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A percolation system with extremely long range connections and node dilution

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HIGHLIGHTS

- We consider the bond-diluted long-range percolation problem on a linear chain.
- Dilution of nodes, which is also considered, competes with long-range connectivity.
- The percolation order parameter only depends on the average connectivity.
- The average connectivity is explicitly computed in terms of the free parameters.

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ABSTRACT

We study the very long-range bond-percolation problem on a linear chain with both node and bond dilution. Very long-range means that the probability p_{ij} for a connection between two nodes i, j at a distance r_{ij} decays as a power-law, i.e. $p_{ij} = \rho/[r_{ij}^{\alpha} N^{1-\alpha}]$ when $\alpha \in [0, 1)$, and $p_{ij} = \rho/[r_{ij} \ln(N)]$ when $\alpha = 1$. Node dilution means that the probability that a node is present in a site is $p_s \in (0, 1]$. The behavior of this model results from the competition between long-range connectivity which enhances the percolation, and node dilution which weakens percolation. The case $\alpha = 0$ with $p_s = 1$ is well-known, being the exactly solvable mean-field model. The percolation order parameter P_∞ is investigated numerically for different values of α, p_s and ρ. We show that in all range $\alpha \in [0, 1]$ the percolation order parameter P_∞ depends only on the average connectivity γ of the nodes, which can be explicitly computed in terms of the three parameters α, p_s and ρ.

1. Introduction

During the last fifty years, percolation theory has brought new understanding and methods to a broad range of topics in physics like materials science, complex networks, surface roughening, epidemiology, geography, and fire propagation (see Refs. [1,2] for a review). This theory was first considered for the optimization of masks supplied to the miners in the coal pits needing a protection which could block poisoning materials, while permitting the passage of air. In other words, it was needed an appropriate dosage of porosity of the material which composed the masks in order to have connected path for air and unconnected path for poisoning materials. After that, the theory was applied to the study of movement and filtering of fluids through porous materials (the most familiar phenomena probably being coffee percolation) and its scope has been progressively extended to all other domains [3–5].

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Nowadays, percolation is still a very active field of research in physics and applied to an always increasing number of phenomena as, for example, fluid flow in random media [6], dielectric breakdown [7] and reaction–diffusion processes in two-dimensional percolating structures [8].

Percolation models have also been increasingly adopted for many phenomena besides physics to understand important features of chemical, biological and social systems. Many of them form complex networks, whose vertices are the elements of the system and whose edges represent their interactions. For example, living systems form a huge genetic network whose vertices are proteins, while the edges represent their chemical exchanges [9]. Equally complex networks occur also in social science, where the vertices are individuals, organizations or countries and the edges characterize their social contacts [10]. Moreover, the effects of the complex connectivity of biological systems can be also studied by percolation theory. Recent advances in this field points to universal laws and offer a new conceptual framework that could potentially revolutionize our view of biology [11].

The effect of long-range connections on percolation is of fundamental interest, since they give rise to a variety of new interesting dynamical and thermodynamical phenomena. In view of that, long-range models have been intensively studied in recent times in different contexts [12–18]. The phenomenology becomes very interesting when long-range connections appear together with node dilution. In this case, in fact, there is competition between long-range connectivity which enhances percolation and node dilution which weaken it [19–21].

In this work we investigate the very long-range percolation problem on a linear chain with both node and bond dilution. Very long-range means that the probability p_{ij} of a connection between two nodes i, j at a distance r_{ij} decays as a power-law, i.e. $p_{ij} = \rho/[r_{ij}^{\alpha} N^{1-\alpha}]$ when $\alpha \in [0, 1)$ and $p_{ij} = \rho/[r_{ij} \ln(N)]$ when $\alpha = 1$. Node dilution means that the probability that a node is present in a site is $p_s \in (0, 1]$. Notice that for this very long-range models, in order to obtain the correct thermodynamic limit, it is compulsory to assume that the probability of a connection decays with the size N of the system as $1/N^{1-\alpha}$ (or as $1/\ln(N)$ in case $\alpha = 1$).

The case $\alpha = 0$, with $p_s = 1$, is well-known, being the exactly solvable mean-field model, while the case $\alpha = 0$ with $p_s < 1$ is its almost trivial extension. In the other regions, the percolation order parameter P_∞ is investigated numerically for different values of α, p_s and ρ. Intuitively, one expects the percolation order parameter P_∞ be reduced by the dilution of nodes [19,20]. Indeed, we will show not only that this is true, but we also show that in all range $\alpha \in [0, 1]$, the percolation order parameter P_∞ depends only on the average connectivity γ of nodes, which we explicitly compute in terms of the three parameters α, p_s and ρ.

In other words, given $\gamma = \gamma(\alpha, \rho, p_s)$, $P_\infty(\gamma)$ is always the same function, independently on the values of α, ρ and p_s. We stress that this result is not only true at the transition, but for all possible values of γ. Therefore, not only we state that the model is the universality class of mean-field bond-percolation (it would be an almost trivial result being it well known when node dilution is absent) but we prove that spatial structures are irrelevant for all values of parameters, being the average connectivity the only relevant aspect.

The paper is organized as follows: In Section 2 we consider the simple case $\alpha = 0$ in the absence and in the presence of dilution. Sections 3 and 4 discuss the cases $\alpha \in (0, 1)$ and $\alpha = 1$ respectively. Finally, our conclusions are in Section 5.

2. Mean-field ($\alpha = 0$)

The percolation order parameter P_∞ is defined as the fraction of nodes of the system that belongs to the infinite cluster. Obviously, P_∞ attains its maximum value ($P_\infty = 1$) when all the nodes are in the infinite cluster, whereas $P_\infty = 0$ below a certain threshold, when the infinite cluster is absent.

A particularly simple model is the mean-field, which corresponds to $\alpha = 0$. We describe below this almost trivial case, first when only bonds are diluted, and afterwards considering dilution for both nodes and bonds.

2.1. Mean-field (bond diluted)

In mean-field bond diluted model ($\alpha = 0,\ p_s = 1$), one assumes that there are N nodes. Any pair of nodes is connected (closed bond) with probability ρ/N and unconnected (open bond) with probability $1 - \rho/N$.

The average connectivity γ of a given node (the average number of connections of a node to the remaining $N - 1$ nodes) is given by

$$\gamma = \frac{\rho}{N}(N - 1) \simeq \rho. \tag{1}$$

This number is simply obtained by multiplying the number $N - 1$ of remaining nodes by the probability that a bond is closed.

Let us call P_∞ the fraction of nodes in the giant component (number of nodes in the giant component divided by the total number of nodes N), which can also be seen as the probability that a node belongs to the giant component itself. The order parameter P_∞ satisfies the self-consistency equation (see, for example, Ref. [22,23])

$$\exp(-\gamma P_\infty) = 1 - P_\infty, \tag{2}$$

whose solution $P_\infty(\gamma)$ is depicted in Fig. 1. The critical value of the control parameter is $\gamma_c = 1$.

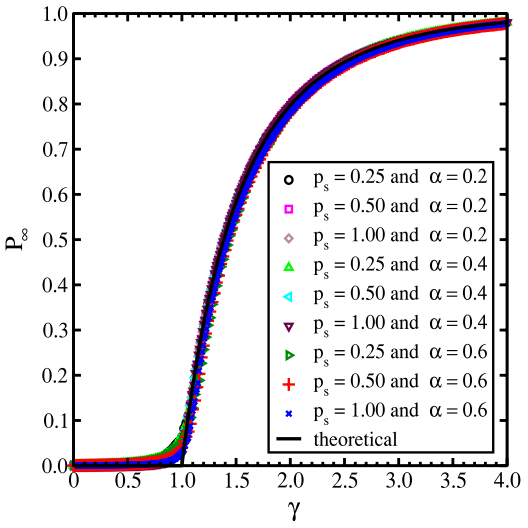

Fig. 1. (Color online) Order parameter P_∞ versus the control parameter $\gamma(\rho, \alpha, p_s)$. For a given choice of the parameters α and p_s, γ only depends on ρ. The figure shows a complete collapse, i.e. the shape is the same for any choice of α and p_s, and coincides with the well-known mean-field result.

2.2. Mean-field (bond and node diluted)

This model has a distribution of bonds as the previous model but node dilution is introduced ($\alpha = 0,\ p_s \in (0, 1)$). It is assumed that a site is active (node is present) with probability p_s, and inactive (node is absent) with probability $1 - p_s$. The number N_s of nodes (active sites) is about $p_s N$.

The average connectivity γ of a given node is

$$\gamma = \frac{\rho}{N}\,(N_s - 1) \simeq \rho\, p_s. \tag{3}$$

This number is simply obtained by multiplying the number $N_s - 1 \simeq p_s\, N$ of remaining active sites with the probability ρ/N that a bond is closed.

It is easy to show that the size $P_\infty(\gamma)$ of the giant component is still the function of the average connectivity given by Eq. (2). In fact, it is sufficient to remark that the size of the giant component is, by definition, the number of active sites in the giant component divided by the total number of active sites. Therefore, it is enough to consider a system composed only by active sites (whose number N_s is about $p_s\, N$) which are connected with probability $\rho/N \simeq \rho p_s/N_s$. In this way, we are re-conduced to the previous model with the difference that ρ and N are replaced by $\rho\, p_s$ and N_s in Eq. (1). Observe that the average connectivity was $\gamma = \rho$ in previous model, while now $\gamma = \rho\, p_s$. So, Eq. (2) must hold also for the present model with γ given by (3).

Notice that γ increases linearly with p_s, while P_∞ is a non-decreasing function of γ. Therefore, dilution decreases the value of P_∞ as expected.

3. Power-law model ($\alpha \in (0, 1)$)

3.1. Definitions

Here we consider a one-dimensional (periodic chain) problem where sites are active according to a given rate $p_s \in (0, 1]$. Two nodes are connected (closed bonds) at a power-law probability depending on their distance.

Assuming that i is the position of a node on the chain, periodic boundary conditions imply that the node i coincides with node $i + N$. Furthermore, given the periodic boundary conditions, the distance r_{ij} between two nodes i and j is $r_{ij} = |i - j|$ if $1 \le |i - j| \le N/2$ and $r_{ij} = N - |i - j|$ if $N/2 < |i - j| < N$.

Therefore, we assume that two active sites i and j of the chain are connected by a closed bond depending upon their distance r_{ij} according to the probability $P(r_{ij})$ which obeys a power-law, i.e.

$$p_{ij} = \frac{\rho}{(r_{ij})^\alpha\, N^{1-\alpha}}, \tag{4}$$

with $\alpha \in (0, 1)$. According to the above prescription, nearest neighbors sites (when both are active) are connected with probability $\rho/N^{1-\alpha}$, thus one has to assume $\rho \in (0, N^{1-\alpha}]$, which reduces to $\rho \in (0, \infty)$ when the thermodynamic limit is reached. Notice that $N^{1-\alpha}$ in the denominator is necessary, otherwise the average connectivity would explode with N

and the correct thermodynamic limit would not be reached. This is the equivalent of the N in the denominator of the bonds probability in the mean-field case $\alpha = 0$.

The exponent $\alpha \in (0, 1)$ controls the range of the network connections, $\rho \in (0, \infty)$ the number of closed bonds and $p_s \in (0, 1]$ the number of active sites.

It must be mentioned that the limiting case $p_s = 1$ (absence of node dilution) was firstly considered in the pioneering work [12].

3.2. Solution

Given a node at position i, the joint probability that there is a node at position j and that it is connected to node in i is

$$\frac{\rho\, p_s}{r_{ij}^{\alpha}\, N^{1-\alpha}}, \tag{5}$$

where r_{ij} is the distance of the two nodes.

Since we assume periodic boundary conditions, there are two nodes at any given distance $1 \leq r \leq N/2$. Therefore, the average connectivity $\gamma = \gamma(\rho, \alpha, p_s)$ is obtained by the sum

$$\gamma = \sum_{r=1}^{N/2} 2\, \frac{\rho\, p_s}{r^{\alpha}\, N^{1-\alpha}} \simeq \frac{2^{\alpha}}{1-\alpha}\, \rho\, p_s, \tag{6}$$

where we have neglected terms which vanish in the thermodynamical limit as $1/N^{1-\alpha}$. We stress again that γ remains finite in the thermodynamic limit due to the $N^{1-\alpha}$ in the denominator of (4). Also we remark that (6) reduces to (3) when $\alpha \to 0$.

The average distance d between two active connected sites of the system is a fraction of the system size. In fact:

$$d = \frac{1}{\gamma} \sum_{r=1}^{N/2} 2\, r\, \frac{\rho\, p_s}{r^{\alpha}\, N^{1-\alpha}} \simeq \frac{1-\alpha}{2\,(2-\alpha)}\, N. \tag{7}$$

Given that P_∞ is the ratio between the number of active sites in the giant component and the total number of active sites, the aim of our numerical work is to show that $P_\infty(\gamma)$ is still given by the solution of Eq. (2), provided that γ is now given by (6).

In practice, for any value of α, ρ and p_s one should compute numerically P_∞ and plot against $\gamma = \gamma(\alpha, \rho, p_s)$ given by (6). Once obtained a curve, one should compare it with the mean-field curve given by Eq. (1). In Fig. 1 we show that, indeed, the shape of $P_\infty(\gamma)$ is the same independently on the numerical parameters, and coincides with the well known mean-field result.

For a given choice of the parameters α and p_s, γ only depends on ρ. Therefore, we have considered various values of p_s and α and plotted $P_\infty(\gamma)$ with respect to γ. For all cases we have considered a system of $N = 10{,}000$ sites, and we have obtained P_∞ as an average over 500 different independent realizations of the network.

Fig. 1 only shows results up to $\alpha = 0.6$ because for larger values of α, a size $N = 10{,}000$ is not enough to reach the thermodynamical limit. This may result counterintuitive, but consider that the next to the leading term in Eq. (6) is of order $1/N^{1-\alpha}$ whose power-law convergence to zero is slower as α approaches 1. Thus, a slowing-down of the convergence can be also expected for $P_N(\gamma)$.

In fact, we are able to show, by a scaling analysis, that the mean-field value is anyway reached in the $N \to \infty$ limit. This can be seen in Fig. 2 where the difference $P_N(\gamma) - P_\infty(\gamma)$ is plotted against N in a log–log scale for the case $\alpha = 0.8$. The function $P_\infty(\gamma)$ is the value calculated analytically, while $P_N(\gamma)$ is the value obtained by a simulation of a network of size N. We observe that the difference $P_N(\gamma) - P_\infty(\gamma)$ converges as a power-law to zero ($P_N(\gamma) - P_\infty(\gamma) \sim 1/N^q$), confirming our data collapse. In Fig. 2 we considered four choices for the parameters ρ and p_s, both upper lines correspond to $\gamma = 1.5$ while both lower lines $\gamma = 3.5$. In all cases, the values of $P_N(\gamma)$ are obtained as an average over 500 different independent realizations of the network, with N ranging from 1000 to 64,000.

4. The case $\alpha = 1$

When $\alpha = 1$ everything goes as before but scaling $1/\ln(N)$ replaces scaling $1/N^{1-\alpha}$ in Eq. (4), i.e.:

$$p_{ij} = \frac{\rho}{r_{ij}\, \ln(N)}. \tag{8}$$

Then, the average connectivity is

$$\gamma = \sum_{r=1}^{N/2} 2\, \frac{\rho\, p_s}{r\, \ln(N)} \simeq 2\, \rho\, p_s. \tag{9}$$

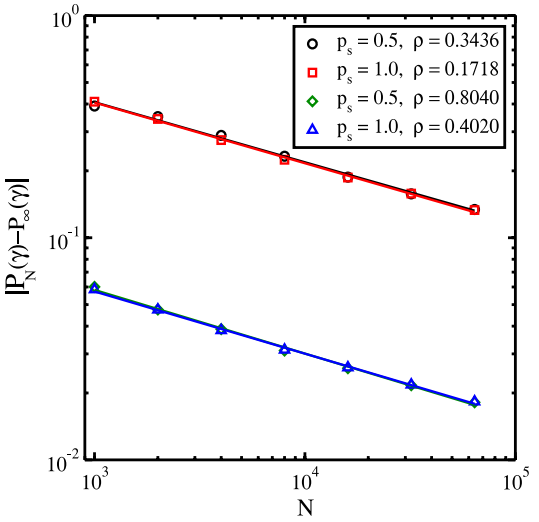

Fig. 2. (Color online) The function $P_N(\gamma) - P_\infty(\gamma)$ plotted against N in a log–log scale for the case $\alpha = 0.8$.

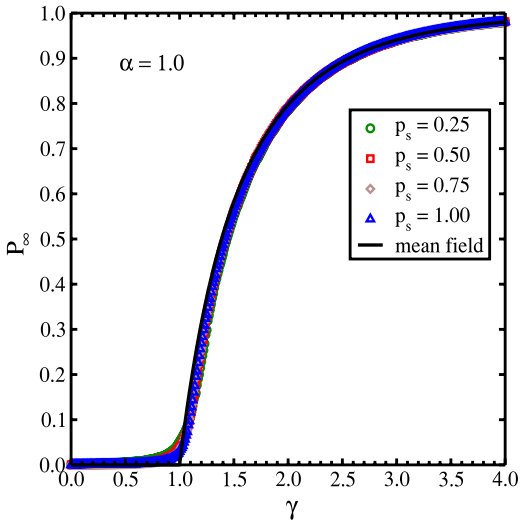

Fig. 3. (Color online) Order parameter P_∞ versus the control parameter $\gamma(\rho, p_s)$ for $\alpha = 1$. For a given choice of the parameter p_s, γ, as in Fig. 1, only depends on ρ. The figure shows again a complete collapse.

We remark that now, at variance with the case $\alpha \in (0, 1)$, the average distance d between two active connected sites is not of the order of the size of the system. In fact:

$$d = \frac{1}{\gamma} \sum_{r=1}^{N/2} 2\, \frac{\rho\, p_s}{\ln(N)} \simeq \frac{1}{2}\, \frac{N}{\ln(N)}, \tag{10}$$

nevertheless, the system still has mean-field properties.

We have checked numerically that $P_\infty(\gamma)$ given by the solution of Eq. (2) is still valid, provided γ is given by (9), as it is shown in Fig. 3, where $P_\infty(\gamma)$ is plotted against γ. Considering $\alpha = 1$, for a given choice of the parameter p_s, γ only depends on ρ. We have taken into account various values of p_s and, for all cases, a system with $N = 10{,}000$ sites. P_∞ is again obtained as an average over 500 different independent realizations of the network.

5. Discussion

We have shown that very long-range percolation ($\alpha \in [0, 1]$), with both node and bond dilution on a linear chain behaves as in the mean-field theory if P_∞ is expressed in terms of the average connectivity γ of a node. In other words, all data collapse on the same universal curve if P_∞ is plotted against $\gamma = \gamma(\alpha, \rho, p_s)$. We stress, that this result holds for any possible bond dilution and, more important, for any node dilution.

Noticeably, collapse is absent when the parameter α is larger then unity [12,19,20]. In this case, in fact, connectivity is not an exhaustive description of the topology of the system, as it can be easily understood. If $\alpha > 1$ one can assume $p_{ij} = \rho/r_{ij}^{\alpha}$ (no rescaling with the system size is necessary now) and then compute the average connectivity

$$\gamma = \sum_{r=1}^{N/2} 2\,\rho\,p_s/r^{\alpha} \simeq 2\,\rho\,p_s\zeta(\alpha), \tag{11}$$

where $\zeta(\alpha) = \sum_{r=1}^{\infty} r^{-\alpha}$ is the zeta Riemann function. In this $\alpha > 1$ case, at variance with the case $\alpha \in [0, 1]$, the probability of connections to nodes at a distance of order unity remains finite in the thermodynamic limit.

This implies that local fluctuations of connectivity and site activity may prevent the emergence of a giant component even when $\gamma > 1$. For example, in the limit $\alpha \to \infty$, which corresponds to the first-neighbors percolation on a linear chain, whenever $\rho < 1$ or $p_s < 1$ percolations is forbidden by a single missing bond or node. On the contrary, we have shown in this paper that spatial structures are irrelevant when $\alpha \in [0, 1]$, being the average connectivity the only relevant aspect for all possible values of the parameters.

We would like to finally stress that while we are always able to explicitly compute γ in terms of α, p_s and ρ for all possible range of these three parameters, the case $\alpha = 0$ is the only one which we are able to completely treat analytically. Our conclusions concerning the region $\alpha \in (0, 1]$ are mainly based on numerical simulations. Our results are very precise and hopefully correct, although a rigorous mathematical proof of our conclusions still remains open.

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