F}, A) \to H^1(N^Q, A) \to 0,$$

where $\bar{F}$ is free on G^*; $\bar{F}$ was denoted F in Section 4. Hence, from (6.2) we obtain

$$\begin{array}{ccccccccc}
0 \longrightarrow & J_2(Q) & \longrightarrow & B_2(Q) & \longrightarrow & B_1(Q) & \longrightarrow & IQ & \longrightarrow 0 \\
& \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\alpha_2} & & \downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\chi} & \\
0 \longrightarrow & \mathrm{Hom}_{N^Q}(\mathbb{Z}, A) & \longrightarrow & \mathrm{Hom}_{N^Q}(\mathbb{Z}\bar{F}, A) & \longrightarrow & \mathrm{Hom}_{N^Q}(I\bar{F}, A) & \longrightarrow & H^1(N^Q, A) & \longrightarrow 0,
\end{array} \tag{6.4}$$

where χ is obtained from α_0 and ψ as in Proposition 6.1. We can now apply Proposition 5.1, where the role of the extension (1.1) is played by

$$1 \to N^Q \to \bar{F} \to Q \to 1,$$

that of e_s (the split extension of A by N) by

$$0 \to A \to A \rtimes N^Q \to N^Q \to 1,$$

that of the map λ by a suitable lifting $\lambda: F \to \bar{F}$ of the obvious projection $F \to Q$, and that of δ by $\delta = \delta_\chi: Q \to H^1(N^Q, A)$, $\delta(q) = \chi(q-1)$, $q \in Q$. Moreover, by Proposition 5.2 we may identify B^δ with the semidirect fibre product $\mathrm{Der}(N^Q, A) \rtimes_{H^1(N^Q,A)} \bar{F}$; here the requisite derivation $d: \bar{F} \to H^1(N^Q, A)$ is the combined map δpr, where $pr: \bar{F} \to Q$ denotes the projection. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
e_{(X;R)}: 0 \longrightarrow & J & \longrightarrow & C & \longrightarrow & F & \longrightarrow & Q & \longrightarrow 1 \\
& \downarrow{\scriptstyle -\sigma} & & \downarrow{\scriptstyle\beta_1} & & \downarrow{\scriptstyle\beta_0} & & \| & \\
\tilde{\delta}: 0 \longrightarrow & A^N & \longrightarrow & A \rtimes N^Q & \longrightarrow & \mathrm{Der}(N^Q, A) \rtimes_{H^1(N^Q,A)} \bar{F} & \longrightarrow & Q & \longrightarrow 1.
\end{array} \tag{6.5}$$

The proof is now completed by the following.

PROPOSITION 6.4. *There is a morphism* $(1, \theta_1, \theta_0, 1): \tilde{\delta} \to \tilde{e}_\psi$ *of crossed 2-fold extensions.*

For the proof we need the following.

LEMMA 6.1. *There is a natural action of the group* $\mathrm{Der}(N^Q, A) \rtimes_{H^1(N^Q,A)} \bar{F}$ *on the middle group* E *of the extension* e, *such that* $(\tau, u) \in \mathrm{Der}(N^Q, A) \rtimes_{H^1(N^Q,A)} \bar{F}$ *induces left translation with* g_u *on* A *and conjugation with* g_u *on* N, *where* $g_u \in G$ *is the image of* $u \in \bar{F}$.

ADDENDUM. *This action induces a commutative diagram*

$$\begin{array}{ccc} \mathrm{Der}(N^Q, A) \underset{H^1(N^Q,A)}{\rtimes} \bar{F} & \longrightarrow & \mathrm{Aut}_G(e) \\ \downarrow{\scriptstyle \pi_\delta} & & \downarrow \\ Q & \xrightarrow{\psi} & \mathrm{Out}_G(e) = H^1(N^Q, A) \underset{T}{\rtimes} Q; \end{array} \tag{6.6}$$

here π_δ *sends* (τ, u) *to the image* $q_u \in Q$ *of* $u \in \bar{F}$.

Proof. Consider the commutative diagram (4.9)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N^G & \xrightarrow{j} & N^Q & \longrightarrow & N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \theta} & & \| & & \\ e: 0 & \longrightarrow & A & \xrightarrow{i} & E & \longrightarrow & N & \longrightarrow & 1 \end{array}$$

already used in the proof of Proposition 4.4, where $\kappa = \pi pr$, and where pr: $N^G \to (N^G)^{Ab} = J_1(G)$ (identified in Section 4.1) is the canonical projection, such that $\alpha_0(1) = \mu \in \mathrm{Hom}_N(J_1(G), A)$ (where α_0: $\mathbb{Z}Q \to \mathrm{Hom}_N(J_1(G), A)$ lifts α: $\mathbb{Z} \to \mathrm{Ext}_N(IG, A)$ as above).

LEMMA 6.2. *The rule* $(a, n) \mapsto a\theta(n)$, $a \in A$, $n \in N^Q$, *describes a projection* π_e: $A \rtimes N^Q \to E$ *such that* π_e *is the coequaliser of*

$$N^G \underset{\kappa}{\overset{j}{\rightrightarrows}} A \rtimes N^Q.$$

Proof. By inspection.

The proof of Lemma 6.1 is now completed as follows: Let $(\tau, u) \in \mathrm{Der}(N^Q, A) \rtimes_{H^1(N^Q,A)} \bar{F}$. Write $g = g_u \in G$ and $q = q_u \in Q$ for the images in G and Q of u, respectively. Define maps α_1: $N^Q \to E$, α_2: $A \to E$ by setting

$$\alpha_1(n) = \tau(unu^{-1})\,\theta(unu^{-1}), \qquad n \in N^Q,$$

$$\alpha_2(a) = {}^g a, \qquad a \in A.$$

Since $(\tau, u) \in \mathrm{Der}(N^Q, A) \rtimes_{H^1(N^Q,A)} \bar{F}$, we have $[\tau] = d(u) \in H^1(N^Q, A)$, hence $h[\tau] = hd(u) = \alpha_0(q_u - 1)$, i.e.,

$$\begin{aligned} \tau | N^G &= (l_g \mu l_g^{-1} - \mu)\, pr \\ &= l_g \kappa i_u^{-1} - \kappa; \end{aligned}$$

here $h: H^1(N^Q, A) \to T$ is the map given by rule (4.7). Consequently, if $n \in N^G$, we have

$$\begin{aligned} \alpha_1(n) &= \tau(unu^{-1}) + \kappa(unu^{-1}) \qquad \text{(using additive notation in } A) \\ &= {}^g\kappa(n) - \kappa(unu^{-1}) + \kappa(unu^{-1}) \\ &= \alpha_2(\kappa(n)). \end{aligned}$$

Thus we obtain a map $A \rtimes N^Q \to E$ given by

$$(a, n) \mapsto \alpha_2(a)\, \alpha_1(n), \qquad a \in A, \quad n \in N^Q,$$

which coequalises j and κ. Hence (τ, u) induces a unique map $\alpha: E \to E$. Clearly, α induces left translation with g_u on A and conjugation with g_u on N whence α is an automorphism of E. Moreover, the rule ${}^{(\tau,u)}x = \alpha(x)$, $x \in E$, describes an action of $\operatorname{Der}(N^Q, A) \rtimes_{H^1(N^Q,A)} \bar{F}$ on E.

Proof of Addendum. If α is obtained as above, i.e., ${}^{(\tau,u)}x = \alpha(x)$, $x \in E$, let (α, g_α) be the corresponding member of $\operatorname{Aut}_G(e)$, where $g_\alpha = g_u$. It is clear that we have a homomorphism $\operatorname{Der}(N^Q, A) \rtimes_{H^1(N^Q,A)} \bar{F} \to \operatorname{Aut}_G(e)$, and, by abuse of language, we denote (α, g_α) by α also. In Proposition 4.4 we constructed a map $\operatorname{Aut}_G(e) \to H^1(N^Q, A) \rtimes_T Q$ given by $\alpha \mapsto ([d_\alpha], q_\alpha)$, $\alpha \in \operatorname{Aut}_G(e)$. Now, if α is the image of some $(\tau, u) \in \operatorname{Der}(N^Q, A) \rtimes_{H^1(N^Q,A)} \bar{F}$, we have

$$\begin{aligned} d_\alpha(n) &= \alpha(\theta(x_g^{-1} n x_g))\, \theta(n^{-1}), \qquad n \in N^Q, \quad g = g_u \in G, \\ &= \alpha_1(x_g^{-1} n x_g)\, \theta(n^{-1}) \\ &= \tau(u x_g^{-1} n x_g u^{-1})\, \theta(u x_g^{-1} n x_g u^{-1})\, \theta(n^{-1}), \end{aligned}$$

where $u x_g^{-1} \in N^G$. Hence

$$d_\alpha(n) = \tau(u x_g^{-1}) + \tau(n) - {}^{\theta(n)}(\tau(u x_g^{-1})) + {}^{(1-\theta(n))}(\kappa(u x_g^{-1})) \in A$$

since N^G acts trivially on A. We obtain

$$d_\alpha(n) = \tau(n) + {}^{(1-\theta(n))}(\tau(u x_g^{-1}) + \kappa(u x_g^{-1})).$$

Consequently, $[d_\alpha] = [\tau] \in H^1(N^Q, A)$, and the Addendum is proved.

Proof of Proposition 6.4. Since B^ψ is the fibre product $\operatorname{Aut}_G(e) \times_{\operatorname{Out}_G(e)} Q$ with respect to $\psi: Q \to \operatorname{Out}_G(e)$, diagram (6.6) induces a unique map $\theta_0: \operatorname{Der}(N^Q, A) \rtimes_{H^1(N^Q,A)} \bar{F} \to B^\psi$. Let $\theta_1 = \pi_e: A \rtimes N^Q \to E$. Then $(1, \theta_1, \theta_0, 1)$ is the desired morphism of crossed 2-fold extensions. Q.E.D.

7. An Example

We offer an example where we determine, by our methods, the differentials $d_2^{0,2}$ and $d_2^{1,1}$; we believe that our example is the simplest possible for producing a non-trivial $d_2^{0,2}$ and $d_2^{1,1}$.

Consider the group extension

$$1 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \xrightarrow{i} \mathbb{Z}/4 \times \mathbb{Z}/2 \to \mathbb{Z}/2 \to 1,$$

where i is the obvious inclusion (hence $N = \mathbb{Z}/2 \times \mathbb{Z}/2$, $G = \mathbb{Z}/4 \times \mathbb{Z}/2$, $Q = \mathbb{Z}/2$). Let $A = \mathbb{Z}/2$.

(i) $d_2 \colon H^2(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2)^{\mathbb{Z}/2} \to H^2(\mathbb{Z}/2, H^1(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2))$. Now $H^2(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2)^{\mathbb{Z}/2} = H^2(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^3$, and there are three groups giving rise to non-trivial extensions of $\mathbb{Z}/2$ by $\mathbb{Z}/2 \times \mathbb{Z}/2$: the group $\mathbb{Z}/4 \times \mathbb{Z}/2 = \langle a, b; a^4, b^2, [a, b] \rangle$, the dihedral group $D_4 = \langle a, b; a^4, b^2, (ab)^2 \rangle$ and the quaternion group $Qu = \langle a, b; a^2 = b^2 = (ab)^2 \rangle$. We write $\mathbb{Z}/2 \times \mathbb{Z}/2 = \langle u, v; u^2, v^2, [u, v] \rangle$ and fix a $\mathbb{Z}/2$-basis $\{e_1, e_2, e_3\}$ of $H^2(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2)$:

$$e_1 \colon 0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \times \mathbb{Z}/2 \xrightarrow{\varphi_1} \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1,$$

$$e_2 \colon 0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \times \mathbb{Z}/2 \xrightarrow{\varphi_2} \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1,$$

$$e_3 \colon 0 \to \mathbb{Z}/2 \to \quad Qu \quad \xrightarrow{\varphi_3} \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1;$$

here $\varphi_1(a) = u$, $\varphi_1(b) = v$, $\varphi_2(a) = v$, $\varphi_2(b) = u$, $\varphi_3(a) = u$, $\varphi_3(b) = v$. By abuse of notation, we do not distinguish between an extension and its class in $H^2(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2)$. Now the extension $e_1 + e_2$ has a $\mathbb{Z}/4 \times \mathbb{Z}/2$ as middle group, and the extensions $e_1 + e_3$, $e_2 + e_3$ and $e_1 + e_2 + e_3$ have the dihedral group as middle group.

We claim: $d_2(e_1) = 0 = d_2(e_2)$; $d_2(e_3) \neq 0$.

Every automorphism of $E = \mathbb{Z}/4 \times \mathbb{Z}/2 = \langle a, b; a^4, b^2, [a, b] \rangle$ fixes a^2. Since $\langle a^2 \rangle = A$, $\mathrm{Aut}^A(E)$ is the full automorphism group of E. But $\mathrm{Aut}_G(N, A)$ is trivial, whence $\mathrm{Aut}_G^A(E) = \mathrm{Hom}(N, A)$. Hence $\mathrm{Aut}_G(e_1) = \mathrm{Hom}(N, A) \times \mathbb{Z}/4 \times \mathbb{Z}/2$. Moreover, $\beta \colon E \to \mathrm{Aut}_G(e_1)$ sends a to $a^2 \in \mathbb{Z}/4 \subset \mathrm{Aut}_G(e_1)$ and b to $b \in \mathbb{Z}/2 \subset \mathrm{Aut}_G(e_1)$. It follows that the extension

$$\bar{e}_1 \colon 0 \to \mathrm{Hom}(N, A) \to \mathrm{Out}_G(e_1) \to \mathbb{Z}/2 \to 1$$

splits. For symmetry reasons, $\bar{e}_2$ also splits.

On the other hand, by the same argument as above, $\mathrm{Aut}_G(e_3) = \mathrm{Hom}(N, A) \times \mathbb{Z}/4 \times \mathbb{Z}/2$, but $\mathrm{Out}_G(e_3)$ is now the cokernel of

$$\begin{aligned} \beta \colon Qu &\to \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/2 \\ &= \langle u, v, a, b; u^2, v^2, a^4, b^2, [u, v] \text{ etc.} \rangle, \end{aligned}$$

where $Qu = \langle x, y; x^2 = y^2 = (xy)^2 \rangle$ and $\beta(x) = va^2$, $\beta(y) = ub$ (note that $\mathrm{Hom}(N, A) = \mathbb{Z}/2 \times \mathbb{Z}/2$ and recall how β was defined in Section 2.2). Now $\mathrm{coker}(\beta) \cong \mathbb{Z}/4 \times \mathbb{Z}/2$ and the extension

$$\bar{e}_3\colon\ 0 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/4 \times \mathbb{Z}/2 \to \mathbb{Z}/2 \to 1$$

does not split.

(ii) $d_2\colon H^1(\mathbb{Z}/2, H^1(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2)) \to H^3(\mathbb{Z}/2, \mathbb{Z}/2)$. As in Section 2.3, let $E = (\mathbb{Z}/2)^3$, let $\{e_1, e_2, e_3\}$ be the obvious $\mathbb{Z}/2$-basis, and consider the split extension

$$e_s\colon\ 0 \to \mathbb{Z}/2(e_1) \to \mathbb{Z}/2(e_1) \times \mathbb{Z}/2(e_2) \times \mathbb{Z}/2(e_3) \to \mathbb{Z}/2(e_2) \times \mathbb{Z}/2(e_3) \to 1.$$

Now $H^1(\mathbb{Z}/2, H^1(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2)) = \mathrm{Hom}(\mathbb{Z}/2, H^1(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2))$ and we identify $H^1(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2)$ with $\mathrm{Aut}_G^A(E)$ as above. Writing $\mathbb{Z}/2 = \langle x; x^2 \rangle$, we choose a basis $\{\eta, \theta\}$ for $H^2(\mathbb{Z}/2, H^1(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2)) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, by setting

$$^{\eta(x)}e_2 = e_1 + e_2, \qquad e_1, e_3 \text{ fixed under } \eta(x),$$

$$^{\theta(x)}e_3 = e_1 + e_3, \qquad e_1, e_2 \text{ fixed under } \theta(x).$$

Now $\mathrm{Aut}_G(e_s) = \langle u, v, a, b; u^2, v^2, a^4, b^2, [u, v] \text{ etc.} \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4 \times \mathbb{Z}/2$, where $^ue_2 = e_1 + e_2$, $^ve_3 = e_1 + e_3$ and all the rest remains fixed under the corresponding elements of $\mathrm{Aut}_G(e_s)$. Maintaining the notation of Section 2.3, the maps η and θ determine crossed 2-fold extensions

$$\tilde{\eta}\colon\ 0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \to B^\eta \to \mathbb{Z}/2 \to 1$$

and

$$\tilde{\theta}\colon\ 0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \to B^\theta \to \mathbb{Z}/2 \to 1;$$

the corresponding ∂'s are the obvious maps. Here $B^\eta = B^\theta = \mathbb{Z}/4 \times \mathbb{Z}/2 = \langle a, b; a^4, b^2, [a, b] \rangle$; B^η acts on $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ by the rule

$$^ae_2 = e_1 + e_2$$

and B^θ acts on $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ by

$$^ae_3 = e_1 + e_3$$

with the convention that everything not written down remains fixed. Clearly, $\tilde{\eta}$ is equivalent to

$$\hat{\eta}\colon\ 0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 1,$$

where $\mathbb{Z}/2 \times \mathbb{Z}/2$ has basis $\{e_1, e_2\}$ and where the generator of $\mathbb{Z}/4$ maps e_2 to $e_1 + e_2$. It follows from the Theorem in [10, Sect. 10] that $[\hat{\eta}] \neq 0 \in H^3(\mathbb{Z}/2, \mathbb{Z}/2)$, since there is no group H of order eight which maps onto $\mathbb{Z}/4$ and contains $\mathbb{Z}/2 \times \mathbb{Z}/2$ as a normal subgroup in such a way that conjugation in H induces the $\mathbb{Z}/4$-action on $\mathbb{Z}/2 \times \mathbb{Z}/2$. On the other hand, if we associate with $\tilde{\theta}$ a crossed 2-fold extension $\hat{\theta}$ in a similar way, it is easy to see that $[\hat{\theta}] = 0 \in H^3(\mathbb{Z}/2, \mathbb{Z}/2)$.

It follows that $d_2[\eta]$ is the generator of $H^3(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$, whence $E_3^{3,0} = 0$, and that θ generates $E_3^{1,1} \cong \mathbb{Z}/2$.

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