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Non-existence of reflexive ideals in Iwasawa algebras of Chevalley type

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Abstract

Let Φ be a root system and let $\Phi(\mathbb{Z}_p)$ be the standard Chevalley $\mathbb{Z}_p$-Lie algebra associated to Φ. For any integer $t \geqslant 1$, let G be the uniform pro-p group corresponding to the powerful Lie algebra $p^t\Phi(\mathbb{Z}_p)$ and suppose that $p \geqslant 5$. Then the Iwasawa algebra Ω_G has no non-trivial two-sided reflexive ideals. This was previously proved by the authors for the root system A_1.

Keywords: Iwasawa algebra; Chevalley type; Reflexive ideal

0. Introduction

0.1. Prime ideals in Iwasawa algebras

One of the main projects in the study of non-commutative Iwasawa algebras aims to understand the structure of two-sided ideals in Iwasawa algebras Λ_G and Ω_G for compact p-adic analytic groups G. A list of open questions in this project was posted in a survey paper by the first author and Brown [AB]. Motivated by its connection to the Iwasawa theory of elliptic curves in arithmetic geometry it is particularly interesting to understand the prime ideals of Λ_G when G

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is an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. A reduction [A] shows that this amounts to understanding the prime ideals of Ω_G when G is an open subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$. In a recent paper we introduced some machinery which allowed us to determine every prime ideal of Ω_G for any open torsionfree subgroup G of $\mathrm{SL}_2(\mathbb{Z}_p)$, see [AWZ, Theorem C]. In this paper the theory developed in [AWZ] will be used to prove that under mild conditions on p, there are no non-zero reflexive ideals in Ω_G when G is a uniform pro-p group of Chevalley type. It follows from this that every two-sided reflexive ideal of $\Lambda_{G\times\mathbb{Z}_p}$ is principal and centrally generated—see [A, Theorem 4.7].

0.2. *Definitions*

Throughout let p be a fixed prime number. Let $\mathbb{Z}_p$ be the ring of p-adic integers and let $\mathbb{F}_p$ be the field $\mathbb{Z}/(p)$. Let G be a compact p-adic analytic group. The *Iwasawa algebra* of G over $\mathbb{Z}_p$ (or the *completed group algebra* of G over $\mathbb{Z}_p$) is defined to be

$$\Lambda_G := \varprojlim \mathbb{Z}_p[G/N],$$

where the inverse limit is taken over the open normal subgroups N of G [L, p. 443], [DDMS, p. 155]. In this paper we use $R[G]$ for the group ring of G over a ring R. For any field K of characteristic p, the *Iwasawa algebra* of G over K (or the *completed group algebra* of G over K) is defined to be

$$K[\![G]\!] := \varprojlim K[G/N],$$

where the inverse limit is taken over the open normal subgroups N of G. If $K=\mathbb{F}_p$, we write Ω_G for $K[\![G]\!]$.

Let A be any algebra and I be a left ideal of A. We call I is *reflexive* if the canonical map

$$I \to \mathrm{Hom}_A\big(\mathrm{Hom}_A(I, A), A\big)$$

is an isomorphism. A reflexive right ideal is defined similarly. We will call a two-sided ideal I of A *reflexive* if it is reflexive as a right and as a left ideal.

0.3. *Main results*

Let Φ be a root system, so that the Dynkin diagram of any indecomposable component of Φ belongs to

$$\big\{A_n(n \geqslant 1), B_n(n \geqslant 2), C_n(n \geqslant 3), D_n(n \geqslant 4), E_6, E_7, E_8, F_4, G_2\big\}.$$

Let $\Phi(\mathbb{Z}_p)$ denote the $\mathbb{Z}_p$-Lie algebra constructed by using a Chevalley basis associated to Φ. For any integer $t \geqslant 1$ (or $t \geqslant 2$ if $p=2$), the $\mathbb{Z}_p$-Lie algebra $p^t\Phi(\mathbb{Z}_p)$ is *powerful*. By [DDMS, Theorem 9.10] there is an isomorphism between the category of uniform pro-p groups and the category of powerful Lie algebras. The uniform pro-p group corresponding to $p^t\Phi(\mathbb{Z}_p)$ is called *of type Φ*, or in general *of Chevalley type* without mentioning Φ. In this case the Iwasawa algebra Ω_G is called *of type Φ* (or *of Chevalley type* in general).

We say that p is a *nice prime* for Φ if $p \geqslant 5$ and if $p \nmid n+1$ when Φ has an indecomposable component of type A_n. Here is our main theorem.

Theorem A. *Let Φ be a root system and let G be a uniform pro-p group of type Φ. If p is a nice prime for Φ, then the Frobenius pair (Ω_G, Ω_{G^p}) satisfies the derivation hypothesis.*

The undefined technical terms in Theorem A will be explained in Section 2. The following corollary was proved in [AWZ], assuming Theorem A. This paper fills in the missing step.

Corollary. *(See [AWZ, Theorems A and B].) Let G be a torsionfree compact p-adic analytic group whose $\mathbb{Q}_p$-Lie algebra $\mathcal{L}(G)$ is split semisimple over $\mathbb{Q}_p$. Suppose that p is a nice prime for the root system Φ of $\mathcal{L}(G)$. Then Ω_G has no non-trivial two-sided reflexive ideals. In particular, every non-zero normal element of Ω_G is a unit.*

It was asked in [AB, Question J] whether an Iwasawa algebra Ω_G of Chevalley type has any non-zero, non-maximal prime ideals. Theorem A says that it has no prime ideals of so-called homological height one and hence provides evidence for a negative answer. Combining this with a result of the first author gives a complete answer to [AB, Question J] in the case when $\Phi = A_1$.

We conjecture that the hypothesis of p being nice is superfluous. When $\Phi = A_1$ we gave a separate proof for $p = 2$ in [AWZ, Section 8] (see also Section 4), which shows the difficulty of dealing with non-nice primes.

0.4. An outline of the paper

In Section 1 we will give a treatment of some elementary material (linear algebra, derivations, Lie algebras) that will form an essential part of the proof of our main result. The reader may wish to skip this material on his first reading and return to it later as needed. Section 2 contains the definitions of some key terms such as derivation hypothesis and Frobenius pair. The proof of Theorem A is given in Section 3. Section 4 contains some remarks about the case when $\Phi = A_n$ and p is not a nice prime.

1. Preparatory results

1.1. A Vandermonde-type determinant

Let $\{w_1, \ldots, w_m\}$ be a basis for an m-dimensional $\mathbb{F}_p$-vector space W. Consider the symmetric algebra

$$B := \mathrm{Sym}(W) \cong \mathbb{F}_p[w_1, \ldots, w_m].$$

We are interested in the following matrix of Vandermonde type:

$$M(w_1, \ldots, w_m; d_1, \ldots, d_m) := \begin{pmatrix} w_1^{p^{d_1}} & w_2^{p^{d_1}} & \cdots & w_m^{p^{d_1}} \\ w_1^{p^{d_2}} & w_2^{p^{d_2}} & \cdots & w_m^{p^{d_2}} \\ \vdots & \vdots & \cdots & \vdots \\ w_1^{p^{d_m}} & w_2^{p^{d_m}} & \cdots & w_m^{p^{d_m}} \end{pmatrix}$$

where $\{d_1, \dots, d_m\}$ is a sequence of non-negative integers. For simplicity we write

$$M(w_1, \dots, w_m) := \begin{pmatrix} w_1 & w_2 & \cdots & w_m \\ w_1^p & w_2^p & \cdots & w_m^p \\ \vdots & \vdots & \cdots & \vdots \\ w_1^{p^{m-1}} & w_2^{p^{m-1}} & \cdots & w_m^{p^{m-1}} \end{pmatrix}.$$

Let $\mathbb{P}(W)$ be the set of all one-dimensional subspaces of W. For each $l \in \mathbb{P}(W)$ we fix a choice of generator $w_l \in l$ so that $l = \langle w_l \rangle$. Define $\Delta(W)$ to be the product $\prod_{l \in \mathbb{P}(W)} w_l$.

Lemma.

(1) *Let* $\{d_1, \dots, d_m\}$ *be a sequence of non-negative integers. Then* $\Delta(W)$ *divides* $\det M(w_1, \dots, w_m; d_1, \dots, d_m)$.
(2) *There exists* $\lambda \in \mathbb{F}_p^\times$ *such that* $\det M(w_1, \dots, w_m) = \lambda \cdot \Delta(W)$.

Proof. (1) Let w be a non-zero element of W; we will show that

$$\det M \in wB,$$

where $M = M(w_1, \dots, w_m; d_1, \dots, d_m)$. Write $w = a_1 w_1 + \cdots + a_m w_m$ for some $a_i \in \mathbb{F}_p$, not all zero. Without loss of generality $a_1 = -1$. Consider the canonical ring homomorphism $\pi : \mathrm{Sym}(W) \to \mathrm{Sym}(W/\langle w \rangle)$; this has kernel exactly wB. Let $u_i = \pi(w_i)$; then

$$\pi(\det M) = \det \begin{pmatrix} a_2 u_2^{p^{d_1}} + \cdots + a_m u_m^{p^{d_1}} & u_2^{p^{d_1}} & \cdots & u_m^{p^{d_1}} \\ a_2 u_2^{p^{d_2}} + \cdots + a_m u_m^{p^{d_2}} & u_2^{p^{d_2}} & \cdots & u_m^{p^{d_2}} \\ \vdots & \vdots & \cdots & \vdots \\ a_2 u_2^{p^{d_m}} + \cdots + a_m u_m^{p^{d_m}} & u_2^{p^{d_m}} & \cdots & u_m^{p^{d_m}} \end{pmatrix}$$

which is zero because the first column is a linear combination of the others. Hence $\det M \in wB$ as claimed.

Now if $l \neq l'$ are two distinct lines then w_l and $w_{l'}$ are coprime in B. Hence $\Delta(W) = \prod_{l \in \mathbb{P}(W)} w_l$ divides $\det M$.

It is well known that $\det M$ is non-zero if and only if $\{d_1, \dots, d_m\}$ are distinct.

(2) Both expressions are polynomials of degree precisely

$$1 + p + p^2 + \cdots + p^{m-1} = \left|\mathbb{P}(W)\right| = \frac{p^m - 1}{p - 1}$$

and the result follows. $\square$

1.2. The adjugate matrix

Later on we will be interested in coming as close as possible to inverting the matrix $M(w_1, \dots, w_m)$. Recall *Cramer's rule*: this says that if A is any square $m \times m$ matrix then

$$\mathrm{adj}(A) \cdot A = A \cdot \mathrm{adj}(A) = \det(A) \cdot I_m$$

where I_m is the identity matrix and $\mathrm{adj}(A)$ is the *adjugate matrix*, defined as follows:

$$\mathrm{adj}(A)_{ij} = (-1)^{i+j} \det C_{ji}$$

where C_{ij} is the matrix A with the ith row and jth column removed.

We will use the following standard piece of notation. Given a list $(x_1, \dots, x_n)$ consisting of n elements, $(x_1, \dots, \widehat{x_j}, \dots, x_n)$ denotes the list consisting of $n-1$ elements, where x_j has been omitted. Thus $M(w_1, \dots, \widehat{w_j}, \dots, w_m)$ is equal to the $(m-1) \times (m-1)$-matrix defined in Section 1.1 with $\{d_1, \dots, d_{m-1}\} = \{0, 1, \dots, m-2\}$. Lemma 1.1 implies the following

Proposition. *Retain the notation of Section* 1.1 *and let $A = M(w_1, \dots, w_m)$. Then for any $j = 1, \dots, m$,* $\det M(w_1, \dots, \widehat{w_j}, \dots, w_m)$ *divides each entry in the jth row of* $\mathrm{adj}(A)$ *in B.*

Proof. We need to show that for all $i = 1, \dots, m$,

$$\det M(w_1, \dots, \widehat{w_j}, \dots, w_m) \mid \det C_{ij}.$$

But $C_{ij} = M(w_1, \dots, \widehat{w_j}, \dots, w_m; 0, \dots, \widehat{i-1}, \dots, m-1)$. The assertion follows from Lemma 1.1(1). □

For each $j = 1, \dots, m$, let W_j be the subspace $\langle w_1, \dots, \widehat{w_j}, \dots, w_m \rangle$ of W. Define

$$\Delta_j := \prod_{l \in \mathbb{P}(W) \setminus \mathbb{P}(W_j)} w_l \in B.$$

By Lemma 1.1(2), we see that for some $\lambda_j \in \mathbb{F}_p^\times$,

$$\Delta_j = \lambda_j \cdot \frac{\det M(w_1, \dots, w_m)}{\det M(w_1, \dots, \widehat{w_j}, \dots, w_m)}.$$

Corollary. *Let D be the diagonal $m \times m$ matrix defined by $D_{ij} = \delta_{ij} \Delta_j$. Then there exists $U \in M_m(B)$ such that $U \cdot A = D$.*

Proof. Let E be the diagonal $m \times m$ matrix defined by

$$\lambda_j E_{ij} = \delta_{ij} \det M(w_1, \dots, \widehat{w_j}, \dots, w_m).$$

By the proposition, there exists $U \in M_m(B)$ such that $E \cdot U = \mathrm{adj}(A)$. Hence $U \cdot A = E^{-1} \cdot \mathrm{adj}(A) \cdot A = E^{-1} \det A = D$, as required. □

1.3. Derivations

Now let V be a finite-dimensional vector space over a field K (soon we will assume that $K = \mathbb{F}_p$). Consider the set $\mathfrak{D}$ of all derivations of $B := \mathrm{Sym}_K(V)$. Note that any $f \in V^* :=$

$\mathrm{Hom}_K(V, K)$ in the dual space of V gives rise to a derivation, which we again denote by f, defined by the rule

$$f(v_1 \cdots v_k) = \sum_{j=1}^{k} v_1 \cdots f(v_j) \cdots v_k$$

for all $v_1, \ldots, v_k \in V$. Next, $\mathfrak{D}$ is naturally a left B-module, with the action given by

$$(b \cdot d)(x) = bd(x)$$

for all $b, x \in B$ and $d \in \mathfrak{D}$. This gives us a K-linear map $\psi : B \otimes V^* \to \mathfrak{D}$, defined by $\psi(b \otimes f) = b \cdot f$. The following lemma is well known.

Lemma. *Let ψ be defined as above. Then ψ is a B-module isomorphism.*

Now we assume that $K = \mathbb{F}_p$. Then $x \mapsto x^{p^r}$ is an $\mathbb{F}_p$-linear endomorphism of B. Hence it extends to an $\mathbb{F}_p$-linear endomorphism, denoted by $(-)^{[p^r]}$, of $B \otimes V^*$ that is determined by

$$(b \otimes f)^{[p^r]} = b^{p^r} \otimes f.$$

Definition. For any $d \in \mathfrak{D}$ and $r \geqslant 0$, let $d^{[p^r]} = \psi(\psi^{-1}(d)^{[p^r]})$ be the corresponding derivation.

Thus $d^{[p^r]}$ is the derivation of B determined by the rule

$$d^{[p^r]}(v) = d(v)^{p^r}$$

for all $v \in V$. We will henceforth identify $B \otimes V^*$ with $\mathfrak{D}$ using ψ.

1.4. A certain module of derivations

The space $\mathrm{End}(V)$ can be canonically identified with $V \otimes V^*$. Since V is contained in B, we will identify $\mathrm{End}(V)$ with $V \otimes V^* \subseteq \mathfrak{D}$.

As in Section 1.1, for each $l \in \mathbb{P}(V)$ choose some $v_l \in V$ such that $l = \langle v_l \rangle$. If $\varphi \in \mathrm{End}(V)$ then $\varphi^* \in \mathrm{End}(V^*)$ is the dual map to φ defined by

$$\varphi^*(g) = g \circ \varphi$$

for all $g \in V^*$.

Proposition. *Let $\varphi \in \mathrm{End}(V)$ and $s \geqslant 0$ be given. Consider the B-submodule*

$$\mathcal{E}_s := \sum_{r \geqslant s} B \cdot \varphi^{[p^r]}$$

of $\mathfrak{D}$, and let $g \in V^*$. Then

$$\left(\prod_{l \in \mathbb{P}(\varphi(V)) \setminus \mathbb{P}(\ker g)} v_l^{p^s} \right) \cdot \varphi^*(g) \in \mathcal{E}_s.$$

Proof. Let $W=\varphi(V)$ and write $m=\dim W$ and $n=\dim V$. Consider the annihilator $(\ker\varphi)^{\perp}$ of $\ker\varphi$ in V^*. This clearly contains $\varphi^*(V^*)$ and is hence equal to it because both spaces have dimension m.

There is nothing to prove if $\varphi^*(g)=0$. Otherwise let $\{f_1,\dots,f_m\}$ be a basis for $(\ker\varphi)^{\perp}$ such that $f_m=\varphi^*(g)$, and extend it to a basis $\{f_1,\dots,f_n\}$ for V^*. Let $\{v_1,\dots,v_n\}$ be the dual basis for V, so that

$$f_i(v_j)=\delta_{ij}\quad\text{for all } i, j.$$

Then $\{v_{m+1},\dots,v_n\}$ is a basis for $\ker\varphi$ and $\{w_1,\dots,w_m\}$ is a basis for $W=\varphi(V)$, where $w_i:=\varphi(v_i)$ for $i=1,\dots,m$.

Inside $\mathfrak{D}$ we have $\varphi=\sum_{i=1}^m w_i\cdot f_i$ by construction, so

$$\varphi^{[p^r]}=\sum_{i=1}^m w_i^{p^r}\cdot f_i \tag{1.4.1}$$

for all $r\geqslant 0$.

Consider the vector space $W^{[p^s]}=\langle w_1^{p^s},\dots,w_m^{p^s}\rangle$ and let $A=M(w_1^{p^s},\dots,w_m^{p^s})$ be the matrix appearing in Section 1.1. Now $\mathfrak{D}$ is a left B-module, so

$$\mathfrak{D}^m:=\begin{pmatrix}\mathfrak{D}\\ \vdots\\ \mathfrak{D}\end{pmatrix}$$

is a left $M_m(B)$-module. Let $\mathbf{e}\in\mathfrak{D}^m$ be the column vector whose rth entry is the derivation $\varphi^{[p^{r+s-1}]}$, and let $\mathbf{f}\in\mathfrak{D}^m$ be the column vector whose rth entry is the derivation f_r, for each $r=1,\dots,m$. Then we can rewrite Eqs. (1.4.1) for $r=s,s+1,\dots,s+m-1$ as

$$A\cdot\mathbf{f}=\mathbf{e}\in\mathcal{E}_s^m$$

inside $\mathfrak{D}^m$. By Corollary 1.2 we can find $U\in M_m(B)$ such that $U\cdot A=D$ is a diagonal matrix whose jth entry is

$$\Delta_j=\prod_{l\in\mathbb{P}(W)\setminus\mathbb{P}(W_j)} v_l^{p^s}.$$

Here $W_j=\langle w_1,\dots,\widehat{w_j},\dots,w_m\rangle$, for all $j=1,\dots,m$. Hence

$$D\cdot\mathbf{f}=U\cdot A\cdot\mathbf{f}=U\cdot\mathbf{e}\in\mathcal{E}_s^m,$$

so in particular $D_m\cdot\varphi^*(g)=D_m\cdot f_m\in\mathcal{E}_s$. Now $g(w_i)=(g\circ\varphi)(v_i)=f_m(v_i)=\delta_{mi}$ for all $i=1,\dots,m$, so

$$W_m=W\cap\ker g.$$

The result follows. $\square$

1.5. $\mathfrak{g}$-*modules*

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra a base field k. By a $\mathfrak{g}$-*module* we mean a left $U(\mathfrak{g})$-module V, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. To give V the structure of a $\mathfrak{g}$-module is the same thing as to give a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

where $\mathfrak{gl}(V) = \mathrm{End}(V)$ is the Lie algebra of all linear endomorphisms of V under the commutator bracket.

If V is a $\mathfrak{g}$-module then so is the dual space V^*, by the rule

$$(x \cdot f)(v) = -f(x \cdot v)$$

for all $x \in \mathfrak{g}$, $f \in V^*$ and $v \in V$. Note that $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\rho^* : \mathfrak{g} \to \mathfrak{gl}(V^*)$ are the corresponding representations then

$$\rho^*(x) = -\rho(x)^*$$

for all $x \in \mathfrak{g}$. Here as in Section 1.4, $\rho(x)^*$ denotes the dual map to $\rho(x) : V \to V$.

1.6. *Invariant bilinear forms*

Let V be a $\mathfrak{g}$-module. Recall that a $\mathfrak{g}$-*invariant form* on V is a bilinear form

$$(,) : V \times V \to k$$

such that $(x \cdot v, w) = -(v, x \cdot w)$ for all $x \in \mathfrak{g}$ and $v, w \in V$. Such a form determines a homomorphism of $\mathfrak{g}$-modules $\beta : V \to V^*$ via the rule $\beta(v)(w) = (v, w)$, and conversely, a $\mathfrak{g}$-module homomorphism $V \to V^*$ defines a $\mathfrak{g}$-invariant form on V. Note that the form $(,)$ is *non-degenerate* if and only if the associated homomorphism β is an isomorphism.

1.7. *The adjoint representation*

Now consider $V = \mathfrak{g}$ as a $\mathfrak{g}$-module via $x \cdot y = [x, y]$ for all $x, y \in \mathfrak{g}$. The following elementary result will be very useful later on.

Lemma. *Suppose that* $\mathfrak{g}$ *has a* $\mathfrak{g}$*-invariant bilinear form* $(,)$*, and let* β *be the associated homomorphism. Then for all* $x, y \in \mathfrak{g}$,

(a) $x \cdot \beta(y) = y \cdot \beta(-x)$, *and*
(b) $[x, \mathfrak{g}] \subseteq \ker \beta(x)$.

Proof. (a) $x \cdot \beta(y) = \beta([x, y]) = \beta([y, -x]) = y \cdot \beta(-x)$.

(b) $\beta(x)([x, \mathfrak{g}]) = (x, [x, \mathfrak{g}]) = ([-x, x], \mathfrak{g}) = 0$. $\square$

1.8. The Killing form

Recall that the *Killing form* on $\mathfrak{g}$ is defined by the rule

$$\mathcal{K}(x, y) = \operatorname{tr}\big(\operatorname{ad}(x)\operatorname{ad}(y)\big)$$

for all $x, y \in \mathfrak{g}$. This is always an example of a $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$. If $\operatorname{char} k = 0$ then Cartan's Criterion states that $\mathfrak{g}$ is semisimple if and only if the Killing form is non-degenerate. However in positive characteristic it may happen that $\mathfrak{g}$ is simple but its Killing form is zero. This can happen even when $\mathfrak{g}$ is of "classical type", meaning that it is a Chevalley Lie algebra over k. There is a way around this problem—see the proof of Theorem 3.4.

2. Frobenius pairs and the derivation hypothesis

In this section we review a minimal amount of material from [AWZ] that is most relevant for this paper. In particular we will recall the derivation hypothesis which plays a key role in [AWZ]. Together with the main theorem of this paper, the theory in [AWZ] leads to a proof of the structure theorem for reflexive ideals in a class of Iwasawa algebras.

2.1. Frobenius pairs

We go back to an arbitrary base field K of characteristic p. Let B be a commutative K-algebra; for example B could be the polynomial algebra

$$B = \operatorname{gr} K[\![G]\!] = \operatorname{Sym}(V \otimes K)$$

for some finite-dimensional $\mathbb{F}_p$-vector space V. The Frobenius map $x \mapsto x^p$ is a ring endomorphism of B and gives an isomorphism of B onto its image

$$B^{[p]} := \big\{b^p \colon b \in B\big\}$$

in B provided that B is reduced. Any derivation $d : B \to B$ is $B^{[p]}$-linear because

$$d\big(a^p b\big) = a^p d(b) + pa^{p-1} d(a) b = a^p d(b)$$

for all $a, b \in B$.

Let t be a positive integer. Whenever $\{y_1, \ldots, y_t\}$ is a t-tuple of elements of B and $\alpha = (\alpha_1, \ldots, \alpha_t)$ is a t-tuple of non-negative integers, we define

$$\mathbf{y}^\alpha = y_1^{\alpha_1} \cdots y_t^{\alpha_t}.$$

Let $[p-1]$ denote the set $\{0, 1, \ldots, p-1\}$ and let $[p-1]^t$ be the product of t copies of $[p-1]$.

Definition. (See [AWZ, Definition 2.2].) Let A be a complete filtered K-algebra and let A_1 be a subalgebra of A. We always view A_1 as a filtered subalgebra of A, equipped with the subspace filtration $F_n A_1 := F_n A \cap A_1$. We say that (A, A_1) is a *Frobenius pair* if the following axioms are satisfied:

(i) A_1 is closed in A,
(ii) $\operatorname{gr} A$ is a commutative noetherian domain, and we write $B = \operatorname{gr} A$,
(iii) the image B_1 of $\operatorname{gr} A_1$ in B satisfies $B^{[p]} \subseteq B_1$, and
(iv) there exist homogeneous elements $y_1, \dots, y_t \in B$ such that

$$B = \bigoplus_{\alpha \in [p-1]^t} B_1 \mathbf{y}^\alpha.$$

2.2. Derivations on B

Let $B_1 \subseteq B$ be commutative rings of characteristic p, such that $B^{[p]} \subseteq B_1$ and

$$B = \bigoplus_{\alpha \in [p-1]^t} B_1 \mathbf{y}^\alpha$$

for some elements $y_1, \dots, y_t$ of B.

Fix $j = 1, \dots, t$ and let ϵ_j denote the t-tuple of integers having a 1 in the jth position and zeros elsewhere. We define a B_1-linear map $\partial_j : B \to B$ by setting

$$\partial_j \left(\sum_{\alpha \in [p-1]^t} u_\alpha \mathbf{y}^\alpha \right) := \sum_{\substack{\alpha \in [p-1]^t \\ \alpha_j > 0}} \alpha_j u_\alpha \mathbf{y}^{\alpha - \epsilon_j}.$$

Let $\mathcal{D} := \operatorname{Der}_{B_1}(B)$ denote the set of all B_1-linear derivations of B. An ideal I of B is called *$\mathcal{D}$-stable* if $\mathcal{D} \cdot I \subseteq I$.

Proposition. *(See [AWZ, Proposition 2.4].)*

(a) *The map ∂_j is a B_1-linear derivation of B for each j.*
(b) $\mathcal{D} = \bigoplus_{j=1}^t B \partial_j$.
(c) *For any $x \in B$, $\mathcal{D}(x) = 0$ if and only if $x \in B_1$.*
(d) *An ideal $I \subseteq B$ is $\mathcal{D}$-stable if and only if it is controlled by B_1:*

$$I = (I \cap B_1) B.$$

If $K = \mathbb{F}_p$ and $B = \operatorname{Sym}(V)$ then $\mathfrak{D} = \mathcal{D}$. Part (a) of the above is similar to Lemma 1.3.

2.3. Inducing derivations on gr A

Let A be a filtered ring with associated graded ring B and let $a \in A$. Suppose that there is an integer $n \geqslant 0$ such that

$$[a, F_k A] \subseteq F_{k-n} A$$

for all $k \in \mathbb{Z}$. This induces linear maps

$$\begin{aligned} \{a,-\}_n\colon \qquad \frac{F_k A}{F_{k-1}A} &\to \frac{F_{k-n}A}{F_{k-n-1}A}, \\ b + F_{k-1}A &\mapsto [a,b] + F_{k-n-1}A \end{aligned}$$

for each $k \in \mathbb{Z}$ which piece together to give a graded derivation

$$\{a,-\}_n : B \to B.$$

Definition. (See [AWZ, Definition 3.2].) A *source of derivations* for a Frobenius pair (A, A_1) is a subset $\mathbf{a} = \{a_0, a_1, a_2, \ldots\}$ of A such that there exist functions $\theta, \theta_1 : \mathbf{a} \to \mathbb{N}$ satisfying the following conditions:

(i) $[a_r, F_k A] \subseteq F_{k-\theta(a_r)}A$ for all $r \geqslant 0$ and all $k \in \mathbb{Z}$,
(ii) $[a_r, F_k A_1] \subseteq F_{k-\theta_1(a_r)}A$ for all $r \geqslant 0$ and all $k \in \mathbb{Z}$,
(iii) $\theta_1(a_r) - \theta(a_r) \to \infty$ as $r \to \infty$.

Let $\mathcal{S}(A, A_1)$ denote the set of all sources of derivations for (A, A_1).

2.4. The derivation hypothesis

Let $\mathbf{a}$ be a source of derivations for a Frobenius pair (A, A_1) and I be a graded ideal of B. We say that the homogeneous element Y of B lies in the $\mathbf{a}$-*closure* of I if $\{a_r, Y\}_{\theta(a_r)}$ lies in I for all $r \gg 0$.

Each source of derivations $\mathbf{a}$ gives rise to a sequence of derivations $\{a_r, -\}_{\theta(a_r)}$ of B, and some or all of these could well be zero. To ensure that we get an interesting supply of derivations of B, we now introduce a condition which holds for Iwasawa algebras of only rather special uniform pro-p groups.

Recall that $\mathcal{D}$ denotes the set of all B_1-linear derivations of B and $\mathcal{S}(A, A_1)$ denotes the set of all sources of derivations for (A, A_1). The derivation hypothesis is really concerned with the action of the derivations induced by $\mathcal{S}(A, A_1)$ on the graded ring B.

Definition. (See [AWZ, Definition 3.5].) Let (A, A_1) be a Frobenius pair. We say that (A, A_1) satisfies the derivation hypothesis if for all homogeneous $X, Y \in B$ such that Y lies in the $\mathbf{a}$-closure of XB for all $\mathbf{a} \in \mathcal{S}(A, A_1)$, we must have $\mathcal{D}(Y) \subseteq XB$.

Using this hypothesis, it is possible to prove the following control theorem for reflexive ideals:

Theorem. *(See [AWZ, Theorem 5.3].) Let (A, A_1) be a Frobenius pair satisfying the derivation hypothesis, such that B and B_1 are UFDs. Let I be a reflexive two-sided ideal of A. Then $I \cap A_1$ is a reflexive two-sided ideal of A_1 and I is controlled by A_1:*

$$I = (I \cap A_1) \cdot A.$$

This is the main technical result of [AWZ], which eventually implies Corollary 0.3.

3. Proof of the main result

3.1. Normalizers of powerful Lie algebras

Recall from [DDMS, §9.4] that a $\mathbb{Z}_p$-Lie algebra L is said to be *powerful* if L is free of finite rank as a module over $\mathbb{Z}_p$ and $[L, L] \subseteq p^\epsilon L$, where

$$\epsilon := \begin{cases} 2 & \text{if } p = 2, \\ 1 & \text{otherwise.} \end{cases}$$

Let L be a powerful $\mathbb{Z}_p$-Lie algebra and let $N = \{x \in \mathbb{Q}_p L\colon\ [x, L] \subseteq L\}$ be the normalizer of L inside $\mathbb{Q}_p L$—this is a $\mathbb{Z}_p$-subalgebra of $\mathbb{Q}_p L$ that contains L as an ideal. Note that N is just the inverse image of $\operatorname{End}_{\mathbb{Z}_p}(L)$ under the homomorphism

$$\operatorname{ad} : \mathbb{Q}_p L \to \operatorname{End}_{\mathbb{Q}_p}(\mathbb{Q}_p L),$$

so N contains the center $Z(\mathbb{Q}_p L)$ of $\mathbb{Q}_p L$ and $N/Z(\mathbb{Q}_p L)$ is a finitely generated $\mathbb{Z}_p$-module. Hence

$$\mathfrak{g} := N/pN$$

is a finite-dimensional $\mathbb{F}_p$-Lie algebra. Define

$$V := L/pL.$$

Then V is naturally a $\mathfrak{g}$-module via the rule

$$(x + pN) \cdot (y + pL) = [x, y] + pL$$

for all $x \in N$ and $y \in L$. Let $\rho : \mathfrak{g} \to \operatorname{End}(V)$ be the associated homomorphism.

Lemma. *Let $x \in N \backslash pN$ and $k \geqslant \epsilon$ be such that $u = p^k x \in L$. Then*

(a) $[u, L] \subseteq p^k L$,
(b) $[u, L] \nsubseteq p^{k+1} L$, *and*
(c) $[u, pL] \subseteq p^{k+1} L$.

Proof. The first and the last assertions are clear. If $[u, L] \subseteq p^{k+1} L$ then $[x, L] \subseteq pL$ and so $p^{-1}x \in N$. But this forces $x \in pN$, which we have assumed not to be the case. □

3.2. Derivations for Iwasawa algebras

By [DDMS, Theorem 9.10] there is a natural assignment

$$G \mapsto \log(G), \qquad L \mapsto \exp(L)$$

which determines an equivalence between the category of uniform pro-p groups and the category of powerful $\mathbb{Z}_p$-Lie algebras.

Now let $G = \exp(L)$ be the uniform pro-p group corresponding to our powerful Lie algebra L, and let K denote an arbitrary field of characteristic p. By [AWZ, Proposition 6.6], $(K[\![G]\!], K[\![G^p]\!])$ is a Frobenius pair, and by [AWZ, Lemma 6.2(d) and Proposition 6.4], there is a canonical isomorphism

$$\mathrm{Sym}(V \otimes K) \xrightarrow{\cong} \mathrm{gr}\, K[\![G]\!].$$

Recall that ρ is the map $\mathfrak{g} \to \mathrm{End}(V) \subseteq \mathrm{End}(V \otimes K)$ defined in Section 3.1.

Proposition. *Let $x \in N \setminus pN$ and let $k \geqslant 1$ be such that $p^k x \in L$. Let $a = \exp(p^k x) \in G$. Then*

$$\{a, -\}_{p^k-1} = \rho(x + pN)^{[p^k]}$$

as derivations of $\mathrm{Sym}(V \otimes K)$.

Proof. This is a rephrasing of [AWZ, Theorem 6.8], using Lemma 3.1. $\square$

3.3. Verifying the derivation hypothesis

We start with a powerful $\mathbb{Z}_p$-Lie algebra L and define $\mathfrak{g}$ and V as in Section 3.1. Let $G = \exp(L)$. We say L *satisfies hypothesis* $(L*)$ if the following hold:

(L0) there exists a $\mathfrak{g}$-module isomorphism $\zeta : \mathfrak{g} \to V$,
(L1) $\sum_\beta \mathfrak{g} \cdot \beta(\mathfrak{g}) = \mathfrak{g}^*$ where the sum runs over all possible $\mathfrak{g}$-module homomorphisms $\beta : \mathfrak{g} \to \mathfrak{g}^*$.

Since $\mathfrak{g} \cdot \beta(\mathfrak{g}) = \beta([\mathfrak{g}, \mathfrak{g}])$, condition (L1) is equivalent to $\sum_\beta \beta([\mathfrak{g}, \mathfrak{g}]) = \mathfrak{g}^*$. Clearly, the following two conditions imply (L1):

- $\mathfrak{g}$ admits a non-degenerate $\mathfrak{g}$-invariant bilinear form $(,)$, and
- $\mathfrak{g}$ is *perfect*: $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Theorem. *Let L be a powerful Lie algebra satisfying hypothesis $(L*)$ and let $G = \exp(L)$. Then the Frobenius pair $(K[\![G]\!], K[\![G^p]\!])$ satisfies the derivation hypothesis.*

Proof. Let X, Y be homogeneous elements of $B = \mathrm{gr}\, K[\![G]\!]$ such that Y lies in the **a**-closure of XB for all $\mathbf{a} \in \mathcal{S}(K[\![G]\!], K[\![G^p]\!])$. Let $x \in \mathfrak{g}$ be a non-zero element, and suppose that $x = x' + pN$ for some $x' \in N \setminus pN$. Let $k \geqslant \epsilon$ be such that $p^k x' \in L$. Then

$$\left(\exp\left(p^k x'\right), \exp\left(p^{k+1} x'\right), \exp\left(p^{k+2} x'\right), \ldots\right)$$

is a source of derivations for $(K[\![G]\!], K[\![G^p]\!])$, by [AWZ, Corollary 6.7]. Hence there exists a large integer $s_x \geqslant k$, such that

$$\left\{\exp\left(p^r x'\right), Y\right\}_{p^r-1} \in XB$$

for all $r \geqslant s_x$. Hence by Proposition 3.2 we see that

$$\rho(x)^{[p^r]}(Y) \in XB$$

for all $r \geqslant s_x$. Since $\mathfrak{g}$ is finite, if we set $s := \max\{s_x\colon x \in \mathfrak{g} \setminus 0\}$ then

$$\rho(x)^{[p^r]}(Y) \in XB$$

for all $r \geqslant s$ and all $x \in \mathfrak{g}$.

Let us identify $\operatorname{Sym}(\mathfrak{g} \otimes K)$ with $\operatorname{Sym}(V \otimes K)$ using the isomorphism ζ in (L0). Then

$$\operatorname{ad}(x)^{[p^r]}(Y) \in XB$$

for all $r \geqslant s$ and all $x \in \mathfrak{g}$.

Let $(\,,\,)$ be any $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$, and let $\beta : \mathfrak{g} \to \mathfrak{g}^*$ be the associated homomorphism.

Fix $x, y \in \mathfrak{g}$, let $\varphi := \operatorname{ad}(x) \in \operatorname{End}(\mathfrak{g})$ and let $g = \beta(y) \in \mathfrak{g}^*$. Then $\varphi^*(g) = -x \cdot g$ by the remarks made in Section 1.5, and $\varphi(\mathfrak{g}) = [x, \mathfrak{g}]$. Using Proposition 1.4 we can deduce that

$$\left(\prod_{l \in \mathbb{P}([x,\mathfrak{g}]) \setminus \mathbb{P}(\ker \beta(y))} v_l^{p^s}\right)\big(x \cdot \beta(y)\big)(Y) \in XB.$$

Swapping x and y, we obtain

$$\left(\prod_{l \in \mathbb{P}([y,\mathfrak{g}]) \setminus \mathbb{P}(\ker \beta(x))} v_l^{p^s}\right)\big(y \cdot \beta(x)\big)(Y) \in XB.$$

Now $x \cdot \beta(y) = -y \cdot \beta(x)$ by Lemma 1.7(a), and

$$\big(\mathbb{P}\big([x, \mathfrak{g}]\big) - \mathbb{P}\big(\ker \beta(y)\big)\big) \cap \big(\mathbb{P}\big([y, \mathfrak{g}]\big) - \mathbb{P}\big(\ker \beta(x)\big)\big) = \emptyset$$

by Lemma 1.7(b). Hence the two products occurring above are coprime, which allows us to deduce that

$$\big(x \cdot \beta(y)\big)(Y) \in XB$$

for all $x, y \in \mathfrak{g}$. Since $\mathfrak{g}^*$ generates $\mathcal{D}$ as a B-module, it will be now enough to show that $\{x \cdot \beta(y)\colon x, y \in \mathfrak{g}\}$ spans $\mathfrak{g}^*$. But this is (L1). $\square$

3.4. *Chevalley Lie algebras over $\mathbb{Z}_p$*

Let Φ be an indecomposable root system, let $C := \Phi(\mathbb{Z}_p)$ be the Lie algebra over $\mathbb{Z}_p$ constructed from a Chevalley basis [CSM, p. 37], let $t \geqslant \epsilon$ and consider the powerful Lie algebra $L = p^t C$. Let $\mathfrak{g} = N/pN$ be the finite-dimensional $\mathbb{F}_p$-Lie algebra constructed from L in Section 3.1.

Recall that p is a *nice prime* for Φ if $p \geqslant 5$ and if $p \nmid n + 1$ when Φ is the root system A_n.

Theorem. *Retain the notation as above and suppose that p is a nice prime for Φ. Then*

(a) $\Phi(\mathbb{F}_p)$ *is a non-abelian simple* $\mathbb{F}_p$*-Lie algebra,*
(b) $N = C$ *and* $\mathfrak{g} = \Phi(\mathbb{F}_p)$*,*
(c) L *satisfies* $(L*)$.

Proof. (a) By construction, $\Phi(\mathbb{F}_p)$ is never abelian. Under our assumptions on p, $\Phi(\mathbb{F}_p)$ is simple [S, p. 181].

(b) Clearly $C \subseteq N$. Let $x \in N \backslash C$, for a contradiction. Then we can find $k > 0$ such that $p^k x \in C \setminus pC$. But now

$$[p^k x, C] \subseteq p^k C \subseteq pC$$

so $p^k x + pC$ is a non-zero central element of $C/pC = \Phi(\mathbb{F}_p)$. This is a contradiction, because $\Phi(\mathbb{F}_p)$ is non-abelian simple by part (a). Hence $N = C$ and $\mathfrak{g} = \Phi(\mathbb{F}_p)$.

(c) Let $\zeta : \mathfrak{g} \to V$ be defined by the obvious rule

$$\zeta(x + pN) = p^t x + pL.$$

This is clearly a $\mathfrak{g}$-module isomorphism.

Consider the *normalized Killing form* on $\mathfrak{g}$, defined by

$$(x + pN, y + pN) = \frac{\operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y))}{2h}$$

for all $x, y \in N = \Phi(\mathbb{Z}_p)$, where h is the Coxeter number for Φ. This form is clearly $\mathfrak{g}$-invariant. By [GN, Proposition 4] this form is non-zero and hence the radical $\mathfrak{r} := \{x \in \mathfrak{g}\colon (x, \mathfrak{g}) = 0\}$ of the form is a proper subspace of $\mathfrak{g}$. But $\mathfrak{r}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}$ is simple, so $\mathfrak{r} = 0$ and hence the form is non-degenerate.

Finally, $\mathfrak{g}$ is perfect because $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$, which must be the whole of $\mathfrak{g}$ since $\mathfrak{g}$ is non-abelian simple by part (a). The assertion follows from the comments made before Theorem 3.3. □

3.5. Proof of Theorem A

Lemma. *Let $L = L_1 \oplus L_2$ where both L_1 and L_2 are powerful $\mathbb{Z}_p$-Lie algebras. If L_i satisfies condition $(L*)$ for $i = 1, 2$, then so does L.*

Proof. Let $N, \mathfrak{g}$ and V be defined as in Section 3.1 for the Lie algebra L (and similar terms for L_1 and L_2). It is clear that $N = N_1 \oplus N_2$; consequently, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $V = V_1 \oplus V_2$. The assertion now follows from the definition of $(L*)$. □

Proof of Theorem A. Applying the lemma and Theorem 3.4, we see that $p^t \Phi(\mathbb{Z}_p)$ satisfies $(L*)$. The result now follows from Theorem 3.3. □

4. Remarks on non-nice primes

4.1. Suppose $\Phi = A_n$ and p divides $n+1$, and let G be a uniform group of type Φ. Let $h_1, \ldots, h_n$ be the co-roots occurring in a Chevalley basis for the $\mathbb{Z}_p$-Lie algebra $\Phi(\mathbb{Z}_p) \cong \mathfrak{sl}_{n+1}(\mathbb{Z}_p)$, and let

$$z := \sum_{i=1}^{n} i h_i \in \Phi(\mathbb{Z}_p).$$

Then $\mathfrak{g} := \Phi(\mathbb{F}_p)$ has a one-dimensional center generated by the image $\bar{z}$ of z in $\mathfrak{g}$, and this fact causes the derivation hypothesis to fail for $(K[\![G]\!], K[\![G^p]\!])$.

However, a version of Corollary 0.3 still holds.

Theorem. *Let G be a uniform pro-p group of type A_n. Then Ω_G has no non-trivial two-sided reflexive ideals.*

The proof is similar to the one given in [AWZ, Section 8] and needs the following lemma.

Lemma. *Let $L = p^t\Phi(\mathbb{Z}_p)$ for some $t \geqslant 1$, let $L_1 = pL + p^t\mathbb{Z}_p z$ and let $L_2 = pL$. Write $G = \exp(L)$, $G_1 = \exp(L_1)$ and $G_2 = \exp(L_2)$. Then*

(a) *The Frobenius pair $(K[\![G]\!], K[\![G_1]\!])$ satisfies the derivation hypothesis.*
(b) *The Frobenius pair $(K[\![G_1]\!], K[\![G_2]\!])$ satisfies the derivation hypothesis.*

Sketch of the proof. (a) Let $\mathfrak{f} := \Phi(\mathbb{F}_p)/\langle\bar{z}\rangle$. This is a simple Lie algebra [S, p. 181] and there is an $\mathfrak{f}$-invariant non-degenerate bilinear form on $\mathfrak{f}$ [J, 6.4(b)]. It induces a $\mathfrak{g}$-invariant bilinear form $(\,,\,)$ on $\mathfrak{g}$. Let $\beta : \mathfrak{g} \to \mathfrak{g}^*$ be the $\mathfrak{g}$-module homomorphism associated to $(\,,\,)$. Then the image of β is equal to $\mathfrak{f}^*$, the annihilator of $\langle\bar{z}\rangle$ in $\mathfrak{g}^*$.

The proof of Theorem 3.3 implies that if Y lies in the $\mathbf{a}$-closure of XB for all sources of derivations $\mathbf{a}$ of $(K[\![G]\!], K[\![G_1]\!])$, then $(B \otimes \mathfrak{g} \cdot \beta(\mathfrak{g}))(Y) \in XB$. By the last paragraph, $\mathfrak{g} \cdot \beta(\mathfrak{g}) = \mathfrak{f}^*$. The assertion is proved by noting that $\mathcal{D} := \mathrm{Der}_{B_1}(B)$ is isomorphic to $B \otimes \mathfrak{f}^*$.

(b) This is an easier case than (a). Since $\mathrm{Der}_{B_2}(B_1)$ is isomorphic to $B_1 \otimes K\bar{z}^* \cong B_1$, we can apply the argument in the second half of the proof of [AWZ, Proposition 8.1], after making the appropriate changes. Therefore $(K[\![G_1]\!], K[\![G_2]\!])$ satisfies the derivation hypothesis. $\square$

Using these techniques, we have verified that Corollary 0.3 holds for all (p, Φ), except for $\{p = 2, \Phi \in \{B_n, C_n, D_n, F_4, E_7\}\}$ and $\{p = 3, \Phi \in \{G_2, E_6\}\}$. We believe that it holds in these exceptional cases as well.

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