The Morse Index of an Isolated Invariant Set Is a Connected Simple System

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1. INTRODUCTION

In [2] C. Conley defines an index for isolated invariant sets of a flow. Given an isolated invariant set of some flow, this index consists of a collection of pointed topological spaces, called index spaces, all of the same homotopy type (the homotopy type defines the homotopy index of an isolated invariant set) and of a collection of homotopy classes of maps between the pointed spaces. Both the spaces and the maps are defined in terms of the flow, and the two collections taken together comprise a category. It is this category which Conley calls the Morse index of the given isolated invariant set. The Morse index of an isolated invariant set generalizes the classical Morse index of a non-degenerate critical point of a vectorfield in that the classical index is a non-negative integer n where n is the dimension of the unstable manifold to the critical point, and considered as an isolated invariant set, the homotopy index of the critical point is the homotopy type of a pointed n-sphere.

As a category, the Morse index enjoys two additional properties which makes it a much stronger invariant than the homotopy index alone. These are (1) the Morse index is a groupoid; i.e., every morphism in the Morse index has an inverse; (2) between any two objects in the Morse index there exists one and exactly one morphism in the Morse index; in particular the only self-morphism of an index space is the homotopy class of the identity map on the index space. These make the Morse index a connected simple system, defined below. The fact that the Morse index is a connected simple system is crucial in developing an appropriate notion of a "map" between Morse indices of different isolated invariant sets where the invariant sets are related by continuation which in turn has application to existence proofs in differential equations, cf. [2], and is also crucial in developing the connection index, a connected simple system whose objects are long coexact sequences of Morse indices of repeller–attractor pairs [5, 6, 2] which again is useful in proving existence theorems, in particular, of orbits connecting one invariant set to another, cf. [2, 7, 8].
The primary purpose of this paper is to give a complete proof of property (2) above. It is immediate from Proposition 1.2 below that to show property (2) it suffices to show that every self-morphism in the Morse index is the homotopy class of the appropriate identity map; this is Theorem 3.5 below. This result was announced by Conley in [2] and an outline of the proof presented here is given there. The proof presented here is that given in the author's P.h.D. dissertation done under the direction of C. Conley.

Besides Theorem 3.5, Theorem 3.4, wherein generalized inclusion induced maps (see 2.11 below) between objects of the Morse index are shown to be equivalences, is another main ingredient in the proof of property (2). A feature of this proof important to the applications (cf. [7, Theorem 3.11]) is the particular homotopy inverse defined for a generalized inclusion induced map between objects of the Morse index.

The description of the morphisms in the Morse index given below differs from the formulation given in [2] and a comparison of the two might at first glance indicate that the collection of morphisms defined here is larger; however, it follows from property (2) above that the collections coincide. This is discussed further after the proof of 3.6 below.

**Notation.** The notation used in this paper is for the most part standard; some possible exceptions: \( \mathbb{R}, \mathbb{R}^+, \text{ and } \mathbb{R}^- \) denote, respectively, the real numbers, the non-negative reals, and the non-positive reals; for \( a, b \in \mathbb{R} \), \([a, b]\) is the open interval from \( a \) to \( b \), \([a, b]\) the corresponding closed interval, and \([a, b]\) and \([a, b]\) the half-open intervals, open on the right and left, respectively; the letter \( I \) is reserved to denote \([0, 1]\); for \( A \) and \( B \) sets, \( A \setminus B = \{ x : x \in A \text{ and } x \notin B \} \). We use \( \text{cl}_x(A) \), \( \text{int}_x(A) \), and \( \partial_x(A) \) to denote the closure, interior, or boundary of a set \( A \) relative to a space \( X \). Other topological notation follows that of Spanier [10]. All homotopies are considered in the category of topological spaces with base point and base point preserving maps. Thus for two maps \( f \) and \( g \), mapping a pointed space \( X \) to another pointed space \( Y \), \( f \sim g \) means there is a homotopy from \( f \) to \( g \) which maps base point to base point throughout the deformation.

We conclude the introduction with the definition of "connected simple system." Section 2 sets the context, defines the collection of index spaces, and sets forth some technical definitions and propositions used to define the morphisms of the Morse index which is done in Section 3 along with showing the Morse index is a connected simple system. Section 4 discusses modifications necessary to treat semi-flows.

**Definition 1.1 (Connected Simple System).**

(1) A groupoid is a small category in which each morphism is an equivalence; call a groupoid simple if for each object \( X \), \( \text{hom}(X, X) = \{1_x\} \).
A simple groupoid is a connected simple system if, and only if, for
each pair of objects $X$ and $Y$ in the groupoid, $\text{hom}(X, Y) \neq \emptyset$.

Once it is proved that the Morse index is a connected simple system, the
following simple proposition shows that there is exactly one morphism
between any two objects of the Morse index.

**Proposition 1.2.** If $X$ and $Y$ are objects of a simple groupoid and if
$\text{hom}(X, Y) \neq \emptyset$, then $\text{hom}(X, Y)$ is a singleton. Consequently in a connected
simple system, for any ordered pair of objects $X$ and $Y$, there is a unique
morphism from $X$ to $Y$.

**Proof.** If $f, g \in \text{hom}(X, Y)$, then $f \circ g^{-1} = 1_y$ and $g^{-1} \circ f = 1_x$ so that $f = (g^{-1})^{-1} = g$. 

### 2. Local Flows, Isolated Invariant Sets, and Index Pairs

The Morse index will be defined in the following context which is the same
as that of [2]. Let $\Gamma$ be an arbitrary topological space with $\Gamma_0 \subset \Gamma$ an open
subset of $\Gamma$ which is Hausdorff in the inherited topology ($\Gamma$ need not be
Hausdorff). We assume that $\Gamma$ admits a flow $\text{f}$; i.e., there exists a continuous
map $\text{f} : \Gamma \times \mathbb{R} \rightarrow \Gamma$ satisfying $\text{f}(\gamma(t_1), t_2) = \text{f}(\gamma(t_1 + t_2))$ for any $\gamma \in \Gamma$,
$t_1, t_2 \in \mathbb{R}$. Whenever convenient we write the flow as a right action of the
additive group of real numbers on the space $\Gamma$, $\text{f}(\gamma, t) = \gamma \cdot t$. For $M \subset \Gamma$ and
$K \subset \mathbb{R}$ we write $M \cdot K$ for $\text{f}(M \times K)$. Also we denote by $\text{f}^*$ the time-reversed
flow induced by $\text{f}$; i.e., $\text{f}^*(\gamma, t) = \text{f}(\gamma, -t)$ for $\gamma \in \Gamma$, $t \in \mathbb{R}$.

**Definition 2.1 (Local flows and semi-flows).**

1. A subset $\Phi \subset \Gamma_0$ is a local semi-flow if, and only if, for each $\gamma \in \Phi$, there exists $\epsilon > 0$ and $U$ open in $\Gamma$ with $\gamma \in U$ so that $(U \cap \Phi) \cdot [0, \epsilon] \subset \Phi$.

2. A subset $\Phi \subset \Gamma_0$ is a local flow if, and only if, for each $\gamma \in \Phi$, there exists $\epsilon > 0$ and $U$ open in $\Gamma$ with $\gamma \in U$ so that $(U \cap \Phi) \cdot [-\epsilon, \epsilon] \subset \Phi$.

For the remainder of this paper it is assumed that all local flows and local
semi-flows are locally compact in the topology inherited from $\Gamma$.

The motivation for the above context is briefly as follows; for details see
[2]. In applications of the Morse index to the study of the qualitative
behavior of autonomous differential equations on $\mathbb{R}^n$, $\Gamma$ is taken to be the
space of curves into $\mathbb{R}^n$ which have an open interval in $\mathbb{R}$ as domain and
which have closed graph in $\mathbb{R} \times \mathbb{R}^n$ with an appropriate version of the
compact-open topology; $\Gamma_0$ is taken to be the subspace of $\Gamma$ of all those
curves which contain zero in their domain. The flow $\text{f}$ is taken to be the tran-
slation of domain flow on the space of curves. Every maximal integral curve of the equation is an element of $\Gamma$, and the collection of those maximal integral curves with zero in their domain form a local flow $\Phi \subset \Gamma_0$ assuming that the domain of the vectorfield of the equation is an open subset $V \subset \mathbb{R}^n$. Moreover $\Phi$ is the embedded image of $V$ in $\Gamma$ via the assignment to each point $x$ of $V$ the maximal integral curve of the equation which passes through $x$ at time zero, and hence $\Phi$ is locally compact. Also this embedding is equivariant with respect to the flow on phase space defined by the equation and the translation flow on the space of curves. Local semi-flows arise similarly from evolution equations where $\Gamma_0$ will be those curves whose domain contains some specified closed interval of non-positive real numbers including zero.

The reason for defining the Morse index in terms of the translation flow on the space of curves rather than directly in terms of the flow on phase space is the resulting simplification of the statements and proofs of various perturbation lemmas involving the Morse index wherein the translation flow acts as a "universal" flow in which all flows on phase space can be embedded and conveniently compared.

Remark. The terminology of Definition 2.1 above differs from that of [2]. In [2] what is here called a local semi-flow is there called a local flow and what is here called a local flow is there called a two-sided local flow. The terminology of 2.1 is used here since it seems to be more in keeping with standard usage of the words "flow" and "semi-flow."

**Definition 2.2.** Recall the following basic definitions. A set $Y \subset \Gamma$ is respectively invariant, positively invariant, negatively invariant as $Y \cdot \mathbb{R}$, $Y \cdot \mathbb{R}^+$, $Y \cdot \mathbb{R}^-$ equals $Y$. The sets invariant under the action of a flow are partially ordered by inclusion and any reference to maximality refers to this partial ordering. Thus define $\omega(Y)$ to be the maximal invariant set of $\text{cl}_{\Gamma}(Y \cdot \mathbb{R}^+)$ and $\omega^*(Y)$ to be the maximal invariant set of $\text{cl}_{\Gamma}(Y \cdot \mathbb{R}^-)$. The set $\omega(Y)$ coincides with the usual notion of the $\omega$-limit set of $Y$ under the action of the flow on $\Gamma$. The set $\omega^*(Y)$ coincides with what is more usually called the $\alpha$-limit set of $Y$.

When $Y \subset \Gamma$ is compact and positively invariant, since $\Gamma$ is not Hausdorff, it may be the case that $\omega(Y) \notin Y$. To simplify the presentation the following definitions are made:

For $Z \subset \Gamma$ and $Y \subset \Gamma$ define $\omega(Z; Y) = \omega(Z) \cap Y$ and $\omega^*(Z; Y) = \omega^*(Z) \cap Y$.

Note that if $Y$ is positively invariant and $Z \subset Y$, then $\omega(Z; Y) = \bigcap_{|t, \infty|: t \in \mathbb{R}^+} \text{cl}_{\Gamma}(Z \cdot t)$. Also if $Z \subset Y \subset N \subset \Gamma$ and $Y$ is $N$-closed and positively invariant, then $\omega(Z; N) = \omega(Z; Y)$. Analogous remarks hold for the $\omega^*$ limit sets.
DEFINITION 2.3 (Isolating neighborhood and isolated invariant set).

(1) If $\Phi \subset \Gamma_0$ is a local (semi-) flow, a compact $\Phi$-neighborhood, $N$, is an isolating neighborhood (relative to $\Phi$) if, and only if, the maximal invariant subset of $N$ is contained in the $\Phi$-interior of $N$.

(2) If $\Phi \subset \Gamma_0$ is a local (semi-) flow, a set $S \subset \Phi$ is an isolated invariant set if, and only if, $S$ is the maximal invariant subset of some isolating $\Phi$-neighborhood.

(3) For $N$ an isolating neighborhood as defined in (1) above, define $A^+(N) = \{y: y \cdot \mathbb{R}^+ \subset N\}$. $A^+(N)$ and $A^-(N)$ are called the forward and backward asymptotic sets of $N$ resp. Note if $S$ is the isolated invariant set of $N$, then $S = A^+(N) \cap A^-(N)$.

DEFINITION 2.4. Let $(N, K)$ be a topological pair in $\Gamma$.

(1) $K$ is positively invariant relative to $N$ if, and only if, for each $y \in K$ and $t \in \mathbb{R}^+$, $y \cdot [0, t] \subset N$ implies $y \cdot [0, t] \subset K$.

(2) For each $t \in \mathbb{R}^+$, define

$$K' = \{y: y \cdot [-t, 0] \subset K\}$$

and define

$$K^{-t} = \{y \in N: \exists s, 0 \leq s \leq t \text{ and } y \cdot [0, s] \subset N \text{ and } y \cdot s \in K\}.$$

Note that $K'$ does not depend on $N$ whereas $K^{-t}$ does; however, the lack of explicit reference to $N$ will not cause any confusion in context.

We now have sufficient definitions at our disposal to define the index spaces associated to an isolated invariant set. Index spaces will be quotient spaces of the following types of pairs.

DEFINITION 2.5 (Index Pairs and Index Spaces). Let $\Phi \subset \Gamma_0$ be a local (semi-) flow, $N$ an isolating neighborhood relative to $\Phi$ with maximal invariant set $S$, and suppose that $(N_1, N_2)$ is an ordered pair of compact subsets of $N$ (no containment relation between $N_1$ and $N_2$ implied); then $(N_1, N_2)$ is an index pair for $S$ relative to $N$ if, and only if, the following three properties hold:

(1) (relative positive invariance property) $N_1$ and $N_2$ are positively invariant relative to $N$,

(2) (isolating property) $S \subset \text{int}_\Phi(N_1 \setminus N_2)$,

(3) (exit property) if $y \in N_1 \setminus A^+(N)$, then $y \in N_2^{-t}$ for some $t \in \mathbb{R}^+$. 
If the containment relation $N_1 \supset N_2$ holds for an index pair $\langle N_1, N_2 \rangle$, then $\langle N_1, N_2 \rangle$ will be called a nested index pair. Note that if $\langle N_1, N_2 \rangle$ is an index pair relative to an isolating neighborhood $N$, it is straightforward to verify that $\langle N_1, N_1 \cap N_2 \rangle$ and $\langle N_1 \cup N_2, N_2 \rangle$ are both nested index pairs relative to $N$.

Given an index pair $\langle N_1, N_2 \rangle$, the quotient space $N_1/N_1 \cap N_2$ is defined to be the index space of the index pair. As $N_1/N_1 \cap N_2 = N_1\setminus N_2 = N_1 \cup N_2/N_2$ and $N_1 \cup N_2/N_2$ are naturally homeomorphic, but for technical reasons the distinction between them will be maintained. For convenience $N_1/(N_1 \cap N_2)$ will be abbreviated to $N_1/N_2$.

The existence of index pairs as defined above is proved by Conley in [2]. The existence of index pairs in the context of semi-flows on a compact metric space was proved by T. G. Young in his thesis [12] under the direction of Conley. Figure 1 illustrates an index pair $\langle N_1, N_2 \rangle$ relative to an isolating neighborhood $N$ for a hyperbolic critical point in the plane. For further examples see [2].

The following series of technical definitions and propositions provides the machinery necessary to define the morphisms of the Morse index which are obtained by deforming index spaces under the action of the flow and to show that the morphisms so defined are equivalences.

**Definition 2.6 (Exit and Entrance Time Maps).** Suppose $N \subseteq I_0$.

1. If $N$ is compact define for each $\gamma \in N$,
   
   $\sigma | N(\gamma) = \sup \{ s \geq 0 : \gamma \cdot [0, s) \subset N \}$

   and

   $\sigma^* | N(\gamma) = \sup \{ s \geq 0 : \gamma \cdot [-s, 0] \subset N \}$;

2. If $N$ has an upper semi-continuous decomposition by compact sets $\{ N_\lambda : \lambda \in \Lambda \}$ (see [11] for the definition of an u.s.c. decomposition) define $\sigma | N = \bigcup \{ \sigma | N_\lambda : \lambda \in \Lambda \}$; i.e., $\sigma | N(\gamma) = \sigma | N_\lambda(\gamma)$ if $\gamma \in N_\lambda$.

![Fig. 1. An index pair for a hyperbolic point.](image)
PROPOSITION 2.7. Suppose $N \subset \Gamma_0$ and has an upper semi-continuous decomposition by compact sets $\{N_\lambda : \lambda \in \Lambda\}$. Then $\sigma | N : N \to [0, \infty]$ and $\sigma^* | N : N \to [0, \infty]$ are both upper semi-continuous. If $N$ itself is compact, then for each $\gamma \in N$ such that $\gamma \cdot \sigma | N(\gamma) < \infty$ (resp. $\gamma \cdot -\sigma^* | N(\gamma) < \infty$), $\gamma \cdot \sigma | N(\gamma) \in \partial_{\Gamma_0} N$ (resp. $\gamma \cdot -\sigma^* | N(\gamma) \in \partial_{\Gamma_0} N$).

Remark. Note that when $N$ is compact the singleton $\{N\}$ is an u.s.c. decomposition of $N$ by compact sets.

Proof. For simplicity the "$N$" in the notation of $\sigma | N$ will be suppressed. Note since $\sigma^*$ is just $\sigma$ defined for the time-reversed flow only that part of the proposition concerning $\sigma$ needs to be proved explicitly.

First suppose $N$ is compact, $\gamma \in N$, and $\sigma(\gamma) < \infty$. As $N$ is $\Gamma_0$-closed, $\partial_{\Gamma_0} N \subset N$, and by definition of $\sigma$, $\gamma \cdot \sigma(\gamma) \in \partial_{\Gamma} N$; hence as $\Gamma_0$ is open, it follows that if $\gamma \cdot \sigma(\gamma) \in N$, then $\gamma \cdot -\sigma^* | N(\gamma) \in \partial_{\Gamma_0} N$.

To show $\gamma \cdot \sigma(\gamma) \in N$, suppose $0 < \sigma(\gamma) < \infty$ (if $\sigma(\gamma) = 0$, there is nothing to show). Choose $U$ open and $\varepsilon > 0$ so that $N \subset U \subset \Gamma_0$ and $U \cdot [-\varepsilon, \varepsilon] \subset \Gamma_0$; this can be done because $N$ is compact, $\Gamma_0$ is open, and the flow is continuous. Choose a positive integer $i_0$ satisfying $1/i_0 < \varepsilon$ and $1/i_0 < \sigma(\gamma)$. Now by definition of $\sigma$, for each integer $i \geq i_0$, $\gamma \cdot \sigma(\gamma) - 1/i \in N$. In particular, $\gamma \cdot \sigma(\gamma) - 1/i_0 \in N$, and as $0 < 1/i_0 < \varepsilon$,

$$
\gamma \cdot \sigma(\gamma) \in \gamma \cdot \sigma(\gamma) - 1/i_0 \cdot [0, \varepsilon] \subset N \cdot [0, \varepsilon] \subset U \cdot [-\varepsilon, \varepsilon] \subset \Gamma_0,
$$

and because the continuity of the flow implies $\gamma \cdot \sigma(\gamma) - 1/i$ converges to $\gamma \cdot \sigma(\gamma)$ as $i$ increases to $\infty$, it follows that $\gamma \cdot \sigma(\gamma)$ is a limit point of $N$ in $\Gamma_0$; whence, $\gamma \cdot \sigma(\gamma) \in N$ because $N$ is closed relative to $\Gamma_0$.

To show the upper semi-continuity of $\sigma$ it suffices to show that $\{\eta : \sigma(\eta) < r\}$ is open for each $r > 0$. Thus suppose $0 < r < \infty$, suppose $\gamma \in N_\lambda \subset N$, and suppose $\sigma | N_\lambda(\gamma) < r$. By what has been shown above, $\gamma \cdot \sigma(\gamma) \in \partial_{\Gamma_0} N_\lambda$. Then by definition of $\sigma | N_\lambda$, choose $\delta > 0$ so that $\sigma(\gamma) + \delta < r$ and $\gamma \cdot \sigma(\gamma) + \delta \in \Gamma_0 \setminus N_\lambda$. As $\Gamma_0$ is open and Hausdorff in $\Gamma$ and as $N_\lambda \subset \Gamma_0$ is compact, choose disjoint open sets $U, V \subset \Gamma_0$ with $\gamma \cdot \sigma(\gamma) + \delta \in U$ and $N_\lambda \subset V$. By the continuity of the flow choose $Q$ open about $\gamma$ so that $Q \cdot \sigma(\gamma) + \delta \subset U$. As $\{N_\lambda : \lambda \in \Lambda\}$ is an u.s.c. decomposition, for some open $W$, $N_\lambda \subset W$ and for each $\mu \in \Lambda$, if $N_\mu \cap W \neq \emptyset$, then $N_\mu \subset V$ (this is the definition of an u.s.c. decomposition). Then if $\eta \in N \cap Q \cap W$, $\eta \cdot \sigma | N_\lambda(\gamma) + \delta \in U$, and for some $\mu \in \Lambda$, $\eta \in N_\mu \subset V$. As $V$ is disjoint from $U$, it follows that $\sigma | N(\eta) = \sigma | N_\mu(\eta) < \sigma | N_\lambda(\gamma) + \delta < r$. Because $\gamma \in N \cap Q \cap W$ and $N \cap Q \cap W$ is open in $N$, this completes the proof. \[\Box\]

Remark 1. Because upper semi-continuous functions assume their supremum on compact sets [3, X1.2.4], in particular, if $N \supset K$ compact and for each $\gamma \in K$, $\sigma | N(\gamma) < \infty$, then $\sigma | N$ is bounded on $K$ and assumes its
finite maximum—analogous remarks hold for \( \sigma^* | N \). This will be used quite often in the sequel.

Remark 2. If \( I \) were assumed Hausdorff, then \( N \) would be closed relative to \( I \), and it would be immediate that \( y \cdot \sigma(y) \in N \) if \( \sigma(y) < \infty \). The rest of the proof could also be simplified somewhat.

Remark 3. Note that if \( N \) is an isolating neighborhood, \( A^+(N) = (\sigma | N)^{-1}(\infty) \) and \( A^-(N) = (\sigma^* | N)^{-1}(\infty) \) so that both \( A^+(N) \) are closed relative to \( N \), hence compact. It follows that the isolated invariant set contained in \( N \) is also compact.

Remark 4. In 1 R. Churchill constructs very special nested index pairs \((B, b^-)\) and \((B, b^+)\), called isolating blocks, in the context of flows on a compact metric space—\((B, b^-)\) is an index pair for the flow and \((B, b^+)\) is an index pair for the time-reversed flow and both isolate the same invariant set. What makes these pairs particularly special is that \( \sigma \cdot B \) and \( \sigma^* \cdot B \) are continuous and \( b^- = (\sigma \cdot B)^{-1}(0) \), \( b^+ = (\sigma^* \cdot B)^{-1}(0) \) and \( b^- \) and \( b^+ \) local sections of the flow.

Churchill's construction can easily be adapted to the context of locally compact Hausdorff local flows \( \Phi \subset I_0 \) using Corollary 2 of 2.8 below, in particular that a \( \Phi \)-isolated invariant set \( S \) is a \( G_\delta \) relative to \( \Phi \). For from this, if \( N \) is an isolating \( \Phi \)-neighborhood of \( S \), as \( N \) is a normal space and \( S \subset \text{int}_\Phi N \), it follows that there exists a Urysohn function \( \rho: N \to [0, 1] \) with \( S = \rho^{-1}(0) \) [cf. 3, VII.4.2]. Churchill uses the distance from \( S \) for \( \rho \), and no other essential use of the metric is made; i.e., his arguments for constructing \( B \subset \text{int}_\Phi N \) with \( \sigma \cdot B \) and \( \sigma^* \cdot B \) continuous go over either by replacing his sequential (semi-) continuity and convergence arguments with arguments using nets or filterbases, or by using the local definitions of (semi-) continuity in terms of open sets. The reader should have little difficulty in supplying the details.

The next proposition and its first corollary are trivialities when \( \Phi = I_0 = I \); the proofs of the general case are left to the reader, or see [4, Proposition 2.4].

**Proposition 2.8.** Suppose \( \Phi \subset I_0 \) is a local (semi-) flow, \( K \subset \Phi \) is compact, and \( V \) is \( I \)-open. If \( t \in \mathbb{R}^+ \) and \( K \cdot [0, t] \subset V \cap \Phi \), then there exist \( U \cap \Phi \) is \( I \)-open and \( \epsilon > 0 \) so that \( K \subset U \) and \( \text{cl}_\Phi(U \cap \Phi) \cdot [0, t + \epsilon] \subset V \cap \Phi \), and \( \text{cl}_\Phi(U \cap \Phi) \) is compact. Furthermore, if \( C \subset H \subset \Phi \) and \( K \subset \text{int}_H(C) \), then \( U \) can be chosen so that it also satisfies \( \text{cl}_\Phi(U \cap \Phi) \cap H \subset \text{int}_H(C) \).

**Corollary 1.** Let \( \Phi \subset I_0 \) be a local flow, \( K \subset \Phi \) compact, and \( V \) open. Suppose \( K \cdot [a, b] \subset \Phi \cap V \). If (i) \( a \geq 0 \) and \( K \cdot [0, a] \subset \Phi \) or (ii) \( b \leq 0 \) and \( K \cdot [b, 0] \subset \Phi \) or (iii) \( 0 \in [a, b] \), then there exist \( U \) open and \( \epsilon > 0 \) so that
$K \subset U$, $\text{cl}_\Phi(U \cap \Phi)$ is compact, and $\text{cl}_\Phi(U \cap \Phi) \cdot [a - \epsilon, b + \epsilon] \subset \Phi \cap V$. Also, if $C \subset H \subset \Phi$ and $K \subset \text{int}_H(C)$, then $\text{cl}_\Phi(U \cap \Phi) \cap H \subset \text{int}_H(C)$.

**Corollary 2.** Let $\Phi \subset \Gamma_0$ be a local (semi-)flow, $N$ an isolating $\Phi$-neighborhood, and $S$ the maximal invariant subset of $N$. Then for each $t \in \mathbb{R}^+$, defining $U_t = \{x \in N : \sigma|N(x) > t\}$ and $U_{t*} = \{x \in N : \sigma^*|N(x) > t\}$, it follows that $\bigcap \{U_t : t \in \mathbb{R}^+\} = A^+(N)$, $\bigcap \{U_{t*} : t \in \mathbb{R}^+\} = A^-(N)$, and for each $t \in \mathbb{R}^+$, $U_t$ is a compact $\Phi$-neighborhood of $S$; if in fact $\Phi$ is a local flow, then $U_{t*}$ is also a compact $\Phi$-neighborhood of $S$, and $S$ is a $G_\delta$ relative to $\Phi$.

**Proof.** Clearly $A^+(N) = \bigcap \{U_t : t \in \mathbb{R}^+\}$ and $A^-(N) = \bigcap \{U_{t*} : t \in \mathbb{R}^+\}$. Also the upper semi-continuity of $\sigma|N$ and $\sigma^*|N$ implies that for each $t \geq 0$, $U_t$ and $U_{t*}$ are closed relative to $N$; hence both are compact as $N$ is.

Let $t > 0$. As observed in Remark 3 after 2.7 above, $S$ is compact. Also, because $N$ is a $\Phi$-neighborhood of $S$, for some $I$-open $V$, $S \subset V \cap \Phi \subset N$, and as $S$ is invariant it follows that $S \cdot [0, t] \subset \Phi \cap \Phi$. Thus applying the proposition yields $U^+$ $\Gamma$-open and $\epsilon > 0$ so that $S \subset U^+$ and $U^+ \cap \Phi \cdot [0, t + \epsilon] \subset \Phi \subset N$. It follows that for each $x \in U^+ \cap \Phi$, $x \in N$ and $\sigma|N(y) > t$ so that $S \subset U^+ \cap \Phi \subset U_t$, which shows that $U_t$ is a $\Phi$-neighborhood of $S$. If in fact $\Phi$ is a local flow, then $\Phi$ is a local semi-flow relative to the time-reversed flow $f^*$, and repeating the above argument in this case yields that $U_{t*}$ is a $\Phi$-neighborhood of $S$. Finally, because $0 < s_1 < s_2$ implies that $U_{s_1} \subset U_{s_2}$ and $U_{s*} \subset U_{s*}$, it follows that $S \subset \bigcap_{i=1}^\infty \text{int}_\Phi(U_{t_i} \cap U_{t*}) \subset \bigcap_{i=1}^\infty U_{t_i} \cap \bigcap_{i=1}^\infty U_{t_i} = A^+(N) \cap A^-(N) = S$, where for each integer $i > 0$, $t_i = i$, which shows that $S$ is a $G_\delta$ relative to $\Phi$.

**Proposition 2.9.** Let $\Phi \subset \Gamma_0$ be a local flow and $(N_1, N_2)$ an index pair relative to some isolating $\Phi$-neighborhood $N$.

1. For each $t \in \mathbb{R}^+$, $(N_1^t, N_2^t)$, $(N_1, N_2^t)$, and $(N_1^t, N_2^{-t})$ are index pairs relative to $N$.

2. For each $t \in \mathbb{R}^+$, $A^+(N) \cap N_2^{-t} = \emptyset$, and for all large enough $t \in \mathbb{R}^+$, $N_1 \cap N_2^{-t}$ is an $N_1$-closed $N_2$-neighborhood of $N_1 \cap N_2$.

3. $\bigcup \{N_1^{s*} \times \{s\} : s \in [0, 1]\}$ and $\bigcup \{N_2^{-s*} \times \{s\} : s \in [0, 1]\}$ are compact subsets of $N \times I$.

**Proof.** The easy verification that $N_1^t$ and $N_2^{-t}$ are positively invariant relative to $N$ for each $t \in \mathbb{R}^+$ is omitted. From the definition of $N_1^t$, it is immediate that $N_1^t = \{y \in N_1 : \sigma|N(y) \geq t\}$. By definition of an index pair $S \subset \text{int}_\Phi(N_1) \subset N_1 \subset N \subset \Phi$ and $N_1$ is compact. Because $N$ is an isolating $\Phi$-neighborhood, it follows that $N_1$ is too; whence by Corollary 2 to Proposition 2.8, it is immediate that $N_1^t$ is a compact $\Phi$-neighborhood of $S$. Define $\tau : N \times I \to [0, t]$ by $\tau(y, s) = \min\{st, \sigma|N(y)\}$, and for each $s \in I$,
define $\tau_t: N \to [0, sr] \mid \tau_t(y) \equiv \tau(y, s)$. Because the infimum of upper semi-continuous functions is upper semi-continuous, it follows that $\tau$ and $\tau_t, s \in I$, are upper semi-continuous. It is straightforward to show that $\mathcal{N} \setminus \mathcal{N}^{-t}_{2} = \{y \in \mathcal{N} : 0 < \tau(y) \leq t\} \cap \mathcal{N} = \emptyset$. By definition of the relative topology, $\phi \setminus \mathcal{N} = \mathcal{V} \cap \phi$ for some $\mathcal{V}$-open $\mathcal{V}$. Thus if $\gamma \in \mathcal{N} \setminus \mathcal{N}^{-t}_{2}$, because $\tau_t(y) = \min\{t, \sigma\} | N(y)\}$, it follows that $\gamma \cdot [0, \tau(y)) \subset V \cap \phi$. By 2.8, for some open $U$ and for some $\varepsilon > 0$, $\gamma \in U$ and $U \cap \phi \setminus [0, \tau(y) + \varepsilon] \subset V \cap \phi$. Because $\tau_t: N \to [0, t]$ is upper semi-continuous, shrinking $U$ if necessary, it can also be assumed that for each $\eta \in U \cap \mathcal{N}$, $\tau_t(\eta) < \tau_t(y) + \varepsilon$; hence for each $\eta \in U \cap \mathcal{N}$, $\gamma \cdot [0, \tau(\eta)) \subset V \cap \phi = \phi \setminus \mathcal{N}$. Thus $\gamma \in U \cap \mathcal{N} \subset \mathcal{N} \setminus \mathcal{N}^{-t}_{2}$, which shows that $\mathcal{N}^{-t}_{2}$ is $\mathcal{N}$-closed, hence compact.

Suppose $\gamma \in A^{-}(N) \cap \mathcal{N}^{-t}_{2}$, and let $S$ be the maximal invariant subset of $\mathcal{N}$. Because $A^{-}(N)$ is positively invariant and compact, $\emptyset \neq \omega(y; A^{+}(N)) \subset A^{+}(N) \subset \mathcal{N}$, and as $\omega(y; A^{+}(N))$ is an invariant set and $\mathcal{N}$ isolates $S$, $\omega(y; A^{+}(N)) \subset \mathcal{S} = \text{int}_{\phi}((\mathcal{N} \setminus \mathcal{N}^{-t}_{2}) \subset \mathcal{N} \setminus \mathcal{N}^{-t}_{2})$. It follows that for all large enough $t$, say $t \geq t_0 > 0$, $\gamma \cdot t \in \mathcal{N} \setminus \mathcal{N}^{-t}_{2}$. However, because $\gamma \cdot \mathcal{R}^{+} \subset \mathcal{N}$ and $\mathcal{N} \setminus \mathcal{N}^{-t}_{2}$ is positively invariant relative to $\mathcal{N}$, $\gamma \cdot t_0 \in \mathcal{N}$, a contradiction. Thus $A^{+}(N) \cap \mathcal{N}^{-t}_{2} = \emptyset$, and it follows that for each $t \in \mathcal{R}^{+}$, $A^{+}(N) \cap \mathcal{N}^{-t}_{2} = \emptyset$; otherwise, for some $t$ and $\gamma \in A^{+}(N) \cap \mathcal{N}^{-t}_{2}$ and for some $\sigma$, $0 < \sigma \leq t$ and $\gamma \cdot [0, \sigma] \subset \mathcal{N}$ and $\gamma \cdot \sigma \in \mathcal{N}^{-t}_{2}$, whence $\gamma \cdot \sigma \in A^{+}(N) \cap \mathcal{N}$, a contradiction. This shows that the first part of conclusion (2) of the proposition holds, and because $S \subset A^{+}(N)$, that for each $t \in \mathcal{R}^{+}$, $S \subset \mathcal{N}^{-t}_{2} = \emptyset$. Since it has already been shown that for each $t \in \mathcal{R}^{+}$, $S \subset \text{int}_{\phi}(\mathcal{N}^{-t}_{2})$ and $\mathcal{N}^{-t}_{2}$ is $\emptyset$-closed, it follows that $S \subset \text{int}_{\phi}(\mathcal{N}^{-t}_{2}) \subset \mathcal{N} \setminus \mathcal{N}^{-t}_{2}$, and completes showing that conditions (1) and (2) in the definition of an index pair are satisfied by each of the pairs in (1) of the current proposition. These pairs also satisfy the exit property of index pairs, for if $\gamma \in \mathcal{N}^{-t}_{1}$, $\gamma \in A^{+}(N)$, and $t \in \mathcal{R}^{+}$, because $\langle \mathcal{N}^{-t}_{1}, \mathcal{N} \setminus \mathcal{N}^{-t}_{2} \rangle$ is an index pair, for some $s$, $\gamma \in \mathcal{N}^{-s-t}_{2}$, and if $s \leq t$, $\mathcal{N}^{-s}_{2} \subset \mathcal{N}^{-t}_{2}$ whereas if $s > t$, setting $s' = s - t$, $\gamma \in (\mathcal{N}^{-t}_{2})^{-t'}$; and this is sufficient to show that the exit property is satisfied for the pairs in (1).

Next, it will be shown that for all large enough $t \geq 0$, $\mathcal{N}^{-t}_{1} \subset \mathcal{N}^{-t}_{2}$ is an $\mathcal{N}$-neighborhood of $\mathcal{N} \setminus \mathcal{N}^{-t}_{2}$. Because $\mathcal{N}$ is a compact Hausdorff space it is normal, and because $A^{+}(\mathcal{N}) \cap \mathcal{N} \subset A^{+}(N) \cap \mathcal{N} = \emptyset$, it follows that for some $U, A^{+}(\mathcal{N}) \subset \mathcal{U} \subset \text{cl}_{\mathcal{N}}(U) \subset \mathcal{N} \setminus \mathcal{N}$. Then, taking complements in $\mathcal{N}$, $\mathcal{N} \setminus \mathcal{U} \supset \mathcal{N} \setminus \text{cl}_{\mathcal{N}}(U) \supset \mathcal{N} \setminus \mathcal{N} \subset \mathcal{N} \setminus \mathcal{N}$. Now $\mathcal{N} \setminus \mathcal{U}$ is compact, and as $A^{+}(\mathcal{N}) \subset \mathcal{V}$, for each $\gamma \in \mathcal{N} \setminus \mathcal{U}$, $s | \mathcal{N}(\gamma) < \infty$. As noted after Proposition 2.7 in Remark 1, this implies that $s | \mathcal{N}$ is bounded on $\mathcal{N} \setminus \mathcal{U}$ and set $t$ to be the supremum on $\mathcal{N} \setminus \mathcal{U}$ plus one. Then from the exit property of index pairs it follows that $\mathcal{N} \setminus \mathcal{U} \subset \mathcal{N}^{-t}_{2}$. Then by the inclusions noted previously, $\mathcal{N} \setminus \mathcal{U} \subset \mathcal{N} \setminus \text{cl}_{\mathcal{N}}(U) \subset \mathcal{N} \setminus \mathcal{U} \subset \mathcal{N} \setminus \mathcal{N}^{-t}_{2}$, which shows that $\mathcal{N} \cap \mathcal{N}^{-t}_{2}$ is an $\mathcal{N}$-neighborhood of $\mathcal{N} \cap \mathcal{N}$ because $\mathcal{N} \setminus \text{cl}_{\mathcal{N}}(U)$ is $\mathcal{N}$-open; hence for each $t \geq t$, $\mathcal{N} \subset \mathcal{N}^{-t}_{2}$ is an $\mathcal{N}$-neighborhood of $\mathcal{N} \subset \mathcal{N}$ because $\mathcal{N}^{-t}_{2} \subset \mathcal{N}^{-t}_{2}$, for $t \geq t$. 
To show (3), set \( C = \bigcup \{ N_i \times \{ s \} : s \in [0, 1] \} \) and \( B = \bigcup \{ N_i^{**} \times \{ s \} : s \in [0, 1] \} \) and let \( \tau: N \times I \to [0, \tau] \) be defined as in the first paragraph of this proof. First, suppose \((y, r) \in N_1 \times \Lambda \). Then \( y \in N_1 \setminus N_2^{**} \) so that \( 0 \leq \sigma_* | N_2(y) < \tau_1. \) Then for some \( \varepsilon > 0, \) \( \sigma_* | N_2(y) < (\tau - \varepsilon) \), and because \( \sigma_* | N_2 \) is upper semi-continuous, for some \( N_1 \)-neighborhood \( Q \) of \( y, \) \( \eta \in Q \) implies \( \sigma_* | N_1(\eta) < (\tau - \varepsilon). \) Then \((y, r) \in Q \times [\tau - \varepsilon, 1] \subset N_1 \times \Lambda \) which shows that \( C \) is closed relative to \( N_1 \times I, \) hence compact.

Next, suppose \((y, r) \in N \times \Lambda \setminus B. \) Then \( y \in N \setminus N_2^{**} \) and it follows that \( y \cdot [0, \tau(y, r)] \cap N_2 = \emptyset. \) Mimicking the proof that \( N_2^{**} \) is closed, but using the fact that \( \tau \) rather than \( \tau_1 \) is upper semi-continuous, then shows that for some open \( U \) about \( y, \) and for some \( I\)-open interval \( J \) about \( r, \) \( (\eta, s) \in U \cap N \times J \) implies \( \eta \cdot [0, \tau(\eta, s)] \cap N_2 = \emptyset. \) Hence that \((y, r) \in U \cap N \times J \subset N \times \Lambda \setminus B. \) Thus \( B \) is closed in \( N \times I, \) hence compact. \( \square \)

The remaining results of this section except for the last lemma will be used to show continuity of the maps which define the homotopy classes of the morphisms in the Morse index. The last lemma is used in showing that several of these maps are homotopy equivalences. The proofs of 2.10, 2.11, 2.12, and 2.13 are all straightforward and are left to the reader, or see [4].

**Proposition 2.10.** Let \( h: X \to Y \) be a continuous map, and let \((A, B)\) and \((C, D)\) be topological pairs in \( X \) and \( Y, \) respectively, satisfying (i) \( B \) is closed relative to \( A, \) (ii) \( h(\partial(A \setminus B)) \subset C, \) (iii) \( h(B) \cap C \setminus D = \emptyset. \) Then \( h \) induces a continuous map \( \bar{h}: A/B \to C/D \) defined by \( \bar{h}[x] = [h(x)] \) if \( x \in A \setminus B, \) \( \bar{h}[B] = [D]. \) Furthermore, if \( h \) is injective and \( h(A \setminus B) \subset C \setminus D, \) then \( \bar{h} \) is injective, and if equality holds \( h \) is bijective.

**Corollary 2.11** (cf. 12). Let \( \Gamma_0 \) be a topological space and \((A, B)\) and \((C, D)\) closed pairs in \( \Gamma_0 \) satisfying (i) \( A \setminus B \subset C, \) (ii) \( B \cap C \setminus D = \emptyset. \) Then the identity map of \( \Gamma_0 \) induces a map \( \bar{i}: A/B \to C/D \) as above.

**Remark.** Such maps will be called inclusion induced maps. When \((A, B) \subset (C, D), \) (i) and (ii) hold and the resulting induced map is called a functorial inclusion induced map because if \((A_1, B_1) \subset (A_2, B_2) \subset \cdots \subset (A_n, B_n)\) is a finite string of inclusions, the composition of the induced maps, \( A_1/B_1 \to A_2/B_2 \to \cdots \to A_n/B_n, \) coincides with the map induced by the inclusion \((A_1, B_1) \subset (A_n, B_n). \) This need not be true for a composition of arbitrary inclusion induced maps; i.e., the composition of inclusion induced maps need not be inclusion induced. An illustration of the more general definition is given in Fig. 2.

**Corollary 2.12.** Let \((C, D)\) be a topological pair and suppose \((A, B)\) is a closed pair in \( C \times I \) with \( B \subset D \times I. \) Let \( \pi: C \times I \to C \) be projection. Then \( \pi \) induces a continuous map \( \bar{\pi}: A/B \to C/D. \)
Lemmas 2.13. Let $X$ be a topological space and $(K, L)$ a closed pair in $X$, and suppose $B$ closed in $K \times I$ with $L \times I \subset B$. Define $j: K/L \times I \to K \times I/B$ by $j(px, t) = q(x, t)$, where $p: K \to K/L$ and $q: K \times I \to K \times I/B$ are the quotient maps. Then $j$ is well defined and continuous.

The proof of the following lemma uses the well-known fact that if a morphism has both a left and right inverse they are equal; hence the morphism has an inverse. The lemma below appears in [12].

Equivalence Lemma 2.14. Let $W, X, Y, Z$ be objects of a category $\mathcal{C}$, and suppose that

$$
\begin{array}{ccc}
W & \xrightarrow{f} & X \\
\downarrow{k} & & \downarrow{g} \\
Y & \xrightarrow{h} & Z
\end{array}
$$

is a commutative diagram of morphisms in $\mathcal{C}$; i.e., $g \circ f = k$ and $h \circ g = l$; and suppose that $k$ and $l$ are equivalences. Then $f$, $g$, and $h$ are equivalences and $f^{-1} = k^{-1}g$, $g^{-1} = l^{-1}h = fk^{-1}$, $h^{-1} = gf^{-1}$.

Proof. $(l^{-1}h)g = l^{-1}(hg) = l^{-1}l = 1_X$ and $g(fk^{-1}) = (gf)k^{-1} = kk^{-1} = 1_Y$; whence $g$ has left inverse $l^{-1}h$ and right inverse $fk^{-1}$. By the remark above, $l^{-1}h = g^{-1} = fk^{-1}$. Then $(k^{-1}g)f = k^{-1}(gf) = k^{-1}k = 1_W$, and $f(k^{-1}g) = (fk^{-1})g = (l^{-1}h)g = l^{-1}l = 1_X$ so that $f$ is an equivalence and $f^{-1} = l^{-1}g$. Similarly $h^{-1} = gf^{-1}$. □

3. The Morse Index

In statements 3.1–3.4 below homotopy equivalences are defined which generate (by composition) a spanning set of representatives for the
morphisms of the Morse index, 3.5 shows that the resulting system is simple, and 3.6 shows that the system is connected.

The results of 3.1 and 3.2 are proved in [2]. The proof of 3.1 given here is cast so as to use 2.10 to prove continuity of the maps in question to help bring out the close inter-relationship among the maps generating the morphisms of the Morse index. The proof of 3.2 is included for completeness. The general inclusion induced maps of 3.4 are not defined in [2].

For 3.1–3.6 below let \( \Phi \subset \Gamma_0 \) be a local flow and \( S \subset \Phi \) a fixed isolated invariant set with isolating \( \Phi \)-neighborhood \( N \) and \( \langle N_1, N_2 \rangle \) a fixed index pair for \( S \) relative to \( N \). Also let \( t \in \mathbb{R}^+ \) be fixed.

**Definition and Proposition 3.1.**

1. Define \( g: N_1/N_2 \to N'_1/N'_2 \) by
   
   \[
   g[x] = \begin{cases} 
   |x \cdot t| & \text{if } x \cdot [0, t] \cap N_2 = \emptyset \\
   |N_2 \cap N'_1| & \text{otherwise.}
   \end{cases}
   \]

   Then \( g \) is well defined and is a homeomorphism of \( N_1/N_2 \) onto \( N'_1/N'_2 \).

2. Define \( f': N_1/N_2 \times I \to N_1/N_2 \) by
   
   \[
   f'(|[x]|, s) = f'_1([x]) = \begin{cases} 
   |x \cdot st| & \text{if } x \cdot [0, st] \cap N_2 = \emptyset \\
   |N_2 \cap N'_1| & \text{otherwise,}
   \end{cases}
   \]

and define \( i: N'_1/N'_2 \to N_1/N_2 \) to be the map induced by the inclusion \( (N'_1, N_2 \cap N'_1) \subset (N_1, N_2 \cap N_1) \). Then \( f' \) is well defined and continuous and is a weak deformation retraction of \( N_1/N_2 \) to \( N'_1/N'_2 \); i.e., \( f'_0 = 1_{N_1/N_2} \), \( f'_1(N_1/N_2) \subset N'_1/N'_2 \), for each \( s \in I \), \( f'_1(N'_1/N'_2) \subset N'_1/N'_2 \), and the end map of the deformation regarded as a map onto \( N'_1/N'_2 \), \( f'_1: N_1/N_2 \to N'_1/N'_2 \), is a homotopy equivalence with homotopy inverse \( t \), with \( t \circ f'_1 \sim 1_{N_1/N_2} \) via \( f' \) and \( f'_1 \circ t \sim 1_{N'_1/N'_2} \) via \( f' \), where \( f' = f' \mid N'_1/N'_2 \times I \) regarded as a map onto \( N'_1/N'_2 \).

**Remark.** As \( g \) is being defined for each \( t \in \mathbb{R}^+ \), and for any index pair, \( \langle N_1, N_2 \rangle \), perhaps the notation should explicitly mention these; however, in favor of simplicity they shall not, but the reference should be clear in context. Analogous remarks hold for \( i \) and also for \( f' \) and \( f'_1 \) as regards the lack of explicit reference to the index pair.

Also, in 3.2 below a map \( \rho: N_1/N_2 \to N_1/N_2 \) will be defined and again because they will be clear from context no explicit reference to \( t \) or the index pair is included in the notation.

**Proof:** With the idea of applying 2.10, set \( A = N_1 \times I \), \( B = \bigcup \{ N_2^{-st} \times \{ s \} : s \in I \} \cap N_1 \times I \), \( C = \bigcup \{ N'_1 \times \{ s \} : s \in I \} \), \( D = N_2 \times I \cap C \), and define
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h: \Gamma \times I \to \Gamma \times I \text{ by } h(\gamma, s) = (\gamma \cdot st, s). \text{ Note that } h \text{ is a homeomorphism, and using 2.10, it will be shown that } h \text{ induces a homeomorphism } \tilde{h}: A/B \to C/D. \text{ Then because } h \text{ preserves "slices," so does } N_1/N_2^{-s} \text{ and } N'_1/N_2 \text{ are the slices for } s = 1 \text{ of } A/B \text{ and } C/D, \text{ respectively, it will then be clear that } g \text{ is the restriction of } \tilde{h} \text{ to } N_1/N_2^{-s} \text{ regarded as a map into } N'_1/N_2. \text{ Finally, using 2.13 and 2.12 it will follow that there are maps } j: N_1/N_2 \times I \to A/B \text{ and } \tilde{\pi}: C/D \to N_1/N_2 \text{ so that } f' = \tilde{\pi} \circ \tilde{h} \circ j.

To begin, because \Phi \times I \text{ is Hausdorff and because 2.9(3) implies that } (A, B) \text{ and } (C, D) \text{ are compact pairs in } \Phi \times I, \text{ it follows that } B \text{ is closed relative to } A. \text{ Next, observe that the exit property and the relative positive invariance of index pairs imply that }

\begin{align*}
(\eta, s) \in N_1 \times I \setminus B & \iff \eta \in N_1 \setminus N_2^{-st} \text{ and } s \in I \\
& \text{iff } \eta \cdot [0, st] \subset N_1 \setminus N_2 \text{ and } s \in I \\
& \text{iff } (\eta \cdot st, s) \in N'_1 \setminus N_2 \times \{s\} \subset C \setminus D.
\end{align*}

It follows that \( h(A \setminus B) = C \setminus D \) and that \( h(B) \cap C \setminus D = \emptyset \). Also, if \((\gamma, r) \in \text{cl}_N \times (N_1 \times I \setminus B), \) but \( \sigma \mid N_1(\gamma) < rt, \) then for some \( \varepsilon, 0 < \varepsilon < r, \sigma \mid N_1(\gamma) < (r - \varepsilon)t, \) and because \( \sigma \mid N_1 \) is upper semi-continuous there exists an \( N_t \)-open \( W \) about \( \gamma \) so that for each \( \eta \in W, \sigma \mid N_1(\eta) < (r - \varepsilon)t, \) and because \( (\gamma, r) \in \text{cl}_N \times (N_1 \times I \setminus B), \) it follows that for some \( (\eta, s), (\eta, s) \in W \times ]r - \varepsilon, 1[ \cap N_1 \times I \setminus B; \) whence by (3), \( \eta \cdot [0, st] \subset N_1 \setminus N_2 \) and \( s \in I, \) which is impossible because \( \sigma \mid N_1(\eta) < (r - \varepsilon)t < st \) and \( \langle N_1, N_1 \rangle \) satisfies the exit property for index pairs; whence for each \( (\gamma, r) \in \text{cl}_N \times (N_1 \times I \setminus B), \sigma \mid N_1(\gamma) \geq rt \) so that \( \gamma \cdot [0, rt] \subset N_1; \) hence \( (\gamma \cdot rt) \in N'_1 \times \{r\} \subset C, \) which shows that \( h(\text{cl}_N(A \setminus B)) \subset C. \) This completes showing that \((A, B), (C, D),\) and \( h \) satisfy the hypotheses of 2.10 including those which guarantee that \( h \) induces a bijection. Because \( A/B \) and \( C/D \) are compact Hausdorff spaces, it follows that \( h \) induces a homeomorphism \( \tilde{h}: A/B \to C/D \) defined by \( \tilde{h}(\gamma, r) = [\gamma \cdot rt, r] \) if \((\gamma, r) \in A \setminus B \) and \( \tilde{h}(B) = [D]; \) by (3) this translates to

\begin{align*}
\tilde{h}(\gamma, r) & = [\gamma \cdot rt, r] \quad \text{ if } \gamma \cdot [0, rt] \cap N_2 = \emptyset \\
& = [D] \quad \text{ otherwise.}
\end{align*}

Then, identifying \( N_1/N_2^{-st} \) with \( N_1 \times \{1\}/N_2^{-st} \times \{1\} \subset A/B \) and \( N'_1/N_2 \) with \( N'_1 \times \{1\}/N_2 \times \{1\} \subset C/D, \) because \( h(\Gamma \times \{s\}) = \Gamma \times \{s\} \) for each \( s \in I, \) it is clear that \( g = \tilde{h} \mid N_1/N_2^{-st} \), where \( \tilde{h} \mid N_1/N_2^{-st} \) is regarded as a map into \( N'_1/N_2 \) and that \( g \) is a homeomorphism because \( \tilde{h} \) is. Finally, because \((A, B)\) and \((C, D)\) are closed pairs in \( \Phi \times I, \) and because \((N_1, N_2 \cap N_1)\) is a closed pair in \( \Phi, \) setting \((K, L) = (N_1, N_2 \cap N_1), \) 2.15 and 2.12 yield continuous maps \( j: N_1/N_2 \times I \to A/B \) and \( \tilde{\pi}: C/D \to N_1/N_2 \) defined by \( j([\gamma], s) = [\gamma, s] \) and
It is then clear from (4) that $f' = \tilde{\pi} \circ \tilde{h} \circ f$, showing that $f'$ is well defined and continuous. Because $(N'_1, N_2 \cap N'_1)$ is also a closed pair in $\Phi$, 2.11 yields that $i: N'_1/N_2 \to N_1/N_2$ is continuous, and from the descriptions of $\tilde{h}$ and $f'$ it follows that $f'$ is a weak deformation retraction of $N_1/N_2$ to $N'_1/N'_2$.

**Definition and Proposition 3.2.** The inclusion $(N_1, N_2 \cap N_1) \subset (N'_1, N'_2 \cap N'_1)$ induces a homotopy equivalence $\rho: N_1/N_2 \to N'_1/N'_2$ with homotopy inverse the composition $N'_1/N'_2 \to N'_1/N_2 \to N_1/N_2$. Hence $g^{-1}$ is homotopic to $\rho \circ i$.

**Proof.** As noted previously $(N_1, N_2 \cap N_1)$ and $(N'_1, N'_2 \cap N'_1)$ are closed pairs in $\Phi$, whence it follows from 2.11 that $\rho$ is continuous. Consider the composition $N'_1/N_2 \circ \rho: N'_1/N'_2 \to N_1/N_2$, and note that $g \circ \rho = f'_1: N_1/N_2 \to N'_1/N'_2$, hence $g \circ \rho$ is a homotopy equivalence by the preceding proposition. Also, $i \circ g$ is an equivalence as both $g$ and $i$ are, again by 3.1. Then by the equivalence lemma, $\rho$ is a homotopy equivalence with homotopy inverse $(f'_1)^{-1}g$, and by 3.1 $(f'_1)^{-1} = i$ yielding $\rho^{-1} = i \circ g$ as claimed. Thus $\rho \sim g^{-1} i^{-1}$; hence $\rho \circ i \sim g^{-1}$.  

**Lemma 3.3.** Let $M$ be an isolating $\Phi$-neighborhood of $S$ (not necessarily distinct from $N$), and suppose that $\langle M_1, M_2 \rangle$ is an index pair for $S$ relative to $M$.

1. If $U^-$ is an $M$-open neighborhood of $A^-(M)$, then there exists $T^- > 0$ so that $s > T^-$ implies $M'_1 \subset U^-$. 
2. If $U^+$ is an $M$-open neighborhood of $A^+(M)$, then there exists $T^+ > 0$ so that $s > T^+$ implies $M'_1 \setminus M'_2^t \subset M_1 \cap U^+$. 
3. If $S \subset V \subset M$, $V$ $\Phi$-open, then there exists $T > 0$ so that $s > T$ implies $M'_1 \setminus M'_2^t \subset V$. 
4. If $M \subset N$, then there exist $s$ and $r$, $s > r > 0$, so that $M'_1 \setminus M'_2^r \subset N_1$ and $M'_1 \cap M'_2^t \cap N_1 \setminus N_2^s = \emptyset$; hence there is an inclusion induced map $M'_1 \setminus M'_2^r \to N_1 \setminus N_2^s$.

**Proof.** Because $A^-(M) \subset U^-$, for each $\gamma \in M \setminus U^-$, $\sigma^* | M(\gamma) < \infty$; whence as $M \setminus U^-$ is closed in $M$, hence compact, by Remark 1 after 2.7, $\sigma^* | M$ is bounded above on $M \setminus U^-$. Set $T^-$ to be the supremum plus one. Then for each $\gamma \in M \setminus U^-$, $\gamma \cdot [T^-, 0] \cap \Gamma_0 \setminus M \neq \emptyset$, and it necessarily follows that if $s \geq T^-$, then $M'_1 \subset U^-$. Analogously, $\sigma | M$ is bounded on $M \setminus U^+$ and setting $T^+$ to be the supremum plus one, for each $\gamma \in M \setminus U^+$, $\gamma \cdot [0, T^+] \cap \Gamma_0 \setminus M \neq \emptyset$. From the exit property of $\langle M_1, M_2 \rangle$ it follows that if $s \geq T^+$, then $M'_1 \setminus U^+ \subset M_1 \cap M'_2^t$. Hence, taking complements in $M_1$, $M'_1 \setminus M'_2^t \subset M_1 \cup U^+$. This completes showing (1) and (2). If $S \subset V \subset N$, $V$ $\Phi$-open, choose $U^+$ and $U^-$ $M$-open so that $A^\pm(M) \subset U^\pm$ and $V = U^+ \cap U^-$. 


Then choose $T^-$ and $T^+$ as guaranteed by (1) and (2) and set $T = \max\{T^-, T^+\}$. For $s > T$, $M_1^{-s} \cap A^+(M) = S \subset V$, $A^-(M) \setminus V$ and $A^+(M) \setminus V$ are disjoint $M$-closed subsets of $M$; hence there are disjoint $M$-open sets $W^- \supset A^-(M) \setminus V$ and $W^+ \supset A^+(M) \setminus V$. Then set $U^\pm = W^\pm \cup V$ to satisfy the claim. This completes showing (3).

To see that (4) holds, suppose $M \subset N$ and set $V = \text{int}_\Phi M \cap \text{int}_\Phi (N \setminus N_2)$. Then $S \subset V \subset M$ and $V$ is $\Phi$-open. Applying (3), for all large enough $r > 0$, $M'_1 \cap M_2^{-r} \subset V \subset N_1$. Then claim that for all large enough $r > 0$, $M'_1 \cap M_2^{-r} \cap A^+(N) = \emptyset$. Assuring this for the moment, choose such an $r > 0$. There then exists by normality of $N$, an $N$-open neighborhood of $A^+(N)$, call it $U^+$, which is disjoint from $M'_1 \cap M_2^{-r}$. Then by applying (2) to $(N_1, N_2)$ and $U^+$, choose $s$, $s > r > 0$, so that $N_1^{-s} \subset N_1 \cap U^+$; whence $M'_1 \cap M_2^{-r} \cap N_1^{-s} \subset M'_1 \cap M_2^{-r} \cap U^+ = \emptyset$.

It remains to show that for all large enough $r > 0$, $M'_1 \cap M_2^{-r} \cap A^+(N) \neq \emptyset$. Suppose not, then for each $r > 0$, $M'_1 \cap A^+(N) \cap \partial_\Phi M \neq \emptyset$; for given $r > 0$, choose $r' > r$ so that for some $\gamma$, $\gamma \in M'_1 \cap M_2^{-r'} \cap A^+(N)$; by (2) $M_2^{-r'} \cap A^+(N) = \emptyset$ so that $\sigma \mid M(\gamma) < \infty$; hence by 2.7 and the relative positive invariance of $(M'_1, M_2^{-r'})$, $\sigma \cdot \sigma \mid M(\gamma) \subset M'_1 \cap M_2^{-r'} \cap A^+(N) \cap \partial_\Phi M$, and as $M'_1 \subset M'_1$, it follows that $M'_1 \cap A^+(N) \cap \partial_\Phi M \neq \emptyset$. Then $\{M'_1 \cap A^+(N) \cap \partial_\Phi M : r > 0\}$ is a nested family of non-empty $M$-closed subsets of $M$; hence no finite intersection is empty; whence because $M$ is compact, $\emptyset \neq \bigcap \{M'_1 \cap A^+(N) \cap \partial_\Phi M : r > 0\} = A^-(M) \cap A^+(N) \cap \partial_\Phi M \subset A^+(N) \cap A^+(N) \cap \partial_\Phi M$ (this last inclusion holding because $M \subset N = S \cap \partial_\Phi M = \emptyset$ because $M$ was assumed to be an isolating $\Phi$-neighborhood of $S$; i.e., $S \subset \text{int}_\Phi M$. This contradiction shows that for all large enough $r > 0$, $M'_1 \cap M_2^{-r} \cap A^+(N) = \emptyset$.

**Proposition 3.4.** Let $(\hat{N}_1, \hat{N}_2)$ be an index pair for $S$ relative to an isolating $\Phi$-neighborhood $\hat{N}$ of $S$ ($\hat{N}$ is not necessarily distinct from $N$), and suppose $i: N_1 \setminus N_2 \rightarrow \hat{N}_1 \setminus \hat{N}_2$ is a continuous inclusion induced map (so $N_1 \setminus N_2 \subset \hat{N}_1$ and $N_1 \cap N_2 \subset \hat{N}_1 \setminus \hat{N}_2 = \emptyset$). Then $i$ is a homotopy equivalence, with homotopy inverse a composition of maps between the index spaces of index pairs with each factor being either an inclusion induced map or an analogue of one of the maps defined in 3.1 or 3.2.

**Remark.** The maps $i: N_1 \setminus N_2 \rightarrow N_1 \setminus N_2$ and $\rho: N_1 \setminus N_2 \rightarrow N_1 \setminus N_2$ are special cases of this proposition.

**Proof:** Set $M = N \cap \hat{N}$, and note $M$ is an isolating $\Phi$-neighborhood for $S$. As $S \subset \text{int}_\Phi (N \setminus N_2)$, choose $V$ $\Phi$-open so that $S \subset V \subset \text{int}_\Phi (M \cap N_1)$. By 3.3 choose $r > 0$ so that $\hat{N}_1^{-r} \subset V$. Then $(\hat{N}_1^{-r} \cap M, \hat{N}_2^{-r} \cap M)$ is an index pair for $S$ relative to $M$: Both members of the pair are clearly compact and
also are positively invariant relative to \( M \) because \( M \subset N \) and \( \tilde{N}_1 \) and \( \tilde{N}_2 \) are both positively invariant relative to \( N \); also, as \( \tilde{N}_1 \cap \tilde{N}_2 = V \subset M \), \( \tilde{N}_1 \cap M \setminus \tilde{N}_2 = \tilde{N}_1 \cap \tilde{N}_2 = V \subset M \) so that \( \langle \tilde{N}_1 \cap M, \tilde{N}_2 \cap M \rangle \) satisfies the isolating property since \( \langle \tilde{N}_1, \tilde{N}_2 \rangle \) does; and finally the exit property is satisfied, for if \( \gamma \in N_1 \cap M \) and \( \sigma \mid M(\gamma) < \infty \), if \( \gamma \cdot [0, \sigma \mid M(\gamma)] \subset \tilde{N}_1 \cap M \setminus \tilde{N}_2 \cap M \), then \( \gamma \cdot [0, \sigma \mid M(\gamma)] \subset \tilde{N}_1 \cap \tilde{N}_2 \subset V \subset M \); whence by 2.8 for some \( \epsilon > 0 \), \( \gamma \cdot [0, \sigma \mid M(\gamma) + \epsilon] \subset V \subset M \) contradicting the definition of \( \sigma \mid M(\gamma) \). Also, as \( q_nM \cap \tilde{N}_2 = \tilde{N}_1 \cap \tilde{N}_2 \), setting \( \langle M_1, M_2 \rangle = \langle \tilde{N}_1 \cap M, \tilde{N}_2 \cap M \rangle \), it follows from 2.10 and 2.11 that there is a natural inclusion induced homeomorphism \( e: \tilde{N}_1 / \tilde{N}_2 \to M_1 / M_2 \), and as \( M \subset N \), by 3.3(4) choose \( r \) and \( s \), \( 0 < r' < s \), so that there is an inclusion induced map \( j: M_1 / M_2 \to N_1 / N_2 \). Then define \( m: N_1 / N_2 \to N_1 / N_2 \) to be the following composition:

\[
\tilde{N}_1 / \tilde{N}_2 \xrightarrow{\tilde{p}} \tilde{N}_1 / \tilde{N}_2 \xrightarrow{j} N_1 / N_2 \xrightarrow{\rho} M_1 / M_2 \xrightarrow{\rho} M_1 / M_2 \xrightarrow{\rho}\]

\[
\tilde{N}_1 / \tilde{N}_2 \xrightarrow{\tilde{k}} M_1 / M_2 \xrightarrow{j} N_1 / N_2 \xrightarrow{\rho} N_1 / N_2 \xrightarrow{\rho} N_1 / N_2 ,
\]

where \( \tilde{p} \) and \( \rho \) are defined as in 3.2 and where \( \tilde{j}_1, \tilde{j}_1', \), \( g \), and \( i \) are defined as in 3.1. Set \( k = m \circ \tilde{i} \) and \( \ell = \tilde{i} \circ m \). The aim is to apply the equivalence lemma to the diagram

\[
\begin{array}{ccc}
N_1 / N_2 & \xrightarrow{i} & \tilde{N}_1 / \tilde{N}_2 \\
\downarrow m & & \downarrow i \\
\tilde{N}_1 / \tilde{N}_2 & \xrightarrow{k} & N_1 / N_2
\end{array}
\]

which requires showing that \( k \) and \( \ell \) are homotopy equivalences. In fact, it will follow from the next proposition that \( k \sim 1_{N_1 / N_2} \) and \( \ell \sim 1_{N_1 / N_2} \); whence it follows from 2.14 that \( \tilde{i}^{-1} = m \), which fact completes the current proposition.

**Proposition 3.5.** Suppose \( \varphi: N_1 / N_2 \to N_1 / N_2 \) is a composition of maps where each factor is a map between index spaces of index pairs for \( S \) of one of the following types:

(i) end map of a deformation as defined in 3.1(2)

\[
M_1 / M_2 \xrightarrow{j_1} M_1 / M_2 \quad \text{or} \quad M_1 / M_2 \xrightarrow{j_1} M_1 / M_2 ;
\]

(ii) a homeomorphism or its inverse as defined in 3.1(1)

\[
M_1 / M_2 \xrightarrow{\rho^{-1}} M_1 / M_2 ;
\]
an inclusion induced map as in 3.4

\[ \frac{M_i/M_2}{\varphi} \to \tilde{M}_1/\tilde{M}_2, \]

where \((M_1, M_3), (\tilde{M}_1, \tilde{M}_3)\) are index pairs for \(S\).

Then \(\varphi \sim \lambda_{N_{1/N_2}}\).

**Proof.** From 3.2 it follows that any map of type \(g: M_i/M_2 \to M_i/M_2\)
has a homotopy inverse the composite \(M_i/M_2 \to \lambda M_i/M_2\)
so that any factor of \(\varphi\) of type \(g^{-1}: M_i/M_2 \to M_i/M_2\)
can be replaced by a composite \(\rho \circ \iota\). Also note \(\tilde{f}_1: M_i/M_2 \to M_i/M_2\)
and \(\tilde{f}_1: M_i/M_2 \to M_i/M_2\)
factor to \(M_i/M_2 \to \rho M_i/M_2\) and \(M_i/M_2 \to \rho M_i/M_2\)
\(M_i/M_2 \to \rho M_i/M_2\), respectively. Also \(f_1 \sim \lambda_{N_{1/N_2}}\). Then since the identity
map of an index pair is a map of type (iii), without loss of generality \(\varphi = j_n \circ g_n \circ j_{n-1} \circ g_{n-1} \circ \ldots \circ j_1 \circ g_1 \circ j_0\), where each \(j_i\) is a composition of maps of
type (iii) and each \(g_i\) is a homeomorphism, \(g_i: M_{i,i}/M_{i,i} \to M_{i,i}/M_{i,i}\). Let
\(\tilde{t} = \sum t_i\). It is then clear that for each \(\gamma \in N_{1}, \varphi[\gamma] = [\gamma \cdot \tilde{t}]\) or \(\varphi[\gamma] = [N_2]\).
Also, for each \(\gamma \in N_1, f_1[\gamma] = [\gamma \cdot \tilde{t}]\) if \(\gamma \in N_{1},\) otherwise \(f_1[\gamma] = [N_2]\). The
proof proceeds by showing that there exists \(r \geq \tilde{t}\) so that \(\rho \circ \varphi = \rho \circ f_1^r\),
where \(\rho: N_1/N_2 \to N_1/N_2\) is defined as in 3.2. Assuming this, then
\[
\varphi = 1_{N_1/N_2} \circ \varphi \sim f_1^r \circ \varphi = (t \circ g \circ \rho) \circ \varphi = (t \circ g)(\rho \circ \varphi)
\]
\[
= (t \circ g)(\rho \circ f_1^r) = (t \circ g \circ \rho) \circ f_1^r
\]
\[
= f_1^r \circ f_1^r \sim 1_{N_1/N_2} \circ 1_{N_1/N_2} = 1_{N_1/N_2},
\]
where \(f_1^r: N_1/N_2 \to N_1/N_2\) is the end map of a deformation as in 3.1(2), and
\(t \circ g \circ \rho\) is the factorization of \(f_1^r\) mentioned above.

To say that for some \(r \geq \tilde{t}\), \(\rho \circ \varphi = \rho \circ f_1^r\) is equivalent to saying
\[
\bigcup \{ p^{-1}\varphi[\gamma] \cup p^{-1}f_1^r[\gamma] : \varphi[\gamma] \neq f_1^r[\gamma] \} \subset N_1 \cap N_2^r, \tag{1}
\]
where \(p: N_1 \to N_1/N_2\) is the quotient map, and by the above observations on
the actions of \(\varphi\) and \(f_1^r\), for \(\gamma \in N_1, \varphi[\gamma] \neq f_1^r[\gamma]\) if, and only if, either (i)
\(\varphi[\gamma] = [N_2] \neq f_1^r[\gamma]\) or (ii) \(\varphi[\gamma] \neq [N_2] = f_1^r[\gamma]\). The proof that (1) holds
proceeds in four steps with the fourth yielding the desired conclusion; the
first two are used in the fourth to show the inclusion
\[
p^{-1}\varphi[\gamma] \cup p^{-1}f_1^r[\gamma] \subset N_1 \cap N_2^r
\]
when (i) holds, and the third is used to show this same inclusion when (ii)
holds.
Observe that if \( \langle M_1, M_2 \rangle \) is an index pair and \( t > 0 \), then \( (M_1 \setminus M_2^{-t}) \cdot t = M_1 \setminus M_2 \). This follows from (3) of 3.1 and will be used in steps two and three below.

**Sublemma I.** Given index pairs \( \langle L_{1,k}, L_{2,k} \rangle \) for \( S, k = 1, \ldots, n + 1 \), and inclusion induced maps

\[
\tilde{t}_k: L_{1,k} / L_{2,k} \to L_{1,k+1} / L_{2,k+1}, \quad k = 1, \ldots, n.
\]

Then

\[
A^+(L_{1,1}) = \bigcap_{k=1}^{n+1} L_{1,k} \setminus L_{2,k};
\]

hence

\[
[L_{2,n+1}] = p_{n+1}(L_{2,n+1}) \notin j p_1 A^+(L_{1,1}),
\]

where \( j = \tilde{t}_n \circ \tilde{t}_{n-1} \circ \ldots \circ \tilde{t}_1 \) and where

\[
p_k: L_{1,k} \to L_{1,k} / L_{2,k}, \quad k = 1, \ldots, n + 1
\]

is the quotient map.

**Proof.** The proof is by induction on \( n \). First recall (2.9(2)) that for any index pair, \( \langle L_1, L_2 \rangle \), \( A^+(L_1) \cap L_2 = \emptyset \), hence, in particular, \( A^+(L_{1,1}) \subseteq L_{1,1} \setminus L_{2,1} \) so that the case \( n = 0 \) holds. For any integer \( m, 0 \leq m < n \), by induction assume

\[
A^+(L_{1,1}) = \bigcap_{k=1}^m L_{1,k} \setminus L_{2,k}.
\]

Then as \( \tilde{t}_m \) is inclusion induced

\[
A^+(L_{1,1}) \subseteq L_{1,m} \setminus L_{2,m} \subseteq L_{1,m+1},
\]

and because \( A^+(L_{1,1}) \) is positively invariant,

\[
A^+(L_{1,1}) \subseteq A^+(L_{1,m+1}) \subseteq L_{1,m+1} \setminus L_{2,m+1}
\]

whence

\[
A^+(L_{1,1}) \subseteq \bigcap_{k=1}^{m+1} L_{1,k} \setminus L_{2,k},
\]

which completes the induction.
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Given \( K \subset L_{1,1} \), clearly by definition of the inclusion induced maps, \([L_{2,n+1}] \in \mathcal{J}p(K)\) if, and only if,

\[
K \cap L_{1,i} \cap L_{2,i} \neq \emptyset
\]

for some \( i, i = 1, \ldots, n + 1 \), so that

\[
[L_{2,n+1}] \in \mathcal{J}p_1 \left( \bigcap_{i=1}^{n+1} L_{1,i} \setminus L_{2,i} \right) = \mathcal{J}p_1(A^+(L_{1,1})). \]

**Sublemma II.** \( |N_2| \in \varphi p(A^+(N_1)) \), where \( p: N_1 \to N_1/N_2 \) is the quotient map.

**Proof:** For convenience in writing what follows set \( t_{n+1} = 0 \) and set

\[
\langle M_{1,n+1}, M_{2,n+1} \rangle = \langle N_1, N_2 \rangle.
\]

Where

\[
\varphi = j_n \circ g_n \circ j_{n-1} \circ \cdots \circ j_1 \circ g_1 \circ j_0,
\]

setting

\[
\varphi_k = j_k \circ g_k \circ \cdots \circ j_1 \circ g_1 \circ j_0
\]

for \( 0 \leq k \leq n \), it must be shown that

\[
[M_{2,k}] \in \varphi_{k-1} p(A^+(N_1))
\]

for \( 1 \leq k \leq n + 1 \) which, using the convention that a summation over an empty index set equals zero, is the same as saying that

\[
A^+(N_1) \cdot \sum_{l=1}^{k-1} t_l \subset M_{1,k} \setminus M_{2,k} \quad (2_k)
\]

for \( 1 \leq k \leq n + 1 \).

Proceed by induction on \( k \). The case \( k = 1 \) follows from Sublemma I applied to \( j_0 = \varphi_0 \). With \( (2_k) \) as induction hypothesis, push both sides of the inclusion of \( (2_k) \) forward under the flow by \( t_k \) obtaining a new inclusion. The resulting left-hand side is the left-hand side of \( (2_{k+1}) \) which is positively invariant and a subset of \( M_{1,k} \setminus M_{2,k} \) by the new inclusion; hence applying Sublemma I to \( j_k \) shows \( (2_{k+1}) \) holds and completes the induction. \( \square \)

**Sublemma III.** If \( \gamma \in N_1 \cap N_2^{-T} \) (so that \( f^T_1[\gamma] = [N_2] \)), then

\[
p^{-1} \varphi[\gamma] \subset N_1 \cap N_2^{-T}.
\]
Proof. First, using the notation and assumptions of Sublemma I, observe that for any \( s \in \mathbb{R}^+ \),
\[
\bigcap_{i=1}^{n-1} (L_{1,i} \setminus L_{2,i}) \cap L_{2,i}^{-} \subset L_{1,n-1} \cap L_{2,n+1}^{-};
\]
if
\[
\gamma \in (L_{1,k} \setminus L_{2,k}) \cap L_{2,k}^{-},
\]
then for some \( r, \) \( 0 \leq r \leq s \), \( \gamma \cdot [0, r] \subset L_{1,k} \), and \( \gamma \cdot r \in L_{2,k} \), and by 2.2
\[
\gamma \cdot r \cdot -\sigma^* | L_{2,k}(\gamma \cdot r) \in L_{2,k};
\]
hence as \( \gamma \in L_{2,k} \), setting \( r_0 = r - \sigma^* | L_{2,k}(\gamma \cdot r) \), \( 0 < r_0 \leq r \leq s \), and \( \gamma \cdot r_0 \in L_{2,k} \), and by the definition of \( \sigma^* | L_{2,k} \) and the relative positive invariance of \( L_{2,k} \) it follows that
\[
\gamma \cdot [0, r_0] \subset L_{1,k} \setminus L_{2,k};
\]
also,
\[
L_{1,k} \setminus L_{2,k} \subset L_{1,k+1}
\]
because
\[
\tilde{r}_k : L_{1,k} \setminus L_{2,k} \rightarrow L_{1,k+1} / L_{2,k+1}
\]
is inclusion induced so that
\[
\gamma \cdot r_0 \in \text{cl}_\Phi(L_{1,k} \setminus L_{2,k}) \subset L_{1,k+1}
\]
as \( L_{1,k+1} \) is \( \Phi \)-closed, being compact; thus
\[
\gamma \cdot r_0 \in L_{1,k} \cap L_{2,k} \cap L_{1,k+1}
\]
which implies that \( \gamma \cdot r_0 \in L_{2,k+1} \) because
\[
L_{1,k} \cap L_{2,k} \cap L_{1,k+1} \setminus L_{2,k+1} = \emptyset
\]
as \( \tilde{r}_k \) is inclusion induced; hence
\[
\gamma \in L_{1,k+1} \cap L_{2,k+1}^{-}
\]
since \( \gamma \cdot [0, r_0] \subset L_{1,k+1} \) and \( \gamma \cdot r_0 \in L_{2,k+1} \) and \( 0 < r_0 \leq s \); by induction it follows that
\[
\bigcap_{i=1}^{n+1} (L_{1,i} \setminus L_{2,i}) \cap L_{2,i}^{-} \subset L_{1,n+1} \cap L_{2,n+1}^{-}.\]
Now suppose that $f^i_1[\gamma] = [N_2]$. If $\varphi[\gamma] = [N_2]$ also, there is nothing to show, so suppose that $\varphi[\gamma] = [\gamma \cdot \bar{t}] \neq [N_2]$. By hypothesis, $\gamma \in (N_1 \setminus N_2) \cap N_2^{-\bar{t}}$. As $\varphi[\gamma] \neq [N_2]$, applying the above observation to $j_0$ yields

$$\gamma \in (M_{1,1} \setminus M_{2,1}^{-\bar{t}}) \cap (M_{2,1}^{-\bar{t}})^{-\bar{t}},$$

whence

$$\gamma \cdot t_1 \in (M_{1,1}^{t_1} \setminus M_{2,1}) \cap M_{2,1}^{-\bar{t}}.$$ 

For induction assume

$$\gamma \cdot \sum_{i=1}^{k} t_i \in (M_{1,k}^{t_1} \setminus M_{2,k}) \cap M_{2,k}^{-\bar{t}}.$$ 

Again, as $\varphi[\gamma] \neq [N_2]$, the above observation applied to $j_k$ yields

$$\gamma \cdot \sum_{i=1}^{k} t_i \in (M_{1,k+1}^{t_1} \setminus M_{2,k+1}) \cap (M_{2,k+1}^{-\bar{t}})^{-\bar{t}},$$

so that

$$\gamma \cdot \sum_{i=1}^{k+1} t_i \in (M_{1,k+1}^{t_1} \setminus M_{2,k+1}) \cap M_{2,k+1}^{-\bar{t}}.$$ 

Hence by induction

$$\gamma \cdot \bar{t} \in (M_{1,n}^{t_1} \setminus M_{2,n}) \cap M_{2,n}^{-\bar{t}}$$

and applying the above observation to $j_n$ gives

$$\gamma \cdot \bar{t} \in (N_1 \setminus N_2) \cap N_2^{-\bar{t}}.$$ 

**Sublemma IV.** There exists $r \geq i$ so that $f^i_1[\gamma] \neq [\gamma]$ implies

$$p^{-1}f^i_1[\gamma] \cup p^{-1}\varphi[\gamma] \subset N_1 \cap N_2^{-r}.$$ 

and hence the composites

$$N_1/N_2 \xrightarrow{\varphi} N_1/N_2 \xrightarrow{p} N_1/N_2^{-r}$$

and

$$N_1/N_2 \xrightarrow{f_1^i} N_1/N_2 \xrightarrow{p} N_1/N_2^{-r}$$

are equal.
Proof. By Sublemma II, \( p^{-1}\phi^{-1}[N_2] \cap A^+(N_1) = \emptyset \). Thus by normality choose \( U \) open relative to \( N_1 \) with \( A^+(N_1) \subset U \) and \( p^{-1}\phi^{-1}[N_2] \subset N_1 \setminus U \). By 3.3(2) choose \( r > t \) so that \( N_1 \setminus N_2^r \subset U \). Then suppose \( y \in N_1 \setminus N_2^r \) and \( \phi[y] \neq f_1^r[y] \). As noted above there are two possibilities. First, suppose

\[
\phi[y] = [N_2] \neq f_1^r[y];
\]

then by choice of \( U \),

\[
y \in N_1 \setminus U \subset N_1 \cap N_2^{-r},
\]

and as \( y \cdot [0, \bar{t}] \subset N_1 \) and as \( N_1 \cap N_2^{-r} \) is positively invariant relative to \( N_1 \),

\[
p^{-1}f_1^r[y] = y \cdot \bar{t} \in N_1 \cap N_2^{-r},
\]

and it follows that the inclusion (3) holds in the first case. In the second case, \( \phi[y] \neq [N_2] = f_1^r[y] \) so that

\[
y \in (N_1 \setminus N_2) \cap N_2^{-r};
\]

then by Sublemma III,

\[
p^{-1}\phi[y] \subset N_1 \cap N_2^{-r} \subset N_1 \cap N_2^{-r},
\]

and it follows that the inclusion (3) holds in this case too. Hence

\[
\bigcup \{ p^{-1}\phi[y] \cup p^{-1}f_1^r[y]: \phi[y] \neq f_1^r[y] \} \subset N_1 \cap N_2^{-r}
\]

and as noted previously this clearly gives the desired equality \( \rho \circ f_1^r = \rho \circ \phi \), concluding the proof of Sublemma IV and also of the proposition.

**Definition and Theorem 3.6.** Let \( \mathcal{S}(S) \) be the category whose objects are the quotient spaces of the index pairs for \( S \) and whose morphisms are the homotopy classes of any defined composition of maps of the types occurring in 3.5. Then \( \mathcal{S}(S) \) is a connected simple system. In particular, all index spaces have the same homotopy type. \( \mathcal{S}(S) \) is called the Morse index of the isolated invariant at \( S \).

Proof. 3.1–3.5 above show that each factor in such a composition is a homotopy equivalence whence so is the composite. Thus \( \mathcal{S}(S) \) is a groupoid and is simple by 3.5. If \( \langle N_1, N_2 \rangle \) and \( \langle \bar{N}_1, \bar{N}_2 \rangle \) are index pairs for \( S \) in the isolating neighborhoods \( N \) and \( \bar{N} \), respectively, let \( M = N \cap \bar{N} \) and let \( m: \bar{N}_1/\bar{N}_2 \to N_1/N_2 \) be defined as in (1) of the proof of 3.4. The homotopy class of \( m \) is a morphism in \( \mathcal{S}(S) \), and it follows that the simple system is connected.
Remark. Conley's construction of the Morse index in [2] utilizes only functorial inclusion induced maps. As a consequence the proof of connectedness of the Morse index given in [2, Section 5.1] is slightly different. First, Conley shows that given two index pairs relative to the same isolating neighborhood there is a homotopy equivalence between them; second, he shows that given two isolating neighborhoods there exist two nested index pairs, one relative to each neighborhood, such that the intersection of either of these index pairs with the intersection of the isolating neighborhoods is an index pair relative to the intersection of the isolating neighborhoods, and each of the resulting inclusion maps of pairs, from an intersection into one of the originals, is an excision which induces a natural homeomorphism between the index spaces. One then can compose maps of the two types to get a homotopy equivalence between index spaces of index pairs relative to different isolating neighborhoods which is the analogue of the map m in the proof of 3.6 above. Conley's construction has the advantage of using only functorial inclusion induced maps, at the cost of introducing two extraneous index spaces geometrically unrelated to two given index spaces one wants to show homotopy equivalent. This would be very inconvenient in certain contexts; e.g. [7]. However, in other contexts it could prove quite useful to note that as a consequence of the two different constructions and 3.5 above, it follows that any general inclusion induced map between index spaces is homotopic to a composition of homeomorphisms g as defined in 3.1(1) and functorial inclusion induced maps.

4. Modifications for Local Semi-Flows

When \( \Phi \subset \Gamma_0 \) is only a local semi-flow, part of Proposition 2.9(1) fails to hold; namely, \( \langle N'_1, N_2 \rangle \) and \( \langle N'_1, N_2^{-1} \rangle \) fail, in general, to be index pairs because the isolating property fails to hold. The following simple counterexample is due to C. Conley (personal communication): Take a flow in the plane which moves every point to the right horizontally except the origin which is a stop point. Take \( \Gamma = \mathbb{R}^2 \) and \( \Phi \) the closed right half-plane and \( S \) the origin which has \( [0, 1] \times [-1, 1] \) as an isolating \( \Phi \)-neighborhood; and set \( N_1 = [0, 1] \times [-1, 1] \), \( N_2 = \{1\} \times [-1, 1] \), and note \( \langle N'_1, N_2 \rangle \) is an (isolating) index pair for \( S \), but \( \text{cl}_\sigma(N'_1 \setminus N_2) \) is not a \( \Phi \)-neighborhood of \( S \) for any \( t > 0 \).

Because the quotient space \( N'_1/N_2 \) occurs naturally as an object of the connected simple system constructed above, in the context of local semi-flows it seems natural albeit a bit cumbersome to widen the definition of index pair to encompass \( \langle N'_1, N_2 \rangle \). To distinguish between the wider definition given below and the previous definition of index pair, we now call
a pair \( \langle N_1, N_2 \rangle \) which satisfies the definition of index pair in 2.5, an isolating index pair. The new definition of index pair is then given by:

**Definition 4.1 (Index Pair for Local Semi-Flows).** Suppose \( \Phi \in \Gamma_0 \) is a local semi-flow and \( N \) an isolating \( \Phi \)-neighborhood with maximal invariant set \( S \). An ordered pair of compact subsets of \( N \), \( \langle N_1, N_2 \rangle \) is called an index pair for \( S \) relative to \( N \) if, and only if,

(i) \( N_1 \) is positively invariant relative to \( N \);
(ii) there exists a compact \( \mathcal{N}_1 \subset N \) satisfying
   
   (a) \( \langle \mathcal{N}_1, N_2 \rangle \) is an isolating index pair for \( S \) relative to \( N \),
   
   (b) \( N_1 \setminus N_2 \subset \mathcal{N}_1 \setminus N_2 \), and
   
   (c) for some \( t \in \mathbb{R}^+ \), \( \mathcal{N}_1 \setminus N_2 \subset N_1 \setminus N_2 \).

With this definition, if \( \langle N_1, N_2 \rangle \) is an isolating index pair, then \( \langle N'_1, N_2 \rangle \) is an index pair where \( N'_1 \) plays the role of \( \mathcal{N}_1 \) in 4.1. Also if "index pair" is defined according to 4.1, then Proposition 2.9 above holds for \( \Phi \in \Gamma_0 \) a local semi-flow with essentially the same proof. Note too that all the results of Section 3 go through for the wider definition of 4.1 with essentially the same proofs except for the following modification to the proof of 3.3(4). Referring to this proof, the fact that \( S \subset \text{int}_\Phi(N_1 \setminus N_2) \) was used. Instead, using the \( \mathcal{N}_1 \) guaranteed by 4.1, \( S \subset \text{int}_\Phi(\mathcal{N}_1 \setminus N_2) \), and for some \( t' > 0 \), \( \mathcal{N}_1' \setminus N_2 \subset N_1 \); so that if \( M_1 \setminus M_2^{-t'} \subset \text{int}_\Phi(M) \cap \text{int}_\Phi(N_1 \setminus N_2) \), then \( M_1^{-t'} \setminus M_2^{-t + t'} \subset \mathcal{N}_1' \setminus N_2 \subset N_1 \setminus N_2 \subset N_1 \), which makes the proof of 3.3(4) go through.

Finally we distinguish between the two types of index pairs (isolating and non-isolating) by altering the notation of 3.6 when in the context of local semi-flows by denoting the category defined in 3.6 which has the collection of all index pairs for \( S \), both isolating and non-isolating, as its collection of objects by \( \mathcal{I}(S) \), and now define \( \mathcal{I}(S) \) to be the full subcategory of \( \mathcal{F}(S) \) which has for its collection of objects the collection of isolating index pairs for \( S \).

**References**


7. H. L. KURLAND, Solutions to boundary value problems of fast–slow systems by continuing homology in the Morse index along a path of isolated invariant sets of the fast-systems, to appear.


