

JOURNAL OF COMBINATORIAL THEORY, Series A **40**, 305–330 (1985)

Regular Incidence-Polytopes with Euclidean or Toroidal Faces and Vertex-Figures

EGON SCHULTE

*Mathematisches Institut, Universität Dortmund,
Postfach 500500 D-46 Dortmund, West Germany*

Communicated by the Managing Editors

Received August 19, 1983

Regular incidence-polytopes are combinatorial generalizations of regular polyhedra. Certain group-theoretical constructions lead to many new regular incidence-polytopes whose faces and vertex-figures are combinatorially isomorphic to classical Euclideanly regular polytopes or regular maps on the torus. © 1985

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1. INTRODUCTION

Regular polytopes have been studied since antiquity, and with the passage of time there have been many changes in point of view about them. In the 20th century the most powerful contributions to a systematic investigation of regular configurations are due to Coxeter; for a detailed survey see, for example, Coxeter [3–6], Coxeter and Moser [9], and Fejes Tóth [16]. In the past few years the notion of a regular polytope was extended in several directions.

For instance, Grünbaum's new regular polyhedra provide a new class of metrically regular configurations in the Euclidean space including also the classical regular polytopes (that is, the Platonic solids), the Kepler–Poinsot polyhedra, and the Petrie–Coxeter polyhedra (cf. Grünbaum [17]). Recently, Dress has made considerable progress in the classification of these objects (cf. [15]).

The study of combinatorial properties of regular polytopes has also led to other fruitful directions of exploration. Danzer's concept of a regular incidence-complex generalizes the notion of a regular polytope in a combinatorial and group-theoretical sense (cf. [13]). The concept is closely related to Grünbaum's notion of a regular polystroma (cf. [18]), but is

more restrictive and rules out many disconnected structures. In particular, each d -dimensional regular incidence-complex is a regular $(d-1)$ -polystroma, but not vice versa. The concept includes all classical regular d -polytopes and all regular complex polytopes as well as many geometries and well-known configurations (cf. Coxeter [6], Shephard [25]).

At the same time the concept allows interactions with the work of Buekenhout on finite simple groups (cf. [1]) growing out of the work of Tits on buildings (cf. [27, 28]), and also with the work of Dress on regular polytopes and equivariant tessellations (cf. [14]). For further references see [13].

One important problem in the theory of regular polytopes is the construction of d -dimensional polytopes with preassigned facets and vertex-figures. The present paper deals with the analogous problem for an interesting class of regular incidence-complexes, namely for incidence-polytopes of Euclidean or toroidal type (cf. Section 2).

Given two regular incidence-polytopes \mathcal{P}_1 and \mathcal{P}_2 isomorphic to a classical regular polytope or a regular map on the torus, we ask whether \mathcal{P}_1 and \mathcal{P}_2 fit together as the facet (or face) and vertex-figure (or co-face) of a higher-dimensional regular incidence-polytope \mathcal{P} , respectively. It turns out that the incidence-polytope (even a finite and non-degenerate one) exists for many choices of \mathcal{P}_1 and \mathcal{P}_2 . The construction is in group-theoretical terms, applying more general methods for constructing regular incidence-complexes to the special situation. These methods also allow one to attack several problems closely related to the above-mentioned problem, for instance, if the torus is replaced by other closed real 2-manifolds.

Analogous problems for regular polystromas were considered by Grünbaum in [18]. In particular, he lists many interesting examples. Among them is a 3-polystroma of type $\{4, 4, 3\}$, for which Coxeter and Shephard found very symmetric realizations in the Euclidean 4-space and 5-space (cf. [10]). In [7, 8] Coxeter describes interesting regular 3-polystromas of types $\{4, 4, 3\}$, $\{3, 5, 3\}$, and $\{5, 3, 5\}$ with few vertices. Further polystromas of type $\{3, 6, 3\}$ were discovered by Weiss (cf. [29]). All these polystromas are also 4-dimensional regular incidence-polytopes, and some of them appear also as the facet-type of a 5-dimensional regular incidence-polytope.

2. REGULAR INCIDENCE-POLYTOPES

For a detailed introduction to the theory of regular incidence-polytopes, or more generally of regular incidence-complexes, the reader is referred to [13]. Many examples of incidence-complexes can be found in Danzer

[11, 12]. Although many results of our paper can be extended to arbitrary incidence-complexes, we shall confine ourselves to the investigation of incidence-polytopes.

First, we shall recall some basic definitions and elementary facts.

An incidence-polytope \mathcal{P} of dimension d (or briefly a d -incidence-polytope) is defined by the properties (I1) to (I4).

(I1) (\mathcal{P}, \leq) is a partially ordered set with elements F_{-1} and F_d such that $F \in \mathcal{P}$ implies $F_{-1} \leq F \leq F_d$.

(I2) Every totally ordered subset of \mathcal{P} is contained in a totally ordered subset with exactly $d+2$ elements, a so-called flag.

The elements of \mathcal{P} are called faces. For convenience, we shall not distinguish a face F and the section-complex $\{G \mid G \leq F\}$ of faces which are majorized by F . The section-complex $\{G \mid F \leq G\}$ of faces which are greater than or equal to F is called the co-face to F (with respect to \mathcal{P}).

With every face F we can associate a dimension $\dim(F)$, where $\dim(F)+2$ is the number of faces in a flag of the complex $\{G \mid G \leq F\}$. In particular, $\dim(F_{-1}) = -1$ and $\dim(F_d) = d$. We call F a vertex, an edge, an i -face or a facet, iff $\dim(F) = 0, 1, i$, or $d-1$, respectively. The co-face to a vertex is also named a vertex-figure.

(I3) \mathcal{P} is connected, which means: if f and g are two different flags of \mathcal{P} and $h := f \cap g$, then there is a finite sequence of flags $f = f_1, f_2, \dots, f_{n-1}, f_n = g$, all containing h , such that f_{m+1} differs from f_m in exactly one face ($1 \leq m \leq n-1$).

The last defining property guarantees a certain homogeneity of \mathcal{P} .

(I4) For any two faces F and G with $F \leq G$ and $\dim(F)+1 = i = \dim(G)-1$, there are exactly two i -faces H of \mathcal{P} with $F \leq H \leq G$ ($i = 0, \dots, d-1$).

In case the partial order induces a lattice we call \mathcal{P} non-degenerate, otherwise degenerate. If nothing else is said, degeneracy is not excluded. One important property of non-degenerate incidence-polytopes is that all faces may be considered as subsets of the set of vertices of \mathcal{P} , or dually of facets of \mathcal{P} .

If \mathcal{P} is a d -incidence-polytope and \leq is replaced by \geq while the set of faces is unchanged, we get the dual d -incidence-polytope \mathcal{P}^* of \mathcal{P} . We call \mathcal{P} self-dual, iff \mathcal{P} and \mathcal{P}^* are isomorphic. Obviously, the dual of a non-degenerate incidence-polytope is non-degenerate too.

A regular incidence-polytope \mathcal{P} is an incidence-polytope, for which the group $A(\mathcal{P})$ of combinatorial automorphisms of \mathcal{P} (that is, of incidence preserving permutations) is flag-transitive. Then, by (I4), $A(\mathcal{P})$ is even sharply flag-transitive (cf. [13]).

The flag-transitivity of $A(\mathcal{P})$ implies that $A(\mathcal{P})$ is also transitive on the i -faces for each i . Later we shall also use the fact that, for a non-degenerate \mathcal{P} , the group $A(\mathcal{P})$ can be regarded as a transitive permutation group on the set of i -faces of \mathcal{P} for each i .

If \mathcal{P} is regular, then so is every face and co-face of \mathcal{P} . Furthermore, faces and co-faces of the same dimension are isomorphic incidence-polytopes, respectively.

With every regular incidence-polytope \mathcal{P} is associated a triangular scheme $(k_{i,j})_{i,j}$, where $k_{i,j}$ denotes the number of flags in the section-complex $\langle F, G \rangle := \{H \mid F \leq H \leq G\}$ belonging to an i -face F and a j -face G incident with F ($-1 \leq i \leq j \leq d$). In particular, $k_{i,i} = k_{i,i+1} = 1$ and $k_{i-1,i+1} = 2$ for all i . As $A(\mathcal{P})$ is sharply flag-transitive, the number k_{ij} coincides with the order of the group of $\langle F, G \rangle$. In particular, $k_{-1,d}$ is the order of $A(\mathcal{P})$.

It is easy to see that the only isomorphism types of 2-incidence-polytopes are the (finite) p -gons $\{p\}$ in the Euclidean plane and the (infinite) apeirogon $\{\infty\}$, and these are also regular. Their groups are the dihedral groups of order $2p$ and the infinite dihedral group, respectively.

As all section-complexes of a d -incidence-polytope \mathcal{P} are also incidence-polytopes, the number $k_{i-1,i+2}$ of \mathcal{P} has the form $k_{i-1,i+2} = 2p_{i+1}$ ($i = 0, \dots, d-2$) with $p_{i+1} \geq 2$ (possibly infinite). In order to avoid extremely degenerate situations we shall always assume that all p_i are greater than or equal to 3. For instance, this is true for non-degenerate incidence-polytopes. In most of the cases we consider the p_i take one of the values 3, 4, 5, or 6.

The family of all regular d -incidence-polytopes considered up to isomorphism, which share the numbers $k_{i,j}$ of the following triangular scheme is called the clan

$$\begin{pmatrix} 2 & & & 2 & \cdots & 2 & & 2 \\ & k_{-1,2} & & & \cdots & & k_{d-3,d} & \\ & & \ddots & & & & & \ddots \\ & & & k_{-1,s} & \cdots & & k_{d-s-1,d} & \end{pmatrix}$$

of degree s . In particular, a clan of degree d (when the scheme becomes complete) is also named a cluster. A clan of degree s is said to be symmetric, iff its scheme is symmetric with respect to its vertical axis. A regular incidence-polytope is called symmetric, iff the cluster it belongs to is symmetric. Obviously, self-dual incidence-polytopes are symmetric, but not vice versa (cf. Section 8).

In the sequel, we shall say that an incidence-polytope \mathcal{P} is of type $\{p_1, \dots, p_{d-1}\}$, if it is a member of the clan

$$\begin{pmatrix} 2 & 2 & \cdots & 2 & 2 \\ 2p_1 & \cdots & 2p_{d-1} \end{pmatrix}$$

of degree 2.

The classical regular polytopes provide examples for regular incidence-polytopes. By a theorem of McMullen (cf. [19]), they are the only isomorphism types of convex polytopes which are regular incidence-polytopes. In particular, the d -polytope with the Schläfli-symbol $\{p_1, \dots, p_{d-1}\}$ is actually of type $\{p_1, \dots, p_{d-1}\}$. We remark without proof that the cluster determines a classical regular polytope up to isomorphism within the class of non-degenerate regular incidence-polytopes.

The classical regular polytopes are also the only finite universal incidence-polytopes in the sense of [22, Sects. 5, 6]. Each regular incidence-polytope of type $\{p_1, \dots, p_{d-1}\}$ can be derived from the respective universal one with Schläfli-symbol $\{p_1, \dots, p_{d-1}\}$ by making suitable identifications. These universal incidence-polytopes were originally found by Tits (cf. [26]) and were rediscovered in [21]. The author is indebted to R. Scharlau for the reference to the paper of Tits and the information that these universal incidence-polytopes are non-degenerate as well as geometrically realizable by convex cones (cf. [20]); both problems remained open in [21, Chap. 4].

In general a regular complex polytope does not lead to a regular incidence-polytope in our sense. In fact, the numbers $k_{i-1, i+2}$ are only 2 for all i , iff the polytope is real (cf. Coxeter [6]).

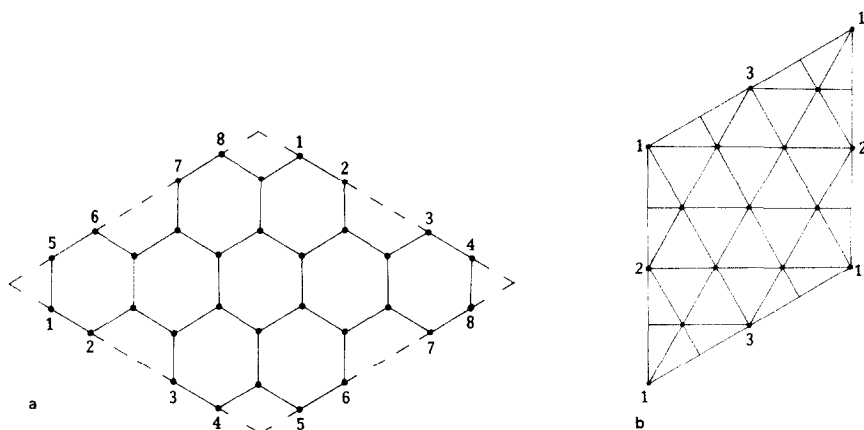
The regular maps on surfaces give further examples of 3-incidence-polytopes; for an extensive account of regular maps see Coxeter and Moser [9]. However, only the reflexible regular maps are regular in our sense. For irreflexible regular maps the automorphism group is not flag-transitive, although it is transitive on the faces of each dimension, that is, on the vertices, edges, and 2-faces, respectively (cf. [9, p. 102]).

In this paper we are concerned only with reflexible regular maps on the torus. There exist only the three infinite series of maps $\{4, 4\}_{b,c}$, $\{6, 3\}_{b,c}$, and $\{3, 6\}_{b,c}$ with $c=0$ or $b=c$. The map $\{4, 4\}_{b,c}$ is self-dual, whereas $\{3, 6\}_{b,c}$ is the dual of $\{6, 3\}_{b,c}$. For later use we have listed some properties of these maps in Table I. The dual $\{3, 6\}_{b,c}$ of $\{6, 3\}_{b,c}$ is omitted, since the duality dictates the necessary changes. The information about the non-degeneracy is also included in the list. As an example, the map $\{6, 3\}_{2,2}$ and its dual are shown in Fig. 1. They will play an important role in Sections 5–8.

TABLE I
The Reflexible Regular Maps on the Torus^a

	$\{4, 4\}_{h,0}$	$\{4, 4\}_{c,c} = \{4, 4\}_{2c}$	$\{6, 3\}_{h,0} = \{6, 3\}_{2b}$	$\{6, 3\}_{c,c}$
Cluster	$\begin{pmatrix} 2 & 2 & 2 \\ 8 & 8 & 8 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 & 2 \\ 8 & 8 & 8 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 & 2 \\ 12 & 6 & 6 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 & 2 \\ 12 & 36c^2 & 6 \end{pmatrix}$
Group	$8b^2$	$16c^2$	$12b^2$	$36c^2$
Group order	$[4, 4]_{h,0}$	$[4, 4]_{c,c} \cong G^{4,4,2c}$	$[6, 3]_{h,0} \cong G^{6,3,2b}$	$[6, 3]_{c,c}$
Extra relation	$8b^2$	$16c^2$	$12b^2$	$36c^2$
Number of vertices	$(\rho_0 \rho_1 \rho_2 \rho_1)^b = 1$	$(\rho_0 \rho_1 \rho_2)^{2c} = 1$	$(\rho_0 \rho_1 \rho_2)^{2b} = 1$	$(\rho_2 \rho_1 \rho_2 \rho_1 \rho_0)^{2c} = 1$
Number of 2-faces	b^2	$2c^2$	$2b^2$	$6c^2$
Non-degenerate, iff	$b \geq 3$	$c \geq 2$	$b \geq 3$	$c \geq 2$
(NA) holds, iff	$b \geq 4$	$c \geq 3$	$b \geq 3$	$c \geq 2$
Length of the Petrie-polygon	$2b$	$2c$	$2b$	$6c$

^aThe dual $\{3, 6\}_{h,c}$ of $\{6, 3\}_{h,c}$ is omitted.

FIG. 1. The maps $\{6, 3\}_{2,2}$ and $\{3, 6\}_{2,2}$ on the torus.

As in the case of a regular polytope the cluster determines a reflexible regular map on the torus up to isomorphism; that is, a map is characterized by its invariants p_1 and p_2 and the order of its group. Note that this does not extend to arbitrary toroidal maps; for example, the reflexible regular map $\{4, 4\}_{5,0}$ and the irreflexible regular map $\{4, 4\}_{7,1}$ have the same group order 200 but are not isomorphic. Also, it is worth mentioning that on the surface of genus 14 there are two non-isomorphic reflexible regular maps $\{7, 3\}_{12}$ and $\{7, 3\}_{14}$, which have not merely the same group order but the same group

$$G^{3,7,12} \simeq G^{3,7,14} \simeq PGL(2, 13)$$

(cf. [9, p. 139]).

The following considerations show that the class of all finite reflexible regular maps on surfaces actually coincides with the class of all finite regular 3-incidence-polytopes. In fact, we can associate with every finite (regular) 3-incidence-polytope \mathcal{P} a closed real 2-manifold $M(\mathcal{P})$ by regarding certain points of the space as the vertices of \mathcal{P} , joining vertices of an edge by a real line and stretching a topological disc in every 2-face. Hence, the study of regular 3-incidence-polytopes is equivalent to the study of reflexible regular maps.

For convenience we shall call a regular d -incidence-polytope \mathcal{P} Euclidean or toroidal (the latter only for $d=3$), if it is combinatorially isomorphic to a classical Euclidean regular convex polytope or a reflexible map on the torus, respectively. Since we are only working with combinatorial configurations and isomorphism types of such, we do not require that \mathcal{P} be geometrically realized in any Euclidean space. In par-

ticular, the automorphisms of \mathcal{P} are only combinatorial automorphisms but not geometric symmetry operations.

Our notation becomes particularly evident in case $d=3$. In fact, the surface $M(\mathcal{P})$ is the 2-sphere, iff either \mathcal{P} is Euclidean or the dihedron $\{p_1, 2\}$ or the hosohedron $\{2, p_2\}$ (cf. Coxeter [5, p. 12]). On the other hand exactly the toroidal 3-incidence-polytopes cover the case $M(\mathcal{P})$ is the torus.

With these preliminaries the general problem underlying this paper can roughly be expressed as follows. Find regular incidence-polytopes with preassigned Euclidean or toroidal faces and co-faces. In some instances, there are simple combinatorial reasons which rule out the possibility of fitting together both types. For example, if two d -incidence-polytopes \mathcal{P}_1 and \mathcal{P}_2 are predescribed for the type of the facets and vertex-figures, respectively, then of course, the facets of \mathcal{P}_2 have to be isomorphic to the vertex-figures of \mathcal{P}_1 . These considerations extend to the general situation where the types of the i -face and of the co-face to a j -face are preassigned.

In accordance with Grünbaum's notation in [18] we denote, for a given dimension d and two given incidence-polytopes \mathcal{P}_1 and \mathcal{P}_2 of dimensions i and $d-j-1$, respectively, the family of all d -incidence-polytopes with i -faces of type \mathcal{P}_1 and co-faces to j -faces of type \mathcal{P}_2 by $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$. In most cases i and j will be $d-1$ and 0, respectively. Since the dimension d will always be clear from the context, we shall not explicitly mark the symbol by an index d . For example, the 4-dimensional 24-cell $\{3, 4, 3\}$ is a member of $\langle \{3, 4\}, \{4, 3\} \rangle$ (cf. Coxeter [5]).

3. AUTOMORPHISM GROUP AND CONSTRUCTION METHODS

The automorphism group of regular incidence-polytopes has been studied in [22, 23]. For later use we shall recall the most important results.

Let \mathcal{P} be a regular d -incidence-polytope of type $\{p_1, \dots, p_{d-1}\}$ and define $U := A(\mathcal{P})$. The automorphism group U is generated by involutions $\rho_0, \dots, \rho_{d-1}$, where ρ_i denotes the unique automorphism that keeps all but the i -face F_i of a certain flag $f := \{F_{-1}, F_0, \dots, F_{d-1}, F_d\}$ of \mathcal{P} fixed ($0 \leq i \leq d-1$). These generators satisfy the relations

$$\left. \begin{aligned} \rho_i^2 &= 1 & (0 \leq i \leq d-1) \\ (\rho_i \rho_j)^2 &= 1 & (0 \leq i < j-1 \leq d-2) \end{aligned} \right\} \quad (1)$$

$$(\rho_i \rho_{i+1})^{p_{i+1}} = 1 \quad (0 \leq i \leq d-2) \quad (2)$$

and in general some additional relations.

For $i=0, \dots, d-1$ and $I \subset \{0, \dots, d-1\}$ define $U_i^- := \langle \rho_j \mid j \leq i \rangle$, $U_i^+ := \langle \rho_j \mid j \geq i \rangle$, and $U_I := \langle \rho_j \mid j \in I \rangle$. In addition, $U_\emptyset := \{1\}$ and

$U_i^- := U_k^+ := \{1\}$ if $i < 0$ and $k > d-1$, respectively. These subgroups have the important property

$$\text{If } I, J \subset \{0, \dots, d-1\}, \quad \text{then } U_I \cap U_J = U_{I \cap J}. \quad (3)$$

For $i \leq j$ the automorphism group of the section-complex $\langle F_i, F_j \rangle := \{F \mid F_i \leq F \leq F_j\}$ of \mathcal{P} belonging to the i -face F_i and the j -face F_j in f is just the subgroup $U_{\{i+1, \dots, j-1\}}$ of U ; its order coincides with the number $k_{i,j}$. In particular, U_{i-1}^- and U_{i+1}^+ are the groups of the i -face F_i and its respective co-face, respectively.

The generators $\rho_0, \dots, \rho_{d-1}$, or more exactly the groups U_I , also allow one to characterize the combinatorial structure of \mathcal{P} . In fact, by the results of [22], we have

$$\varphi(F_i) \leq \psi(F_j) \iff (-1 \leq i \leq j \leq d \wedge \psi^{-1}\varphi \in U_{i-1}^- U_{j+1}^+), \quad (4)$$

where the case $i = j$ describes equality of faces. Note that, by the transitivity properties of $U = A(\mathcal{P})$, $\varphi(F_i)$ and $\psi(F_j)$ run over all i -faces and j -faces of \mathcal{P} , if φ and ψ run over U , respectively.

In the case of a Euclidean incidence-polytope \mathcal{P} , or more generally a universal incidence-polytope $\{p_1, \dots, p_{d-1}\}$, the relations (1) and (2) suffice for the definition of the group (cf. Coxeter [5], and [22]). Thus, the group is just the Coxeter-group with linear diagram

$$\bullet \text{---} \bullet \text{---} \dots \text{---} \bullet$$

p_1
 p_{d-1}

For toroidal 3-incidence-polytopes (that is, reflexible regular maps on the torus), one extra relation has to be added (cf. Coxeter and Moser [9, Sect. 8]). It is given in Table I; the relation for $\{3, 6\}_{b,c}$ is obtained from that of $\{6, 3\}_{b,c}$ by replacing ρ_0 by ρ_2 and vice versa.

The most important feature of properties (1) and (3) is that they characterize the groups of regular d -incidence-polytopes. In fact, if A is a group generated by involutory generators $\rho_0, \dots, \rho_{d-1}$ with properties (1) and (3), then A is the automorphism group of a regular d -incidence-polytope $\mathcal{P}(A)$ ($\mathcal{H}(A)$ in the notation of [22]) and $\rho_0, \dots, \rho_{d-1}$ the respective generators of A (with respect to a suitable flag of $\mathcal{P}(A)$). In particular, (2) holds with certain numbers p_1, \dots, p_{d-1} determining the type of $\mathcal{P}(A)$. It is worth mentioning that property (4) originally served for the definition of the partial order in $\mathcal{P}(A)$ (cf. [22]).

Moreover, if the group A is just the group of a regular d -incidence-polytope \mathcal{P} generated by the respective generators (that is, $U = A$), then $\mathcal{P}(A)$ is actually isomorphic to \mathcal{P} . Therefore we can conclude that the investigation of regular incidence-polytopes is completely equivalent to the investigation of groups of the above-mentioned type.

In the sequel we shall use the notation A_i^- , A_i^+ , and A_i with respect to A analogously to U_i^- , U_i^+ , and U_i with respect to U .

In view of the above-mentioned results our problem is transformed into an embedding problem for groups. That means, instead of fitting together the incidence-polytopes \mathcal{P}_1 and \mathcal{P}_2 as facets and vertex-figures of a suitable higher-dimensional incidence-polytope, we can as well embed the automorphism groups $A(\mathcal{P}_1)$ and $A(\mathcal{P}_2)$ into a suitable group A which gives rise to an incidence-polytope $\mathcal{P}(A)$.

For instance, if \mathcal{P}_1 and \mathcal{P}_2 are Euclidean or toroidal regular 3-incidence-polytopes, we have to search for a group A such that all properties are satisfied together with $A_2^- = A(\mathcal{P}_1)$ and $A_1^+ = A(\mathcal{P}_2)$. As Euclidean and toroidal regular 3-incidence-polytopes are completely characterized by their group (and even by the group order) up to duality, these properties would imply that the facets and vertex-figures of $\mathcal{P}(A)$ are isomorphic to \mathcal{P}_1 and \mathcal{P}_2 , respectively.

In a sense this problem has a "universal solution." Indeed, if \mathcal{P} is a regular 4-incidence-polytope with facets and vertex-figures of types \mathcal{P}_1 and \mathcal{P}_2 , respectively, then the generators ρ_0, \dots, ρ_3 of $A(\mathcal{P})$ have to satisfy certain relations dictated by the defining relations of the subgroups $\langle \rho_0, \rho_1, \rho_2 \rangle = A(\mathcal{P}_1)$ and $\langle \rho_1, \rho_2, \rho_3 \rangle = A(\mathcal{P}_2)$ of $A(\mathcal{P})$. By applying the general construction of [22] to the group A abstractly defined by these relations, we get a 4-dimensional partially ordered set $\mathcal{P}(A)$ with properties (I1), (I2), and (I4) and a certain connectivity property, which admits A as a flag-transitive group. But in general it is by no means clear that $\mathcal{P}(A)$ is actually an incidence-polytope in our sense (that is, has property (I3)), and has facets and vertex-figures isomorphic to \mathcal{P}_1 and \mathcal{P}_2 , respectively; or in other words, that A satisfies property (3) together with $A_2^- = A(\mathcal{P}_1)$ and $A_1^+ = A(\mathcal{P}_2)$. Even if both should hold true the finiteness of $\mathcal{P}(A)$ (that is, of A) is still undecided. In any case we can conclude that every 4-incidence-polytope solving our problem is obtained from the "universal solution" by making suitable identifications. Of course, these ideas extend to dimensions greater than 4.

For later reference we shall sum up the results in Theorem 1. As the starting point we take the facet type \mathcal{P}_1 and try to embed $A(\mathcal{P}_1)$ into a suitable group A ; then, the type of the vertex-figures is determined by A_1^+ .

THEOREM 1. *Let \mathcal{P}_1 be a regular d -incidence-polytope of type $\{p_1, \dots, p_{d-1}\}$ and $\rho_0, \dots, \rho_{d-1}$ the generators of $U := A(\mathcal{P}_1)$ belonging to a flag $f := \{F_{-1}, F_0, \dots, F_d\}$. Suppose that U is contained in a group A generated by $\rho_0, \dots, \rho_{d-1}$ and one additional involution ρ_d such that ρ_0, \dots, ρ_d satisfy the relations (1) (with d replaced by $d+1$) and*

$$A_i^+ \cap U = U_i^+ \quad (0 \leq i \leq d). \quad (3')$$

Then, the regular $(d+1)$ -incidence-polytope $\mathcal{P} := \mathcal{P}(A)$ has the following properties.

(a) A is the automorphism group of \mathcal{P} and ρ_0, \dots, ρ_d the respective generators. \mathcal{P} is finite, iff A is finite.

(b) The facets of \mathcal{P} are isomorphic to \mathcal{P}_1 .

(c) The co-face to an i -face of \mathcal{P} is isomorphic to the incidence-polytope $\mathcal{P}(A_{i+1}^+)$ derived from A_{i+1}^+ in the same way as \mathcal{P} from A ($i = 0, \dots, d-1$).

(d) If p_d is the order of $\rho_{d-1} \cdot \rho_d$ in A , then \mathcal{P} is of type $\{p_1, \dots, p_{d-1}, p_d\}$.

Moreover, if \mathcal{P}_1 is non-degenerate while A satisfies the condition

$$\left. \begin{array}{l} \text{Let } 0 \leq i \leq j < k \leq d \text{ and } \tau \in U_{k-1}^-. \text{ If } F_k \text{ is the supremum of} \\ F_j \text{ and } \tau(F_i) \text{ in } \mathcal{P}_1, \text{ then } A_{j+1}^+ \cap \tau A_{i+1}^+ U \subset A_{k+1}^+ U. \end{array} \right\} \quad (5)$$

then,

(e) \mathcal{P} is non-degenerate.

Theorem 1 is a combined version of [23, Satz 1, 2] applied to the special situation. Note in particular, that (3') is a simplification of (3). Whereas in many instances it is easy to verify property (3'), it is on the other hand extremely difficult to check (5). However, (5) cannot be weakened, since it exactly describes the non-degeneracy.

4. PETRIE-POLYGONS

In the classical theory of regular polytopes and of regular maps on surfaces Petrie-polygons play an important role; see, for example, Coxeter [5], Coxeter and Moser [9]. In many instances, the length of the Petrie-polygon gives rise to a defining relation of the automorphism group.

In this section we generalize the concept of Petrie-polygons to regular incidence-polytopes. We shall confine ourselves to non-degenerate regular incidence-polytopes, although the considerations extend to some classes of degenerate incidence-polytopes. So, throughout this section all incidence-polytopes are supposed to be non-degenerate.

Following the inductive definition of Petrie-polygons in Coxeter [5, Sect. 12.4], we declare that the Petrie-polygon of a 2-incidence-polytope is that incidence-polytope itself (see Section 2). For $d \geq 3$, a Petrie-polygon of a d -incidence-polytope \mathcal{P} is a path within the 1-skeleton of \mathcal{P} (that is, a path with vertices and edges in \mathcal{P}) such that any $d-1$ consecutive edges, but no d , belong to a Petrie-polygon of a $(d-1)$ -face (facet) of \mathcal{P} .

First, we remark that Petrie-polygons are completely determined by their vertices. In fact, the non-degeneracy of \mathcal{P} implies that any two vertices of \mathcal{P} lie in at most one edge of \mathcal{P} . Moreover, if two vertices F and G lie in an edge H , then H is also incident with each face of \mathcal{P} containing F and G . These considerations show that we could have said " d consecutive vertices" instead of " $d-1$ consecutive edges" too. But in the case of a degenerate incidence-polytope these things can be quite different.

The proof for the following results is the combinatorial analog of the proof for the corresponding results on Petrie-polygons of regular polytopes. In order to avoid needless duplication we shall restrict ourselves to a summary of the main facts.

Petrie-polygons do actually exist for every non-degenerate regular d -incidence-polytope \mathcal{P} and have at least $d+1$ vertices. By the regularity they are all alike; that is, any Petrie-polygon of \mathcal{P} can be mapped onto any other Petrie-polygon by an automorphism of \mathcal{P} . Therefore, we are justified in speaking of *the* Petrie-polygon of \mathcal{P} .

Each path with d vertices belonging to a Petrie-polygon of a facet of \mathcal{P} can be uniquely extended to a Petrie-polygon of \mathcal{P} . Hence, Petrie-polygons are uniquely determined by d consecutive vertices.

Let $\pi = \cdots G_{-1}G_0G_1\cdots G_{d-2}G_{d-1}\cdots$ be a Petrie-polygon of \mathcal{P} , where the G_i denote the consecutive vertices. For $i=0, \dots, d-1$, the path $G_0G_1\cdots G_i$ belongs to a Petrie-polygon of a unique i -face F_i of \mathcal{P} . Clearly, $f := \{F_{-1}, F_0, \dots, F_{d-1}, F_d\}$ (with F_{-1}, F_d as in (11)) is a flag of \mathcal{P} which gives rise to a system of generators $\rho_0, \dots, \rho_{d-1}$ (cf. Section 3). By σ we shall denote the automorphism $\rho_0 \cdot \rho_1 \cdot \cdots \cdot \rho_{d-1}$, and by h its order. Hence,

$$(\rho_0 \cdot \rho_1 \cdot \cdots \cdot \rho_{d-1})^h = 1. \quad (6)$$

Then, σ permutes the vertices of π . Actually, they are shifted forward, which means $\sigma(G_j) = G_{j+1}$ for each j . Defining h' to be the number of vertices in π , we observe that σ^l keeps all vertices of π fixed, iff l is a multiple of h' . As automorphisms of non-degenerate incidence-polytopes are completely determined by their effect on the vertices, this implies the equality $h = h'$. Therefore, the order of σ is just the length of the Petrie-polygons in \mathcal{P} .

The relation (6) is of course well known for regular maps on surfaces and regular polytopes and honeycombs. Important classes of such configurations denoted by $\{p, q\}_r$ (with $h=r$) and $\{p, q, r, \dots\}_t$ (with $h=t$), respectively, have groups abstractly defined by the relations (1), (2), and (6) (cf. Coxeter [2], Coxeter and Moser [9]).

In accordance with Coxeter [7] we shall also use the notation $\langle \{p, q, r\}_t \rangle$ for a 4-incidence-polytope in the class $\langle \{p, q\}_s, \{q, r\}_t \rangle$. Also, we write

$\langle \mathcal{P}_1, \mathcal{P}_2 \rangle_h$ for the subclass of $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ consisting of those incidence-polytopes, for which the automorphism $\sigma = \rho_0 \cdot \rho_1 \cdot \dots \cdot \rho_{d-1}$ has order h . So, in the case of a non-degenerate member \mathcal{P} of $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle_h$ the subscript h is just the length of the Petrie-polygon of \mathcal{P} . For instance, the arrangement $\{3, 5, 3\}_5$ of 11 hemi-icosahedra described in Coxeter [8] and Grünbaum [18] is a non-degenerate member of the class $\langle \{3, 5\}_5, \{5, 3\}_5 \rangle_6$ (with a hexagonal Petrie-polygon). It is actually the universal incidence-polytope of this class, and its group is the simple group $PSL(2, 11) \simeq G^{5.5.5}$ of order 660.

5. INCIDENCE-POLYTOPES WITH A SYMMETRIC GROUP

In this section we remind the reader of the construction in [23, Satz 3] providing incidence-polytopes \mathcal{P} with preassigned facet-type \mathcal{P}_1 . In particular, we shall study some additional properties of \mathcal{P} which were not pointed out there. For example, we shall see that the group of \mathcal{P} is the symmetric group of degree “number of facets of \mathcal{P}_1 plus 1.”

In order to construct \mathcal{P} we shall suitably apply Theorem 1 to the given facet-type \mathcal{P}_1 . Here, our starting point will be the presentation of the group of \mathcal{P}_1 on the set of facets of \mathcal{P}_1 .

We shall restrict our attention to finite and non-degenerate regular d -incidence-polytopes \mathcal{P}_1 . So, let \mathcal{P}_1 be such an incidence-polytope, $f := \{F_{-1}, \dots, F_d\}$ a fixed flag of \mathcal{P}_1 , and $U := A(\mathcal{P}_1)$ the automorphism group of \mathcal{P}_1 with generators $\rho_0, \dots, \rho_{d-1}$ determined by f (cf. Section 3). By L_i we denote the set of all facets of \mathcal{P}_1 incident with F_i , and by m_i its cardinality ($i = -1, 0, \dots, d-1$). For brevity we write $L := L_{-1}$ and $m := m_{-1}$, so that L and m are the set respectively the number of all facets of \mathcal{P}_1 . In addition we choose one element a not in \mathcal{P}_1 and define $M := L \cup \{a\}$ and $M_i := L_i \cup \{a\}$ ($i = -1, 0, \dots, d-1$). Then, M and M_i have cardinality $m+1$ and m_i+1 , respectively.

Since we have assumed non-degeneracy for \mathcal{P}_1 , each automorphism of \mathcal{P}_1 is uniquely determined by its effect on the elements of L . So, we can regard U as a subgroup of the symmetric group S_M fixing the element a of M . Now, we turn to the construction of the group A with the properties of Theorem 1.

We define A to be the subgroup of S_M generated by U and the transposition $\rho_d := (F_{d-1}a)$ of S_M . Since $\rho_0, \dots, \rho_{d-2}$ keep the facet F_{d-1} of \mathcal{P}_1 fixed, we immediately deduce the relations

$$(\rho_i \rho_d)^2 = 1 \quad (i = 0, \dots, d-2).$$

As an example we consider the 3-cube $\{4, 3\}$ with facets $1, \dots, 6$ numbered

in such a way that (1, 2), (3, 4), and (5, 6) are pairs of opposite facets (cf. Fig. 2). Then, its group $[4, 3]$ is generated by

$$\rho_0 = (3\ 4), \quad \rho_1 = (1\ 4)(2\ 3), \quad \rho_2 = (1\ 5)(2\ 6). \quad (7)$$

Renaming F_{d-1} by 5 and a by 7, our group A will be a subgroup of S_7 generated by ρ_0, ρ_1, ρ_2 and

$$\rho_3 = (5\ 7). \quad (8)$$

Here, easy calculations reveal a regular 4-incidence-polytope of type $\langle \{4, 3\}, \{3, 6\}_{2,2} \rangle_7$ with group S_7 , 35 vertices, and 105 facets (see also Section 7). However, this incidence-polytope degenerates, since there are pairs of vertices which are antipodal with respect to at least two cubes. But it can be shown that it has a heptagonal Petrie-polygon.

For a general incidence-polytope \mathcal{P}_1 it has been proved in [23, Satz 3] that A actually satisfies the condition (3') of Theorem 1 and thus gives rise to the regular $(d+1)$ -incidence-polytope $\mathcal{P} := \mathcal{P}(A)$ with properties (a), ..., (d) of Theorem 1. However, the non-degeneracy, that is, the property (5) of A , depends on the following condition (NA) on the group U of \mathcal{P}_1 .

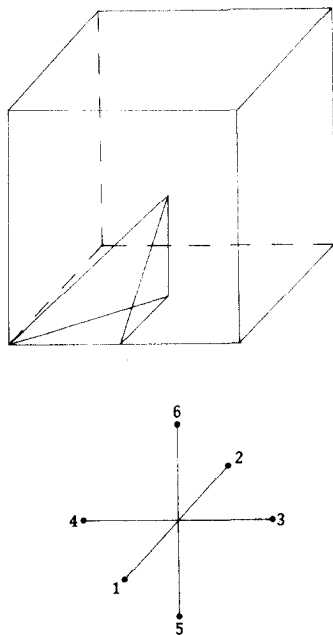


FIG. 2. The 3-cube (with fundamental region and suitably numbered facets).

But this condition is rather weak and also satisfied in the most important cases. In fact, one can prove that for any dimension d there are only finitely many regular d -incidence-polytopes for which (NA) fails.

For any two vertices G and H in \mathcal{P}_1 , the identity map is the only automorphism φ with the following property. Any facet of \mathcal{P}_1 not containing G or H is kept fixed by φ , and any facet containing G but not H is mapped onto a facet which also contains G . (NA)

In order to analyse the structure of \mathcal{P} and its co-faces we proceed with the investigation of the group A and its subgroups $A_{i+1}^+ := \langle \rho_{i+1}, \dots, \rho_d \rangle$. By Theorem 1, the latter are just the automorphism groups of the co-faces to i -faces of \mathcal{P} .

Here and also in Section 6, we shall make use of the following simple fact about permutation groups. Let Y, Z be finite sets with $Y \subset Z$ and z be in Z but not in Y . If a subgroup B of the symmetric group S_Z on Z contains both a transposition $\rho = (yz)$ with $y \in Y$ and a subgroup B' which acts transitively on the elements in Y while keeping z fixed, then, B contains also the subgroup $S_{Y \cup \{z\}}$. In particular, if $Z = Y \cup \{z\}$, then $B = S_Z$. In fact, by the transitivity of B' on Y , for each y' in Y there is a permutation φ in B' with $\varphi(y) = y'$. Hence, $(y'z) = \varphi \rho \varphi^{-1}$ is in B . But, the transpositions $(y'z)$ with $y' \in Y$ generate the group $S_{Y \cup \{z\}}$, so that $S_{Y \cup \{z\}} \subset B$.

Applying these considerations to the case $Z = M$, $Y = L$, $z = a$, $B = A \subset S_M$, and $B' = U \subset S_L$ we immediately get $A = S_M \simeq S_{m+1}$. In other words, the automorphism group of \mathcal{P} is the symmetric group of degree "number of facets of \mathcal{P}_1 plus 1." We remark that for the d -simplex A becomes S_{d+2} and \mathcal{P} the $(d+1)$ -simplex.

Turning to the subgroups A_{i+1}^+ we see that $A_d^+ = \langle \rho_d \rangle$ has order 2 and that, in case \mathcal{P}_1 is not the d -simplex, $A_{d-1}^+ = \langle \rho_{d-1}, \rho_d \rangle$ is the dihedral group $D_6 \simeq S_3 \times S_2$ of order 12. The latter reveals hexagonal co-faces to $(d-2)$ -faces of \mathcal{P} .

Now, let $0 \leq i \leq d-3$ and φ be in A_{i+1}^+ . Then, φ can be written in the form $\varphi = \varphi_1 \rho_d \varphi_2 \rho_d \cdots \rho_d \varphi_k$ with $\varphi_1, \dots, \varphi_k \in U_{i+1}^+$. Recalling that each automorphism in U_{i+1}^+ leaves the i -face F_i of f invariant and thus permutes the facets within L_i as well within $L \setminus L_i$, we observe that φ can be split uniquely into two permutations ψ_1 and ψ_2 in S_{M_i} and $S_{M \setminus M_i} = S_{L \setminus L_i}$, respectively. Here, ψ_2 is just the restriction of the automorphism $\varphi_1 \cdot \varphi_2 \cdot \cdots \cdot \varphi_k$ in U_{i+1}^+ to $L \setminus L_i$.

Taking into account the transitivity of U_{i+1}^+ on the facets in L_i , a suitable application of the above-mentioned fact about permutation groups ($Z = M$, $Y = L_i$, $z = a$, $B = A_{i+1}^+ \subset S_M$, $B' = U_{i+1}^+ \subset S_L$) shows that S_{M_i} is a subgroup of A_{i+1}^+ . Consequently, writing C for the group consisting of all

Incidence-Polytopes Obtained b

Dimension	Type	Group	Group order	<i>h</i>
4	$\langle \{6, 3\}_{b,0}, \{3, 6\}_{2,2} \rangle$ for $b \geq 3$	S_{b^2+1}	$(b^2+1)!$	21, if $b=3$ $2b(2b+1)$, if $b \geq 4$
		$S_{b^2+1} \times G^{6,3,2b}$	$(b^2+1)! \cdot 12b^2$	$2b(2b+1)$
4	$\langle \{6, 3\}_{c,c}, \{3, 6\}_{2,2} \rangle$ for $c \geq 2$	S_{3c^2+1}	$(3c^2+1)!$	13, if $c=2$ 171, if $c=3$ $6c(6c+1)$, if $c \geq 4$
		$S_{3c^2+1} \times [6, 3]_{c,c}$	$(3c^2+1)! \cdot 36c^2$	$6c(6c+1)$
4	$\langle \{5, 3\}, \{3, 6\}_{2,2} \rangle$	S_{13}	13!	22
		$S_{13} \times A_5 \times S_2$	$13! \cdot 120$	110
4	$\langle \{5, 3\}_5, \{3, 6\}_{2,2} \rangle$	S_7	5040	6
		$S_7 \times A_5$	$7! \cdot 60$	30
5	$\langle \{3, 4, 3\}, \{3, 6\}_{2,2} \rangle$	S_{25}	25!	156
		$S_{25} \times [3, 4, 3]$	$25! \cdot 1152$	156
5	$\langle \{5, 3, 3\}, \{3, 6\}_{2,2} \rangle$	S_{121}	121!	930
		$S_{121} \times [5, 3, 3]$	$121! \cdot 14400$	930
5	$\langle {}_5\{3, 5, 3\}_5, \{3, 6\}_{2,2} \rangle$	S_{12}	12!	42
		$S_{12} \times PSL(2, 11)$	$12! \cdot 660$	42
5	$\langle \mathcal{L}_{3,0}, \{3, 6\}_{2,2} \rangle$	S_{21}	21!	110
		$S_{21} \times S_6 \times S_2$	$21! \cdot 6! \cdot 2$	110
$d+1$ (≥ 4)	$\langle \{4, 3^{d-2}\}, \{3, 6\}_{2,2} \rangle$	S_{2d+1}	$(2d+1)!$	$2d+1$
		$S_{2d+1} \times [4, 3^{d-2}]$	$(2d+1)! \cdot 2^d \cdot d!$	$2d(2d+1)$
$d+1$ (≥ 4)	$\langle \{3^{d-1}\}, \{3, 6\}_{2,2} \rangle$	$S_{d+2} \times S_{d+1}$	$(d+2)! \cdot (d+1)!$	$(d+2)(d+1)$
$d+1$ (≥ 4)	$\langle \{3^{d-1}\}, \{3, 6\}_{2,0} \rangle$	$S_{d+2} \times S_2$	$(d+2)! \cdot 2$	$d+2$, if $d+1$ odd $2d+4$, if $d+1$ even

^a In the sixth column the symbols +, e(·), *, or ? indicate the existence of a Petrie-polygon of length the existence of a Petrie-polygon whose length is the number in brackets (with ? if the length is unknown), the existence of hamiltonian Petrie-polygons, or that the existence of Petrie-polygons is unknown, respectively. In the last column we use the symbols + for non-degeneracy, - for degeneracy, or ? if this is known.

tructions of Sections 5, 6, and 8^a

Petrie-polygon	Number of vertices	Number of facets	Non-degenerate/ degenerate
+	$(b^2 + 1)!/144$	$(b^2 + 1)!/12b^2$	+
+	$(b^2 + 1)! \cdot b^2/12$	$(b^2 + 1)!$	+
+	$(3c^2 + 1)!/144$	$(3c^2 + 1)!/36c^2$	+
+	$(3c^2 + 1)! \cdot c^2/4$	$(3c^2 + 1)!$	+
+	$13!/144$	$13!/120$	+
+	$13! \cdot 5/6$	$13!$	+
e(?)	35	84	?
+	2100	5040	+
+	$25!/48 \cdot 7!$	$25!/1152$	+
+	$25!/210$	$25!$	+
+	$121!/2880$	$121!/14400$	+
+	$121! \cdot 5$	$121!$	+
?	1584	$9! \cdot 2$?
+	$12! \cdot 11/7!$	$12!$	+
?	$20!/16 \cdot 6!$	$21!/2 \cdot 6!$?
+	$20!/8$	$21!$	+
e(7) if $d = 3$, ? if $d \geq 4$	$(2d + 1)!/(d + 1)! \cdot d!$	$(2d + 1)!/2^d \cdot d!$	—
+	$(2d + 1)! \cdot 2^d/(d + 1)!$	$(2d + 1)!$	+
+, *	$(d + 2)(d + 1)$	$(d + 2)!$	+
+ if $d + 1$ odd, $d + 2$ if $d + 1$ even, * in both cases	$d + 2$	$2d + 4$	—

restrictions of automorphisms in U_{i+1}^+ to $L \setminus L_i$, we see that C is a subgroup of A_{i+1}^+ . Hence, ψ_1 can run over all of S_{M_i} while ψ_2 can run over all of C . If all automorphisms in U_{i+1}^+ are uniquely determined by their effect on $L \setminus L_i$, then C is actually isomorphic to U_{i+1}^+ , implying that $A_{i+1}^+ \simeq S_{M_i} \times C \simeq S_{m_i+1} \times U_{i+1}^+$. For instance, this is true for all i , if the condition (NA) holds for \mathcal{P}_1 . However, a weaker condition would suffice.

Summing up all information we can easily calculate the cluster $(k_{ij})_{ij}$ of \mathcal{P} . While the numbers k_{ij} with $i, j \leq d$ are determined by \mathcal{P}_1 , the remaining numbers $k_{i,d+1}$ are given by

$$k_{i,d+1} = \begin{cases} \left(\frac{k_{i,d}}{k_{i,d-1}} + 1 \right)!, & \text{if } i = -1 \\ \left(\frac{k_{i,d}}{k_{i,d-1}} + 1 \right)! \cdot k_{i,d}, & \text{if } i = 0, \dots, d-2 \text{ and} \end{cases}$$

$$A_{i+1}^+ \simeq S_{m_i+1} \times U_{i+1}^+.$$

For a d -incidence-polytope \mathcal{P}_1 of type $\{p_1, \dots, p_{d-1}\}$ our $(d+1)$ -incidence-polytope \mathcal{P} is of type $\{p_1, \dots, p_{d-1}, 6\}$ with $k_{d-3,d+1} = (p_{d-1} + 1)!$ or $(p_{d-1} + 1)! \cdot 2p_{d-1}$ if $d = 2$ or ≥ 3 , respectively.

Of particular importance is the case $p_{d-1} = 3$. Then, the co-faces to $(d-3)$ -faces of \mathcal{P} turn out to be of type $\{3, 6\}$ with group $A_{d-2}^+ \simeq S_4 \times S_3$ of order $k_{d-3,d+1} = 144$. By the considerations of Section 2, this implies isomorphism with the toroidal regular map $\{3, 6\}_{2,2}$.

Applying our results to more specific classes of incidence-polytopes we get a number of interesting new regular configurations. Some of them are listed in Table II, together with information about their type, group, group order, number of vertices and facets, non-degeneracy or degeneracy, and length of Petrie-polygons.

6. CONSTRUCTIONS OF NON-DEGENERATE INCIDENCE-POLYTOPES

The non-degeneracy of the incidence-polytopes in Section 5 depends on the condition (NA) which is satisfied in most cases but not in general. In this section we shall modify the construction so as to obtain non-degenerate incidence-polytopes \mathcal{P} for any preassigned non-degenerate facet-type \mathcal{P}_1 . However, this can be done only by enlarging the automorphism group and turning A from S_{m+1} into $S_{m+1} \times A(\mathcal{P}_1)$. The new construction has also interesting connections with that of Section 5.

Again, let the finite and non-degenerate regular d -incidence-polytope \mathcal{P}_1 be given and let $U = A(\mathcal{P}_1)$. Assume that $f, \rho_i, L_i, L, M_i, M, m_i, m$, and a are defined as in Section 5.

We start from a new presentation of U as a permutation group on the facets F in L and on the elements F' of a certain set L' . We take care that U acts on the elements of L' in the same way as on the facets in L .

To be precise we consider a set L' and a bijection $\Phi: L \rightarrow L'$. For F in L we define F' by $F' := \Phi(F)$ and, for $i = -1, 0, \dots, d-1$, L'_i by $L'_i := \Phi(L_i)$. Then, $L' = L'_{-1}$.

Each automorphism φ of \mathcal{P}_1 regarded as a permutation on L induces in a natural way a permutation on L' , namely by $\varphi(F') := (\varphi(F))'$ for F' in L' . In this way φ becomes a permutation on $L \cup L'$ and, therefore, U a subgroup of $S_{L \cup L'}$.

Next, we adjoin the element a but do not add any element a' corresponding to a . So, U is considered a subgroup of $S_{M \cup L'}$ leaving a invariant. Again, we define the group A to be the subgroup of $S_{M \cup L'}$ generated by U and the transposition $\rho_d := (F_{d-1} a)$ of $S_{M \cup L'}$. Note in particular that ρ_d keeps all elements in L' fixed.

As an example we consider again the 3-cube. With the notation as in Section 5 our group A will become a subgroup of $S_{\{1, \dots, 7, 1', \dots, 6'\}} \simeq S_{13}$ generated by

$$\begin{aligned} \rho_0 &= (3 \ 4)(3' \ 4'), & \rho_2 &= (1 \ 5)(2 \ 6)(1' \ 5')(2' \ 6'), \\ \rho_1 &= (1 \ 4)(2 \ 3)(1' \ 4')(2' \ 3'), & \rho_3 &= (5 \ 7). \end{aligned}$$

Here, it turns out that the group A is just $S_{\{1, \dots, 7\}} \times [4, 3] \simeq S_7 \times [4, 3]$ of order $48 \cdot 7!$ and belongs to a non-degenerate regular 4-incidence-polytope of type $\langle \{4, 3\}, \{3, 6\}_{2,2} \rangle_{42}$ with 1680 vertices and 5040 facets.

In the general situation any element φ in A , $\varphi = \varphi_1 \rho_d \varphi_2 \rho_d \cdots \rho_d \varphi_k$ with $\varphi_1, \dots, \varphi_k \in U$, can be split uniquely into two elements ψ and τ of S_M and $S_{L'}$, respectively. In this sense, A becomes a subgroup of $S_M \times S_{L'}$. Since ρ_d is the identity map when restricted to L' , the element τ in $S_{L'}$ is actually the automorphism $\varphi_1 \cdot \varphi_2 \cdot \cdots \cdot \varphi_k$ of \mathcal{P}_1 expressed as a permutation on L' . As in Section 5 the transitivity of U on L implies that S_M is a subgroup of A ; here $Z = M \cup L'$, $Y = L$, $z = a$, $B = A$, and $B' = U$. So, ψ and τ can run over all of S_M and U , respectively. Therefore, $A \simeq S_M \times U \simeq S_{m+1} \times A(\mathcal{P}_1)$.

For later use we remark also that for $j \leq i$ the subgroup $A_{i+1}^+ = \langle \rho_{i+1}, \dots, \rho_d \rangle$ of A permutes the elements within M_j , $M \setminus M_j = L \setminus L_j$, L'_j as well as within $L' \setminus L'_j$, respectively ($i = 0, \dots, d-1$). This follows directly from the fact that the automorphisms in U_{i+1}^+ fix the j -face F_j in the flag f and thus permute the elements within L_j as well as within L'_j .

Since the generators $\rho_0, \dots, \rho_{d-2}$ fix both the facet F_{d-1} of \mathcal{P}_1 and the element a in M , we immediately deduce the relations

$$(\rho_i \ \rho_d)^2 = 1 \quad (i = 0, \dots, d-2).$$

By Theorem 1, the existence of a regular $(d+1)$ -incidence-polytope $\mathcal{P} := \mathcal{P}(A)$ will be secure, if we can verify the assumption (3'), that is, $A_i^+ \cap U = U_i^+$ for $i=0, \dots, d$.

But this is now immediate. In fact, if the permutation φ in $U = A(\mathcal{P}_1)$ is also in A_i^+ , then it permutes the elements within L'_j for $j \leq i-1$. On the other hand, if an automorphism of \mathcal{P}_1 permutes the elements within L'_j and thus within L_j for $j \leq i-1$, then it leaves the j -faces F_j in f invariant. Hence, φ is in U_i^+ , the stabilizer of F_0, \dots, F_{i-1} .

The proof of the non-degeneracy of \mathcal{P} , that is, of property (5) of Theorem 1, is almost the same as the proof for the incidence-polytopes considered in [23, Satz 3]. In order to avoid needless duplication we shall not give it here. In fact, the only serious change concerns the conclusion " $\sigma = e$ " in the proof of [23, Satz 3], where the crucial condition (NA) enters. Now, this condition is no longer needed. Indeed, it turns out that the automorphism $\sigma = \hat{\varphi} \rho \hat{\psi}^{-1} \tau$ of \mathcal{P}_1 (notation as in [23, p.70]) is the identity when restricted to L' , implying that it is also the identity in $S_{M \cup L'}$. So, one can complete the proof as in [23].

The structure of the co-face to an i -face of \mathcal{P} is determined by its group A_{i+1}^+ . Here, we shall prove $A_{i+1}^+ \simeq S_{m_i+1} \times U_{i+1}^+$ ($i=0, \dots, d-2$). In other words, for any given i -face of \mathcal{P} , the group of its co-face with respect to \mathcal{P} is the direct product of S_{m_i+1} with the group of its co-face with respect to \mathcal{P}_1 . In particular, for $i=d-2$, $A_{d-1}^+ \simeq S_3 \times S_2 \simeq D_6$, implying again hexagonal co-faces to $(d-2)$ -faces of \mathcal{P} .

Now, let i be given and φ be in A_{i+1}^+ , $\varphi = \varphi_1 \rho_d \varphi_2 \rho_d \cdots \rho_d \varphi_k$ with $\varphi_1, \dots, \varphi_k \in U_{i+1}^+$. By the above remark, we can split φ uniquely into $\varphi = \psi_1 \psi_2 \tau$ with $\psi_1 \in S_{M_i}$, $\psi_2 \in S_{M \setminus M_i} = S_{L \setminus L_i}$, and $\tau \in S_{L'}$. With respect to the transitivity of U_{i+1}^+ on L_i arguments similar to those in Section 5 show that S_{M_i} is a subgroup of A_{i+1}^+ ; here $Z = M \cup L'$, $Y = L_i$, $z = a$, $B = A_{i+1}^+$, and $B' := U_{i+1}^+$. Hence, ψ_1 can run over all S_{M_i} . Again, τ is actually the automorphism $\varphi_1 \varphi_2 \cdots \varphi_k$ of \mathcal{P}_1 expressed as a permutation on L' and is therefore in U_{i+1}^+ . The permutation ψ_2 is the restriction of the same automorphism to $L \setminus L_i$. Hence, the elements $\psi_2 \tau$ form a subgroup C of A_{i+1}^+ isomorphic to U_{i+1}^+ . So, we can conclude $A_{i+1}^+ \simeq S_{M_i} \times C \simeq S_{m_i+1} \times U_{i+1}^+$, establishing the desired equality.

We sum up the results in the following theorem.

THEOREM 2. *Let \mathcal{P}_1 be a finite and non-degenerate regular d -incidence-polytope with group $U = A(\mathcal{P}_1) = \langle \rho_0, \dots, \rho_{d-1} \rangle$. Then, \mathcal{P}_1 can be realized as the facet of a finite and non-degenerate regular $(d+1)$ -incidence-polytope \mathcal{P} with group $S_{m+1} \times U$. For $i=0, \dots, d-2$, the group of the co-face to an i -face of \mathcal{P} is $S_{m_i+1} \times U_{i+1}^+$, giving D_6 in case $i=d-2$. The number of vertices and facets of \mathcal{P} is $v \cdot (m+1)!/(m_0+1)!$ and $(m+1)!$, respectively, v denoting the number of vertices of \mathcal{P}_1 .*

Again, Theorem 2 is of particular interest for d -incidence-polytopes \mathcal{P}_1 of type $\{p_1, \dots, p_{d-2}, 3\}$. Then, $A_{d-2}^+ = \langle \rho_{d-2}, \rho_{d-1}, \rho_d \rangle$ turns out to be $S_4 \times S_3$, implying again isomorphism of the co-faces to $(d-3)$ -faces of \mathcal{P} with the toroidal map $\{3, 6\}_{2,2}$. For interesting applications of Theorem 2 to special incidence-polytopes see Table II and Section 7.

Concluding this section we note without proof some further properties of the incidence-polytopes constructed above.

First, we remark that in a sense our construction commutes with the operation of passing from an incidence-polytope to a co-face. In other words, applying first Theorem 2 to \mathcal{P}_1 and then passing from the resulting incidence-polytope \mathcal{P} to the co-face to an i -face F of \mathcal{P} (with respect to \mathcal{P}) gives the same isomorphism type of an incidence-polytope as passing first from \mathcal{P}_1 to the co-face to F (with respect to \mathcal{P}_1) and then applying Theorem 2 to it. This fact is plausible at least from the equality of the groups; in both cases the group is $S_{m+1} \times U_{i+1}^+$.

Furthermore, comparing the construction of this section with that of the preceding section we observe that in most cases the co-faces of the resulting incidence-polytopes have isomorphic groups, namely $S_{m+1} \times U_{i+1}^+$. It can be proved at least under the assumption (NA) for \mathcal{P}_1 (cf. Section 5) that the co-faces do not merely have the same group but are actually isomorphic. In this case the constructions of Sections 5 and 6 provide two different types \mathcal{P} and \mathcal{P}' of $(d+1)$ -incidence-polytopes, respectively, both with the same facets and the same vertex-figures. It deserves mentioning that \mathcal{P} is a "contraction" of \mathcal{P}' , which means that \mathcal{P} is obtained from \mathcal{P}' by making suitable identifications. The normal subgroup of $A(\mathcal{P}') \simeq S_{m+1} \times U$ responsible for this identification is just that subgroup of A determining the second factor U of the direct product.

Finally, the construction of this section seems to be closely related to the construction of [24, Theorem 3]. It is likely that both lead to the same incidence-polytopes. However, the new embedding of the group is elementary while the other involves the heavy machinery of free products with amalgamation.

We remark also that the construction of this section can be generalized to arbitrary regular incidence-complexes (see note added in proof). But we do not use this here.

7. EUCLIDEAN AND TOROIDAL FACES AND CO-FACES

In this section we shall evaluate the constructions of the preceding sections when applied to certain classes of regular incidence-polytopes. For the most part we shall consider incidence-polytopes whose faces and co-faces are Euclidean or toroidal or at least related to one of these types.

Table II contains our most important examples, together with further information on the type, group, and number of faces. Non-degeneracy or degeneracy is also indicated, if it is known. The table also includes the values for the order h of the automorphism $\rho_0 \rho_1 \cdots \rho_{d-1}$. In the case of a non-degenerate incidence-polytope h is just the length of the Petrie-polygon (cf. Section 4).

Starting with the 4-incidence-polytopes of type $\langle \{6, 3\}_{b,c}, \{3, 6\}_{2,2} \rangle$, not $b = c = 2$, we note that they provide a negative answer to Problem 1 in Danzer [11]. They show that a symmetric clan of degree 2 can well contain an incidence-polytope with an asymmetric cluster. Of course, such an incidence-polytope cannot be self-dual. Although the clusters become symmetric for $b = c = 2$, the incidence-polytopes do not become self-dual (for instance, the automorphisms $\rho_0 \rho_1 \rho_3$ and $\rho_3 \rho_2 \rho_0$ have distinct orders). So, the symmetry of a cluster does not imply the self-duality of its members. This is also true in dimension 3. Here, we get an example of type $\{6, 6\}$ with group S_7 from the construction of Section 5 applied to the hexagon $\{6\}$.

Of particular interest are the infinite sequences of $(d+1)$ -incidence-polytopes of type $\langle \{4, 3^{d-2}\}, \{3, 6\}_{2,2} \rangle$ and $\langle \{3^{d-1}\}, \{3, 6\}_{2,2} \rangle$, respectively ($d+1 \geq 4$). Here, the i -dimensional incidence-polytope of the latter type is just the vertex-figure of the $(i+1)$ -dimensional incidence-polytope of the same type as well as of the two $(i+1)$ -dimensional incidence-polytopes of the former type ($i \geq 4$). The existence of such sequences was conjectured in Grünbaum [18]. Comparing the length of the Petrie-polygon with the number of vertices, we also see that the incidence-polytopes of type $\langle \{3^{d-1}\}, \{3, 6\}_{2,2} \rangle$ have hamiltonian Petrie-polygons. The 4-dimensional member of the sequence is probably the dual of Grünbaum's \mathcal{H}_{22} (cf. [18]).

We shall study the 4-incidence-polytope of type $\langle \{4, 3\}, \{3, 6\}_{2,2} \rangle$ with group S_7 in more detail (see also Section 5). It is degenerate, has 35 vertices and 105 facets, and has a heptagonal Petrie-polygon. For brevity, we shall denote it by \mathcal{C}_4 .

The search for a universal 4-incidence-polytope of type $\langle \{4, 3\}, \{3, 6\}_{2,2} \rangle$ involves analysis of the "rotation" group $\langle \alpha, \beta, \gamma \rangle$ defined by

$$\alpha^4 = \beta^3 = \gamma^6 = (\alpha\beta)^2 = (\alpha\beta\gamma)^2 = (\beta\gamma)^2 = (\beta\gamma^{-2}\beta^{-1}\gamma^2)^2 = 1, \quad (9)$$

where $\alpha := \rho_0 \rho_1$, $\beta := \rho_1 \rho_2$, and $\gamma := \rho_2 \rho_3$. Here, the "rotation" group $\langle \beta, \gamma \rangle$ of $\{3, 6\}_{2,2}$ (of order 72) seems to have infinitely many cosets. But the number is reduced to 70 by means of the extra relation

$$(\alpha\gamma)^7 = 1 \quad (\text{that is, } (\rho_0 \rho_1 \rho_2 \rho_3)^7 = 1), \quad (10)$$

which then defines the group S_7 . The enumeration of cosets was carried out by H. S. M. Coxeter and A. Sinkov using an electronic computer. They kindly communicated the results to the author.

The "rotation" group of our incidence-polytope \mathcal{C}_4 actually coincides with the full group $A(\mathcal{C}_4) = S_7$. With respect to the generators (7) and (8) of S_7 this gives a presentation of S_7 in terms of the generators

$$\alpha = (1\ 3\ 2\ 4), \quad \beta = (1\ 5\ 4)(2\ 6\ 3), \quad \gamma = (1\ 5\ 7)(2\ 6)$$

and the relations (9) and (10).

Coxeter and Sinkov have also verified the existence of the "universal" 4-incidence-polytope in the class $\langle \{4, 3\}, \{3, 6\}_{2,2} \rangle_7$ with group $S_7 \times S_2$ generated by the permutations

$$\rho_0 = (3\ 4)(8\ 9), \quad \rho_2 = (1\ 5)(2\ 6)(8\ 9),$$

$$\rho_1 = (1\ 4)(2\ 3)(8\ 9), \quad \rho_3 = (5\ 7)(8\ 9).$$

Its "rotation" subgroup is again S_7 (with the same generators α, β, γ as above), and it has twice the number of vertices and facets as \mathcal{C}_4 . Our incidence-polytope \mathcal{C}_4 is obtained from it as an "elliptic" contraction, that is, by deleting the transposition (8 9) in the generating permutations (see (7) and (8)).

We remark that Coxeter and Sinkov have also found the universal 4-incidence-polytope in the class $\langle \{4, 3\}, \{3, 6\}_{2,2} \rangle_8$, which has 432 vertices and 1296 facets.

The self-dual 24-cell $\{3, 4, 3\}$ gives rise to two non-degenerate 5-incidence-polytopes of type $\langle 3, 4, 3 \rangle, \{3, 6\}_{2,2}$. When our constructions are applied to their duals, we also achieve regular 6-incidence-polytopes of type $\{6, 3, 4, 3, 6\}$. In this way the 5-incidence-polytope with group $S_{25} \times [3, 4, 3]$ is realized as the vertex-figure and its dual as the facet of a 6-incidence-polytope of this type. On the other hand, there is also a 6-incidence-polytope of type $\{6, 3, 4, 3, 6\}$ whose facets (with group $S_{25} \times [3, 4, 3]$) are not the duals of the vertex-figures (with group S_{25}). However, the 4-faces are the duals of the co-faces to 1-faces, the latter being isomorphic to our 4-incidence-polytope \mathcal{C}_4 .

Coxeter's and Grünbaum's ${}_5\{3, 5, 3\}_5$ with group $PSL(2, 11)$ (cf. [8, 18]) can also serve as a section-complex of a non-degenerate regular 6-incidence-polytope. Here, we get the type $\{6, 3, 5, 3, 6\}$.

The author does not know whether the honeycombs ${}_6\{3, 6, 3\}_6$ and ${}_8\{3, 6, 3\}_8$ discovered by Weiss (cf. [29]) are actually non-degenerate. If this is true, then our methods would also give regular 6-incidence-polytopes of type $\{6, 3, 6, 3, 6\}$.

Moreover, we remark that the infinite sequence of d -incidence-polytopes of type $\{3^{d-2}, 6\}$ gives rise to an infinite sequence of $(d+1)$ -incidence-polytopes of type $\{6, 3^{d-2}, 6\}$.

The regular 4-incidence-polytope $\mathcal{L}_{3,0}$ was independently found by Grünbaum (cf. [18]) and Coxeter and Shephard (cf. [10]). It is a non-degenerate member of the class $\langle \{4, 4\}_{3,0}, \{4, 3\} \rangle_{10}$ with group $S_6 \times S_2$, 30 vertices, and 20 facets. It also gives rise to an "elliptic" contraction $\mathcal{L}_{3,0}/2$ in the class $\langle \{4, 4\}_{3,0}, \{4, 3\} \rangle_5$ with group S_6 and half the number of vertices and facets. The incidence-polytope $\mathcal{L}_{3,0}/2$ appears also as an "elliptic" contraction of Coxeter's $\{4, 4, 3\}_5$, which is the universal member in the class $\langle \{4, 4\}_{3,3}, \{4, 3\} \rangle_5$ and has group $S_6 \times S_2$ (cf. [7]). However, $\{4, 4, 3\}$ and also its contraction $\mathcal{L}_{3,0}/2$ are degenerate, so that our methods cannot be applied. But from $\mathcal{L}_{3,0}$ we can deduce two 5-incidence-polytopes of type $\langle \mathcal{L}_{3,0}, \{3, 6\}_{2,2} \rangle$.

As pointed out at the end of the last section our list will not give further isomorphism types of incidence-polytopes by passing from an incidence-polytope included in the table to any of its co-faces. For instance, the vertex-figure of the 5-incidence-polytopes of type $\langle \{5, 3, 3\}, \{3, 6\}_{2,2} \rangle$ is the 4-incidence-polytope of type $\langle \{3, 3\}, \{3, 6\}_{2,2} \rangle$ with group $S_5 \times S_4$.

Concluding this section we remark that our constructions lead also to a number of interesting incidence-polytopes, if they are applied to a wider class of structures. For instance, if a reflexible regular map of type $\{p, q\}$ on a surface is given, then we immediately get a 4-incidence-polytope of type $\{p, q, 6\}$. If $q = 3$, then the vertex-figure will be the toroidal map $\{3, 6\}_{2,2}$. For example, for the well-known map $\{7, 3\}_8$ discovered by Klein (cf. Coxeter and Moser [9]) we achieve two members in $\langle \{7, 3\}_8, \{3, 6\}_{2,2} \rangle$ with groups S_{25} and $S_{25} \times PGL(2, 7)$, respectively.

8. FURTHER CONSTRUCTIONS

Here, we present some further constructions which do not fit into the general considerations of Sections 5 and 6. Again, we refer to Theorem 1.

In [18] Grünbaum conjectured that every toroidal regular map $\{6, 3\}_{b,c}$ is the facet of a 4-dimensional incidence-polytope in $\langle \{6, 3\}_{b,c}, \{3, 3\} \rangle$. Extending this problem to higher dimensions we may ask for incidence-polytopes in the class $\langle \{6, 3\}_{b,c}, \{3, \dots, 3\} \rangle$ or dually $\langle \{3, \dots, 3\}, \{3, 6\}_{b,c} \rangle$. For the case $b = c = 2$ the existence of an infinite sequence was verified, even with the nice property that its i -dimensional member is the vertex-figure of the $(i + 1)$ -dimensional one. These results can now be generalized to the class $\langle \{3^{d-1}\}, \{3, 6\}_{2,0} \rangle$ of $(d + 1)$ -incidence-polytopes ($d + 1 \geq 4$).

We start from the d -simplex $\{3^{d-1}\}$ and consider its group U as a permutation group on the vertices $0, 1, \dots, d$ such that the generator ρ_i becomes $\rho_i = (i \ i + 1)$ for $i = 0, \dots, d - 1$. Next, we embed U into the permutation group A on the numbers $0, 1, \dots, d, d + 1, d + 2, d + 3$ generated by the elements of U and by $\rho_d := (d \ d + 1)(d + 2 \ d + 3)$. Then, it turns out that

A has the required properties of Theorem 1. This guarantees the existence of a regular $(d+1)$ -incidence-polytope of type $\{3^{d-1}, 6\}$ with group A and simplicial facets. Easy calculations show also $A_i^+ \simeq S_{\{i, \dots, d+1\}} \times S_{\{d+2, d+3\}} \simeq S_{d+2-i} \times S_2$ for $i=0, \dots, d-1$, implying $A \simeq S_{d+2} \times S_2$ for $i=0$ and $A_{d-2}^+ \simeq S_4 \times S_2$. The latter reveals toroidal co-faces to $(d-3)$ -faces of type $\{3, 6\}_{2,0}$. As $\{3, 6\}_{2,0}$ is degenerate, our incidence-polytope is of course degenerate too. Therefore, we obtain an infinite sequence of degenerate incidence-polytopes of type $\langle \{3^{d-1}\}, \{3, 6\}_{2,0} \rangle$, and by construction, the i -dimensional member of the sequence is the vertex-figure of the $(i+1)$ -dimensional one ($i \geq 4$). It seems that the 4-dimensional member is just the dual of Grünbaum's $\mathcal{H}_{2,0}$ (cf. [18]).

It is also noteworthy that the incidence-polytopes of type $\langle \{3^{d-1}\}, \{3, 6\}_{2,0} \rangle$ have Petrie-polygons; these are hamiltonian and thus have length $d+2$. This shows in particular that in case of degeneracy the number h need not coincide with the length of the Petrie-polygon.

Our methods are less effective for incidence-polytopes of type $\{p_1, \dots, p_{d-1}\}$ with $p_{d-1} \neq 3$. However, in some instances, there are other possibilities for embedding the automorphism group into a suitable permutation group. For instance, the author has proved that, for $b \geq 3$, $c=0$, and $b=c \geq 3$, the toroidal map $\{4, 4\}_{b,c}$ can be realized as the facet-type of a finite regular member in $\langle \{4, 4\}_{b,c}, \{4, 4\}_{4,0} \rangle$. Also for $d \geq 4$, the map $\{4, 4\}_{2,2}$ appears as the 3-face of a finite regular d -incidence-polytope of type $\langle \{4, 4\}_{2,2}, \{4, 3^{d-3}\} \rangle$. Moreover, each map $\{6, 3\}_{b,c}$ is the facet-type of a finite regular 4-incidence-polytope with octahedral vertex-figures $\{3, 4\}$. However, in all these cases the group is rather complicated so that its order is just as little known as whether the respective incidence-polytope is degenerate or not.

Finally, we note that other finite and non-degenerate regular d -incidence-polytopes of the above-mentioned type can be obtained from the results in Danzer [11, 12], namely from the construction of the incidence-polytopes $2^{\mathcal{P}}$. Whereas our constructions provide incidence-polytopes of type $\{p_1, \dots, p_{d-1}, 6\}$, Danzer's methods give incidence-polytopes of type $\{p_1, \dots, p_{d-1}, 4\}$.

ACKNOWLEDGMENTS

The author is indebted to Professors H. S. M. Coxeter and A. Sinkov for both stimulating discussion and many helpful suggestions.

Note added in proof. Further incidence-polytopes of the type in question can be obtained from the constructions in the author's paper "Extensions of regular complexes," *Proc. Finite Geometries*, Winnipeg, July, 1984.

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