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# Nonlocal anisotropic dispersal with monostable nonlinearity

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**Abstract**

We study the travelling wave problem

$$J \star u - u - cu' + f(u) = 0 \quad \text{in } \mathbb{R}, \qquad u(-\infty) = 0, \qquad u(+\infty) = 1$$

with an asymmetric kernel $J$ and a monostable nonlinearity. We prove the existence of a minimal speed, and under certain hypothesis the uniqueness of the profile for $c \neq 0$. For $c = 0$ we show examples of nonuniqueness.

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## 1. Introduction and main results

During the past ten years, much attention has been drawn to the study of the following nonlocal equation

$$\frac{\partial U}{\partial t} = \mathcal{J} \star U - U + f(U) \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{1.1}$$

$$U(x, 0) = U_0(x), \tag{1.2}$$

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where $\mathcal{J}$ is a probability density on $\mathbb{R}^N$ and $f$ a given nonlinearity. Such kind of equations appears in various applications ranging from population dynamics to Ising models as seen in [1,6, 12,13,15,16,19,23,24] among many references. Here we will only be concerned with probability densities $\mathcal{J}$ which satisfy the following assumption:

$$\mathcal{J} \in C\big(\mathbb{R}^n\big), \qquad \mathcal{J}(z) \geqslant 0, \qquad \int_{\mathbb{R}^n} \mathcal{J}(z)\,dz = 1, \qquad \int_{\mathbb{R}^n} |z|\mathcal{J}(z)\,dz < \infty,$$

and nonlinearities $f$ of monostable type, e.g.

(f1) $f \in C^1(\mathbb{R})$, which satisfies $f(0) = f(1) = 0$, $f'(1) < 0$, $f|_{(0,1)} > 0$ and $f|_{\mathbb{R}\setminus[0,1]} \leqslant 0$.

Such nonlinearities are commonly used in population dynamics to describe the interaction (birth, death, ...) of a species in its environment as described in [14,17].

Our analysis in this paper will mainly focus on the travelling wave solutions of Eq. (1.1). These particular type of solutions are of the form $U_e(x,t) := u(x.e + ct)$ where $e \in \mathbb{S}^{n-1}$ is a given unit vector, the velocity $c \in \mathbb{R}$ and the scalar function $u$ satisfy

$$J \star u - u - cu' + f(u) = 0 \quad \text{in } \mathbb{R}, \tag{1.3}$$

$$u(-\infty) = 0, \tag{1.4}$$

$$u(+\infty) = 1, \tag{1.5}$$

where $u(\pm\infty)$ denotes the limit of $u(x)$ as $x \to \pm\infty$ and $J$ is the real function defined as

$$J(s) := \int_{\Pi_s} \mathcal{J}(y)\,dy,$$

where $\Pi_s = \{y \in \mathbb{R}^N \colon \langle y, e\rangle = s\}$. Thus we shall assume that the kernel $J$ satisfies

(j1) $$J \in C(\mathbb{R}), \qquad J(z) \geqslant 0, \qquad \int_{\mathbb{R}} J(z)\,dz = 1, \qquad \int_{\mathbb{R}} |z|J(z)\,dz < \infty.$$

We will call a solution $u \in L^\infty(\mathbb{R})$ to (1.3)–(1.5) a travelling wave or travelling front if it is nondecreasing.

The first works to study travelling fronts in this setting are due to Schumacher [24] and in related nonlocal problems by Weinberger [25,26] who constructed travelling fronts satisfying some exponential decay for $J$ symmetric and particular monostable nonlinearities, the so-called KPP nonlinearity, e.g.

(f2) $f$ is monostable and satisfies $f(s) \leqslant f'(0)s$.

Then, Harris, Hudson and Zinner [18] and more recently Carr and Chmaj [4], Chen and Guo [5] and Coville and Dupaigne [11] extended and completed the work of Schumacher to more general monostable nonlinearities and dispersal kernels $J$ satisfying what is called in the literature the Mollison condition [21–23]:

$$\text{(j2)} \quad \exists\lambda > 0 \quad \text{such that} \quad \int_{-\infty}^{\infty} J(-z)e^{\lambda z}\,dz < +\infty.$$

More precisely, they show that

**Theorem 1.1.** *(See [4,5,11,18,24].) Let $f$ be a monostable nonlinearity, $J$ be a symmetric function satisfying* (j1)–(j2). *Then there exists a constant $c^* > 0$ such that for all $c \geqslant c^*$, there exists an increasing function $u$, such that $(u, c)$ is a solution of* (1.3)–(1.5) *and for any $c < c^*$, there exists no increasing solution of* (1.3)–(1.5). *Moreover, if in addition $f'(0) > 0$, then any bounded solution $(u, c)$ of* (1.3)–(1.5) *is unique up to translation.*

Furthermore, as in the classical case, when the nonlinearity is KPP the critical speed $c^*$ can be precisely evaluated by means of a formula.

**Theorem 1.2.** *(See [4,5,18,24,25].) Let $f$ be a KPP nonlinearity and $J$ be a symmetric function satisfying* (j1)–(j2). *Then the critical speed $c^*$ is given by*

$$c^* = \min_{\lambda>0} \frac{1}{\lambda}\left(\int_{\mathbb{R}} J(x)e^{\lambda x}\,dx + f'(0) - 1\right).$$

In Theorems 1.1 and 1.2 the dispersal kernel $J$ is assumed to be symmetric. This corresponds to the situation where the dispersion of the species is isotropic. Since the dispersal of an individual can be influenced in many ways (wind, landscape, ...), it is natural to ask what happens when the kernel $J$ is nonsymmetric. In this direction, we have the following result:

**Theorem 1.3.** *Let $f$ be a monostable nonlinearity satisfying* (f1) *and $J$ be a dispersal kernel satisfying* (j1). *Assume further that there exists $(w, \kappa)$ with $w \in C(\mathbb{R})$ a super-solution of* (1.3)–(1.5) *in the sense*:

$$\begin{aligned} J \star w - w - \kappa w' + f(w) &\leqslant 0 \quad \text{in } \mathbb{R}, \\ w(-\infty) &\geqslant 0, \\ w(+\infty) &\geqslant 1 \end{aligned} \tag{1.6}$$

*and such that $w(x_0) < 1$ for some $x_0 \in \mathbb{R}$. Then there exists a critical speed $c^* \leqslant \kappa$, such that for all $c \geqslant c^*$ there exists a nondecreasing solution $(u, c)$ to* (1.3)–(1.5) *and for $c < c^*$ there exists no nondecreasing travelling wave with speed $c$.*

We emphasize that in the above theorem we do not require monotonicity of the super-solution $w$. The first consequence of Theorem 1.3 is to relate the existence of a minimal speed $c^*$ and the existence of a travelling front for any speed $c \geqslant c^*$ to the existence of a super-solution. In other words, we have the following necessary and sufficient condition:

**Corollary 1.4.** *Let $f$ and $J$ be such that* (f1) *and* (j1) *hold. Then there exists a nondecreasing solution with minimal speed $(u, c^*)$ of* (1.3)–(1.5) *if and only if there exists a super-solution $(w, \kappa)$ of* (1.3)–(1.5).

The existence of a super-solution in Theorem 1.3 is automatic under extra assumptions on $J$. For instance, we have

**Theorem 1.5.** *Let $f$ be a monostable nonlinearity and $J$ satisfy* (j1) *and Mollison's condition* (j2). *Then there exists a critical speed $c^*$ such that for all $c \geqslant c^*$ there exists a nondecreasing function $u$ such that $(u, c)$ is a solution of* (1.3)–(1.5), *while there is no nondecreasing travelling wave with speed $c < c^*$.*

Next we examine the validity of Theorem 1.2 for nonsymmetric $J$. Let $c^1$ denote the following quantity

$$c^1 := \inf_{\lambda>0} \frac{1}{\lambda}\left(\int_{\mathbb{R}} J(-x)e^{\lambda x}\,dx + f'(0) - 1\right).$$

For $c \geqslant c^1$ we denote $\lambda(c)$ the unique minimal $\lambda > 0$ such that

$$-c\lambda + \int_{\mathbb{R}} J(-x)e^{\lambda x}\,dx + f'(0) - 1 = 0.$$

We generalize a result of Carr and Chmaj [4] to the case when $J$ is nonsymmetric.

**Theorem 1.6.** *Let $f$ be a monostable nonlinearity satisfying* (f1), *$f'(0) > 0$, $f \in C^{1,\gamma}$ near* 0 *and there are $m \geqslant 1$, $\delta > 0$, $A > 0$ such that*

$$\left|u - f(u)\right| \geqslant Au^m \quad \text{for all } 0 \leqslant u < \delta. \tag{1.7}$$

*Let $J$ be a dispersal kernel satisfying* (j1), *$J \in C^1$ and is compactly supported. Then $c^1 \leqslant c^*$. Moreover, if $u$ is a solution of* (1.3), (1.4), *$0 \leqslant u \leqslant 1$, $u \not\equiv 0$ then, when $c = c^1$*

$$0 < \lim_{x\to-\infty} \frac{u(x)}{|x|e^{\lambda(c^*)x}} < \infty, \tag{1.8}$$

*and when $c > c^1$*

$$0 < \lim_{x\to-\infty} \frac{u(x)}{e^{\lambda(c)x}} < \infty. \tag{1.9}$$

In Theorem 1.6 we do not need to assume that the solution $u$ to (1.3), (1.4) is monotone.

**Corollary 1.7.** *If $f$ and $J$ satisfy the hypotheses of Theorem* 1.6 *and $f$ satisfies also* (f2) *then*

$$c^* = c^1.$$

Observe that when $J$ is symmetric, by Jensen's inequality $c^1 > 0$. On the other hand, it is not difficult to construct examples of nonsymmetric $J$ such that $c^1 \leqslant 0$. This fact should not be surprising. Indeed, let us recall a connection between the nonlocal problem (1.1) and a local version which arises by considering a family of kernels that approaches a Dirac mass, that is, $J_\varepsilon(x) = \frac{1}{\varepsilon} J(\frac{x}{\varepsilon})$ with $\varepsilon > 0$. Assuming that $u$ is smooth and $J$ decays fast enough, expanding $J_\varepsilon \star u - u$ in powers of $\varepsilon$ we see that

$$\begin{aligned} J_\varepsilon \star u(x) - u(x) &= \frac{1}{\varepsilon} \int_{\mathbb{R}} J\left(\frac{x-y}{\varepsilon}\right)\big(u(y) - u(x)\big)\, dy = \int_{\mathbb{R}} J(-z)\big(u(x+\varepsilon z) - u(x)\big)\, dz \\ &= \varepsilon \beta u'(x) + \varepsilon^2 \alpha u''(x) + o\big(\varepsilon^2\big) \end{aligned} \tag{1.10}$$

as $\varepsilon \to 0$, where

$$\alpha = \frac{1}{2} \int_{\mathbb{R}} J(z) z^2\, dz \quad \text{and} \quad \beta = \int_{\mathbb{R}} J(-z) z\, dz.$$

Thus there is a formal analogy between $J \star u - u$ and $\beta u'(x) + \varepsilon \alpha u''(x)$. When $J$ is symmetric then $\beta = 0$ and the results for travelling waves of (1.3)–(1.5) are similar to those for travelling wave solutions of

$$\tilde{\alpha} u'' - c u' + f(u) = 0 \quad \text{in } \mathbb{R}, \qquad du(-\infty) = 0, \qquad u(+\infty) = 1, \tag{1.11}$$

where $\tilde{\alpha} > 0$. For (1.11) there exists a minimal speed $c^* > 0$ such that travelling front solutions exist if and only if $c \geqslant c^*$ (see [20]). For general asymmetric $J$ we see from (1.10) that a better analogue than (1.11) for (1.3)–(1.5) is the problem

$$\tilde{\alpha} u'' - (c - \tilde{\beta}) u' + f(u) = 0 \quad \text{in } \mathbb{R}, \qquad u(-\infty) = 0, \qquad u(+\infty) = 1$$

for some $\tilde{\alpha} \geqslant 0$ and $\tilde{\beta} \in \mathbb{R}$. This equation is the same as (1.11) with a shift in the speed, that is, the minimal speed is $c^* + \tilde{\beta}$ where $c^*$ is the old minimal speed in (1.11). This new minimal speed can be either positive or negative depending on the size and sign of $\tilde{\beta}$, which is related to the asymmetry of $J$.

Regarding the uniqueness of the profile of the travelling waves we prove:

**Theorem 1.8.** *Assume $f$ and $J$ satisfy the hypotheses of Theorem* 1.6 *and $J$ satisfies*:

$$\exists a < 0 < b \quad \textit{such that} \quad J(a) > 0,\ J(b) > 0. \tag{1.12}$$

*Then for $c \neq 0$ the solution of the problem* (1.3)–(1.5) *is unique up to translation.*

We notice if $c \neq 0$ then any solution to (1.3) is continuous. In the case $c = 0$, the same argument used to prove Theorem 1.8 gives uniqueness for continuous solutions of (1.3)–(1.5) provided that this problem admits a continuous solution (see Remark 6.4). In the case $c = 0$ one sufficient condition for a solution $0 \leqslant u \leqslant 1$ to (1.3) to be continuous is that

$$u - f(u) \text{ is strictly increasing in } [0, 1].$$

In Proposition 6.7 we give examples of $f$ and nonsymmetric $J$ such that no solution of (1.3)–(1.5) is continuous, and this problem admits infinitely many solutions.

Our results also have implications in the study of solutions to

$$J \star u - u + f(u) = 0 \tag{1.13}$$

which corresponds to (1.3) with velocity $c = 0$. In [10] it was shown that if $f(u)/u$ is decreasing and $J$ is symmetric then any nontrivial bounded solution of (1.13) is identically 1. The symmetry of $J$ was important in the argument and it was conjectured that if the kernel $J$ is not even (1.13) may have more than one solution. For this discussion we shall assume that $f$ and $J$ satisfy the hypotheses of Theorem 1.6 and $f$ also satisfies (f2). We observe that when the dispersal kernel is not even, the critical velocity $c^*$ can be nonpositive. If $c^* \leqslant 0$ we obtain that Eq. (1.13) has a nonconstant positive solution satisfying (1.4)–(1.5). Similarly, Eq. (1.13) has positive solutions satisfying

$$\lim_{x\to-\infty} u(x) = 1, \qquad \lim_{x\to+\infty} u(x) = 0, \qquad u \text{ is nonincreasing}$$

if and only if $c_* \leqslant 0$ where

$$c_* = \min_{\lambda>0} \frac{1}{\lambda}\left(\int_{\mathbb{R}} J(x)e^{\lambda x}\,dx + f'(0) - 1\right).$$

Observe that by Jensen's inequality we have $c^* > 0$ or $c_* > 0$. In summary, besides $u \equiv 0$ and $u \equiv 1$ Eq. (1.13) has travelling wave solutions if $c^* \leqslant 0$ or $c_* \leqslant 0$. One may wonder whether other types of solutions may exist, maybe not monotone or with other behavior at $\pm\infty$. Under some additional conditions on $f$ we have a complete classification result for (1.13), in the sense that we do not require the boundary conditions at $\pm\infty$, continuity nor the monotonicity of the solutions. This result can be shown by slightly modifying the arguments for Theorem 2.1 in [4].

**Theorem 1.9.** *Suppose $f$ and $J$ satisfy the hypotheses of Theorem* 1.6*, $J$ satisfies* (1.12) *and $f'(r) \leqslant f'(0)$ for $r \in (0,1)$. Then any solution $0 \leqslant u \leqslant 1$ of problem* (1.13) *is one of the following*: (1) $u \equiv 0$ *or* $u \equiv 1$, (2) *a nondecreasing travelling wave, or* (3) *a nonincreasing travelling wave. Moreover in cases* (2) *and* (3) *the profile is unique up to translation.*

Regarding Mollison's condition (j2) let us mention that recently Kot and Medlock in [21] have shown that for a one-dimensional problem when the dispersal kernel $J$ is even with a fat tails and $f(s) := s(1-s)$, the solutions of the initial value problem (1.1) do not behave like travelling waves with constant speed but rather like what they called *accelerating* waves. Moreover, they predict the apparition of accelerating waves for (1.1). More precisely, supported by numerical evidence and analytical proof, they conjecture that (1.1) admits travelling wave solutions if and only if for some $\lambda > 0$

$$\int_{-\infty}^{+\infty} J(z)e^{\lambda z}\,dz < +\infty.$$

It appears from our analysis on nonsymmetric dispersal kernels, that the existence of travelling waves with constant speed is more related to

$$\int_0^{+\infty} J(z)e^{\lambda z}\,dz < +\infty \quad \text{for some } \lambda > 0$$

if we look at fronts propagating from the left to the right and

$$\int_0^{+\infty} J(-z)e^{\lambda z}\,dz < +\infty \quad \text{for some } \lambda > 0$$

if we look at fronts propagating from the right to the left. As a consequence, for asymmetric kernels, it may happen that in one direction, the solution behave like a front with finite speed and in the other like an accelerating wave.

The outline of this paper is the following. In Section 2, we recall some results on front solutions for ignition nonlinearities, then in Section 3 we construct increasing solution of for $J$ compactly supported. Section 4 is devoted to the proofs of Theorems 1.3 and 1.5. Section 5 contains the proofs of Theorem 1.6 and Corollary 1.7. In Section 6 we prove the uniqueness of the profile Theorem 1.8 and Theorem 1.9.

## 2. Approximation by ignition type nonlinearities

The proof of Theorem 1.3 essentially relies on some estimates and properties of the speed of fronts for problem (1.1) with ignition type nonlinearities $f$. We say that $f$ is of ignition type if $f \in C^1([0,1])$ and

(f3) there exists $\rho \in (0,1)$ such that $f|_{[0,\rho]} \equiv 0$, $f|_{(\rho,1)} > 0$ and $f(1) = 0$.

Consider the following problem

$$\begin{cases} J \star u - u - cu' + f(u) = 0 & \text{in } \mathbb{R}, \\ u(-\infty) = 0, \\ u(+\infty) = 1, \end{cases} \tag{2.1}$$

where $c \in \mathbb{R}$ and $f$ is either an ignition nonlinearity or a monostable nonlinearity.

The main result in this section is the following:

**Proposition 2.1.** *Let $f$ be a monostable nonlinearity and assume that $J$ is a nonnegative continuous function of unit mass. Assume further that there exists $(w,\kappa)$ a super-solution of* (1.3)–(1.5). *Let $(f_k)_{k\in\mathbb{N}}$ be any sequence of ignition functions which converges pointwise to $f$ and satisfies $\forall k \in \mathbb{N}$, $f_k \leqslant f_{k+1} \leqslant f$ and let $c_k$ be the unique speed of fronts associated to* (2.1). *Then*

$$\lim_{k\to+\infty} c_k = c^* \tag{2.2}$$

*exists and is independent of the sequence $f_k$. Furthermore, $c^* \leqslant \kappa$, there exists a nondecreasing solution $(u,c^*)$ of* (1.3)–(1.5) *and for $c < c^*$ there are no nondecreasing solutions to* (1.3)–(1.5).

The fact that for (2.1) with ignition type nonlinearity there exists a unique speed of fronts has been recently established by one of the authors in [7–9] and holds also for the following perturbation of (2.1)

$$\begin{cases} \varepsilon u'' + J \star u - u - cu' + f(u) = 0 & \text{in } \mathbb{R}, \\ u(-\infty) = 0, \\ u(+\infty) = 1, \end{cases} \tag{2.3}$$

where $\varepsilon \geqslant 0$, $c \in \mathbb{R}$.

**Theorem 2.2.** *(See [9, Theorem 1.2] and [7, Theorem 3.2].) Let $f$ be an ignition nonlinearity and assume that $J$ satisfies* (j1). *Then there exists a nondecreasing solution $(u, c)$ of* (2.3). *Furthermore the speed $c$ is unique. Moreover, if $(v, c')$ is a super-solution of* (2.3)*, then $c \leqslant c'$. The inequality becomes strict when $v$ is not a solution of* (2.3).

We remark that in this results the super-solution $v$ is not required to be monotone.

**Corollary 2.3.** *Let $f_1 \geqslant f_2$, $f_1 \not\equiv f_2$ be two ignition nonlinearities and assume that $J$ is a nonnegative continuous function of unit mass with finite first moment. Then $c_1 > c_2$ where $c_1$ and $c_2$ are the corresponding unique speeds given by Theorem* 2.2.

We also recall some useful results on solutions of (2.3), which can be found in [9,11].

**Lemma 2.4.** *(See [9, Lemma 2.1].) Suppose $f$ satisfies* (f1) *and $J$ satisfies* (j1). *Assume $\varepsilon \geqslant 0$, $c \in \mathbb{R}$ and let $0 \leqslant u \leqslant 1$ be an increasing solution of* (2.3). *Then*

$$f(l^{\pm}) = 0,$$

*where $l^{\pm}$ are the limits of $u$ at $\pm\infty$.*

**Lemma 2.5.** *(See [9, Lemma 2.2].) Let $f$ and $J$ be as in Theorem* 2.2. *Then the following holds*

$$\mu c^2 - \nu|c| \leqslant 0,$$

*where the constants $\mu, \nu$ are defined by*

$$\mu := \inf\{\rho, 1 - \rho\}, \qquad \nu := \int_{\mathbb{R}} J(z)|z|\,dz.$$

**Proof of Proposition 2.1.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of ignition functions which converges pointwise to $f$ and satisfies $\forall n \in \mathbb{N}$, $f_n \leqslant f_{n+1} \leqslant f$. Let $(u_n, c_n)$ denote the corresponding solution given by Theorem 2.2. By Corollary 2.3, $(c_n)_{n\in\mathbb{N}}$ is an increasing sequence. Next, we see that $c_n \leqslant \kappa$. Since $w$ satisfies

$$J \star w - w - \kappa w' + f_n(w) \leqslant 0 \quad \text{in } \mathbb{R}$$

by Theorem 2.2 we get

$$c_n \leqslant \kappa.$$

Let us observe that we can normalize the sequence of solutions $u_n$ by $u_n(0) = \frac{1}{2}$. Indeed, when $c^* = 0$ since $c_n < c^*$ the solution $u_n$ is smooth. Since any translation of $u_n$ is a solution of the problem and $u_n(-\infty) = 0$, $u_n(+\infty) = 1$ we can normalize it by $u_n(0) = \frac{1}{2}$. When $c^* \neq 0$, since $c_n \to c^*$ the sequence $u_n$ is smooth for all $n$ sufficiently large. Thus the same normalization can be also taken in this situation.

Since $(u_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence of increasing functions, using Helly's lemma there exists a subsequence which converges pointwise to a nondecreasing function $u$. Moreover, $u$ satisfies in the distribution sense

$$J \star u - u - c^* u' + f(u) = 0 \quad \text{in } \mathbb{R},$$

and by the monotonicity and the normalization of $u_n$

$$u(x) \leqslant \frac{1}{2} \quad \text{for all } x \leqslant 0, \qquad u(x) \geqslant \frac{1}{2} \quad \text{for all } x \geqslant 0. \tag{2.4}$$

Observe that when $c^* \neq 0$, using $C^1_{\mathrm{loc}}$ regularity, we get that $u \in C^1_{\mathrm{loc}}$ and satisfies the above equation in a strong sense. Otherwise, when $c^* = 0$, a standard argument shows that $u$ satisfies almost everywhere the equation

$$J \star u - u + f(u) = 0.$$

Observe that by (2.4) $u$ is nontrivial. It remains to show that $u$ satisfies the right boundary conditions. Now, since $u$ is nondecreasing and bounded, the following limits are well defined:

$$l^- := \lim_{x\to-\infty} u(x),$$
$$l^+ := \lim_{x\to+\infty} u(x).$$

We get $l^+ = 1$ and $l^- = 0$ using Lemma 2.4, the definition of $f$ and the monotonicity of $u$.

To finish we need to prove that $c^*$ is independent of the sequence $f_n$. So consider another sequence $\tilde{f}_n$ of ignition functions such that $\tilde{f}_n \leqslant \tilde{f}_{n+1} \leqslant f$ and $\tilde{f}_n \to f$ pointwise. Let $(\tilde{u}_n, \tilde{c}_n)$ denote the front solution and speed of (2.1) with nonlinearity $\tilde{f}_n$ and let

$$\tilde{c} = \lim_{n\to\infty} \tilde{c}_n.$$

Since $u = \lim_{n\to\infty} u_n$ satisfies

$$J \star u - u - c^* u' + \tilde{f}_n(u) \leqslant 0$$

by Theorem 2.2 we have $\tilde{c}_n \leqslant c^*$. Hence $\tilde{c} \leqslant c^*$ and reversing the roles of $f_n$ and $\tilde{f}_n$ we get $c^* \leqslant \tilde{c}$.

Finally observe that for $c < c^*$ there is no monotone solution to (1.4)–(1.5). Otherwise this solution would be a super-solution of (2.1) with $f_n$ instead of $f$. By Theorem 2.2 we would have $c_n \leqslant c$ for all $n$, which is a contradiction. □

## 3. Construction of solutions of (1.3)–(1.5) when $J$ is compactly supported

In this section we construct monotone solutions of (1.3)–(1.5) when $J$ is compactly supported. More precisely we prove the following

**Proposition 3.1.** *Let $f$ be a monostable nonlinearity and $J$ be continuous compactly supported which satisfies* (j1). *Assume further that there exists $a \in \mathbb{R}$ such that $\{a, -a\} \subset \operatorname{supp}(J)$. Then there exists a critical speed $c^*$, such that for all $c \geqslant c^*$ there exists a nondecreasing function $u$ such that $(u, c)$ is a solution of* (1.3)–(1.5). *Moreover, there is no nondecreasing travelling wave with speed $c < c^*$.*

To prove the above result we proceed following the strategy developed in [11]. It is based on the vanishing viscosity technique, a priori estimates, the construction of adequate super- and sub-solutions and the characterization of the critical speed obtained in Section 2. Let us first briefly explain how we proceed.

*Step 1*. For convenience, let us first rewrite problem (2.3) in the following way:

$$\begin{cases} \mathcal{M}[u] + f(u) = 0 \quad \text{in } \mathbb{R}, \\ u(-\infty) = 0, \\ u(+\infty) = 1, \end{cases} \tag{3.1}$$

where the operator $\mathcal{M}$ is defined for a given $\varepsilon > 0$, $c \in \mathbb{R}$ by

$$\mathcal{M}[u] = \mathcal{M}(\varepsilon, c)u = \varepsilon u'' + J \star u - u - cu'. \tag{3.2}$$

For problem (3.1), for small $\varepsilon$, we construct a super-solution which is independent of $\varepsilon$. More precisely we show the following

**Lemma 3.2.** *Let $J$ and $f$ be as in Proposition* 3.1. *Then there exist $\varepsilon_0 > 0$ and $(w, \kappa)$ such that $\forall 0 < \varepsilon \leqslant \varepsilon_0$, $(w, \kappa)$ is a super-solution of* (3.1).

*Step 2.* Using the above super-solution and a standard approximation scheme, for fixed $0 < \varepsilon \leqslant \varepsilon_0$, we prove the following

**Proposition 3.3.** *Fix $0 < \varepsilon \leqslant \varepsilon_0$ and let $J$ and $f$ be as in Proposition* 3.1. *Then there exists $c^*(\varepsilon)$ such that $\forall c \geqslant c^*(\varepsilon)$, there exists an increasing function $u_\varepsilon$ such that $(u_\varepsilon, c)$ is a solution of* (3.1). *Moreover $c^*(\varepsilon) \leqslant \kappa$ where $(w, \kappa)$ is the super-solution of Lemma* 3.2.

*Step 3.* We study the singular limit $\varepsilon \to 0$ and prove Proposition 3.1.

Some of the arguments developed in [11], on which this procedure is based, do not use the symmetry of $J$. Hence in some cases we will skip details in our proofs, making appropriate references to [11].

We divide this section in 3 subsections, each one devoted to one step.

### *3.1. Step 1. Existence of a super-solution*

We start with the construction of a super-solution of (3.1) for speeds $c \geqslant \bar{\kappa}$ for some $\bar{\kappa} > 0$ which is independent of $\varepsilon$ for $0 < \varepsilon \leqslant 1$.

**Lemma 3.4.** *Assume $J$ has compact support and let $\varepsilon > 0$. There exist a real number $\bar{\kappa} > 0$ and an increasing function $\bar{w} \in C^2(\mathbb{R})$ such that, given any $c \geqslant \bar{\kappa}$ and $0 < \varepsilon \leqslant 1$*

$$\begin{cases} \mathcal{M}[\bar{w}] + f(\bar{w}) \leqslant 0 & \text{in } \mathbb{R}, \\ \bar{w}(-\infty) = 0, \\ \bar{w}(+\infty) = 1, \end{cases}$$

*where $\mathcal{M} = \mathcal{M}(\varepsilon, c)$ is defined by* (3.2)*. Furthermore, $\bar{w}(0) = \frac{1}{2}$.*

The construction of the super-solution is an adaptation of the one proposed in [11]. The essential difference lies in the computation of the super-solution in a neighborhood of $-\infty$.

**Proof.** As in [11], fix positive constants $N, \lambda, \delta$ such that $\lambda > \delta$.

Let $\bar{w} \in C^2(\mathbb{R})$ be a positive increasing function satisfying

- $\bar{w}(x) = e^{\lambda x}$ for $x \in (-\infty, -N]$,
- $\bar{w}(x) \leqslant e^{\lambda x}$ on $\mathbb{R}$,
- $\bar{w}(x) = 1 - e^{-\delta x}$ for $x \in [N, +\infty)$,
- $\bar{w}(0) = \frac{1}{2}$.

Let $x_0 = e^{-\lambda N}$ and $x_1 = 1 - e^{-\delta N}$. We have $0 < x_0 < x_1 < 1$.

We now construct a positive function $g$ defined on $(0, 1)$ which satisfies $g(\bar{w}) \geqslant f(\bar{w})$. Since $f$ is smooth near 0 and 1, we have for $c$ large enough, say $c \geqslant \kappa_0$,

$$\lambda(c - \lambda)s \geqslant f(s) \quad \text{for } s \in [0, x_0] \tag{3.3}$$

and

$$\delta(c - \delta)(1 - s) \geqslant f(s) \quad \text{for } s \in [x_1, 1]. \tag{3.4}$$

Therefore we can achieve $g(s) \geqslant f(s)$ for $s$ in [0,1], with $g$ defined by

$$g(s) = \begin{cases} \lambda(\kappa_0 - \lambda)s & \text{for } 0 \leqslant s \leqslant x_0, \\ l(s) & \text{for } x_0 < s < x_1, \\ \delta(\kappa_0 - \delta)(1 - s) & \text{for } x_1 \leqslant s \leqslant 1, \end{cases} \tag{3.5}$$

where $l$ is any smooth positive function greater than $f$ on $[x_0, x_1]$ such that $g$ is of class $C^1$.

According to (3.5), for $x \leqslant -N$, i.e. for $w \leqslant e^{-\lambda N}$, we have

$$\begin{aligned} \mathcal{M}[\bar{w}] + g(\bar{w}) &= \varepsilon \bar{w}'' + J \star \bar{w} - \bar{w} - c\bar{w}' + g(\bar{w}) \\ &= \varepsilon \lambda^2 e^{\lambda x} + J \star \bar{w} - e^{\lambda x} - \lambda c e^{\lambda x} + \lambda(\kappa_0 - \lambda) e^{\lambda x} \end{aligned}$$

$$\begin{aligned}
&\leqslant \varepsilon\lambda^2 e^{\lambda x} + J \star e^{\lambda x} - e^{\lambda x} - \lambda c e^{\lambda x} + \lambda(\kappa_0 - \lambda)e^{\lambda x} \\
&\leqslant e^{\lambda x}\left[\int_{\mathbb{R}} J(-z)e^{\lambda z}\,dz - 1 - \lambda(c - \kappa_0) - \lambda^2(1-\varepsilon)\right] \\
&\leqslant 0,
\end{aligned}$$

for $c$ large enough, say

$$c \geqslant \kappa_1 = \frac{\int_{\mathbb{R}} J(-z)e^{\lambda z}\,dz - 1 + \lambda\kappa_0 - \lambda^2(1-\varepsilon)}{\lambda}.$$

In the open set $(x_1, +\infty)$, the computation of the super-solution is identical to the one in [11]. So, we end up with

$$\mathcal{M}[\bar{w}] + g(\bar{w}) \leqslant 0 \quad \text{in } (x_1, +\infty)$$

for $c$ large enough, say $c \geqslant \kappa_2$.

Therefore, by taking $c \geqslant \sup\{\kappa_0, \kappa_1, \kappa_2\}$, we achieve

$$g(\bar{w}) \geqslant f(\bar{w}) \quad \text{and} \quad \mathcal{M}[\bar{w}] + g(\bar{w}) \leqslant 0 \quad \text{for } 0 \leqslant \bar{w} \leqslant e^{-\lambda N} \text{ and } \bar{w} \geqslant 1 - e^{-\delta N}.$$

For the remaining values of $\bar{w}$, i.e. for $x \in [-N, N]$, $\bar{w}' > 0$ and we may increase $c$ further if necessary, to achieve

$$\mathcal{M}[\bar{w}] + g(\bar{w}) \leqslant 0 \quad \text{in } \mathbb{R}.$$

The result follows for

$$\bar{\kappa}(\varepsilon) := \sup\{\kappa_0, \kappa_1, \kappa_2, \kappa_3\},$$

where

$$\kappa_3 = \sup_{x \in [-N,N]}\left\{\frac{\varepsilon|\bar{w}''| + |J \star \bar{w} - \bar{w}| + g(\bar{w})}{\bar{w}'}\right\}. \qquad \square$$

Now, note that $\bar{\kappa}(\varepsilon)$ is a nondecreasing function of $\varepsilon$, therefore for all nonnegative $\varepsilon \leqslant 1$, $(\bar{w}, \bar{\kappa})$ with $\bar{\kappa} = \bar{\kappa}(1)$, will be a super-solution of (3.1), which ends Step 1.

**Remark 3.5.** The above construction of a super-solution also works if we only assume that for some positive $\lambda$, the following holds

$$\int_0^{+\infty} J(-z)e^{\lambda z}\,dz < +\infty.$$

### *3.2. Step 2. Construction of a solution when $\varepsilon > 0$*

To prove Proposition 3.3 we follow the strategy used in [11] relying on the following approximation scheme.

We first prove existence and uniqueness of a monotone solution for

$$\begin{cases} \mathcal{S}[u] + f(u) = -h_r(x) & \text{in } \omega, \\ u(-r) = \theta, \\ u(+\infty) = 1, \end{cases} \tag{3.6}$$

where $\varepsilon > 0$, $r \in \mathbb{R}$, $c \in \mathbb{R}$ and $\theta \in (0, 1)$ are given, and

$$\omega = (-r, +\infty), \tag{3.7}$$

$$\mathcal{S}[u] = \mathcal{S}(\varepsilon, r, c)[u] = \varepsilon u'' + \int_{-r}^{+\infty} J(x - y)u(y)\,dy - u - cu', \tag{3.8}$$

$$h_r(x) = \theta \int_{-\infty}^{-r} J(x - y)\,dy. \tag{3.9}$$

More precisely, we show

**Proposition 3.6.** *Assume $f$ and $J$ are as in Proposition* 3.1. *For any $\varepsilon > 0$, $\theta \in [0, 1)$, $r > 0$ so that* supp $J \subset (-r, +\infty)$ *and $c \in \mathbb{R}$ there exists a unique positive increasing solution $u_c$ of* (3.6).

To prove this proposition we use a construction introduced by one of the authors [8,9] which consists first to obtain a solution of the following problem:

$$\begin{cases} \mathcal{L}[u] + f(u) + h_r + h_R = 0 & \text{for } x \in \Omega, \\ u(-r) = \theta, \\ u(+R) = 1, \end{cases} \tag{3.10}$$

where $\Omega = (-r, +R)$ and $\mathcal{L} = \mathcal{L}(\varepsilon, J, r, R, c)$, $h_r$ and $h_R$ are defined by

$$\mathcal{L}[u] = \mathcal{L}(\varepsilon, J, r, R, c)[u] = \varepsilon u'' + \left[ \int_{-r}^{+R} J(x - y)u(y)\,dy - u \right] - cu',$$

$$h_r(x) = \theta \int_{-\infty}^{-r} J(x - y)\,dy,$$

$$h_R(x) = \int_{+R}^{+\infty} J(x - y)\,dy. \tag{3.11}$$

Namely, we have

**Proposition 3.7.** *Assume $f$ and $J$ are as in Proposition* 3.1. *For any $\varepsilon > 0$, $\theta \in [0, 1)$, $r < R$ so that* $\operatorname{supp} J \subset (-r, R)$ *and $c \in \mathbb{R}$ there exists a unique positive increasing solution $u_c$ of* (3.10).

**Proof.** The construction of a solution uses the super- and sub-solution iterative scheme presented in [9]. To produce a solution, we just have to construct ordered sub- and super-solutions. An easy computation shows that $\underline{u} = \theta$ and $\bar{u} = 1$ are respectively a sub- and a super-solution of (3.10). Indeed,

$$\begin{aligned}\mathcal{L}[\underline{u}] + f(\underline{u}) + h_r + h_R &= \int_{-r}^{R} J(x-y)\theta\,dy - \theta + \theta\int_{-\infty}^{-r} J(x-y)\,dy + \int_{R}^{+\infty} J(x-y)\,dy + f(\theta)\\ &= (1-\theta)\int_{R}^{+\infty} J(x-y)\,dy + f(\theta) \geqslant 0\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}[\bar{u}] + f(\bar{u}) + h_r + h_R &= \int_{-r}^{R} J(x-y)\,dy - 1 + \theta\int_{-\infty}^{-r} J(x-y)\,dy + \int_{R}^{+\infty} J(x-y)\,dy + f(1)\\ &= (\theta-1)\int_{-\infty}^{-r} J(x-y)\,dy \leqslant 0.\end{aligned}$$

The uniqueness and the monotonicity of such solutions have been already established in [8], so we refer to this reference for interested reader. □

We are now in a position to prove Proposition 3.6.

**Proof of Proposition 3.6.** Let us now construct a solution of (3.6). Fix $\varepsilon > 0$, $c \in \mathbb{R}$ and $r > 0$ such that $\operatorname{supp}(J) \subset \omega$. Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of reals which converges to $+\infty$. Since $J$ has compact support, without loosing generality we may also assume that $\operatorname{supp}(J) \subset (-r, R_n)$ for all $n \in \mathbb{N}$. Let us denote $(u_n, c)$ the corresponding solution given by Proposition 3.7. Clearly, $h_{R_n} \to 0$ pointwise, as $n \to \infty$. Observe now that $(u_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence of increasing functions. Since $\varepsilon > 0$, using local $C^{2,\alpha}$ estimates, up to a subsequence, $u_n$ converges in $C^{2,\alpha}_{\mathrm{loc}}$ to a nondecreasing function $u$. Therefore $u \in C^{2,\alpha}$ and satisfies

$$\begin{cases} \varepsilon u'' + \displaystyle\int_{-r}^{+\infty} J(x-y)u(y)\,dy - u - cu' + f(u) + h_r = 0 \quad \text{in } \omega,\\ u(-r) = \theta. \end{cases} \tag{3.12}$$

To complete the construction of the solution, we prove that $u(+\infty) = 1$. Indeed, since $u$ is uniformly bounded and nondecreasing, $u$ achieves its limit at $+\infty$. Using Lemma 2.4 yields $u(+\infty) = 1$. □

**Proof of Proposition 3.3.** By Lemma 3.4 there exist $\bar{\kappa}$ and a function $\bar{w}$ which is a super-solution to (3.1) for any $c \geqslant \bar{\kappa}$ and any $0 < \varepsilon \leqslant 1$. If $c \geqslant \bar{\kappa}$, following the approach in [11], we can take the limit as $r \to \infty$ in the problem (3.6) to obtain a solution of (3.1).

Finally one can also verify, see [11], that there exists a monotone solution $u_\varepsilon$ with the following speed

$$c^*(\varepsilon) := \inf\{c \mid (3.1) \text{ admits a monotone solution with speed } c\}.$$

The proof of these claims are straightforward adaptations of [11], since in this reference the author makes no use of the symmetry of $J$ for this part of the proof, and essentially relies on the maximum principle and Helly's theorem. We point the interested reader to [11] for the details. □

**Remark 3.8.** Note that from the previous comments we get the following uniform estimates

$$\forall 0 < \varepsilon \leqslant \varepsilon_0, \quad c^*(\varepsilon) \leqslant \bar{\kappa}.$$

*3.3. Step 3. Proof of Proposition 3.1*

We essentially use the ideas introduced in [11].

First, we remark that since $J$ has a compact support, using the super-solution of Step 1, we get from Proposition 2.1 a monotone solution $(u, c^*)$ of (1.3)–(1.5). Furthermore, there exists no monotone solution of (1.3)–(1.5) with speed $c < c^*$ and we have the following characterization:

$$\lim_{k\to\infty} c_k = c^*,$$

where $c_k$ is the unique speed of fronts associated with an arbitrary sequence of ignition functions $(f_k)_{k\in\mathbb{N}}$ which converges pointwise to $f$ and satisfies $\forall k \in \mathbb{N}$, $f_k \leqslant f_{k+1} \leqslant f$.

Also observe that from Remark 3.8 we have a uniform bound from above on $c^*(\varepsilon)$.

**Lemma 3.9.** *For all $\varepsilon \leqslant \varepsilon_0$ we have $c^*(\varepsilon) \leqslant \bar{\kappa}$.*

For any speed $c \geqslant \bar{\kappa} > 0$, there exists a monotone solution $(u_\varepsilon, c)$ of (3.1) for any $\varepsilon \leqslant \varepsilon_0$. Normalizing the functions by $u_\varepsilon(0) = \frac{1}{2}$ and letting $\varepsilon \to 0$, using Helly's theorem, a priori bounds and some regularity we end up with a solution $(u, c)$ of (1.3)–(1.5). Repeating this limiting process for any speed $c \geqslant \bar{\kappa}$, we end up with a monotone solution of (1.3)–(1.5) for any speed $c \geqslant \bar{\kappa}$.

Define now the following critical speed

$$c^{**} = \inf\{c \mid \forall c' \geqslant c \ (1.3)\text{–}(1.5) \text{ has a positive monotone solution of speed } c'\}.$$

**Remark 3.10.** Observe that from the uniform bounds we easily see that

$$c^{**} \leqslant \liminf_{\varepsilon\to 0} c^*(\varepsilon). \tag{3.13}$$

Obviously, we have $c^* \leqslant c^{**} \leqslant \bar{\kappa}$. To complete the proof of Proposition 3.1, we are then led to prove that $c^{**} = c^*$. To prove this equality, we use some properties of the speed of the following approximated problem

$$\begin{cases} \varepsilon u'' + J \star u - u - cu' + f\eta_\theta(u) = 0 \quad \text{in } \mathbb{R}, \\ u(-\infty) = 0, \\ u(+\infty) = 1, \end{cases} \tag{3.14}$$

where $\theta > 0$, $\eta_\theta(u) = \eta(u/\theta)$ and $\eta \in C^\infty(\mathbb{R})$ is such that

$$0 \leqslant \eta \leqslant 1, \qquad \eta' \geqslant 0, \qquad \eta(s) = 0 \quad \text{for } s \leqslant 1, \qquad \eta(s) = 1 \quad \text{for } s \geqslant 2.$$

Then $\eta_\theta$ has the following properties:

- $\eta_\theta \in C^\infty(\mathbb{R})$,
- $0 \leqslant \eta_\theta \leqslant 1$,
- $\eta_\theta(s) \equiv 0$ for $s \leqslant \theta$ and $\eta_\theta(s) \equiv 1$ for $s \geqslant 2\theta$,
- if $0 < \theta_1 \leqslant \theta_2$ then $\eta_{\theta_1} \leqslant \eta_{\theta_2}$.

For (3.14), we have the following results:

**Lemma 3.11.** *Let $c^\theta$ be the unique speed of front solutions to* (2.1) *and nonlinearity $f\eta_\theta$. Let $c_\varepsilon^\theta$, $c^*(\varepsilon)$ be respectively the unique* (*minimal*) *speed solution of* (3.1) *with the nonlinearity $f\eta_\theta$ and $f$. Then the following holds*:

(a) *For fixed $\theta > 0$, $\lim_{\varepsilon \to 0} c_\varepsilon^\theta = c^\theta$.*
(b) *For fixed $\varepsilon$ so that $\varepsilon_0 \geqslant \varepsilon > 0$, $\lim_{\theta \to 0} c_\varepsilon^\theta = c^*(\varepsilon)$.*

**Proof.** The first limit, as $\varepsilon \to 0$ when $\theta > 0$ is fixed, has been already obtained in [9], so we refer to this reference for a detailed proof. The second limit, for fixed $\varepsilon > 0$, is obtained using a similar argument as in the proof of Proposition 2.1 to obtain the characterization of $c^*$. $\square$

**Proof of Proposition 3.1.** Assume by contradiction that $c^* < c^{**}$. Then choose $c$ such that $c^* < c < c^{**}$. By (3.13) we may fix $\varepsilon_0 > 0$ small such that

$$c < c^*(\varepsilon) \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{3.15}$$

Now consider any sequence $\bar{\theta}_n \to 0$. Since $c^{\bar{\theta}_n} < c$, using Lemma 3.11(a) there exists $0 < \varepsilon_n < \varepsilon_0$, $\varepsilon_n \to 0$, such that

$$c_{\varepsilon_n}^{\bar{\theta}_n} < c. \tag{3.16}$$

Then, using the continuity of the map $\theta \mapsto c_{\varepsilon_n}^\theta$, (3.16), (3.15) and Lemma 3.11(b) we conclude that there exists $0 < \theta_n < \bar{\theta}_n$ such that

$$c = c_{\varepsilon_n}^{\theta_n}.$$

Note that $\theta_n \to 0$. Let $u_n$ be the associated solution to (3.1) with $\varepsilon = \varepsilon_n$, speed $c$ and nonlinearity $f\eta_{\theta_n}$. We normalize $u_n$ by $u_n(0) = 1/2$. Using Helly's theorem we get a solution $\bar{u}$ of (1.3)–(1.5) with speed $c$. This contradicts the definition of $c^{**}$. $\square$

## 4. Construction of solution in the general case: Proofs of Theorems 1.3 and 1.5

Theorem 1.5 is a direct consequence of Theorem 1.3. Indeed, since $J$ satisfies the Mollison condition, the construction in Section 3 (Step 1, Section 3.1) of a smooth super-solution $(w, \kappa)$ with $w(0) = \frac{1}{2}$ holds. Therefore, Theorem 1.5 is a direct application of Theorem 1.3.

In the rest of the section we prove Theorem 1.3, that is, we construct solutions of (1.3)–(1.5) only assuming that there exists a super-solution $(w, \kappa)$ of (1.3)–(1.5). The construction uses a standard procedure of approximation of $J$ by kernels $J_n$ with compact support and the characterization of the minimal speed $c^*$ obtained in Section 2.

Let us describe briefly our proof. From Proposition 2.1, there exists a monotone solution $(u, c^*)$ of (1.3)–(1.5) with critical speed. Then we construct monotone solution of (1.3)–(1.5) for any $c > c^*$, $c \neq 0$, using a sequence $(J_n)_{n\in\mathbb{N}}$ of approximated kernels and the same type of arguments developed in Step 3 of the above section. Let us first construct the approximated kernel and get some uniform lower bounds for $c_n^*$.

### 4.1. The approximated kernel and related problems

First, let $j_0$ be a positive symmetric function defined by

$$j_0(x) = \begin{cases} e^{\frac{1}{x^2-1}} & \text{for } x \in (-1, 1), \\ 0 & \text{elsewhere.} \end{cases} \tag{4.1}$$

Now, let $(\chi_n)_{n\in\mathbb{N}}$ be the following sequence of "cut-off" function:

- $\chi_n \in C_0^\infty(\mathbb{R})$,
- $0 \leqslant \chi_n \leqslant 1$,
- $\chi_n(s) \equiv 1$ for $|s| \leqslant n$ and $\chi_n(s) \equiv 0$ for $|s| \geqslant 2n$.

Define

$$J_n := \frac{1}{m_n}\left(\frac{j_0}{n} + J(z)\chi_n(z)\right),$$

where $m_n := \frac{1}{n}\int_{\mathbb{R}} j_0(z)\,dz + \int_{\mathbb{R}} J\chi_n(z)\,dz$. Observe that since $\int_{\mathbb{R}} j_0 > 0$, $J_n$ is well defined and $J_n(z) \to J(z)$ pointwise.

Since $J_n$ satisfies the assumption of Proposition 3.1, there exists for each $n \in \mathbb{N}$ a critical speed $c_n^*$ for the problem (4.2) below:

$$\begin{cases} J_n \star u - u - cu' + f(u) = 0 & \text{in } \mathbb{R}, \\ u(-\infty) = 0, \\ u(+\infty) = 1. \end{cases} \tag{4.2}$$

Before going to the proof of Theorem 1.3, we prove some a priori estimates on $c_n^*$. Namely we have the following

**Proposition 4.1.** *Let $c_n^*$ be the critical speed defined above, then there exists a positive constant $\kappa_1$ such that*

$$-\kappa_1 \leqslant c_n^*.$$

**Proof.** Let $f_\theta$ be a fixed function of ignition type such that $f_\theta \leqslant f$. Using Theorem 2.2, we have $c_n^\theta \leqslant c_n^*$. To obtain our desired bound, we just have to bound from below $c_n^\theta$. The later is obtained using Lemma 2.5. Indeed, for each $n \in \mathbb{N}$, we have

$$\mu\left(c_n^\theta\right)^2 - \nu_n\left|c_n^\theta\right| \leqslant 0,$$

with $\nu_n := \int_{\mathbb{R}} J_n(z)|z|\,dz$ and $\mu$ is independent of $n$. Since $\nu_n \leqslant \bar{\nu} := \sup_{n\in\mathbb{N}}\{\nu_n\} < \infty$, we end up with

$$\mu\left(c_n^\theta\right)^2 - \bar{\nu}\left|c_n^\theta\right| \leqslant 0.$$

Hence,

$$\left|c_n^\theta\right| \leqslant \kappa_1. \qquad \square$$

Let us also recall some properties of the following approximated problem:

$$\begin{cases} J_n \star u - u - cu' + f\eta_\theta(u) = 0 & \text{in } \mathbb{R}, \\ u(-\infty) = 0, \\ u(+\infty) = 1, \end{cases} \tag{4.3}$$

where $\theta > 0$ and $\eta_\theta$ is such that

- $\eta_\theta \in C_0^\infty(\mathbb{R})$,
- $0 \leqslant \eta_\theta \leqslant 1$,
- $\eta_\theta(s) \equiv 0$ for $s \leqslant \theta$ and $\eta_\theta(s) \equiv 1$ for $s \geqslant 2\theta$.

For such kind of problem we have

**Lemma 4.2.** *Let $c^\theta$ and $c_n^\theta$ be the unique speed solutions of* (2.1) *with the nonlinearity $f\eta_\theta$ and respectively the kernels $J$ and $J_n$ and let $c_n^*$ be the critical speed solution of* (4.3) *with the nonlinearity $f$ and the kernel $J_n$. Then the following holds*:

(a) *For fixed $\theta$,* $\lim_{n\to\infty} c_n^\theta = c^\theta$.
(b) *For a fixed $n$, then* $\lim_{\theta\to 0} c_n^\theta = c_n^*$.

Part (b) of this lemma is contained in Proposition 2.1. Part (a) can be proved using similar arguments as in Proposition 2.1.

### 4.2. Construction of the solutions: Proof of Theorem 1.3

We are now in position to prove Theorem 1.3. From Proposition 2.1, we already know that there exists a travelling front to (1.3)–(1.5) with a critical speed $c^*$. To complete the proof, we have to construct nondecreasing solution for any speed $c \geqslant c^*$. We emphasize that since $(w, \kappa)$ is not a super-solution of (1.3)–(1.5) with the approximated kernel $J_n$, there is no uniform upper bound directly available for the speed $c_n^*$ and the argumentation in the above section cannot directly be applied.

From Proposition 4.1, we have the following dichotomy: either $\liminf(c_n^*)_{n\in\mathbb{N}} < +\infty$ or $\liminf(c_n^*)_{n\in\mathbb{N}} = +\infty$. We prove that in both situations there exists a front solution for any speed $c \geqslant c^*$.

*Case 1:* $\liminf(c_n^*)_{n\in\mathbb{N}} < +\infty$. In this case, the same argument as in Proposition 3.1 in Section 3.3 works. Indeed, up to a subsequence $c_n^* \to \tilde{c}$ and we must have $c^* \leqslant \tilde{c}$. To prove that $c^* = \tilde{c}$ we proceed as in Section 3.3, using Lemma 4.2 instead of Lemma 3.11.

Let now turn our attention to the other situation.

*Case 2:* $\liminf(c_n^*)_{n\in\mathbb{N}} = +\infty$. In this case $\lim_{n\to\infty} c_n^* = +\infty$ we argue as follows. Fix $c > c^*$, $c \neq 0$, where $c^*$ is defined by Proposition 2.1. We will show that for such $c$ there is a monotone solution to (1.3)–(1.5). When $c^* \leqslant 0$ and $c = 0$ then a standard limiting procedure will show that a monotone solution exists with this speed.

Again, by Theorem 2.2 and Proposition 2.1, we have $c^\theta < c^*$ for every positive $\theta$. Therefore,

$$\forall \theta > 0, \quad c^\theta < c^* < c.$$

Fix $\theta > 0$. Since $c_n^\theta \to c^\theta$, one has on the one hand $c_n^\theta < c$ for $n \geqslant n_0$ for some integer $n_0$. On the other hand, $c_n^* \to +\infty$, thus there exists an integer $n_1$ such that $c < c_n^*$ for all $n \geqslant n_1$. Therefore, we may achieve for $n \geqslant \sup\{n_0, n_1\}$,

$$c_n^\theta < c < c_n^*.$$

From this last inequality, and according to Theorem 2.2 and Lemma 4.2, for each $n \geqslant \sup\{n_0, n_1\}$ there exists a positive $\theta(n) \leqslant \theta$ such that $c = c_n^{\theta(n)}$.

Let $u_n$ be the nondecreasing solution of (4.2) associated with $\theta(n)$. Since $\theta(n)$ is bounded, we can extract a subsequence still denoted $(\theta(n))_{n\in\mathbb{N}}$ which converges to some $\bar{\theta}$. We claim that

**Claim.** $\bar{\theta} = 0$.

Assume for the moment that the claim is proved. Using the translation invariance, we may assume that for all $n$, $u_n(0) = \frac{1}{2}$. Using now that $u_n$ is uniformly bounded and Helly's theorem, up to a subsequence $u_n \to u$ pointwise, where $u$ is a solution of (1.3)–(1.5) with speed $c$.

In this way we get a nontrivial solution of (1.3)–(1.5) for any speed $c \geqslant c^*$.

Let us now turn our attention to the proof of the above claim.

**Proof of the Claim.** We argue by contradiction. If not, then $\bar{\theta} > 0$ and the speed $c^{\bar{\theta}}$ of the corresponding nondecreasing front solution of (2.1) satisfies

$$c^{\bar{\theta}} < c^* < c.$$

Let us now consider, $u_n$ the solution associated with $\theta(n)$, normalized by $u_n(0) = \theta(n)$. Using uniform a priori estimates, Helly's theorem we can extract a converging sequence of function and get a solution $u$ with speed $c$ of the following:

$$J \star u - u - cu' + f_{\bar{\theta}}(u) = 0 \quad \text{in } \mathbb{R}.$$

Using the arguments developed in [9, Section 5.1] to prove Theorem 1.2 of that reference, one can show that $u$ satisfies the boundary conditions

$$u(+\infty) = 1, \qquad u(-\infty) = 0.$$

According to Proposition 2.1, we get the contradiction

$$c = c^{\bar{\theta}} < c^* < c.$$

Hence $\bar{\theta} = 0$. $\square$

## 5. Characterization of the minimal speed and asymptotic behavior

Throughout this section we will assume the hypotheses of Theorem 1.6, namely $f$ satisfies (f1), $f'(0) > 0$, $f \in C^{1,\gamma}$ near 0 and (1.7), and $J$ satisfies (j1), $J \in C^1$ and is compactly supported.

Let us consider the following equation

$$\begin{aligned} J \star u - u - cu' + f(u) = 0 \quad \text{in } \mathbb{R}, \\ \lim_{x \to -\infty} u(x) = 0. \end{aligned} \tag{5.1}$$

We need to establish some estimates on bounded solutions of (5.1) that we constantly use along this section.

**Lemma 5.1.** *Let $u$ be a nonnegative bounded solution of* (5.1)*, then the following holds*:

(i) $\int_y^x \int_{\mathbb{R}} J(s-t)[u(t) - u(s)]\,dt\,ds = \int_0^1 \int_{\mathbb{R}} J(-z)z[u(x+z\eta) - u(y+z\eta)]\,dz\,d\eta,$
(ii) $f(u) \in L^1(\mathbb{R})$,
(iii) $u, J \star u \in L^1(\mathbb{R}^-)$,
(iv) $v(x) := \int_{-\infty}^x u(s)\,ds$ *satisfies* $v(x) \leqslant K(1 + |x|)$ *for some positive* $K$ *and* $v(x) \in L^1(\mathbb{R}^-)$.

**Proof.** We start with the proof of (i). Let $(u_n)_n$ be a sequence of smooth ($C^1$) functions which converges pointwise to $u$. Using the Fundamental Theorem of Calculus and Fubini's Theorem, we have

$$\begin{aligned} \int_y^x \int_{\mathbb{R}} J(s-t)\big[u_n(t) - u_n(s)\big]\,dt\,ds &= \int_y^x \int_0^1 \int_{\mathbb{R}} J(-z)zu_n'(s+z\eta)\,dz\,d\eta\,ds \\ &= \int_0^1 \int_{\mathbb{R}} J(-z)z\big[u_n(x+z\eta) - u_n(y+z\eta)\big]\,dz\,d\eta. \end{aligned}$$

Since $|J(-z)zu_n(y+\eta z)| \leqslant K|J(-z)z| \in L^1(\mathbb{R}\times[0,1])$ and $u_n$ converges pointwise to $u$, passing to the limit in the above equation yields

$$\int_y^x \int_{\mathbb{R}} J(s-t)\big[u(t)-u(s)\big]\,dt\,ds = \int_0^1 \int_{\mathbb{R}} J(-z)z\big[u(x+z\eta)-u(y+z\eta)\big]\,dz\,d\eta.$$

To obtain (ii), we argue as follows. Integrating (5.1) from $y$ to $x$, it follows that

$$c\big(u(x)-u(y)\big) - \int_y^x \int_{\mathbb{R}} J(s-t)\big[u(t)-u(s)\big]\,dt\,ds = \int_y^x f\big(u(s)\big)\,ds. \tag{5.2}$$

Using (i), we end up with

$$c\big(u(x)-u(y)\big) - \int_0^1 \int_{\mathbb{R}} J(-z)z\big[u(x+z\eta)-u(y+z\eta)\big]\,dz\,d\eta = \int_y^x f\big(u(s)\big)\,ds. \tag{5.3}$$

Again, since $|J(-z)zu(y+\eta z)| \leqslant K|J(-z)z| \in L^1(\mathbb{R}\times[0,1])$, we can pass to the limit $y \to -\infty$ in the above equation using Lebesgue dominated convergence theorem. Therefore, we end up with

$$cu(x) - \int_0^1 \int_{\mathbb{R}} J(-z)zu(x+z\eta)\,dz\,d\eta = \int_{-\infty}^x f\big(u(s)\big)\,ds.$$

Thus,

$$\int_{-\infty}^x f\big(u(s)\big)\,ds \leqslant K\left(|c| + \int_{\mathbb{R}} J(z)|z|\,dz\right)$$

which proves (ii). From Eq. (5.2), we have

$$c\big(u(x)-u(y)\big) - \int_y^x f\big(u(s)\big)\,ds + \int_y^x u(s)\,ds = \int_y^x J\star u(s)\,ds.$$

Thus $J\star u \in L^1(\mathbb{R}^-)$ will immediately follow from $u \in L^1(\mathbb{R}^-)$ and (ii). Observe now that since $f'(0) > 0$, and $u(-\infty) = 0$, for $x \ll -1$, we have $f(u) > \alpha u$ for some positive constant $\alpha$. Therefore,

$$\alpha \int_{-\infty}^x u(s)\,ds \leqslant \int_{-\infty}^x f\big(u(s)\big)\,ds$$

and (iii) is proved.

To obtain (iv) we argue as follows. From (i)–(iii), $v$ is a well-defined nondecreasing function such that $v(-\infty) = 0$. Moreover, $v$ is smooth provide $u$ is continuous. By definition of $v$, we easily see that $v(x) \leqslant C(|x| + 1)$ for all $x \in \mathbb{R}$. Indeed, we have

$$v(x) \leqslant \int_{\infty}^{0} u(s)\,ds + \int_{0}^{|x|} u(s)\,ds \leqslant K\big(1 + |x|\big),$$

where $K = \sup\{\int_{-\infty}^{0} u(s)\,ds;\ \|u\|_{L^\infty(\mathbb{R})}\}$.

Now, integrating (5.1) on $(-\infty, x)$, we easily see that

$$cv'(x) = J \star v(x) - v(x) + \int_{-\infty}^{x} f\big(u(s)\big)\,ds. \tag{5.4}$$

Since $f'(0) > 0$ we can choose $R \ll -1$ so that for $s \leqslant R$, $f(u(s)) \geqslant \alpha u(s)$ for some $\alpha > 0$. Fixing now $x < R$ and integrating (5.4) between $y$ and $x$, we obtain

$$c\big(v(x) - v(y)\big) \geqslant \int_{y}^{x} \big(J \star v(s) - v(s)\big)\,ds + \alpha \int_{y}^{x} v(s)\,ds. \tag{5.5}$$

Proceeding as above, we get that $v \in L^1(-\infty, R)$. □

Following the idea of Carr and Chmaj [4], we now derive some asymptotic behavior of the nonnegative bounded solution $u$ of (5.1). More precisely, we show the following

**Lemma 5.2.** *Let $u$ be a nonnegative bounded continuous solution of* (5.1). *Then there exist two positive constants $M, \beta$, such that $v(x) = \int_{-\infty}^{x} u(s)\,ds$ satisfies*

$$v(x) \leqslant Me^{\beta x}. \tag{5.6}$$

**Proof.** The proof uses ideas from [11]. Let us first show that for some positive constants $C$, $R$, we have

$$\int_{-\infty}^{-R} v(x)e^{-\beta x}\,dx < C, \tag{5.7}$$

for some $\beta > 0$ small.

Consider $R > 0$ and $\beta > 0$ constants to be chosen later. Let $\zeta \in C^\infty(\mathbb{R})$ be a nonnegative nondecreasing function such that $\zeta \equiv 0$ in $(-\infty, -2]$ and $\zeta \equiv 1$ in $[-1, \infty)$. For $N \in \mathbb{N}$, let $\zeta_N = \zeta(x/N)$. Multiplying (5.4) by $e^{-\beta x}\zeta_N$ and integrating over $\mathbb{R}$, we get

$$\int_{\mathbb{R}} (J \star v - v)\big(e^{-\beta x}\zeta_N\big)\,dx - \int_{\mathbb{R}} cv'\big(e^{-\beta x}\zeta_N\big)\,dx + \int_{\mathbb{R}} \int_{-\infty}^{x} f\big(u(s)\big)\,ds\big(e^{-\beta x}\zeta_N\big)\,dx = 0. \tag{5.8}$$

Note that by the monotonicity of $\zeta_N$ we have

$$\int_{\mathbb{R}} J \star v(x)\zeta_N(x)e^{-\beta x}\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)e^{-\beta x}\zeta_N(x)v(y)\,dz\,dy$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} J(z)e^{-\beta(z+y)}\zeta_N(z+y)v(y)\,dz\,dy$$
$$\geqslant \int_{\mathbb{R}} v(y)e^{-\beta y}\left(\int_{-R}^{\infty} J(z)e^{-\beta z}\zeta_N(y-R)\,dz\right)dy.$$

Therefore, we have

$$\int_{\mathbb{R}} (J\star v - v)\big(e^{-\beta x}\zeta_N\big)\,dx \geqslant \int v(x)e^{-\beta x}\left(\int_{-R}^{\infty} J(z)e^{-\beta z}\,dz\,\zeta_N(x-R) - \zeta_N(x)\right)dx. \quad (5.9)$$

Let us now choose our adequate $R > 0$. First pick $0 < \alpha < f'(0)$ and $R > 0$ so large that

$$f(u)(x) \geqslant \alpha u(x) \quad \text{for } x \leqslant -R. \quad (5.10)$$

Next, one can increase $R$ further if necessary so that $\int_{-R}^{\infty} J(y)\,dy > (1-\alpha/2)$. By continuity we obtain for some $\beta_0 > 0$ and all $0 < \beta < \beta_0$,

$$\int_{-R}^{\infty} J(y)e^{-\beta y}\,dy \geqslant (1-\alpha/2)e^{\beta R}. \quad (5.11)$$

Collecting (5.9) and (5.11), we then obtain

$$\int_{\mathbb{R}} (J\star v - v)\big(e^{-\beta x}\zeta_N\big) \geqslant \int_{\mathbb{R}} v(x)e^{-\beta x}\big((1-\alpha/2)e^{\beta R}\zeta_N(x-R) - \zeta_N(x)\big)\,dx$$
$$\geqslant (1-\alpha/2)\int_{\mathbb{R}} v(x+R)e^{-\beta x}\zeta_N(x)\,dx - \int_{\mathbb{R}} v(x)e^{-\beta x}\zeta_N(x)\,dx$$
$$\geqslant -\alpha/2\int_{\mathbb{R}} v(x)e^{-\beta x}\zeta_N(x)\,dx, \quad (5.12)$$

where we used the monotone behavior of $v$ in the last inequality.

We now estimate the second term in (5.8):

$$\int_{\mathbb{R}} v'\zeta_N e^{-\beta x}\,dx = \beta\int_{\mathbb{R}} v\zeta_N e^{-\beta x} - \int_{\mathbb{R}} v\zeta_n' e^{-\beta x}\,dx \leqslant \beta\int_{\mathbb{R}} v\zeta_N e^{-\beta x}. \quad (5.13)$$

Finally using (5.10), the last term in (5.8) satisfies

$$\int_{\mathbb{R}}\left(\int_{-\infty}^{x} f\big(u(s)\big)\,ds\right)\zeta_N e^{-\beta x}\,dx = \int_{-\infty}^{-R}\left(\int_{-\infty}^{x} f\big(u(s)\big)\,ds\right)\zeta_N e^{-\beta x}\,dx - C$$
$$\geqslant \alpha \int_{-\infty}^{-R} v\zeta_N e^{-\beta x}\,dx - C. \tag{5.14}$$

By (5.8), (5.12)–(5.14), we then obtain

$$|c|\beta \int_{\mathbb{R}} u\zeta_N e^{-\beta x}\,dx \geqslant \alpha \int_{-\infty}^{-R} u\zeta_N e^{-\beta x}\,dx - C - \alpha/2 \int_{\mathbb{R}} v\zeta_N e^{-\beta x}\,dx,$$
$$\big(\alpha/2 - |c|\beta\big) \int_{-\infty}^{-R} u\zeta_N e^{-\beta x}\,dx \leqslant \tilde{C}.$$

Choosing $\beta < \alpha/(2|c|)$ and letting $N \to \infty$ proves (5.7).

Using the monotonicity of $v$ we can conclude that

$$v(x) \leqslant Ce^{\beta x}, \tag{5.15}$$

for some constant $C$. Indeed, if (5.15) does not hold, then for a sequence $x_n \to -\infty$ we have $v(x_n) \geqslant ne^{\beta x_n}$. Extracting a subsequence if necessary, we can assume that $x_{n+1} < x_n - 1$, thus since $v$ is increasing we have

$$\int_{-\infty}^{x_0} v(x)e^{-\beta x}\,dx \geqslant \sum_{n\geqslant 1} \int_{x_n}^{x_{n-1}} ne^{\beta x_n} e^{-\beta x}\,dx$$
$$\geqslant \sum_{n\geqslant 1} n\frac{1-e^{-\beta(x_n - x_{n-1})}}{\beta}$$
$$\geqslant \sum_{n\geqslant 1} n\frac{1-e^{-\beta}}{\beta} = \infty$$

which is a contradiction. $\square$

In the next result we establish that the bounded solution $u$ of (5.1) also decays exponentially as $x \to -\infty$.

**Lemma 5.3.** *Suppose that $u$ is bounded solution of* (5.1). *If for some $M, \beta > 0$ we have that $v(x) \leqslant Me^{\beta x}$ for all $x$ then there exist $M_1, \alpha > 0$ such that*

$$u(x) \leqslant M_1 e^{\alpha x} \quad \textit{for all } x \in \mathbb{R}. \tag{5.16}$$

**Proof.** When $c \neq 0$ then by (5.4) we have the following estimates

$$
\begin{aligned}
|c|u(x) &= \left| J \star v - v + \int_{\infty}^{x} f\big(u(s)\big)\, ds \right| \\
&\leqslant J \star v + v + \int_{\infty}^{x} \frac{f(u(s))}{u(s)} u(s)\, ds \\
&\leqslant J \star v + (K+1)v,
\end{aligned}
$$

where $K$ is the Lipschitz constant of $f$. Now since

$$
J \star v(x) \leqslant C \int_{\mathbb{R}} J(x-y)e^{\beta y} \leqslant C' e^{\beta x},
$$

we easily see that (5.6) holds.

When $c = 0$ the estimate does not directly comes from (5.4) and we have to distinguish several cases.

Let first observe that for $x < 0$ since $u$ is bounded by some constant $C$, $J \star u$ satisfies the following

$$
\begin{aligned}
J \star u(x) &= \int_{\infty}^{\frac{\alpha}{\beta}x} J(x-y)u(y)\, dy + \int_{\frac{\alpha}{\beta}x}^{+\infty} J(x-y)u(y)\, dy \\
&\leqslant \|J\|_{\infty} \int_{-\infty}^{\frac{\alpha}{\beta}x} u(y)\, dy + C \int_{x(\frac{\alpha}{\beta}-1)}^{\infty} J(-z)\, dz \\
&\leqslant \|J\|_{\infty} v\left(\frac{\alpha}{\beta}x\right) + Ce^{(\beta-\alpha)x} \int_{x(\frac{\alpha}{\beta}-1)}^{\infty} J(-z)e^{\beta z}\, dz.
\end{aligned}
$$

Choosing $\alpha = \frac{\beta}{2}$ in the above equation, we end up with

$$
J \star u(x) \leqslant Ce^{\frac{\beta}{2}x}, \tag{5.17}
$$

for some constant $C$. Observe also that since $f$ is smooth and $f(0) = 0$, we have for small $\varepsilon > 0$ and $s > 0$ small,

$$
\left| \frac{f(s)}{s} - f'(0) \right| \leqslant \varepsilon.
$$

Therefore from (5.1), for $\varepsilon > 0$ small there exists $K(\varepsilon) > 0$ such that for $x < -K(\varepsilon)$ we have

$$
u\big(1 - f'(0) + \varepsilon\big) \geqslant J \star u = u\left(1 - \frac{f(u)}{u}\right) \geqslant u\big(1 - f'(0) - \varepsilon\big). \tag{5.18}
$$

Observe now that if $f'(0) > 1$, we get a contradiction. Indeed, choose $\varepsilon$ so that $(1 - f'(0) + \varepsilon) < 0$, then we have the following contradiction when $x < -K(\varepsilon)$

$$0 > u\bigl(1 - f'(0) + \varepsilon\bigr) \geqslant J \star u \geqslant 0.$$

Thus, when $f'(0) > 1$, there is no positive solution of (5.1) with zero speed.

Let us now look at the other cases. Assume now that $f'(0) < 1$ and choose $\varepsilon$ small so that $(1 - f'(0) - \varepsilon) > 0$ then from (5.18) for $x < -K(\varepsilon)$ there exists a positive constant $C$ so that

$$u \leqslant CJ \star u \leqslant Ce^{\frac{\beta}{2}x}.$$

Finally, when $f'(0) = 1$ recall that $f$ satisfies (1.7). Thus, for $x \ll -1$

$$J \star u(x) = u - f(u) \geqslant Au^m,$$

where $A > 0$, $m \geqslant 1$. Using (5.17), yields

$$u \leqslant \frac{C}{A} e^{\frac{\beta}{2m}x}. \qquad \square$$

**Remark 5.4.** From the above proof, we easily conclude that for any $0 < \alpha < \bar{\alpha}$, where $\bar{\alpha}$ depends only on $\beta$ and $\gamma$, there exists $M_1 > 0$ such that (5.16) holds.

As in [4], for $u$ a solution of (5.1) we define the function $U(\lambda) = \int_{\mathbb{R}} e^{-\lambda x} u(x)\, dx$ which by Lemma 5.3 is defined and analytic in the strip $0 < \operatorname{Re}\lambda < \alpha$. Note that

$$\int_{\mathbb{R}} J \star u(x) e^{-\lambda x} = \int_{\mathbb{R}} u(y) e^{-\lambda y}\, dy \int_{\mathbb{R}} J(-z) e^{\lambda z}\, dz$$

and using integration by parts

$$c \int_{\mathbb{R}} u' e^{-\lambda x}\, dx = \lambda c \int_{\mathbb{R}} u(y) e^{-\lambda y}\, dy.$$

Using the above identities, if we multiply (5.1) by $e^{-\lambda x}$ and integrate in $\mathbb{R}$ we obtain

$$U(\lambda)\bigl(-c\lambda + m(\lambda)\bigr) = \int_{\mathbb{R}} e^{-\lambda x}\bigl(f'(0)u(x) - f\bigl(u(x)\bigr)\bigr)\, dx, \tag{5.19}$$

where the function $m(\lambda) = \int_{\mathbb{R}} J(-x) e^{-\lambda x} dx + f'(0) - 1$ is analytic in $\mathbb{C}$.

Let $c^1$ be the following quantity

$$c^1 := \min_{\lambda > 0} \frac{1}{\lambda}\biggl(\int_{\mathbb{R}} J(-x) e^{\lambda x}\, dx + f'(0) - 1\biggr).$$

**Proposition 5.5.** *If $c < c^1$ then* (5.1) *does not have any solution.*

**Proof.** Since $u > 0$ we deduce, from a property of Laplace transform [27, Theorem 5b, p. 58] and Lemma 5.3, that the function $U(\lambda)$ is analytic in $0 < \operatorname{Re}\lambda < B$, where $B \geqslant \alpha$, and $U(\lambda)$ has a singularity at $\lambda = B$. Observe that if $c < c^1$ then for some $\delta > 0$

$$-c\lambda + m(\lambda) > \delta\lambda, \quad \text{for all } \lambda > 0. \tag{5.20}$$

Observe that since $f \in C^{1,\gamma}$ near 0 and using Lemma 5.3 we have that for some constant $C > 0$

$$\begin{aligned}\int_{\mathbb{R}} e^{-\lambda x}\big|f'(0)u(x) - f\big(u(x)\big)\big|\,dx &= \int_{\infty}^{-K} e^{-\lambda x}\big|f'(0)u(x) - f\big(u(x)\big)\big|\,dx \\ &\quad + \int_{-K}^{+\infty} e^{-\lambda x}\big|f'(0)u(x) - f\big(u(x)\big)\big|\,dx \\ &\leqslant \int_{\infty}^{-K} e^{-\lambda x}\big|Au^{1+\gamma} + o\big(u^{1+\gamma}\big)\big|\,dx + C\int_{-K}^{+\infty} e^{-\lambda x}u(x)\,dx \\ &\leqslant C\int_{\mathbb{R}} e^{-\lambda x}u^{1+\gamma}(x)\,dx \\ &\leqslant C\int_{\mathbb{R}} e^{(-\lambda+\gamma\alpha)x}u(x)\,dx.\end{aligned}$$

From the above computation, it follows that $\int_{\mathbb{R}} e^{-\lambda x}|f'(0)u(x) - f(u(x))|\,dx$ is analytic in the region $0 < \operatorname{Re}\lambda < B + \gamma\alpha$. Since $\gamma > 0$, using Eq. (5.19), we get $U(\lambda)$ defined and analytic for $0 < \operatorname{Re}\lambda < B + \gamma\alpha$. Bootstrapping this argumentation we can extend analytically $U(\lambda)$ to $\operatorname{Re}\lambda > 0$. Then for all $\lambda > 0$

$$\int_{\mathbb{R}} e^{-\lambda x}\big|f'(0)u(x) - f\big(u(x)\big)\big|\,dx \leqslant \big(f'(0) + k\big)\int_{\mathbb{R}} e^{-\lambda x}u(x) = CU(\lambda).$$

Therefore for all $\lambda > 0$, using (5.19), it follows that $-c\lambda + m(\lambda) \leqslant C$ contradicting (5.20). □

**Remark 5.6.** We should point out that the above proposition holds as well if the kernel $J$ instead of being compactly supported, is only assumed to satisfy:

$$\exists M, \lambda_0 > 0 \quad \text{such that} \quad \int_0^{+\infty} J(-x)e^{\lambda_0 x} \leqslant M.$$

Let us now establish the exact asymptotic behavior, as $x \to -\infty$, of a solution $u$ of (5.1). We proceed as follows. First, we obtain the exact behavior of $v = \int_{-\infty}^{x} u(s)\,ds$, proceeding as in [4] and then we conclude the behavior of $u$.

For $c \geqslant c^1$ we denote $\lambda(c)$ the unique minimal $\lambda > 0$ such that $-c\lambda + m(\lambda) = 0$. It can be easily verified that $\lambda(c)$ is a simple root of $-c\lambda + m(\lambda)$ if $c > c^1$, and it is a double root when $c = c^1$.

**Proof of Theorem 1.6.** Since there is a monotone solution $(u, c^*)$ of (1.3)–(1.5) with critical speed, it is a bounded solution of (5.1). Thus by Proposition 5.5, $c^* \geqslant c^1$.

It remains to prove (1.8) and (1.9). The proof follows from a modified version of Ikehara's Theorem (see [27]). We define $F(\lambda) = \int_{-\infty}^{0} v(y)e^{-\lambda y}$. Since $v$ is monotone, we can obtain the appropriate asymptotic behavior of $v$ if $F$ has the representation

$$F(\lambda) = \frac{H(\lambda)}{(\lambda - \alpha)^{k+1}}, \tag{5.21}$$

with $H$ analytic in the strip $0 < \operatorname{Re} \lambda \leqslant \alpha$, and $k = 0$ when $c > c^*$, $k = 1$ when $c = c^*$.

Using (5.4), we have that

$$\int_{-\infty}^{0} v(x)e^{-\lambda x}\,dx = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{x} f(u(s)) - f'(0)u(s)\,ds\,e^{-\lambda x}\,dx}{c\lambda - m(\lambda)} - \int_{0}^{\infty} v(x)e^{-\lambda x},$$

thus, using that either $c \neq 0$ or $f'(0) < 1$ holds, we have that by Lemma 5.3, (5.21) holds replacing $u$ by $v$ with $\alpha = \lambda(c)$ described above, since it can be checked that $-c\lambda + m(\lambda)$ has only two real roots which are simple when $c > c^1$ and double when $c = c^1$.

It remains to conclude that (5.21) holds for $u$. First suppose that $c = c^1$ and denote $\lambda = \lambda(c^1)$. If $c \neq 0$ then using (5.4) we have that

$$cu = J \star v(x) - \bigl(1 - f'(0)\bigr)v(x) + \int_{-\infty}^{x} f\bigl(u(s)\bigr) - f'(0)u(s)\,ds.$$

By Remark 5.4 and since $f$ is $C^{1,\gamma}$ near 0 we have that

$$\frac{\int_{-\infty}^{x} f(u(s)) - f'(0)u(s)\,ds}{|x|e^{-\lambda(c^1)x}} \to 0, \tag{5.22}$$

as $|x| \to -\infty$. Therefore, we just have to prove that

$$\lim_{x\to-\infty} \frac{J \star v(x) - (1 - f'(0))v(x)}{|x|e^{\lambda(c^1)x}} = L \neq 0. \tag{5.23}$$

Observe that since $v$ satisfies (1.8) we have that for $\eta = \lim_{x\to-\infty} \frac{v(x)}{|x|e^{\lambda(c^1)x}}$ and $\operatorname{supp} J \subset [-k, k]$

$$\frac{J \star v}{|x|e^{\lambda(c^1)x}} = \frac{1}{|x|}\int_{-k}^{k} J(-z)\bigl(\eta + O(1/x)\bigr)e^{\lambda(c^1)z}\bigl(|x| + z\bigr)\,dz,$$

therefore

$$\frac{J \star v(x) - (1 - f'(0))v(x)}{|x|e^{\lambda(c^1)x}} \to \eta m\big(\lambda\big(c^1\big)\big) = \eta c^1 \lambda\big(c^1\big) \neq 0,$$

which gives the desired result.

When $c^1 = 0$, we proceed in a slightly different way. Observe that in this case $f'(0) < 1$,

$$\big(1 - f'(0)\big)u = J \star u + f(u) - f'(0)u, \tag{5.24}$$

and by Remark 5.4 and since $f \in C^{1,\gamma}$ near 0 we have that (5.22) holds. Also, by (j2) we have that $J \star u = J' \star v$ and

$$\begin{aligned}\frac{J' \star v}{|x|e^{\lambda(c^1)x}} &= \frac{1}{|x|}\int_{-k}^{k} J'(-z)\big(\eta + O(1/x)\big)e^{\lambda(c^1)z}\big(|x| + z\big)\,dz \\ &= \eta \int_{\mathbb{R}} J(-z)e^{\lambda(c^1)z}\,dz + O(1/x),\end{aligned}$$

with $\eta > 0$ as above. Hence, we obtain the desired result.

Finally, the case $c > c^1$ is analogous. □

**Proof of Corollary 1.7.** Observe now that in the case of a KPP nonlinearity $f$, the function $w := e^{\lambda x}$ is a super-solution of (1.3)–(1.5), provided that $\lambda > 0$ is chosen such that $-c\lambda + m(\lambda) = 0$. The existence of such $\lambda > 0$ is guaranteed since $c \geqslant c^1$. The existence of a monotone travelling wave for any $c \geqslant c^1$ is then provided by Theorem 1.3. Therefore $c^* \leqslant c^1$ and we conclude $c^* = c^1$. □

## 6. Uniqueness of the profile

In this section we deal with the uniqueness up to translation of solution of (1.3)–(1.5). Our proof follows ideas of [7] and is mainly based on the sliding methods introduced by Berestycki and Nirenberg [2,3] (see also [7]).

In the sequel, given a function $u : \mathbb{R} \to \mathbb{R}$ and $\tau \in \mathbb{R}$ we define its translation by $\tau$ as

$$u_\tau(x) = u(x + \tau) \tag{6.1}$$

and sometimes we shall write $u^\tau(x) = u(\tau + x)$.

Let $L$ denote the operator

$$Lu = J \star u - u - cu'.$$

**Proposition 6.1** *(Nonlinear Comparison Principle). Let $J$ satisfy* (j1), (1.12) *and let $f$ be a monostable nonlinearity so that $f'(1) < 0$. Let $u$ and $v$ be two continuous functions in $\mathbb{R}$ such that*

$$Lu + f(u) \leqslant 0 \quad on\ \mathbb{R}, \tag{6.2}$$

$$Lv + f(v) \geqslant 0 \quad on\ \mathbb{R}, \tag{6.3}$$

$$\lim_{x\to-\infty} u(x) \geqslant 0, \qquad \lim_{x\to-\infty} v(x) \leqslant 0, \tag{6.4}$$

$$\lim_{x\to+\infty} u(x) \geqslant 1, \qquad \lim_{x\to+\infty} v(x) \leqslant 1. \tag{6.5}$$

*Assume further that either $u$ or $v$ is monotone and that $u \geqslant v$ in some interval $(-\infty, K)$. Then there exists $\tau \in \mathbb{R}$ such that $u_\tau \geqslant v$ in $\mathbb{R}$. Moreover, either $u_\tau > v$ in $\mathbb{R}$ or $u_\tau \equiv v$.*

**Remark 6.2.** Observe that by the maximum principle and since $f(s) \geqslant 0$ $\forall s \leqslant 0$, the super-solution $u$ is necessarily positive. Similarly, since $f(s) \leqslant 0$ $\forall s \geqslant 1$, the maximum principle implies that $v < 1$.

**Proof of Proposition 6.1.** Note that if $\inf_{\mathbb{R}} u \geqslant \sup_{\mathbb{R}} v$, the theorem trivially holds. In the sequel, we assume that $\inf_{\mathbb{R}} u < \sup_{\mathbb{R}} v$.

Let $\varepsilon > 0$ be such that

$$f'(p) \leqslant 0 \quad \text{for } 1 - \varepsilon < p < 1. \tag{6.6}$$

Now fix $0 < \delta \leqslant \frac{\varepsilon}{2}$ and choose $M > 0$ sufficiently large so that

$$1 - u(x) < \frac{\delta}{2} \quad \forall x > M, \tag{6.7}$$

$$v(x) < \frac{\delta}{2} \quad \forall x < -M, \quad \text{and} \tag{6.8}$$

$$v(x) \leqslant u(x) \quad \forall x < -M. \tag{6.9}$$

*Step 1.* There exists a constant $D$ such that for every $b \geqslant D$

$$u(x+b) > v(x) \quad \forall x \in [-M-1-b, M+1]. \tag{6.10}$$

Indeed, since $u > 0$ in $\mathbb{R}$ and $\lim_{x\to+\infty} u(x) \geqslant 1$ we have

$$c_0 := \inf_{[-M-1,\infty)} u > 0.$$

Since $\lim_{x\to-\infty} v(x) \leqslant 0$ there is $L > 0$ large such that

$$v(x) < c_0 \quad \forall x \leqslant -L.$$

Then for all $b > 0$

$$u(x+b) > v(x) \quad \forall x \in [M-1-b, -L].$$

Now, since $\sup_{[-L,M+1]} v < 1$ and $\lim_{x\to+\infty} u(x) \geqslant 1$ we deduce (6.10).

*Step 2.* There exists $b \geqslant D$ such that

$$u(x+b)+\frac{\delta}{2} > v(x) \quad \forall x \in \mathbb{R}. \tag{6.11}$$

If not then we have

$$\forall b \geqslant D \quad \text{there exists } x(b) \quad \text{such that} \quad u(x(b)+b)+\frac{\delta}{2} \leqslant v\big(x(b)\big). \tag{6.12}$$

Since $u$ is nonnegative and $v$ satisfies (6.4) there exists a positive constant $A$ such that

$$u(x+b)+\frac{\delta}{2} > v(x) \quad \text{for all } b>0 \text{ and } x \leqslant -A. \tag{6.13}$$

Take now a sequence $(b_n)_{n\in\mathbb{N}}$ which tends to $+\infty$. Let $x(b_n)$ be the point defined by (6.12). Thus we have for that sequence

$$u\big(x(b_n)+b_n\big)+\frac{\delta}{2} \leqslant v\big(x(b_n)\big). \tag{6.14}$$

According to (6.13) we have $x(b_n) \geqslant -A$. Therefore the sequence $x(b_n)+b_n$ converges to $+\infty$. Pass to the limit in (6.14) to get

$$1+\frac{\delta}{2} \leqslant \lim_{n\to+\infty} u\big(x(b_n)+b_n\big)+\frac{\delta}{2} \leqslant \limsup_{n\to+\infty} v\big(x(b_n)\big) \leqslant 1,$$

which is a contradiction. This proves our claim (6.11).

*Step 3.* We observe that as a consequence of (6.10) and (6.11), and using that either $u$ or $v$ is monotone we in fact have

$$\begin{gathered} u(x+b) \geqslant v(x) \quad \forall x \leqslant M+1, \\ u(x+b)+\frac{\delta}{2} > v(x) \quad \forall x \geqslant M+1. \end{gathered} \tag{6.15}$$

Indeed, it only remains to verify that $u(x+b) > v(x)$ for $x \leqslant M-1-b$. If $u$ is monotone from (6.9) we have $u(x+b) > u(x) > v(x)$ for $x < -M$. If $v$ is monotone $u(x) > v(x) > v(x-b)$ for $x < -M$.

*Step 4.* Now we claim that

$$u(x+b) \geqslant v(x) \quad \forall x \in \mathbb{R}. \tag{6.16}$$

To prove this, consider

$$a^* = \inf\big\{a>0 \;\big|\; u(x+b)+a \geqslant v(x)\ \forall x \in \mathbb{R}\big\} \tag{6.17}$$

which is well defined by (6.11).

If $a^*=0$ then (6.16) follows. Suppose $a^*>0$. Then, since

$$\lim_{x\to\pm\infty} u(x+b)+a^*-v(x)\geqslant a^*>0,$$

there exists $x_0\in\mathbb{R}$ such that $u(x_0+b)+a^*=v(x_0)$.

Let $w(x):=u(x+b)+a^*-v(x)$ and note that

$$0=w(x_0)=\min_{\mathbb{R}} w(x). \tag{6.18}$$

Observe that $w$ also satisfies the following equations:

$$Lw\leqslant f\bigl(v(x)\bigr)-f\bigl(u(x+b)\bigr), \tag{6.19}$$

$$w(+\infty)\geqslant a^*, \tag{6.20}$$

$$w(-\infty)\geqslant a^*. \tag{6.21}$$

Since $w\geqslant 0$, $w\not\equiv 0$, using the strong maximum principle for some global minimum $x_0$ of $w$ we have

$$Lw(x_0)>0. \tag{6.22}$$

By (6.15) we necessarily have $x_0>M+1$.

At $x_0$ we have

$$f\bigl(u(x_0+b)+a^*\bigr)-f\bigl(u(x_0+b)\bigr)\leqslant 0, \tag{6.23}$$

since $f$ is nonincreasing for $s\geqslant 1-\varepsilon$, $a^*>0$ and $1-\varepsilon<1-\frac{\delta}{2}\leqslant u$ for $x>M$. Combining (6.19), (6.22) and (6.23) yields the contradiction

$$0<Lw(x_0)\leqslant f\bigl(u(x_0+b)+a^*\bigr)-f\bigl(u(x_0+b)\bigr)\leqslant 0.$$

*Step 5.* Finally it remains to prove that either $u_\tau>v$ or $u_\tau\equiv v$. Let $w:=u_\tau-v$, then either $w>0$ or $w(x_0)=0$ at some point $x_0\in\mathbb{R}$. In the latter case we have $w(x)\geqslant w(x_0)=0$ and

$$0\leqslant Lw(x_0)\leqslant f\bigl(v(x_0)\bigr)-f\bigl(u(x_0+\tau)\bigr)=f\bigl(v(x_0)\bigr)-f\bigl(v(x_0)\bigr)=0. \tag{6.24}$$

Then using the maximum principle, we obtain $w\equiv 0$, which means $u_\tau\equiv v$. □

**Proposition 6.3.** *Let $J$ satisfy* (j1)*,* (1.12) *and let $f$ be a monostable nonlinearity so that $f'(1)<0$. Let $u_1$ and $u_2$ be respectively super- and sub-solutions of* (1.3)–(1.5) *which are continuous. If $u_1\geqslant u_2$ in some interval $(-\infty,K)$ and either $u_1$ or $u_2$ is monotone then $u_1\geqslant u_2$ everywhere. Moreover either $u_1>u_2$ or $u_1\equiv u_2$.*

**Proof.** Assume first that $\inf_{\mathbb{R}} u_1<\sup_{\mathbb{R}} u_2$. Otherwise there is nothing to prove. Without losing generality we can assume that $u_1$ is monotone. Using Proposition 6.1, $u_1^\tau\geqslant u_2$ for some $\tau\in\mathbb{R}$, so the following quantity is well defined

$$\tau^*:=\inf\bigl\{\tau\in\mathbb{R}\bigm| u_1^\tau\geqslant u_2\bigr\}.$$

We claim that

$$\tau^* \leqslant 0. \tag{6.25}$$

Observe that by showing that $\tau^* \leqslant 0$, we end the proof. To prove (6.25) we argue by contradiction. Assume that $\tau^* > 0$, then since $u_i$ are continuous functions, we will have $u_1^{\tau^*} \geqslant u_2$ in $\mathbb{R}$. Let $w := u_1^{\tau^*} - u_2 \geqslant 0$. Since $\tau^* > 0$ and $u_1$ is monotone then $w > 0$ in $(-\infty, K)$. Now observe that $w > 0$ in $\mathbb{R}$ or $w(x_0) = 0$ for some point $x_0$ in $\mathbb{R}$. In the latter case

$$0 \leqslant (J \star w - w)(x_0) \leqslant f\big(u_2(x_0)\big) - f\big(u_1^{\tau^*}(x_0)\big) = 0.$$

Thus, using the maximum principle, $w \equiv 0$, which contradicts that $w > 0$ in $(-\infty, K)$. Now since $u_1$ is monotone and $\tau^* > 0$ for small $\varepsilon > 0$, we have $u_1^{\tau^* - \varepsilon} > u_2$ in $(-\infty, M)$. Arguing as in Step 4 of the proof of Proposition 6.1 we deduce $u^{\tau^* - \varepsilon} > u_2$ in $\mathbb{R}$ which contradicts the definition of $\tau^*$. □

**Remark 6.4.** With minor modifications the proofs of Propositions 6.1 and 6.3 hold if only one of the functions $u_1$ or $u_2$ is continuous. For the proof of this statement we need the strong maximum principle for solutions in $L^\infty$, which can be found in [10].

**Theorem 6.5.** *Assume $J$ satisfies* (j1), (1.12) *and let $c \in L^\infty(\mathbb{R})$. If $u \in L^\infty(\mathbb{R})$ satisfies $u \leqslant 0$ a.e. and $J \star u - u + c(x)u \geqslant 0$ a.e. in $\mathbb{R}$, then* $\operatorname{ess\,sup}_K u < 0$ *for all compact $K \subset \mathbb{R}$ or $u = 0$ a.e. in $\mathbb{R}$.*

**Proof of Theorem 1.8.** The case of $c \neq c^1$ and $c = c^1$ being similar, we present only the case $c \neq c^1$. Let $u_1$ and $u_2$ be two solutions of (1.3)–(1.5) with the same speed $c \neq 0$. Since $c \neq 0$ the functions $u_i$ are uniformly continuous. From Theorem 1.3, we can assume that $u_1$ is a monotonic function. Since, $u_i$ solve the same equation and $u_1$ is monotone, using the translation invariance of the equation and (1.9) we see that up to a translation

$$u_1 = e^{\lambda(c)x} + o\big(e^{\lambda(c)x}\big), \quad \text{as } x \to -\infty, \tag{6.26}$$

$$u_2 = e^{\lambda(c)x} + o\big(e^{\lambda(c)x}\big), \quad \text{as } x \to -\infty. \tag{6.27}$$

Let us first recall the following notation, $u^\tau(.) := u(. + \tau)$. Then, by monotonicity of $u_1$ and (6.26)–(6.27) for some positive $\tau$ we have $u_1^\tau \geqslant u_2$ in some interval $(-\infty, -K)$. Using Proposition 6.3, it follows that $u_1^\tau \geqslant u_2$ for possibly a new $\tau$. Define now the following quantity:

$$\tau^* := \inf\big\{\tau > 0 \,\big|\, u_1^\tau \geqslant u_2\big\}.$$

Observe that form the above argument $\tau^*$ is well defined. We claim

**Claim.** $\tau^* = 0$.

Observe that proving the claim ends the proof of the uniqueness up to translation of the solution. Indeed, assume for a moment that the claim is proved then we end up with $u_1 \geqslant u_2$. Observe now that in the above argumentation the role of $u_1$ and $u_2$ can be interchanged, so we easily see that we have $u_1 \leqslant u_2 \leqslant u_1$ which ends the proof of the uniqueness. □

Let us now prove the Claim.

**Proof of the Claim.** If not, then $\tau^* > 0$. Let $w := u_1^{\tau^*} - u_2 \geqslant 0$. Then either there exists a point $x_0$ where $w(x_0) = 0$ or $w > 0$. In the first case, at $x_0$, $w$ satisfies:

$$0 \leqslant J \star w(x_0) - w(x_0) = f\big(u_2(x_0)\big) - f\big(u_1^{\tau^*}(x_0)\big) = 0.$$

Using the strong maximum principle, it follows that $w \equiv 0$. Thus $u_1^{\tau^*} \equiv u_2$, which contradicts (6.26)–(6.27). Therefore, $u_1^{\tau^*} > u_2$. Using (6.26), since $\tau^* > 0$ we have for $u_1^{\tau^*}$ the following behavior near $-\infty$:

$$u_1^{\tau^*} := e^{\tau^*} e^{\lambda(c)x} + o\big(e^{\lambda(c)x}\big).$$

Therefore, for some $\varepsilon > 0$ small, we still have $u_1^{\tau^*-\varepsilon} \geqslant u_2$ in some neighborhood $(-\infty, -K)$ of $-\infty$. Using Proposition 6.3, we end up with $u_1^{\tau^*-\varepsilon} \geqslant u_2$ everywhere, contradicting the definition of $\tau^*$. Hence, $\tau^* = 0$. $\square$

Regarding Theorem 1.9 we need the following result:

**Lemma 6.6.** *Assume that $J$ and $f$ satisfy* (j1), (j2), (1.12) *and* (f1), (f2), *respectively. Let* $0 \leqslant u \leqslant 1$ *be a solution to* (1.3).

(a) *Then*

$$\lim_{x \to -\infty} u(x) = 0 \quad or \quad \lim_{x \to -\infty} u(x) = 1,$$

*and*

$$\lim_{x \to \infty} u(x) = 0 \quad or \quad \lim_{x \to \infty} u(x) = 1.$$

(b) *If* $u(-\infty) = 1$ *and* $u(+\infty) = 1$ *then* $u \equiv 1$.

Note that in this lemma we do not assume that $u$ is continuous.

**Proof.** (a) Let $0 \leqslant u \leqslant 1$ be a solution to (1.13). We first note that by (5.3) any bounded solution $u$ of (1.3) satisfies

$$\int_{-\infty}^{\infty} f(u)\,du < \infty. \tag{6.28}$$

Let $g(u) = u - f(u)$ and note that

$$J \star u = g(u) \quad \text{in } \mathbb{R}, \tag{6.29}$$

and that the hypotheses on $f$ imply $g'(u) \geqslant g'(0)$ and $g(u) \leqslant u$ for $u \in [0, 1]$.

If $f'(0) < 1$ then $g'(0) > 0$ and then $g$ is strictly increasing. This together with (6.29) implies that $u$ is uniformly continuous and using (6.28) we see that $u(-\infty) = 0$ or $u(-\infty) = 1$ and the same at $+\infty$ which is the desired conclusion. Therefore in the sequel we assume $f'(0) \geqslant 1$, that is, $g'(0) \leqslant 0$.

Since both limits at $-\infty$ and $+\infty$ are analogous we concentrate on the case $x \to -\infty$.

We will establish the conclusion of part (a) by proving

$$\liminf_{x\to-\infty} J \star u(x) = 0 \quad \Longrightarrow \quad \lim_{x\to-\infty} u(x) = 0, \tag{6.30}$$

and

$$\liminf_{x\to-\infty} J \star u(x) > 0 \quad \Longrightarrow \quad \lim_{x\to-\infty} u(x) = 1. \tag{6.31}$$

We start with (6.30). Suppose that $f'(0) > 1$. Then there is $\delta > 0$ such that $g(u) < 0$ for $u \in (0, \delta)$ and from (6.29) we deduce that $u(x) \geqslant \delta$ for all $x$, so regarding (6.30) there is nothing to prove.

Suppose $f'(0) = 1$. Then $g$ is nondecreasing and by (1.7) we have, for some $A > 0$, $m \geqslant 1$, $\delta_1 > 0$

$$g(u) \geqslant Au^m \quad \forall 0 \leqslant u \leqslant \delta_1. \tag{6.32}$$

Assume that $\liminf_{x\to-\infty} J \star u(x) = 0$ and let us show first that

$$\lim_{x\to-\infty} J \star u(x) = 0. \tag{6.33}$$

Otherwise, set $\bar{l} = \limsup_{x\to-\infty} J \star u(x) > 0$. Choose $l \in (0, \bar{l})$ such that $g'(l) > 0$ and then pick a sequence $x_n \to -\infty$ such that $J \star u(x_n) = g(l)$ for all $n$. Then there is some $\sigma > 0$ such that for $x \in (x_n - \sigma, x_n + \sigma)$ we have $f(u(x)) \geqslant c > 0$ for some uniform $c$. This contradicts (6.28) and we deduce (6.33). This combined with (6.32) implies that $\lim_{x\to-\infty} u(x) = 0$, and this establishes (6.30).

We prove now (6.31). Let us assume

$$\underline{l} := \liminf_{x\to-\infty} J \star u(x) > 0.$$

Since $J \star u = g(u) \leqslant u$ it is enough to show that

$$\lim_{x\to-\infty} J \star u(x) = 1. \tag{6.34}$$

Assume the contrary, that is,

$$0 < \underline{l} < 1. \tag{6.35}$$

Observe that

$$\liminf_{x\to-\infty} u(x) > 0.$$

This is direct if $f'(0) > 1$ and follows from (6.29), (6.32) and $\underline{l} > 0$ if $f'(0) = 1$. Therefore $\limsup_{x\to-\infty} u(x) = 1$, otherwise (6.28) cannot hold. Hence

$$\limsup_{x\to-\infty} J \star u(x) = 1. \tag{6.36}$$

Chose now $\alpha \in (\underline{l}, 1)$ a regular value of the function $g$. By (6.35), (6.36) and the continuity of $J \star u$ there exists a sequence $x_n \to -\infty$ such that $J \star u(x_n) = \alpha$. Note that the set $\{u \in [0,1]/g(u) = \alpha\}$ is discrete and hence finite and does not contain 0 nor 1. Hence, for sufficiently small $\varepsilon > 0$ we have $\{u \in [0,1]/\alpha - \varepsilon < g(u) < \alpha + \varepsilon\} \subseteq [\varepsilon, 1-\varepsilon]$. Since $J \star u$ is uniformly continuous there is $\sigma > 0$ such that for $x \in (x_n - \sigma, x_n + \sigma)$ we have $\varepsilon \leqslant u(x_n) \leqslant 1 - \varepsilon$. This contradicts the integrability condition (6.28), and we deduce the validity of (6.34).

(b) Assume that $\lim_{x\to\infty} u(x) = \lim_{x\to-\infty} u(x) = 1$ and set $\gamma^* = \sup\{0 < \gamma < 1/u > \gamma\}$. For the sake of contradiction assume that $u$ is nonconstant. Then $0 < \gamma^* < 1$. Since $f(\gamma^*) > 0$ we have that $v = u - \gamma^* \geqslant 0$ satisfies

$$J \star v - v - cv' + \frac{f(u) - f(\gamma^*)}{u - \gamma^*}(u - \gamma^*) < 0. \tag{6.37}$$

If $c \neq 0$ then $v$ reaches its global minimum at some $x_0 \in \mathbb{R}$ which satisfies $v(x_0) = 0$. Thus, evaluating (6.37) at $x_0$ we obtain a contradiction. If $c = 0$ we reach again a contradiction applying Theorem 6.5. □

**Proof of Theorem 1.9.** Assume $0 \leqslant u \leqslant 1$ is a solution of (1.13) such that $u \not\equiv 0$ and $u \not\equiv 1$. By Lemma 6.6, $u(-\infty) = 0$ or $u(+\infty) = 0$. Then we may apply Theorem 1.6 and deduce the exact asymptotic behavior of $u$ at either $-\infty$ or $+\infty$ and that $c^* \leqslant 0$ or $c_* \leqslant 0$. Let $u_0$ denote a nondecreasing travelling wave with speed $c = 0$ if $c^* \leqslant 0$ or a nonincreasing one if $c_* \leqslant 0$. Then, by slightly modifying the proof of Theorem 2.1 in [4] we deduce that for a suitable translation we have $u^\tau \equiv u_0$. In particular the profile of the travelling wave $u_0$ is unique. □

Next we address the issues of nonuniqueness and discontinuities of solutions when $c = 0$. We consider $f$ such that

$$f \text{ is smooth}, \quad 0 < f'(0) < 1, \quad f'(1) < 0 \quad \text{and} \quad f \text{ is KPP}. \tag{6.38}$$

We are interested in the case where $u - f(u)$ is not monotone, and for simplicity we shall assume that setting

$$g(u) = u - f(u)$$

there exist $0 < \alpha < \beta < 1$ such that

$$\begin{aligned} g'(u) > 0 \quad \forall u \in [0,\alpha) \cup (\beta, 1], \\ g'(u) < 0 \quad \forall u \in (\alpha, \beta). \end{aligned} \tag{6.39}$$

**Proposition 6.7.** *Assume $f$ satisfies* (6.38), (6.39). *Then there exists $J$ such that no solution of* (1.3)–(1.5) *is continuous, and this problem admits infinitely many solutions.*

**Proof.** Let us choose $J \in C^1$, with compact support and satisfying (j1) and (1.12), and such that $c^1 \leqslant 0$. Then by Corollary 1.7 we have $c^* = c^1 \leqslant 0$. Thus there exists a monotone travelling wave solution $u_1$ of (1.3)–(1.5) with speed $c = 0$. If (1.3)–(1.5) has a continuous solution $u_2$, then by Theorem 1.8 and Remark 6.4 we have $u_1 \equiv u_2$. Hence $u_1$ is monotone and continuous. Then $J \star u_1$ is monotone which implies that $u_1 - f(u_1)$ is monotone in $\mathbb{R}$. This is impossible if $u_1$ is continuous and $u - f(u)$ is not monotone.

For the construction of infinitely many solutions we follow closely the work of [1]. Since $g'(0) > 0$ and $g'(1) > 0$ there are $a < b$ such that

$$g \text{ is increasing in } [0, a], \quad g \text{ is increasing in } [b, 1],$$

$$g(a) = g(b) \quad \text{and} \quad g \text{ is not monotone in } [a, b].$$

Define

$$\tilde{g}(u) = \begin{cases} g(u) & \text{if } u \in [0, a] \text{ or } u \in [b, 1], \\ g(a) & \text{if } u \in [a, b]. \end{cases}$$

Let $g_n : [0, 1] \to \mathbb{R}$ be smooth such that $g_n \to g$ uniformly in $[0, 1]$, $g_n \equiv g$ in a neighborhood of 0 and 1, $g_n' > 0$ and $u - g_n(u)$ is KPP. Then by Corollary 1.7 the problem (1.3)–(1.5) with nonlinearity $f_n = u - g_n(u)$ has critical speed $c^* \leqslant 0$ independent of $n$, and hence there exists a monotone solution $u_n$

$$J \star u_n = g_n(u_n), \qquad u_n(-\infty) = 0, \qquad u_n(+\infty) = 1.$$

Notice that any solution to this problem is continuous and hence we may choose

$$u_n(0) = a.$$

By Helly's theorem there is a subsequence which converges pointwise to a solution $u$ of the following problem

$$J \star u = \tilde{g}(u) \quad \text{in } \mathbb{R}.$$

Remark that $u(0) = a$, and $u(-\infty) = 0$, $u(+\infty) = 1$ by Lemma 2.4. Note that $u$ is continuous in $(-\infty, 0]$ since $u \leqslant a$ in $(-\infty, 0]$ and $g$ is strictly increasing in $[0, a]$.

We will show that $u$ has a discontinuity at 0 and $u(0^+) = b$. As in [1], choose $\delta_n > 0$ such that $u_n(\delta_n) = b$. Let $\delta = \liminf \delta_n$ and note that $u \geqslant b$ in $(\delta, \infty)$. Let us show that $\delta = 0$. If not, then $\tilde{g}(u(x)) = g(a)$ for $x \in (0, \delta)$ and this implies $J \star u = \text{const}$ in $(0, \delta)$. Then for $0 < \tau < \delta/2$ we have $J \star (u - u(\cdot - \tau)) \geqslant 0$ and vanishes in a nonempty interval. By the maximum principle $u \equiv u(\cdot - \tau)$ and this implies that $u$ is constant, which is a contradiction. Thus $\delta = 0$ and $u$ has a jump discontinuity at 0. Hence $u$ is a solution to (1.3)–(1.5). We conclude that $u(0^+) = b$ because $J \star u$ is continuous. $\square$

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